A STUDY ON DIFFERENTIAL GEOMETRY OF RIEMANNIAN MANIFOLD AND ALMOST CONTACT MANIFOLDS

By

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Thesis submitted in fulfillment for the requirement of the Degree of Doctor of Philosophy in Mathematics

To

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CERTIFICATE

This is to certify that the thesis entitled "A STUDY ON DIFFERENTIAL GEOMETRY OF RIEMANNIAN MANIFOLD AND ALMOST CONTACT MANIFOLDS " submitted by Archana Singh, for the degree of Doctor of Philosophy of the Mizoram University, Aizawl, embodies the record of original investigations carried out by her under my supervision. she has been duly registered and the thesis presented is worthy of being considered for the award of the Ph.D degree. This work has not been submitted for any degree of any other universities.

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CANDIDATE'S DECLARATION

I, Archana Singh, hereby declare that the subject matter of this thesis entitled "A STUDY ON DIFFERENTIAL GEOMETRY OF RIEMANNIAN MANIFOLD AND ALMOST CONTACT MANIFOLDS " is the record of work done by me, that the contents of this thesis do not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in other University/Institute.

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PREFACE

The present thesis entitled "A STUDY ON DIFFERENTIAL GEOMETRY OF RIEMANNIAN MANIFOLD AND ALMOST CONTACT MANIFOLDS" is an outcome of the researcher carried out by the author under the supervision of Dr. Jay Prakash Singh, Department of Mathematics & Computer science, Mizoram University, Aizawl, Mizoram and Dr. Rajesh Kumar, Department of Mathematics, Pachhunga University College, Aizawl, Mizoram.

This thesis has been divided into six chapters and each chapter is subdivided into a number of articles. The first chapter is introductory in which we have defined Differentiable manifolds, Tangent Vector, Tangent space and Vector field, Tensors, Lie-bracket, Covarient derivatives, Lie derivative and Exterior derivatives, Connection, Riemannian manifolds, Torsion tensor, Ricci tensor, Curvature tensors on Riemannian manifolds, Almost contact manifold, Almost paracontact metric manifold, Lorentzian paracontact manifold, Lorentzian α-Sasakian manifold Submanifold, Almost r-paracontact Submanifold.

The second chapter is related with the characterization of some curvature conditions on LP-Sasakian manifold admitting a quarter symmetric non-metric connection. In this chapter we have studied an LP-Sasakian manifold admitting a quarter symmetric non-metric connection satisfying $\overline{L} \cdot \overline{S} = 0$ is an η-Einstein manifold. We also prove that an n-dimensional LP-Sasakian manifold is ξ-conharmonically flat with respect to the quarter symmetric non-metric connection if and only if the manifold is also ξ-conharmonicaly flat with respect to the Riemannian connection provided the vector fields X and Y are horizontal vector fields. We also discussed Projective Ricci tensor with respect to quarter symmetric non-metric connection ∇ in an LP-Sasakian manifold.

The third chapter deals with study of LP-Sasakian manifolds. In this chapter we have studied certain curvature conditions on LP-Sasakian manifolds and obtained some interesting results.

The fourth chapter we have discussed semi-generalized Concircularly and M-projectively recurrent manifolds and obtained some interesting results. Semi-generalized recurrent Lorentzian α-Sasakian manifolds and Semi-generalized , semi-generalized φ-recurrent Lorentzian α-Sasakian manifolds and semi-generalized φ-recurrent P-Sasakian manifolds are also discussed in this chapter.

In the fifth chapter we studied the almost r-paracontact structure and obtained the several results. We have shown that the M^{n-1} be a submanifold tangent to the structure vector field $\widetilde{\xi}_\alpha$ of an almost r-paracontact metric manifold M^{n+1}. If M^{n-1} is totally umbilical then M^{n-1} is totally geodesic.

The last chapter is summary and conclusion.

In the end, the references of the papers of the authors have been given with surname of the author and their years of the publication, which are decoded in chronological order in the Bibliography.

A good portion of present thesis has been already published in National/International journals. A brief account of published chapters is given in the list of publications.

Contents

Chapter 1

Introduction

1.1 Differentiable Manifold

A topological space M^n is said to be a differentiable manifold of dimension n if it satisfies the following:

(1) M^n is a Housdorff space,

(2) each point $x \in M^n$ has a neighbourhood U which is homeomorphic to an open subset V of R^n, i.e., M^n is locally Euclidean,

(3) M^n is a second countable space i.e., M^n has a countable basis of open sets,

(4) M^n is endowed with a collection $\{(U_\alpha, \varphi_\alpha) : \alpha \in \Lambda\}$ of coordinate charts such that

(i) $\cup_{\alpha \in \Lambda} U_\alpha = M^n$ i.e., $\{U_\alpha : \alpha \in \Lambda\}$ covers M^n,

(ii) the mapping $\varphi_\alpha o \varphi_\beta^{-1}$ or $\varphi_\beta o \varphi_\alpha^{-1}$ is C^∞ for all $\alpha, \beta \in \Lambda$ and

(iii) if (U, φ) is any other coordinate charts for which $\varphi o \varphi_\alpha^{-1}$ and $\varphi_\alpha o \varphi^{-1}$ are C^∞, then $(U, \varphi) \in \{(U_\alpha, \varphi_\alpha) : \alpha \in \Lambda\}$.

The properties (1)-(3) are the conditions of a topological manifold while the property (4) (with (i)-(iii)) is the condition of a differentiable structure. If M^n has a C^∞- differentiable structure, then M^n is called an analytic manifold.

1.2 Tangent Vector, Tangent Space and Vector Field

Tangent Vector: Let M^n be an n-dimensional differentiable manifold and $p \in M^n$ and $C^\infty(p)$ be the set of all real valued C^∞ function on some neighbourhood $\cup$ of p. Let us consider a vector X at p such that
(i) $X \in M^n$, $f \in C^\infty(p)$ implies that $Xf \in C^\infty(p)$,
(ii) $X(f+g) = Xf + Xg$, $\qquad f, g \in C^\infty(p)$,
(iii) $X(fg) = f(Xg) + (Xf)g$, and
(iv) $X(af) = a(Xf)$, $a \in \mathbb{R}$,
then X is called a tangent vector to M^n at p.

Tangent Space: The set of all tangent vectors at p with operation addition '+' satisfying

$$(X+Y)f = Xf + Yf \tag{1.2.1}$$

and an operation of scalar multiplication '$\cdot$' satisfying

$$(f \cdot X)g = f \cdot (Xg) \tag{1.2.2}$$

is a vector space and this space is called the tangent space to M^n at the point p and is denoted by $T_{(p)}$. The basis of $T_{(p)}$ with respect to coordinate system $(x^1, x^2, ...x^n)$ is $(\frac{\partial}{\partial x^i}), i = 1, 2, ...n$.
Let $T^*_{(p)}$ be the dual space of $T_{(p)}$ whose basis with respect to the basis $(\frac{\partial}{\partial x^i})$ is $(dx^1, dx^2, ...dx^n)$. We observe that the elements of $T_{(p)}$ are the contravariant vectors and elements of $T^*_{(p)}$ are the covariant vectors with respect to the basis of $T_{(p)}$.

Vector Field: A vector field X on a set A is a mapping that assigns to each $p \in A$ to a vector X_p in $T_{(p)}$. A vector field X is C^∞ on A, if
(i) A is open, and
(ii) The function Xf at p is C^∞ on $A \cap M^n$, f being C^∞ real valued function on M^n.

1.3 Tensors

We consider an n-dimensional C^∞ manifold M^n. Let p be a point in M^n. A tensor of the type (r, s) at p is $(r+s)$- linear real valued function on $(T_{(p)})^r \otimes (T^*_{(p)})^s$ and vector space of these tensors is denoted by $T^r_{(p)s}$.

A tensor Q of type (r, o) is said to be symmetric in the h^{th} and k^{th} places if

$$S_{h,k}(Q) = Q \tag{1.3.1}$$

and skew-symmetric in the h^{th} and k^{th} places if

$$S_{h,k}(Q) = -Q \tag{1.3.2}$$

where $(1 \leq h \leq k \leq r)$ and $S_{h,k}$ is a linear mapping which interchanges the vectors at the h^{th} and k^{th} places in the tensor product of the r-covariant vectors.
A tensor Q of type (r, o) is said to be symmetric if (1.3.1) hold, for all indices h and k and it is said to be skew symmetric if (1.3.2) holds, for all indices h and k.

1.4 Contracted Tensors

The linear mapping

$$C_k^h : T_s^r \rightarrow T_{s-1}^{r-1} \qquad (i \leq h \leq r \qquad i \leq k \leq s)$$

such that

$$\begin{aligned} C_k^h(\lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_r \otimes \alpha_1 \otimes \cdots \otimes \alpha_s) &= \alpha_k(\lambda_1 \otimes \cdot \cdot \otimes \lambda_{h-1} \otimes \lambda_{h+1} \cdots \otimes \lambda_r \\ &\quad \otimes \alpha_1 \otimes \alpha_2 \otimes \cdot \cdot \alpha_{k-1} \otimes \lambda_{k+1} \otimes \cdot \cdot \alpha_s). \end{aligned}$$

where $\lambda_1, \lambda_2 ... \lambda_r \in T_{(p)}$ and $\alpha_1, \alpha_2 .... \alpha_s \in T^*_{(p)}$ and $\otimes$ denote tensor product, is called contraction with respect to h^{th} contravariant and k^{th} covariant places. A tensor obtained after contraction is called a contracted tensor.

1.5 Lie-Bracket and Covariant Derivative

Lie-Bracket: Let X and Y be arbitrary C^∞ vector field of M^n, then their Lie-bracket is a mapping $[\quad , \quad] : M^n \times M^n \rightarrow M^n$ such that

$$[X, Y]f = X(Yf) - Y(Xf), \tag{1.5.1}$$

where f is a C^∞-function. The Lie-bracket has the following properties:

$$[X, Y](f_1 + f_2) = [X, Y]f_1 + [X, Y]f_2, \tag{1.5.2}$$

$$[X,Y](f_1 \cdot f_2) = f_1[X,Y]f_2 + f_2[X,Y]f_1, \tag{1.5.3}$$

$$[X,Y] + [Y,X] = 0, \qquad \text{(skew-symmetry)} \tag{1.5.4}$$

$$[X+Y,Z] = [X,Z] + [Y,Z], \qquad \text{(bilinear)} \tag{1.5.5}$$

$$[X,[Y+Z]] + [Y,[Z+X]] + [Z,[X,Y]] = 0, \text{(Jacobian identity)} \tag{1.5.6}$$

and

$$[f_1X, f_2Y] = f_1f_2[X,Y] + f_1(Xf_2)Y - f_2(Yf_1)X. \tag{1.5.7}$$

Covariant Derivative: A linear affine connection on M^n is a function

$$\nabla : T_p(M^n) * T_p(M^n) \to T_p(M^n) \text{ such that}$$

$$\nabla_{fX+gY}Z = f(\nabla_X Z) + g(\nabla_Y Z), \tag{1.5.8}$$

$$\nabla_X f = Xf, \tag{1.5.9}$$

$$\nabla_X(fY + gZ) = f(\nabla_X Y) + g(\nabla_X Z) + (Xf)Y + (Xg)Z, \tag{1.5.10}$$

for arbitrary vector fields X, Y, Z and smooth function f, g on M^n. ∇_X is called covariant derivative operator and $\nabla_X Y$ is called covariant derivative of Y with respect to X.
The covariant derivative of a 1-form w is given by

$$(\nabla_X w)(Y) = X(w(Y)) - w(\nabla_X Y).$$

1.6 Lie Derivative and Exterior Derivative

Lie Derivative: Let X be a C^∞ vector field on an open set A of M^n. An operator L_X is called the Lie derivative along the vector field X if it is a type preserving mapping

$L_X : T_s^r \to T_s^r$, such that

$$L_X f = Xf, \qquad f \in F, \tag{1.6.1}$$

$$L_X a = 0, \qquad a \in \mathbb{R}, \tag{1.6.2}$$

$$L_X Y = [X, Y], \qquad X, Y \in T_{(p)}, \tag{1.6.3}$$

$$(L_X A)(Y) = X(A(Y)) - A([X, Y]), \qquad A \in T^*_{(p)} \tag{1.6.4}$$

and

$$\begin{aligned}(L_X P)(A_1, ...A_r, X_1, ...X_s) &= X(P(A_1, ...A_r, X_1, ...X_s)) \\ &- P(L_X A_1, ...A_r, X_1, ...X_s).... \\ &- P(A_1, ...A_r, [X, X_1], X_2.....X_s)..... \\ &\quad .... \\ &- P(A_1..........X_{s-1}, [X, X_s]), P \in T_s^r,\end{aligned} \tag{1.6.5}$$

where f is a C^∞ function, $X_1, ...X_s$ are vector fields, $A_1, ...A_r$ are 1-forms and P is a tensor field of type (r, s), is called Lie differentiation with respect to X and $L_X P$ is called Lie derivative of P with respect to X.

Exterior Derivative: Let V_p be the set of all C^∞ p-forms on an open set A. Then the mapping $d : V_p \to V_{p+1}$ given by

$$(df)(X) = Xf, \qquad X \in T_{(p)}, \qquad f \in F \tag{1.6.6}$$

and

$$\begin{aligned}(dA)(X_1, ..., X_{p+1}) &= X_1 (A(X_2, ..., X_{p+1})) \\ &- X_2 (A(X_1, X_3, ...X_{p+1})) \\ &+ X_3 (A(X_1, X_2, X_4, ..., X_{p+1})) .... \\ &- A([X_1, X_2], X_3, ..., X_{p+1})\end{aligned}$$

$$\begin{aligned} &+ \quad A([X_1, X_3], X_2, X_4, ..., X_{p+1}) \\ &- \quad A([X_2, X_3], X_1, X_4, ..., X_{p+1}) + ... \end{aligned} \tag{1.6.7}$$

for arbitrary C^∞ vector fields $X'^s \in V^1$ and $A \in V_p$, is called the exterior derivative.

1.7 Connection

Let us consider a C^∞-manifold M^n and $p \in M^n$ be a point of M^n. Let $T_(p)$ be a tangent space to M^n at the point p. Let $T^r_{(p)s}$ be a vector space whose elements are the tensors of the type (r, s). A connection ∇ is a type preserving mapping $\nabla : T_{(p)} \otimes T^r_s \to T^r_s$, which assigns to each pair of C^∞-vector field (X, P), $X \in T_p, P \in T^r_s$, a C^∞-vector fields $\nabla_X P$, such that

$$\nabla_X f = Xf, \qquad f \quad \text{is} \quad C^\infty\text{-function} \tag{1.7.1}$$

$$\nabla_X a = 0, \qquad a \in \mathbb{R}, \tag{1.7.2}$$

$$\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z, \tag{1.7.3}$$

$$\nabla_X(fY) = f(\nabla_X Y) + (Xf)Y, \tag{1.7.4}$$

$$\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z, \tag{1.7.5}$$

$$\nabla_{fX} Y = f(\nabla_X Y), \tag{1.7.6}$$

$$(\nabla_X A)Y = X(A(Y)) - A(\nabla_X Y), \tag{1.7.7}$$

and

$$\begin{aligned} (\nabla_X P)(A_1, ..., A_r, X_1, ...X_s) &= X(P(A_1, ...A_r, X_1, ...X_s)) \\ &- P(\nabla_X A_1, A_2, ...A_r, X_1, ...X_s)... \\ &- P(A_1......A_r, X_1, ...\nabla_X X_s). \end{aligned} \tag{1.7.8}$$

1.8 Riemannian Manifold

Let us consider a C^∞ real valued, bilinear symmetric, non-singular positive definite function g on the ordered pair $X, Y \in T_(p)$ at a point $p \in M^n$, such that

$$g(X,Y) \text{ is a real number,} \tag{1.8.1}$$

$$g \text{ is symmetric} \Rightarrow g(X,Y) = g(Y,X),$$

$$g \text{ is non-singular i.e. } g(X,Y) = 0, \forall Y \neq 0 \Rightarrow X = 0.$$

$$g \text{ is positive definite i.e. } g(X,Y) > 0, \forall X \neq 0.$$

and

$$g(X+Y,Z) = g(X,Z) + g(Y,Z),$$

$$g(aX,Z) = ag(X,Z), a \in \mathbb{R}$$

then g is said to be Riemannian metric tensor.

The manifold M^n with a Riemannian metric is called a Riemannian manifold and its geometry is called a Riemannian geometry.

1.9 Torsion Tensor

A vector valued, skew-symmetry, bilinear function T of the type $(1,2)$ defined by

$$T(X,Y) \overset{def}{=} \nabla_X Y - \nabla_Y X - [X,Y] \tag{1.9.1}$$

is called a torsion tensor of the connection ∇ in a C^∞-manifold M^n.
If the torsion tensor of a connection ∇ vanishes, it is said to be symmetric or torsion free.
A connection ∇ is said to be Riemannian, if

$$T(X,Y) = 0 \tag{1.9.2}$$

and

$$\nabla_X g = 0. \tag{1.9.3}$$

1.10 Riemannian Connection

A connection D is said to be Riemannian connection if
(i) It is symmetric i.e. $D_X Y - D_Y X = [X, Y]$,
(ii) g is covariant constant with respect to D i.e.

$$Xg(Y, Z) = g(D_X Y, Z) + g(Y, D_X Z). \tag{1.10.1}$$

Hence, we can say that a linear connection is symmetric and metric if and only if it is the Riemannian connection.

1.11 Quarter Symmetric Non-Metric Connection

Golab, (1975) defined and studied quarter symmetric connection in a differentiable manifold with affine connection as a linear connection ∇ on an n-dimensional Riemannian manifold (M^n, g) is called a quarter symmetric connection if its torsion tensor T of the connection ∇

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \tag{1.11.1}$$

satisfies

$$T(X, Y) = \eta(Y)\varphi X - \eta(X)\varphi Y, \tag{1.11.2}$$

where η is 1-form and φ is a (1,1) tensor field. In particular, if $\varphi(X) = X$, then the quarter symmetric connection reduces to a semi symmetric connection. Thus the notion of quarter symmetric connection generalizes the notion of semi symmetric connection.
Moreover, if a quarter symmetric connection ∇ satisfies the condition

$$(\nabla_X g)(Y, Z) = 0, \tag{1.11.3}$$

for all $X, Y, Z \in T_p(M^n)$, where $T_p(M^n)$ is the Lie algebra of vector fields of the manifold M^n, then ∇ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter symmetric non-metric connection.

1.12 Curvature Tensor

The curvature tensor R of type (1,3) with respect to the Riemannian connection ∇ is defined by the mapping

$$R : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \longrightarrow T_p(M^n)$$

given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1.12.1}$$

for all $X, Y, Z \in T_p(M^n)$.

Let $'R$ be the associative curvature tensor of the type $(0,4)$ of the curvature tensor R. Then

$$'R(X,Y,Z,W) = g(R(X,Y,Z)W), \tag{1.12.2}$$

$'R$ is called the Riemann-Christoffel curvature tensor of first kind.
The following identities are satisfied by associative curvature tensor $'R$:
$'R$ is skew-symmetric in first two slot

$$i.e., \quad 'R(X,Y,Z,W) = -\ 'R(Y,X,Z,W) \tag{1.12.3}$$

$'R$ is skew-symmetric in last two slot

$$i.e., \quad 'R(X,Y,Z,W) = -\ 'R(X,Y,W,Z) \tag{1.12.4}$$

$'R$ is symmetric in two pair of slot

$$i.e., \quad 'R(X,Y,Z,W) = \ 'R(Z,W,X,Y) \tag{1.12.5}$$

$'R$ satisfies Bianchi's first identities

$$i.e., \quad 'R(X,Y,Z,W) + \ 'R(Y,Z,X,W) + \ 'R(Z,X,Y,W) = 0 \tag{1.12.6}$$

and $'R$ satisfies Bianchi's second identities

$$i.e., \ (\nabla_X\ 'R)(Y,Z,W,V) + (\nabla_Y\ 'R)(Z,X,W,V) + (\nabla_Z\ 'R)(X,Y,W,V) = 0. \tag{1.12.7}$$

1.13 Ricci-Tensor

Let M^n is a Riemannian manifold with a Riemannian connection ∇. Then the Ricci tensor field S is the covariant tensor field of degree 2 defined as $Ric(Y, Z) = S(Y, Z) =$ Trace of the linear map $X \to R(X, Y)Z$ for all $X, Y, Z \in T_p(M^n)$.

If $\{e_1, ..., e_n\}$ is an orthonormal basis of the tangent space $T_p, p \in M^n$ and R is the Riemannian curvature tensor of the Riemannian manifold (M^n, g), then

$$S(X, Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i) \tag{1.13.1}$$

$$= \sum_{i=1}^{n} R(e_i, X, Y, e_i) \tag{1.13.2}$$

$$= \sum_{i=1}^{n} R(X, e_i, e_i, Y) = g(R(X, e_i)e_i, Y),$$

where R is the Riemannian curvature tensor of the manifold of type $(0, 4)$. The linear map Q of the type $(1, 1)$ defined by

$$g(QX, Y) \overset{def}{=} S(X, Y) \tag{1.13.3}$$

is called a Ricci-map. It is self-adjoint,

$$i.e., \quad g(QX, Y) = g(X, QY). \tag{1.13.4}$$

The scalar r defined by

$$r \overset{def}{=} (C_1^1 R) \tag{1.13.5}$$

is called the scalar curvature of M^n at the point p.

A Riemannian manifold M^n is said to be Einstein manifold, if

$$S(X, Y) = \frac{r}{n} g(X, Y). \tag{1.13.6}$$

A Riemannian manifold M^n is said to be η-Einstein manifold, if

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y). \tag{1.13.7}$$

where α and β are smooth functions.
A Riemannian manifold M^n is said to be flat manifold, if

$$R(X,Y,Z) = 0. \tag{1.13.8}$$

1.14 Important Curvature Tensors on Riemannian Manifold

The concircular curvature tensor $'L$ of type (0,4), is given by

$$\begin{aligned} 'L(X,Y,Z,W) &= \ 'R(X,Y,Z,W) - \frac{r}{n(n-1)}\Big\{g(Y,Z)g(X,W) \\ &- \ g(X,Z)g(Y,W)\Big\} \end{aligned} \tag{1.14.1}$$

It satisfies the following algebraic properties

$$\begin{aligned} &(a)\ 'L(X,Y,Z,W) = -'L(Y,X,Z,W), \\ &(b)\ 'L(X,Y,Z,W) = -'L(X,Y,W,Z), \\ &(c)\ 'L(X,Y,Z,W) = \ 'L(Z,W,X,Y), \\ &(d)\ 'L(X,Y,Z,W) + \ 'L(Y,Z,X,W) + \ 'L(Z,X,Y,W) = 0, \end{aligned}$$

where

$$'L(X,Y,Z,W) = g(L(X,Y,Z),W).$$

The conharmonic curvature tensor $'H$ of the type (0,4), is defined as follows

$$\begin{aligned} 'H(X,Y,Z,W) &= \ 'R(X,Y,Z,W) - \frac{1}{n-1}\Big\{S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &+ \ S(X,W)g(Y,Z) - S(Y,W)g(X,Z)\Big\} \end{aligned} \tag{1.14.2}$$

It satisfies the following properties

$$(a)\ 'H(X,Y,Z,W) = -\ 'H(Y,X,Z,W),$$

(b) $'H(X,Y,Z,W) = \, 'H(X,Y,W,Z)$,

(c) $'H(X,Y,Z,W) = \, 'H(Z,W,X,Y)$,

(d) $'H(X,Y,Z,W) + \, 'H(Y,Z,X,W) + \, 'H(Z,X,Y,W) = 0$

where

$$'H(X,Y,Z,W) = g(H(X,Y,Z),W).$$

The projective curvature tensor $'P$ of the type $(0,4)$, is defined by

$$\begin{aligned}'P(X,Y,Z,W) &= \, 'R(X,Y,Z,W) - \frac{1}{n-1}\Big\{S(Y,Z)g(X,W) \\ &- S(X,Z)g(Y,W)\Big\}. \end{aligned} \tag{1.14.3}$$

The projective curvature tensor $'P$ satisfies the following identities

(a) $'P(X,Y,Z,W) = - \, 'P(Y,X,Z,W)$,

(b) $C_1^1 P = C_2^1 P = C_3^1 P = 0$,

(c) $'P(X,Y,Z,W) + \, 'P(Y,Z,X,W) + \, 'P(Z,X,Y,W) = 0$,

where

$$'P(X,Y,Z,W) = g(P(X,Y,Z),W).$$

The M-projective curvature tensor $'W^*$ of the type $(0,4)$, is defined by

$$\begin{aligned}'W^*(X,Y,Z,W) &= \, 'R(X,Y,Z,W) - \frac{1}{2(n-1)}\Big\{g(X,W)S(Y,Z) - g(Y,W)S(X,Z) \\ &+ S(X,W)g(Y,Z) - S(Y,W)g(X,Z)\Big\}. \end{aligned} \tag{1.14.4}$$

It satisfies the following algebraic properties

(a) $'W^*(X,Y,Z,W) = \, 'W^*(Z,W,X,Y)$,

(b) $'W^*(X,Y,Z,W) = - \, 'W^*(Y,X,W,Z)$,

(c) $'W^*(X,Y,Z,W) = - \, 'W^*(X,Y,W,Z)$,

(d) $'W^*(X,Y,Z,W) + \, 'W^*(Y,Z,X,W) + \, 'W^*(Z,X,Y,W) = 0$

where

$$'W^*(X,Y,Z,W) = g(W^*(X,Y,Z),W).$$

The conformal curvature tensor C of the type (0,4), is defined as

$$\begin{aligned} 'C(X,Y,Z,W) &= R(X,Y,Z,W) - \frac{1}{(n-2)}\Big[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &+ g(Y,Z)S(X,W) - g(X,Z)S(Y,W)\Big] \\ &+ \frac{r}{(n-1)(n-2)}\Big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\Big]. \end{aligned} \tag{1.14.5}$$

It satisfies the following properties

$$\begin{aligned} &(a)\ 'C(X,Y,Z,W) = -'C(Y,X,Z,W), \\ &(b)\ 'C(X,Y,Z,W) = -'C(X,Y,W,Z), \\ &(c)\ 'C(X,Y,Z,W) = \ 'C(Z,W,X,Y), \\ &(d)\ 'C(X,Y,Z,W) + \ 'C(Y,Z,X,W) + \ 'C(Z,X,Y,W) = 0 \end{aligned}$$

where

$$'C(X,Y,Z,W) = g(C(X,Y,Z),W).$$

Finally the Quasi-conformal curvature tensor $\overline{C}$ of the type (0,4), is defined as

$$\begin{aligned} '\overline{C}(X,Y,Z,W) &= a\ R(X,Y,Z,W) + b\ \Big[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &+ g(Y,Z)S(X,W) - g(X,Z)S(Y,W)\Big] \\ &+ \frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)\ \Big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\Big]. \end{aligned} \tag{1.14.6}$$

It satisfies the following identities

$$\begin{aligned} &(a)\ '\overline{C}(X,Y,Z,W) = -'\overline{C}(Y,X,Z,W), \\ &(b)\ '\overline{C}(X,Y,Z,W) = -'\overline{C}(X,Y,W,Z), \\ &(c)\ '\overline{C}(X,Y,Z,W) = \ '\overline{C}(Z,W,X,Y), \\ &(d)\ '\overline{C}(X,Y,Z,W) + \ '\overline{C}(Y,Z,X,W) + \ '\overline{C}(Z,X,Y,W) = 0 \end{aligned}$$

where

$$'\overline{C}(X,Y,Z,W) = g(\overline{C}(X,Y,Z),W).$$

1.15 Almost Contact Metric Manifold

If M^n be an odd dimensional differentiable manifold on which there are defined a real vector valued linear function φ, a 1-form η and a vector field ξ satisfying for arbitrary vectors $X, Y, Z, ...$

$$\varphi^2 X = -X + \eta(X)\xi, \tag{1.15.1}$$

$$\eta(\xi) = 1, \tag{1.15.2}$$

$$\varphi(\xi) = 0, \tag{1.15.3}$$

$$\eta(\varphi X) = 0, \tag{1.15.4}$$

and

$$rank(\varphi) = n - 1, \tag{1.15.5}$$

is called an almost contact manifold (Sasaki, 1965) and the structure (φ, η, ξ) is called an almost contact structure (Hatakeyama *et al.* (1963); Sasaki and Hatakeyama (1960, 1961)).

An almost contact manifold M^n on which a Riemannian metric tensor g satisfying

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{1.15.6}$$

and

$$g(X,\xi) = \eta(X), \tag{1.15.7}$$

is called an almost contact metric manifold and the structure (φ, ξ, η, g) is called an almost contact metric structure (Sasaki, 1960).

The fundamental 2-form $'F$ of an almost contact metric manifold M^n is defined by

$$'F(X,Y) = g(\varphi X, Y). \tag{1.15.8}$$

From the equations (1.15.6) and (1.15.8), we have

$$'F(X,Y) = -\,'F(Y,X). \tag{1.15.9}$$

If in an almost contact metric manifold

$$2\,'F(X,Y) = (\nabla_X \eta)(Y) - (\nabla_Y \eta)(X). \tag{1.15.10}$$

then M^n is called an almost Sasakian manifold.

1.16 Lorentzian Paracontact Metric Manifold

Let M^n be an n-dimensional differentiable manifold endowed with a tensor field φ of the type (1,1), a vector field ξ, a 1-form η and a Lorentzian metric g satisfying

$$\varphi^2 X = X + \eta(X)\xi, \tag{1.16.1}$$

$$\eta(\xi) = -1, \tag{1.16.2}$$

$$g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{1.16.3}$$

$$g(X,\xi) = \eta(X), \tag{1.16.4}$$

for arbitrary vector field X and Y, then M^n is called a Lorentzian paracontact (LP-Contact) manifold and the structure (φ, ξ, η, g) is called the Lorentzian paracontact structure (Matsumoto, 1989).

Let M^n be a Lorentzian paracontact manifold with stucture (φ, ξ, η, g). Then it satisfy

$$(a)\ \varphi(\xi) = 0, \quad (b)\ \eta(\varphi X) = 0, \quad (c)\ rank(\varphi) = n-1. \tag{1.16.5}$$

A Lorentzian paracontact manifold is called a Lorentzian Para-Sasakian manifold if (Matsumoto and Mihai, 1988)

$$D_X\xi = \varphi X, \tag{1.16.6}$$

$$(D_X\varphi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\,\eta(X)\eta(Y)\xi, \tag{1.16.7}$$

where D denotes the covariant differentiation with respect to g.

Let us put $'F(X,Y) = g(\varphi X, Y)$. Then the tensor field $'F$ is symmetric.

$$i.e.\ \ 'F(X,Y) = \ 'F(Y,X), \tag{1.16.8}$$

and

$$'F(X,Y) = (D_X\eta)(Y). \tag{1.16.9}$$

Also, in an LP-Sasakian manifold the following relation holds

$$'R(X,Y,Z,\xi) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{1.16.10}$$

and

$$S(X,\xi) = (n-1)\eta(X). \tag{1.16.11}$$

A differentiable manifold M of dimension n is said to be a Lorentzian α-Sasakian manifold if it admits a (1,1)-tensor field φ, a vector field ξ, a one form η, and Lorentzian metric g satisfy (Prakasha and Yildiz, 2010)

$$\eta(\xi) = -1, \qquad \varphi\xi = 0, \qquad \eta\circ\varphi = 0, \tag{1.16.12}$$

$$\varphi^2 = I + \eta\otimes\xi. \tag{1.16.13}$$

$$g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{1.16.14}$$

$$g(X,\xi) = \eta(X). \tag{1.16.15}$$

Lorentzian α-Sasakian manifold M^n satisfying the following:

$$(\nabla_X \eta)(Y) = \alpha \; g(\varphi X, Y), \tag{1.16.16}$$

$$\nabla_X \xi = \alpha \; \varphi X, \tag{1.16.17}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is constant.
On Lorentzian α-Sasakian manifolds M^n the following relations hold:

$$\eta(R(X,Y)Z) = \alpha^2 \; [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{1.16.18}$$

$$R(\xi, X)Y = \alpha^2 \; [g(X,Y)\xi - \eta(Y)X], \tag{1.16.19}$$

$$R(X,Y)\xi = \alpha^2 \; [\eta(Y)X - \eta(X)Y], \tag{1.16.20}$$

$$R(\xi, X)\xi = \alpha^2 \; [\eta(X)\xi + X], \tag{1.16.21}$$

$$S(X, \xi) = (n-1) \; \alpha^2 \; \eta(X), \tag{1.16.22}$$

$$Q\xi = (n-1) \; \alpha^2 \xi, \tag{1.16.23}$$

$$S(\varphi X, \varphi Y) = S(X,Y) + (n-1)\alpha^2 \; \eta(X)\eta(Y), \tag{1.16.24}$$

for all vector fields X, Y, Z where S is the Ricci tensor and Q is the Ricci operator given by

$$S(X,Y) = g(QX, Y).$$

An Lorentzian α-Sasakian manifolds M^n is said to be Einstein if its Ricci tensor S is of the form

$$S(X,Y) = \lambda \; g(X,Y), \tag{1.16.25}$$

for any vector fields X and Y, where λ is a function on M^n.

1.17 Almost Paracontact Metric Manifold

Let M^n be an n-dimensional C^∞-manifold. If there exist in M^n a tensor field φ of the type (1,1), consisting of a vector field ξ and 1-form η in M^n satisfying

$$\varphi^2 X = X - \eta(X)\xi, \tag{1.17.1}$$

$$\varphi(\xi) = 0, \qquad \eta(\xi) = 1, \tag{1.17.2}$$

then M^n is called an almost paracontact manifold.
Let g the Riemannian metric satisfying

$$\eta(X) = g(X, \xi), \qquad \eta(\varphi X) = 0, \tag{1.17.3}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.17.4}$$

then the structure (φ, ξ, η, g) satisfying (1.17.1) to (1.17.4) is called an almost paracontact Riemannian structure. The manifold with such structure is called an almost paracontact Riemannian manifold (Sato and Matsumoto, 1976).
If we define $'F(X, Y) = g(\varphi X, Y)$, then the following relations are satisfied

$$'F(X, Y) = \ 'F(Y, X), \tag{1.17.5}$$

and

$$'F(\varphi X, \varphi Y) = \ 'F(X, Y). \tag{1.17.6}$$

If in M^n the relation

$$(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0, \tag{1.17.7}$$

$$d\eta(X, Y) = 0, \quad \textit{i.e. } \eta \textit{ is closed.} \tag{1.17.8}$$

$$(\nabla_X {}'F)(Y,Z) = -g(X,Z)\eta(Y) - g(X,Y)\eta(Z) + 2\,\eta(X)\eta(Y)\eta(Z), \tag{1.17.9}$$

$$(\nabla_X \eta)(Y) + (\nabla_X \eta)(X) = 2\,{}'F(X,Y), \tag{1.17.10}$$

and

$$\nabla_X \xi = \varphi X, \tag{1.17.11}$$

hold good then (M^n, g) is called Para-Sasakian manifold or briefly P-Sasakian manifold.

1.18 Recurrent Manifold

Let M^n be an n-dimensional smooth Riemannian manifold and $T_p(M^n)$ denotes the set of differentiable vector fields on M^n. Let $X, Y \in T_p(M^n)$; $\nabla_X Y$ denotes the covariant derivative of Y with respect to X and $R(X,Y,Z)$ be the Riemannian curvature tensor of type (1,3).

A Riemannian manifold M^n is said be recurrent (Kobayashi and Nomizu, 1963) if

$$(\nabla_U R)(X,Y,Z) = \alpha(U)R(X,Y,Z), \tag{1.18.1}$$

where $X, Y \in T_p(M^n)$ and α is a non-zero 1-form known as recurrence parameter. If the 1-form α is zero in (1.18.1), then the manifold reduces to symmetric manifold (Singh and Khan, 1999).

A Riemannian manifold (M^n, g) is said to be semi-symmetric if it satisfies the relation (Szabo, 1982)

$$(R(X,Y).R)(U,V)W = 0, \tag{1.18.2}$$

where $R(X,Y)$ is considered as the tensor algebra at each point of the manifold i.e. $R(X,Y)$ is curvature transformation or curvature operator.

A Riemannian manifold (M^n, g) is said to be Ricci-recurrent if it satisfies the relation

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) \tag{1.18.3}$$

for all $X, Y, Z \in T_p(M^n)$, where ∇ denotes the Riemannian connection and A is a 1-form on M^n. If the 1-form A vanishes identically on M^n, then a Ricci-recurrent manifold becomes a Ricci-symmetric manifold.

A Riemannian manifold (M^n, g) is called a generalized recurrent manifold (De and Guha, 1991) if its curvature tensor R satisfies the condition:

$$(\nabla_X R)(Y, Z)U = A(X)R(Y, Z)U + B(X)[g(Z, U)Y - g(Y, U)Z]. \qquad (1.18.4)$$

where A and B are two 1-forms, B is non-zero and these are defined by

$$A(X) = g(X, P_1), \qquad B(X) = g(X, P_2), \qquad (1.18.5)$$

P_1 and P_2 are vector fields associated with 1-forms A and B, respectively.

A Riemannian manifold (M^n, g) is called a semi-generalized recurrent manifold (Prasad, 2000) if its curvature tensor R satisfies the condition:

$$(\nabla_X R)(Y, Z)W = A(X)R(Y, Z)W + B(X)g(Z, W)Y, \qquad (1.18.6)$$

where A and B are two 1-forms, B is non-zero and these are given by the equation (1.18.5).

A Riemannian manifold (M^n, g) is said to be φ-recurrent manifold if there exists a non zero 1-form A such that

$$\varphi^2((\nabla_X R)(Y, Z)W) = A(X)R(Y, Z)W, \qquad (1.18.7)$$

for arbitrary vector fields X, Y, Z, W.

A Riemannian manifold (M^n, g) is called generalized φ-recurrent if its curvature tensor R satisfies the condition

$$\begin{aligned} \varphi^2((\nabla_W R)(Y, Z)U) &= A(W)R(Y, Z)U \\ &+ B(W)[g(Z, U)Y - g(Y, U)Z] \qquad (1.18.8) \end{aligned}$$

where A and B are two 1-forms, B is non-zero and these are defined by earlier.

1.19 Submanifold

let M^n be a C^∞-Riemannian manifold. A C^∞ manifold $M^m (m \leq n)$ is called a submanifold of M^n, if for each point in M^n, there is a coordinate neighborhood $\overline{U}$ of M^n with coordinate function $\{y_\alpha : \alpha = 1, 2, ..., n\}$ such that for the set

$$U = \{p \in \overline{U} : y_{m+1} = ... = y_n = 0 \quad at \quad p\}$$

is a coordinate neighborhood of P in M^m with coordinate functions

$$x_i = y_\alpha | U, i = 1, 2, ..., m$$

Let

$$b : M^m \longrightarrow M^n$$

be the inclusion map such that $p \in M^m \Rightarrow bp \in M^n$.

The map b induces a linear transformation B called the Jaccobian map such that

$$b : T_p^m \longrightarrow T_p^n$$

where T_P^m is the tangent space to M^m at point p and T_p^n is the tangent space to M^n at bp, such that

$$X \in M^n \quad at \quad p \Rightarrow BX \in M^n \quad at \quad bp$$

. Let G be the metric tensor at M^n and g the induced metric tensor of M^m at bp relative to the metric tensor G of M^n at bp. Let X, Y be arbitrary vector fields to M^n. Then

$$g(X, Y) = (G(BX, BY)) \circ b. \tag{1.19.1}$$

A C^∞ vector field N of M^n satisfying

$$\begin{aligned} &(a) \quad G(N, BX) \circ b = 0 \\ &(b) \quad G(N, N) \circ b = 1, \end{aligned} \tag{1.19.2}$$

for arbitrary vector field X is called field of normal.

Let $\underset{x}{N}, x = m + 1, ..., n.$ be a system of C^∞-orthogonal unit normal vector fields to M^m. Then

$$\begin{aligned} &(a) \quad (G(\underset{x}{N}, BX)) \circ b = 0 \\ &(b) \quad G(\underset{x}{N}, \underset{y}{N}) = \delta xy. \end{aligned} \tag{1.19.3}$$

Let D be the Riemannian connection in M^n and E be the induced connection in M^m. Then the Gauss and the Weingarten equation can be written as

$$\begin{aligned}
&(a) \quad D_{BX}BY = BE_XY + {}'\underset{x}{H}(X,Y)\underset{y}{N} \qquad (\textit{Gauss Equation}) \\
&(b) \quad D_{BX}\underset{x}{N} = -B\underset{x}{H}X + \overset{y}{\underset{x}{I}}(X)\underset{y}{N} \qquad (\textit{Weingarten Equation}) \qquad (1.19.4)
\end{aligned}$$

where

$$\begin{aligned}
&(a) \quad g(\underset{x}{H}, Y) = {}'\underset{x}{H}(X,Y) = {}'\underset{x}{H}(Y,X) \\
&(b) \quad \overset{y}{\underset{x}{I}} + \overset{x}{\underset{y}{I}} = 0. \qquad (1.19.5)
\end{aligned}$$

${}'H$ is called second fundamental magnitudes in M^m.

1.20 Almost r-paracontact Riemannian manifold

Let M^n be an n-dimensional Riemannian manifold with a positive definite metric g. If there exist a tensor field ψ of type $(1,1)$, r- vector fields $\xi_1, \xi_2, ...\xi_r$ $(n > r)$, r 1-forms $\eta^1, \eta^2, ..., \eta^r$ such that

$$\begin{aligned}
&(i) \quad \eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \alpha, \beta\epsilon(r) = 1,2,..,r, \\
&(ii) \quad \psi^2(X) = X - \eta^\alpha(X)\xi_\alpha, \\
&(iii) \quad \eta^\alpha(X) = g(X, \xi_\alpha), \alpha \in (r), \\
&(iv) \quad g(\psi X, \psi Y) = g(X,Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y),
\end{aligned}$$

where X and Y are vector fields on M^n and $a^\alpha b_\alpha \overset{def}{=} \sum_\alpha a^\alpha b_\alpha$, then the structure $\sum = (\psi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ is said to be an almost r-paracontact structure on M^n and M^n is an almost r-paracontact Riemannian manifold (Ahmad *et al.*, 2011).

1.21 Review of Literature

Friedman and Schouten (1924) introduced the idea of semi-symmetric linear connection on a differentiable manifold. Hayden (1932) defined a semi-symmetric metric connection on a Riemannian manifold and this was further developed by Yano (1970). Agashe and Chafle (1992) introduced a semi symmetric non-metric connection on a Riemannian manifold and this was further studied by Prasad (1994), Ojha and Prasad (1994), Sengupta *et al.* (2000), Pandey and Ojha (2001), Tripathi and Kakar (2001a, b), Prasad and Kumar (2002), Chaturvedi and Pandey (2008), Murathan and Özgür (2008), Chaubey (2011), Singh (2014) and many others. Sharfuddin and Hussain (1976) defined a semi-symmetric metric connection in an almost contact manifold. De and Sengupta (2001) investigated the curvature tensor of an almost contact metric manifold that admit a type of semi-symmetric metric connection and studied the curvature properties of conformal curvature tensor and projective curvature tensor. It is also studied by many geometers like as Hatakeyama (1963), Hatakeyama *et al.* (1963), Sato (1976), Sasaki and Hatakeyama (1961), Oubina (1985) and Tripathi *et al.* (2008) the structure of some classes of contact metric manifolds. Golab (1975) introduced and studied quarter symmetric connection in a Riemannian manifold with an affine connection which generalizes the idea of semi-symmetric metric connection. Mishra and Pandey (1980) studied quarter symmetric metric connection on a Riemannian, Kahlerian and Sasakian manifolds. It is also studied by many geometers like as Yano and Imai (1982), Rastogi (1987, 2012), Mukhopadhya and Barua (1991), Biswas and De (1997), Pandey and Ojha (2001), Nivas and Verma (2005), Mondal and De (2009), De and De (2011), Singh (2013) and many others.

Matsumoto (1989) introduced the notion of Lorentzian Para Sasakian manifold. Mihai and Rosca (1992) already introduced the same notion independently and they obtained several results on this manifold. Lorentzian Para-Sasakian manifolds have also been studied by Matsumoto and Mihai (1988), Mihai *et al.* (1999a, b), De *et al.* (1999), Shaikh and De (2000), De and Sengupta (2002), Özgür (2003), Shaikh and Biswas (2004), Venkatesha and Bagewadi (2008), Dhruwa *et al.* (2009), Perktas and Tripathi (2010), Taleshian and Asghari (2010), Venkatesha *et al.* (2011), Prakash *et al.* (2011), Taleshian and Asghari (2011) and Singh (2013, 2015) obtained some results on Lorentzian Para-Sasakian manifolds.On the other hand an LP-Sasakian manifold and obtained some properties of this connection.

Pokhariyal and Mishra (1971) have introduced new curvature tensor called M-projective curvature tensor in a Riemannian manifold and studied its properties. Ojha (1975) studied a note on the M-projective curvature tensor. Pokhariyal (1982) has studied some properties of this curvature tensor in a Sasakian manifold. Ojha (1986), Chaubey (2012), Singh (2009, 2012, 2016), Devi and Singh (2015) and many others geometers have studied this curvature tensor.

The idea of recurrent manifolds was introduced by Walker (1950). On the other hand De and Guha (1991) introduced generalized recurrent manifold with the non zero 1-form A and another non-zero associated 1-form B. Such a manifold has been denoted by GK_n.

If the associated 1-form B becomes zero, then the manifold GK_n reduces to a recurrent manifold introduced by Ruse (1951) which is denoted by K_n. Prasad (2000) introduced the idea of semi generalized recurrent manifold. Yildiz and Murathan (2005) studied Lorentzian α-Sasakian manifolds and proved that conformally flat and quasi conformally flat Lorentzian α-Sasakian manifolds are locally isometric with a sphere. Lorentzian α-Sasakian manifolds have been studied by De and Tripathi (2003), Prakasha *et al.* (2008), Yildiz and Turan (2009), Yildiz *et al.* (2009), Prakasha and Yildiz (2010), Lokesh, *et al.* (2012), Teleshian and Asghari (2012), Yadav and Suthar (2012), Bhattacharya and Patra (2014), Berman (2014), and many others. Adati and Matsumoto (1977) defined P-Sasakian and Special Para Sasakian manifold, which are special classes of an almost para-contact manifold introduced by Sato (1976). Para Sasakian manifolds have been studied by Matsumoto (1977), Adati and Miyazawa (1979), Matsumoto *et.al.* (1986), De and Pathak (1994), Özgür and Tripathi (2007), De and Sarkar (2009), Shukla and Shukla (2010), Berman (2013), Singh (2014) and many others.

Takahashi (1977) introduced the notion of φ-symmetric Sasakian manifold and obtained some interesting properties. Many authors like Shaikh and De (2000), De and Pathak (2004) and Venkatesha and Bagewadi (2006) have extended this notion to 3-dimensional LP-Sasakian manifold, 3-dimensional Kenmatsu manifold and 3-dimensional trans-Sasakian manifolds respectively. De and Kamilya (1994) studied the generalized concircular recurrent manifolds. and De *et al.* (1995) studied the generalized Ricci-recurrent manifolds. Generalizing the notion of recurrency the author Khan (2004) introduced the notion of generalized recurrent Sasakian manifold. Prasad (2000) introduced the notion of semi-generalized recurrent manifold and obtained some interesting results. Jaiswal and Ojha (2009) studied generalized φ-recurrent and generalized concircular φ-recurrent LP-Sasakian manifolds. Sreenivasa *et al.* (2009) define φ-recurrent Lorentzian β-Kenmatsu manifold and Prove that a concircular φ-recurrent Lorentzian β-Kenmatsu manifold is an Einstein manifold. Singh (2012) introduced the M-projective recurrent Riemannian manifold with interesting results. Debnath and Bhattacharya (2013) studied the generalized φ-recurrent trans-Sasakian manifolds.

Bucki (1985) defined an almost r-paracontact structure and studied some properties of invariant hypersurfaces of an almost r-paracontact structure. Bucki (1998) studied product submanifolds of almost r-paracontact Riemannian manifolds of P-Sasakian type. Ahmad *et al.* (2009, 2010) studied the properties of hypersurfaces and submanifold on r-paracontact Riemannian manifold with connection. Al and Nivas (2000) studied on submanifolds of a manifold with quarter-symmetric connection. Yano and Kon (1977) studied anti invariant submanifold of Sasakian space forms.

The present thesis is devoted to investigate the properties of Quarter symmetric non-metric connection on LP-Sasakian manifolds. Some properties of quasi conformal curvature tensor and M-projective curvature tensor on LP-Sasakian manifolds. The properties of concircular curvature tensor and M-projective curvature tensor are discussed in a semi-generalized recurrent manifolds. Certain properties of the almost r-paracontact submanifold are also discussed.

Chapter 2

Quarter Symmetric Non-Metric Connection on An LP-Sasakian Manifold

2.1 Introduction

In an n-dimensional LP-Sasakian manifold with structure (φ, ξ, η, g) defined in (1.16.1-1.16.11) also hold the following relations (Matsumoto, 1989; Mihai and Rosca, 1972)

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{2.1.1}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.1.2}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.1.3}$$

$$\eta(R(X,Y,Z)) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.1.4}$$

$$S(\varphi X, \varphi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.1.5}$$

$$(D_X\eta)(Y) = g(X,\varphi Y) = g(\varphi X, Y), \tag{2.1.6}$$

for any vector fields X, Y, Z, where R is the curvature tensor, S is the Ricci tensor. Here we consider a quarter symmetric non-metric connection ∇ on LP-Sasakian manifold

$$\nabla_X Y = D_X Y - \eta(X)\varphi Y \tag{2.1.7}$$

given by (Mishra and Pandey, 1980) which satisfies

$$(\nabla_X g)(Y, Z) = 2\ \eta(X)\ g(\varphi Y, Z). \tag{2.1.8}$$

The curvature tensor with respect to a quarter symmetric non-metric connection ∇ and the curvature tensor R with respect to Riemannian connection D in an LP-Sasakian manifold are related as

$$\begin{aligned} \overline{R}(X,Y)Z &= R(X,Y)Z + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X. \end{aligned} \tag{2.1.9}$$

Contracting (2.1.9) with respect to X, we get

$$\overline{S}(Y,Z) = S(Y,Z) - g(Y,Z) - n\ \eta(Y)\eta(Z), \tag{2.1.10}$$

where $\overline{S}$ is the Ricci tensor of M^n with respect to quarter symmetric non-metric connection.
This gives

$$\overline{Q}Y = Q\ Y - Y - n\ \eta(Y)\xi. \tag{2.1.11}$$

Contracting the above equation, we get

$$\overline{r} = r, \tag{2.1.12}$$

where $\overline{r}$ and r are the scalar curvatures of the connection ∇ and D respectively.

2.2 Locally $W^*-\varphi-$ symmetric and $\xi-W^*-$projectively flat LP-Sasakian manifolds with respect to the quarter symmetric non-metric connection

Definition 2.2.1 *An n-dimensional LP-Sasakian manifold M^n is said to be locally $W^*-\varphi$-symmetric if*

$$\varphi^2\big((D_U W^*)(X,Y)Z\big) = 0, \tag{2.2.1}$$

for all vector fields X, Y, Z, U orthogonal to ξ. This notion was introduced by (Takahashi, 1977), for a Sasakian manifold. The W^* is M-projective curvature tensor (Pokhariyal and Mishra, 1970) given as

$$\begin{aligned} W^*(X,Y)Z &= R(X,Y)Z - \frac{1}{2(n-1)}\Big\{S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\Big\}. \end{aligned} \tag{2.2.2}$$

Analogous to the definition of locally $W^* - \varphi$-symmetric LP-Sasakian manifold with respect to the Riemannian connection, we define a locally $\overline{W^*} - \varphi$-symmetric LP-Sasakian manifolds with respect to the quarter symmetric non-metric connection as

$$\varphi^2\big((\nabla_U \overline{W^*})(X,Y)Z\big) = 0, \tag{2.2.3}$$

for all vector fields X, Y, Z, U orthogonal to ξ, where $\overline{W^*}$ is the M-projective curvature tensor with respect to a quarter symmetric non-metric connection given by

$$\begin{aligned} \overline{W^*}(X,Y)Z &= \overline{R}(X,Y)Z - \frac{1}{2(n-1)}\Big\{\overline{S}(Y,Z)X - \overline{S}(X,Z)Y \\ &+ g(Y,Z)\overline{Q}X - g(X,Z)\overline{Q}Y\Big\}. \end{aligned} \tag{2.2.4}$$

Definition 2.2.2 *An n-dimensional LP-Sasakian manifold M^n is said to be $\xi - W^*$-projectively flat if*

$$W^*(X,Y)\xi = 0,$$

for all vector fields X, Y on M^n.

Analogous to the definition of $\xi - W^*-$projectively flat LP-Sasakian manifold with respect *to quarter symmetric non-metric connection by*

$$\overline{W^*}(X,Y)\xi = 0,$$

for all vector fields X, Y on M^n.

Theorem 2.2.1 *An n-dimensional LP-Sasakian manifold M^n is locally $\overline{W^*}-\varphi$-symmetric with respect to the quarter symmetric non-metric connection ∇ if and only if it is so with respect to the Riemannian connection D.*

Proof: From (2.1.7), we have

$$(\nabla_U\overline{W^*})(X,Y)Z = (D_U\overline{W^*})(X,Y)Z - \eta(U)\varphi(\overline{W^*}(X,Y)Z). \tag{2.2.5}$$

Now, differentiating (2.2.4) covariantly with respect to U, we get

$$\begin{aligned}(D_U\overline{W^*})(X,Y)Z &= (D_U\overline{R})(X,Y)Z \\ &- \frac{1}{2(n-1)}\Big\{(D_U\overline{S})(Y,Z)X - (D_U\overline{S})(X,Z)Y \\ &+ g(Y,Z)(D_U\overline{Q})X - g(X,Z)(D_U\overline{Q})Y\Big\}.\end{aligned}$$

Making use of (2.1.9), (2.1.10), (2.1.11) and (2.2.2) in the above equation, we obtain

$$\begin{aligned}(D_U\overline{W^*})(X,Y)Z &= (D_UW^*)(X,Y)Z \\ &+ \frac{3n-2}{2(n-1)}\Big[\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\varphi(U) \\ &+ \{g(Y,Z)g(\varphi X,U) - g(X,Z)g(\varphi Y,U)\}\xi \\ &- \{\eta(X)Y - \eta(Y)X\}g(\varphi Z,U)\Big] \\ &+ \frac{n-2}{2(n-1)}\Big\{g(\varphi X,U)Y - g(\varphi Y,U)X\Big\}\eta(Z).\end{aligned} \tag{2.2.6}$$

Now, using equation (2.2.6) in equation (2.2.5), we get

$$\begin{aligned}(\nabla_U\overline{W^*})(X,Y)Z &= (D_UW^*)(X,Y)Z \\ &+ \frac{3n-2}{2(n-1)}\Big[\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\varphi(U) \\ &+ \{g(Y,Z)g(\varphi X,U) - g(X,Z)g(\varphi Y,U)\}\xi\end{aligned}$$

$$
\begin{aligned}
&- \{\eta(X)Y - \eta(Y)X\}g(\varphi Z, U)\Big] \\
&+ \frac{n-2}{2(n-1)}\Big\{g(\varphi X, U)Y - g(\varphi Y, U)X\Big\}\eta(Z) \\
&- \eta(U)\varphi(W^*(X,Y)Z). \qquad (2.2.7)
\end{aligned}
$$

Applying φ^2 on both sides of the above equation and using the equation (1.16.1) and (1.16.5), we get

$$
\begin{aligned}
\varphi^2((\nabla_U\overline{W^*})(X,Y)Z) &= \varphi^2((D_UW^*)(X,Y)Z) \\
&+ \frac{3n-2}{2(n-1)}\Big[\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\varphi^2(\varphi(U)) \\
&+ \{g(Y,Z)g(\varphi X, U) - g(X,Z)g(\varphi Y, U)\}\varphi^2(\xi) \\
&- \{\eta(X)(\varphi^2 Y) - \eta(Y)(\varphi^2 X)\}g(\varphi Z, U)\Big] \\
&+ \frac{n-2}{2(n-1)}\Big\{g(\varphi X, U)(\varphi^2 Y) - g(\varphi Y, U)(\varphi^2 X)\Big\}\eta(Z) \\
&- \eta(U)\varphi^3(W^*(X,Y)Z). \qquad (2.2.8)
\end{aligned}
$$

Consider X, Y, Z and U are orthogonal to ξ, then equation (2.2.8) yields

$$
\varphi^2((\nabla_U\overline{W^*})(X,Y)Z) = \varphi^2((D_UW^*)(X,Y)Z). \qquad (2.2.9)
$$

This completes the proof.

Theorem 2.2.2 *An n-dimensional LP-Sasakian manifold is $\xi-\overline{W^*}$-projectively flat with respect to the quarter symmetric non-metric connection if and only if the manifold is $\xi-W^*$-projectively flat with respect to the Riemannian connection provided that the vector fields X and Y are orthogonal to ξ.*

Proof: Using (2.1.9) in (2.2.4), we get

$$
\begin{aligned}
\overline{W^*}(X,Y)Z &= R(X,Y)Z + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\
&+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
&- \frac{1}{2(n-1)}\Big\{\overline{S}(Y,Z)X - \overline{S}(X,Z)Y \\
&+ g(Y,Z)\overline{Q}X - g(X,Z)\overline{Q}Y\Big\}. \qquad (2.2.10)
\end{aligned}
$$

In consequence of (2.1.10), (2.1.11) and (2.2.2) the above equation becomes

$$\begin{aligned}\overline{W^*}(X,Y)Z &= W^*(X,Y)Z \\ &+ \frac{3n-2}{2(n-1)}\Big[\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \\ &+ \{\eta(X)Y - \eta(Y)X\}\ \eta(Z)\Big] \\ &- \frac{1}{n-1}\Big\{g(X,Z)Y - g(Y,Z)X\Big\}. \end{aligned} \tag{2.2.11}$$

Putting $Z = \xi$ in (2.2.11) and using (1.16.2) and (1.16.4), it follows that

$$\overline{W^*}(X,Y)\xi = W^*(X,Y)\xi - \frac{3\ n}{2(n-1)}\Big\{\eta(Y)X - \eta(X)Y\Big\}. \tag{2.2.12}$$

Suppose X and Y are orthogonal to ξ, then from (2.2.12), we obtain

$$\overline{W^*}(X,Y)\xi = W^*(X,Y)\xi. \tag{2.2.13}$$

Hence, prove the theorem.

2.3 Locally $W_2 - \varphi$-symmetric and $\xi - W_2$-projectively flat LP-Sasakian manifold with respect to the quarter symmetric non-metric connection

Definition 2.3.1 *An n-dimensional LP-Sasakian manifold M^n is said to be locally $W_2 -$ φ-symmetric if*

$$\varphi^2((D_U W_2)(X,Y)Z) = 0, \tag{2.3.1}$$

for all vector fields X, Y, Z and U orthogonal to ξ, where W_2-projective curvature tensor is given by (Pokhariyal and Mishra, 1971)

$$\begin{aligned} W_2(X,Y)Z &= R(X,Y)Z \\ &- \frac{1}{n-1}\Big\{g(Y,Z)QX - g(X,Z)QY\Big\}. \end{aligned} \tag{2.3.2}$$

Analogous to the definition of locally φ-symmetric LP-Sasakian manifolds with respect to the Riemannian connection, we define a locally $\overline{W_2} - \varphi$-symmetric LP-Sasakian manifolds *with respect to the quarter symmetric non-metric connection by*

$$\varphi^2((\nabla_U\overline{W_2})(X,Y)Z) = 0, \tag{2.3.3}$$

for all vector fields X, Y, Z and U orthogonal to ξ, where the W_2-curvature tensor with respect to the quarter symmetric non-metric connection given as

$$\begin{aligned}\overline{W_2}(X,Y)Z &= \overline{R}(X,Y)Z \\ &- \frac{1}{n-1}\Big\{g(Y,Z)\overline{Q}X - g(X,Z)\overline{Q}Y\Big\}.\end{aligned} \tag{2.3.4}$$

Definition 2.3.2 *An n-dimensional LP-Sasakian manifold M^n is said to be $\xi - W^*$-projectively flat if*

$$W_2(X,Y)\xi = 0,$$

for all vector fields X, Y on M^n.
Analogous to the definition of $\xi - W^* -$projectively flat LP-Sasakian manifold with respect to quarter symmetric non-metric connection by

$$\overline{W_2}(X,Y)\xi = 0,$$

for all vector fields X, Y on M^n.

Theorem 2.3.1 *An n-dimensional LP-Sasakian manifold M^n is locally $\overline{W_2}-\varphi$-symmetric with respect to the quarter symmetric non-metric connection ∇ if and only if it is locally $W_2 - \varphi$-symmetric with respect to the Riemannian connection D.*

Proof: Using (2.1.7), we can write

$$(\nabla_U\overline{W_2}(X,Y)Z) = (D_U\overline{W_2})(X,Y)Z - \eta(U)\varphi(\overline{W_2}(X,Y)Z). \tag{2.3.5}$$

Now, differentiating (2.3.4) covariantly with respect to U, we obtain

$$\begin{aligned}(D_U\overline{W_2})(X,Y)Z &= (D_U\overline{R})(X,Y)Z - \frac{1}{n-1}\Big\{g(Y,Z)(D_U\overline{Q})(X) \\ &- g(X,Z)(D_U\overline{Q})(Y)\Big\}.\end{aligned} \tag{2.3.6}$$

In view of (2.1.9) and (2.1.11) the equation (2.3.6), takes the form

$$\begin{aligned}(D_U\overline{W_2})(X,Y)Z &= (D_UR)(X,Y)Z\\ &- \{\eta(X)Y-\eta(Y)X\}g(\varphi Z,U)\\ &+ \{g(\varphi X,U)g(Y,Z)-g(\varphi Y,U)g(X,Z)\}\xi\\ &+ \{\eta(X)g(Y,Z)-\eta(Y)g(X,Z)\}\varphi U\\ &+ \{g(\varphi X,U)Y-g(\varphi Y,U)X\}\eta(Z)\\ &- \frac{1}{n-1}\Big[\{g(Y,Z)(D_UQ)X-g(X,Z)(D_UQ)Y\}\\ &- n\ g(Y,Z)\{g(\varphi X,U)\xi+\eta(X)\varphi U\}\\ &+ n\ g(X,Z)\{g(\varphi Y,U)\xi+\eta(Y)\varphi U\}\Big]\xi. \end{aligned} \tag{2.3.7}$$

Taking account of (2.3.2) and (2.3.5) the above equation can be written as

$$\begin{aligned}(\nabla_U\overline{W_2})(X,Y)Z &= (D_UW_2)(X,Y)Z\\ &+ \{g(\varphi X,U)g(Y,Z)-g(\varphi Y,U)g(X,Z)\}\xi\\ &+ \{\eta(X)g(Y,Z)-\eta(Y)g(X,Z)\}\varphi(U)\\ &+ \{g(\varphi X,U)Y-g(\varphi Y,U)X\}\eta(Z)\\ &- \frac{1}{n-1}\Big[n\ g(X,Z)\{g(\varphi Y,U)\xi+\eta(Y)\varphi U\}\\ &- n\ g(Y,Z)\{g(\varphi X,U)\xi+\eta(X)\varphi U\}\Big]\xi\\ &- \eta(U)\varphi(\overline{W_2}(X,Y)Z). \end{aligned} \tag{2.3.8}$$

By operating φ^2 both sides and using (1.16.1), (1.16.5) in above equation, we obtain

$$\begin{aligned}\varphi^2((\nabla_U\overline{W_2})(X,Y)Z) &= \varphi^2((D_UW_2)(X,Y)Z)\\ &- \{\eta(X)g(Y,Z)-\eta(Y)g(X,Z)\}\varphi^2(\varphi U)\\ &+ \{g(\varphi X,U)(\varphi^2Y)-g(\varphi Y,U)(\varphi^2X)\}\eta(Z)\\ &- \eta(U)\varphi^3(\overline{W_2}(X,Y)Z). \end{aligned} \tag{2.3.9}$$

If we consider X,Y,Z and U are orthogonal to ξ, then equation (2.3.9) reduces to

$$\varphi^2((\nabla_U\overline{W_2})(X,Y)Z)=\varphi^2((D_UW_2)(X,Y)Z). \tag{2.3.10}$$

This completes the proof.

Theorem 2.3.2 *An n-dimensional LP-Ssasakian manifold is $\xi-\overline{W_2}$-projectively flat* *with respect to the quarter symmetric non-metric connection if and only if the manifold is $\xi - W_2$-projectively flat with respect to the Riemannian connection provided that the vector fields X and Y are orthogonal to ξ.*

Proof: Using (2.1.9) in (2.3.4), we get

$$\begin{aligned}\overline{W_2}(X,Y)Z &= R(X,Y)Z + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \\ &- \frac{1}{n-1}\Big\{g(Y,Z)\overline{Q}X - g(X,Z)\overline{Q}Y\Big\} \\ &+ \{\eta(X)Y - \eta(Y)X\}\eta(Z). \end{aligned} \tag{2.3.11}$$

Again using (2.1.11) and (2.3.2) in (2.3.11), we get

$$\begin{aligned}\overline{W_2}(X,Y)Z &= W_2(X,Y)Z \\ &+ \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \\ &+ \{\eta(X)Y - \eta(Y)X\}\eta(Z) \\ &- \frac{1}{n-1}\Big\{g(X,Z)Y - g(Y,Z)X\Big\} \\ &- \frac{n}{n-1}\Big\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\Big\}\xi. \end{aligned} \tag{2.3.12}$$

Putting $Z = \xi$ in (2.3.12), and using (1.16.4), (1.16.2), it follows that

$$\overline{W_2}(X,Y)\xi = W_2(X,Y)\xi - \frac{n}{n-1}\Big\{\eta(X)\ Y - \eta(Y)X\Big\}. \tag{2.3.13}$$

Suppose X and Y are orthogonal to ξ, then from (2.3.13), we obtain

$$\overline{W_2}(X,Y)\xi = W_2(X,Y)\xi. \tag{2.3.14}$$

This completes the proof.

2.4 Einstein manifold with respect to quarter symmetric non-metric connection ∇ in an LP-Sasakian manifold

Theorem 2.4.1 *In an LP-Sasakian manifolds M^n with quarter symmetric non-metric connection if the relation*

$$g(X,Y) + n\ \eta(X)\ \eta(Y) = 0.$$

then the manifold is an Einstein manifold for the Riemannian connection if and only if it is an Einstein manifold for the quarter symmetric non-metric connection ∇.

Proof: A Riemannian manifold M^n is called an Einstein manifold with respect to Riemannian connection if

$$S(X,Y) = \frac{r}{n}\, g(X,Y). \tag{2.4.1}$$

Analogous, to this definition, we define Einstein manifold with respect to quarter symmetric non-metric connection ∇ by

$$\overline{S}(X,Y) = \frac{\overline{r}}{n}\, g(X,Y). \tag{2.4.2}$$

From (2.1.10), (2.1.12) and (2.4.2), we have

$$\begin{aligned} \overline{S}(X,Y) - \frac{\overline{r}}{n}\, g(X,Y) &= S(X,Y) - \frac{r}{n}\, g(X,Y) \\ &- g(X,Y) - n\, \eta(X)\, \eta(Y). \end{aligned} \tag{2.4.3}$$

If

$$g(X,Y) + n\, \eta(X)\, \eta(Y) = 0. \tag{2.4.4}$$

Then from (2.4.3), we get

$$\overline{S}(X,Y) - \frac{\overline{r}}{n}\, g(X,Y) = S(X,Y) - \frac{r}{n}\, g(X,Y). \tag{2.4.5}$$

This completes the proof.

2.5 Projective Ricci tensor with respect to quarter symmetric non-metric connection ∇ in an LP-Sasakian manifold

Theorem 2.5.1 *If an LP-Sasakian manifold admits a skew symmetric with respect to quarter symmetric non-metric connection ∇ then a necessary and sufficient condition for the projective Ricci tensor of ∇ to be skew symmetric with respect to quarter symmetric* *non-metric is that the Ricci tensor of the Riemannian connection D is given by*

$$S(X,Y) = \big(1 + \frac{r}{n}\big)g(X,Y) + n\ \eta(X)\eta(Y).$$

Proof: Projective Ricci tensor in a Riemannian manifold is defined by (Chaki and Saha, 1994) as follows

$$P^{*}(X,Y) = \frac{n}{n-1}\Big[S(X,Y) - \frac{r}{n}\ g(X,Y)\Big]. \tag{2.5.1}$$

Analogous to this definition, we define projective Ricci tensor with respect to quarter symmetric non-metric connection ∇, given by

$$\overline{P}^{*}(X,Y) = \frac{n}{n-1}\Big[\overline{S}(X,Y) - \frac{\overline{r}}{n}\ g(X,Y)\Big]. \tag{2.5.2}$$

From (2.1.10), (2.1.12) and (2.5.2), we have

$$\begin{aligned} \overline{P}^{*}(X,Y) &= \frac{n}{n-1}\Big[S(X,Y) - g(X,Y) \\ &- n\ \eta(X)\eta(Y) - \frac{r}{n}g(X,Y)\Big]. \end{aligned} \tag{2.5.3}$$

From (2.5.3) we have

$$\begin{aligned} \overline{P}^{*}(Y,X) &= \frac{n}{n-1}\Big[\ S(Y,X) - g(Y,X) \\ &- n\ \eta(Y)\ \eta(X) - \frac{r}{n}\ g(Y,X)\Big]. \end{aligned} \tag{2.5.4}$$

From equation (2.5.3) and (2.5.4), we have

$$\begin{aligned} \overline{P}^{*}(X,Y) + \overline{P}^{*}(Y,X) &= \frac{2n}{n-1}\Big[S(X,Y) - g(X,Y) \\ &- n\ \eta(X)\ \eta(Y) - \frac{r}{n}\ g(X,Y)\Big]. \end{aligned} \tag{2.5.5}$$

If $\overline{P}^{*}(X,Y)$ is skew symmetric with respect to quarter symmetric non-metric connection ∇ then the left hand side vanishes and we get

$$S(X,Y) = \big(1 + \frac{r}{n}\ \big)g(X,Y) + n\ \eta(X)\eta(Y). \tag{2.5.6}$$

Moreover if $S(X,Y)$ is given by (2.5.6), then from (2.5.5), we get

$$\overline{P}^{*}\ (X,Y) + \overline{P}^{*}\ (Y,X) = 0, \tag{2.5.7}$$

i.e. projective Ricci tensor of ∇ is skew symmetric with respect to quarter symmetric non-metric connection ∇.

2.6 φ-Conharmonicaly flat LP-Sasakian manifolds with respect to the quarter symmetric non-metric connection

Definition 2.6.1 *Conharmonic curvature tensor with respect to Riemannian connection D is defined as*

$$\begin{aligned} H(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}\Big[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\Big] \end{aligned} \tag{2.6.1}$$

and the conharmonic curvature tensor with respect to quarter symmetric non-metric connection ∇ is given as

$$\begin{aligned} \overline{H}(X,Y)Z &= \overline{R}(X,Y)Z - \frac{1}{n-2}\Big[\overline{S}(Y,Z)X - \overline{S}(X,Z)Y \\ &+ g(Y,Z)\overline{Q}X - g(X,Z)\overline{Q}Y\Big]. \end{aligned} \tag{2.6.2}$$

Definition 2.6.2 *An n-dimensional LP-Sasakian manifold M^n satisfying the equation*

$$\varphi^2(H(\varphi X, \varphi Y)\varphi Z) = 0, \tag{2.6.3}$$

is called φ-Conharmonicaly flat (Sharfuddin and Hussain, 1976).

Analogous to the above an n-dimensional LP-Sasakian manifolds is said to be φ-conharmonicaly flat with respect to quarter-symmetric non-metric connection if it satisfies

$$\varphi^2(\overline{H}(\varphi X, \varphi Y)\varphi Z) = 0, \tag{2.6.4}$$

where $\overline{H}$ is the conharmonic curvature tensor of the manifold with respect to quarter symmetric non-metric connection.

Theorem 2.6.1 *Let M^n be an n-dimensional φ-conharmonicaly flat LP-Sasakian manifolds admitting a quarter-symmetric non-metric connection, then M^n is an η-Einstein manifold with the scalar curvature $r = -\frac{2(n-1)}{n-2}$ with respect to the Riemannian connection.*

Proof: Suppose M^n is φ-conharmonicaly flat LP-Sasakian manifold with respect to quarter symmetric non-metric connection. It is easy to see that $\varphi^2(\overline{H}(\varphi X, \varphi Y)\varphi Z) = 0$ holds if and only if

$$g(\overline{H}(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0, \tag{2.6.5}$$

for any vector fields X, Y, Z, W. Taking account of above equation the equation (2.6.2) assume the form as

$$\begin{aligned} g(\overline{R}(\varphi X, \varphi Y)\varphi Z, \varphi W) &= \frac{1}{(n-2)}\Big[\overline{S}(\varphi Y, \varphi Z)g(\varphi X, \varphi W) \\ &- \overline{S}(\varphi X, \varphi Z)g(\varphi Y, \varphi W) + g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) \\ &- g(\varphi X, \varphi Z)S(\varphi Y, \varphi W)\Big]. \end{aligned} \tag{2.6.6}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of the vector fields in M^n. It is obvious that $\{\varphi e_1, \varphi e_2, ..., \varphi e_{n-1}, \xi\}$ is also a local orthonormal basis.
Now, taking a frame field and contraction over X and W, we get

$$\begin{aligned} \sum_{i=1}^{n-1} g(\overline{R}(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) &= \sum_{i=1}^{n-1} \frac{1}{(n-2)}\Big[\overline{S}(\varphi Y, \varphi Z)g(\varphi e_i, \varphi e_i) \\ &- \overline{S}(\varphi e_i, \varphi Z)g(\varphi Y, \varphi e_i) + g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i) \\ &- g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i)\Big]. \end{aligned} \tag{2.6.7}$$

Also, it can be seen that (Agashe and Chafle, 1992)

$$\sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) = S(\varphi Y, \varphi Z) + g(\varphi Y, \varphi Z), \tag{2.6.8}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi e_i) = n + 1, \tag{2.6.9}$$

$$\sum_{i=1}^{n-1} S(\varphi e_i, \varphi e_i) = r + n - 1, \tag{2.6.10}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi Z) S(\varphi Y, \varphi e_i) = S(\varphi Y, \varphi Z), \tag{2.6.11}$$

and

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi Z) g(\varphi Y, \varphi e_i) = g(\varphi Y, \varphi Z). \tag{2.6.12}$$

Hence by virtue of the equations (2.6.8) to (2.6.12) the equation (2.6.7) takes the form

$$S(\varphi Y, \varphi Z) = (r+1)\; g(\varphi Y, \varphi Z). \tag{2.6.13}$$

Now, using the equations (1.16.3) and (2.1.5) in above equation, we obtain

$$S(Y, Z) = (r+1) g(Y, Z) + (r - n + 2)\eta(Y)\eta(Z). \tag{2.6.14}$$

Contracting the above equation, we obtain

$$r = -\frac{2(n-1)}{n-2},$$

which implies M^n is an η-Einstein manifold with the scalar curvature $r = -\frac{2(n-1)}{n-2}$ with respect to the Riemannian connection.
Hence, proves the theorem.

2.7 ξ-Conharmonicaly flat and ξ-Concircularly flat LP-Sasakian manifolds with respect to the quarter symmetric non-metric connection

Theorem 2.7.1 *An n-dimensional LP-Sasakian manifold is ξ-Conharmonicaly flat with respect to the quarter symmetric non-metric connection if and only if the manifold is also ξ-Conharmonicaly flat with respect to the Riemannian connection.*

Proof: Using (2.6.2) and (2.1.9), we get

$$\begin{aligned} \overline{H}(X,Y)Z &= R(X,Y)Z + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X - \frac{1}{n-2}\Big[g(Y,Z)\overline{Q}X \\ &- g(X,Z)\overline{Q}Y + \overline{S}(Y,Z)X - \overline{S}(X,Z)Y\Big]. \end{aligned} \tag{2.7.1}$$

Making use of (2.1.10), (2.1.11) and (2.6.1) in (2.7.1), we get

$$\begin{aligned} \overline{H}(X,Y)Z &= H(X,Y)Z - \frac{2}{n-2}\Big[g(X,Z)Y - g(Y,Z)X\Big] \\ &- \frac{2}{n-2}\Big[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\Big] \\ &+ \frac{2(n-1)}{n-2}\Big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\Big]\xi. \end{aligned} \tag{2.7.2}$$

Putting $Z = \xi$ in (2.7.2) and using (1.16.2) and (1.16.4) it follows that

$$\overline{H}(X,Y)\xi = H(X,Y)\xi.$$

This completes the proof.

Theorem 2.7.2 *An n-dimensional LP-Sasakian manifold is ξ-Concircularly flat with respect to the quarter symmetric non-metric connection if and only if the manifold is ξ-Concircular flat with respect to the Riemannian connection provided that the vector fields X and Y are orthogonal to ξ.*

Proof: The Concircular curvature tensor is defined as

$$L(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\Big[g(Y,Z)X - g(X,Z)Y\Big], \tag{2.7.3}$$

and the concircular curvature tensor with respect to quarter symmetric non-metric connection is given as

$$\overline{L}(X,Y)Z = \overline{R}(X,Y)Z - \frac{\overline{r}}{n(n-1)}\Big[g(Y,Z)X - g(X,Z)Y\Big]. \tag{2.7.4}$$

In consequence of the equation (2.1.9) the equation (2.7.4) becomes

$$\begin{aligned} \overline{L}(X,Y)Z &= R(X,Y)Z + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X - \frac{\overline{r}}{n(n-1)}\Big[g(Y,Z)X - g(X,Z)Y\Big]. \end{aligned} \tag{2.7.5}$$

Again using (2.1.12) and (2.7.3) in the equation (2.7.5), we obtain

$$\begin{aligned} \overline{L}(X,Y)Z &= L(X,Y)Z + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X. \end{aligned} \tag{2.7.6}$$

Putting $Z = \xi$ in (2.7.6) and using (1.16.2) and (1.16.4) it follows that

$$\overline{L}(X,Y)\xi = L(X,Y)\xi - [\eta(X)Y - \eta(Y)X]. \tag{2.7.7}$$

Suppose X and Y are are orthogonal to ξ, then from (2.7.7), we obtain

$$\overline{L}(X,Y)\xi = L(X,Y)\xi.$$

This completes the proof .

2.8 LP-Sasakian manifold admitting a quarter symmetric non-metric connection satisfying $\overline{P}\cdot\overline{S} = 0$, $\overline{R}\cdot\overline{S} = 0$ and $\overline{L}\cdot\overline{S} = 0$

Theorem 2.8.1 *An LP-Sasakian manifold admitting a quarter symmetric non-metric connection satisfying $\overline{P}\cdot\overline{S} = 0$ is an η-Einstein manifold.*

Proof: The projective curvature tensor is defined as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\Big[S(Y,Z)X - S(X,Z)Y\Big], \tag{2.8.1}$$

and the projective curvature tensor with respect to quarter symmetric non-metric connection is given as

$$\overline{P}(X,Y)Z = \overline{R}(X,Y)Z - \frac{1}{n-1}\Big[\overline{S}(Y,Z)X - \overline{S}(X,Z)Y\Big]. \tag{2.8.2}$$

Consider LP-Sasakian manifolds with respect to a quarter symmetric non-metric connection satisfying

$$(\overline{P}(X,Y)\cdot\overline{S})(Z,U) = 0. \tag{2.8.3}$$

Then, we have

$$\overline{S}(\overline{P}(X,Y)Z,U) + \overline{S}(Z,\overline{P}(X,Y)U) = 0. \tag{2.8.4}$$

Putting $X = \xi$ in the equation (2.8.4), we have

$$\overline{S}(\overline{P}(\xi,Y)Z,U) + \overline{S}(Z,\overline{P}(\xi,Y)U) = 0. \tag{2.8.5}$$

In view of the equation (2.8.2), we have

$$\overline{P}(\xi, Y)Z = \overline{R}(\xi, Y)Z - \frac{1}{n-1}\Big[\overline{S}(Y, Z)\xi - \overline{S}(\xi, Z)Y\Big]. \tag{2.8.6}$$

By putting $X = \xi$ in (2.1.9) and $Y = \xi$ in (2.1.10), we get the following equations

$$\begin{aligned} \overline{R}(\xi, Y)Z &= R(\xi, Y)Z - 2\eta(Y)\eta(Z)\xi \\ &- \eta(Z)Y - g(Y, Z)\xi, \end{aligned} \tag{2.8.7}$$

and

$$\overline{S}(\xi, Z) = S(\xi, Z) + (n-1)\eta(Z). \tag{2.8.8}$$

By virtue of the equations (2.8.7), (2.8.8) and (2.1.10), the equation (2.8.6) yields

$$\begin{aligned} \overline{P}(\xi, Y)Z &= \frac{1}{n-1}\Big\{g(Y, Z) - S(Y, Z)\Big\}\xi \\ &+ \Big(\frac{n-2}{n-1}\Big)\eta(Y)\eta(Z)\xi. \end{aligned} \tag{2.8.9}$$

Now using the equation (2.8.9) in the equation (2.8.5), we get

$$\begin{aligned} g(Y, Z)\eta(U) &+ g(Y, U)\eta(Z) + (n-2)\eta(Y)\eta(Z)\eta(U) \\ &- [S(Y, Z)\eta(U) + S(Y, U)\eta(Z)] = 0. \end{aligned} \tag{2.8.10}$$

Putting $U = \xi$ in the above equation and using equations (1.16.2) and (1.16.11), we get

$$S(Y, Z) = g(Y, Z) + 2(n-2)\ \eta(Y)\eta(Z). \tag{2.8.11}$$

This show that LP-Sasakian manifolds is an η-Einstein manifold.

Theorem 2.8.2 *An LP-Sasakian manifold admitting a quarter symmetric non-metric connection satisfying $\overline{R} \cdot \overline{S} = 0$ is an η-Einstein manifold.*

Proof: Consider LP-Sasakian manifolds with respect to a quarter symmetric non-metric connection satisfying

$$(\overline{R}(X, Y) \cdot \overline{S})(Z, U) = 0. \tag{2.8.12}$$

Then we have

$$\overline{S}(\overline{R}(X,Y)Z,U) + \overline{S}(Z,\overline{R}(X,Y)U) = 0. \tag{2.8.13}$$

Putting $X = \xi$ in the equation (2.8.13), we have

$$\overline{S}(\overline{R}(\xi,Y)Z,U) + \overline{S}(Z,\overline{R}(\xi,Y)U) = 0. \tag{2.8.14}$$

By virtue of the equations (1.16.2) and (1.16.4), the equation (2.1.9) yields

$$\overline{R}(\xi,Y)Z = -2\ \{\eta(Y)\eta(Z)\xi + \eta(Z)Y\}. \tag{2.8.15}$$

By virtue of the equations (2.8.15) and (2.8.14), we get

$$\begin{aligned} &3(n-1)\ \eta(Y)\eta(Z)\eta(U) + S(Y,U)\eta(Z) + S(Y,Z)\eta(U) \\ &-g(Y,U)\eta(Z) - g(Y,Z)\eta(U) = 0. \end{aligned} \tag{2.8.16}$$

Putting $U = \xi$ in the above equation and using equations (1.16.2), (1.16.4) and (1.16.11), we get

$$S(Y,Z) = g(Y,Z) - (2n-1)\eta(Y)\eta(Z). \tag{2.8.17}$$

This shows that LP-Sasakian manifolds is an η-Einstein manifold.

Theorem 2.8.3 *An LP-Sasakian manifold admitting a quarter symmetric non-metric connection satisfying $\overline{L}\cdot\overline{S} = 0$ is an η-Einstein manifold.*

Proof: Consider LP-Sasakian manifolds with respect to a quarter symmetric non-metric connection satisfying

$$(\overline{L}(X,Y)\cdot\overline{S})(Z,U) = 0. \tag{2.8.18}$$

Then we have

$$\overline{S}(\overline{L}(X,Y)Z,U) + \overline{S}(Z,\overline{L}(X,Y)U) = 0. \tag{2.8.19}$$

Putting $X = \xi$ in the above equation (2.8.19), we have

$$\overline{S}(\overline{L}(\xi,Y)Z,U) + \overline{S}(Z,\overline{L}(\xi,Y)U) = 0. \tag{2.8.20}$$

In view of the equation (2.7.4), we have

$$\overline{L}(\xi, Y)Z = \overline{R}(\xi, Y)Z - \frac{\overline{r}}{n(n-1)}\Big[g(Y,Z)\xi - g(\xi, Z)Y\Big]. \tag{2.8.21}$$

Using (2.1.9) and (2.1.12) in the equation (2.8.21), we obtain

$$\begin{aligned}\overline{L}(\xi, Y)Z &= -2\,[\eta(Z)Y + \eta(Y)\eta(Z)\xi] \\ &- \frac{\overline{r}}{n(n-1)}\Big[g(Y,Z)\xi - \eta(Z)Y\Big].\end{aligned} \tag{2.8.22}$$

Now using the equation (2.8.22) in (2.8.20), we get

$$\begin{aligned}&- 2\eta(Y)\eta(Z)\overline{S}(\xi,U) - 2\eta(Z)\overline{S}(Y,U) - \frac{r}{n(n-1)}\,g(Y,Z)\overline{S}(\xi,U) \\ &+ \frac{r}{n(n-1)}\eta(Z)\overline{S}(Y,U) - 2\,\eta(Y)\eta(U)\overline{S}(Z,\xi) - 2\eta(U)\overline{S}(Y,Z) \\ &- \frac{r}{n(n-1)}g(Y,U)\overline{S}(Z,\xi) + \frac{r}{n(n-1)}\eta(U)\overline{S}(Y,Z) = 0.\end{aligned} \tag{2.8.23}$$

Putting $U = \xi$ in the above equation and using equations (1.16.2), (1.16.4) and (2.1.10), we obtain

$$S(Y,Z) = [(2n-1)r - 2n(n-1)]\;g(Y,Z) + n\;[2n(n-2) + 2 + r]\;\eta(Y)\eta(Z)$$

or

$$S(Y,Z) = a\;g(Y,Z) + b\;\eta(Y)\eta(Z),$$

where $a = \big[r(2n-1) - 2n(n-1)\big]$ and $b = n\;\big[2n(n-2) + 2 + r\big]$.
This shows that LP-Sasakian manifolds an η-Einstein manifold.

2.9 Skew symmetric condition of Ricci tensor of ∇ in an LP-Sasakian manifold

Theorem 2.9.1 *If an LP-Sasakian manifold admits a quarter symmetric non-metric connection ∇ then a necessary and sufficient condition for the Ricci tensor of ∇ to be skew symmetric is that the Ricci tensor of the Riemannian connection D is given by*

$$S(Y,X) = g(Y,X) + n\;\eta(Y)\eta(X).$$

Proof: Replacing Z by X in (2.1.10), we get

$$\overline{S}(Y,X) = S(Y,X) - g(Y,X) - n\ \eta(Y)\eta(X). \tag{2.9.1}$$

Interchanging X and Y in (2.9.1), we have

$$\overline{S}(X,Y) = S(X,Y) - g(X,Y) - n\ \eta(X)\eta(Y). \tag{2.9.2}$$

Adding the equations (2.9.1) and (2.9.2) we get

$$\overline{S}(Y,X) + \overline{S}(X,Y) = 2\ S(Y,X) - 2\ g(Y,X) - 2\ n\ \eta(Y)\eta(X). \tag{2.9.3}$$

If $\overline{S}(Y,X)$ is skew symmetric then the left hand side of equation (2.9.3) vanishes, and we get

$$S(Y,X) = g(Y,X) + n\ \eta(Y)\eta(X). \tag{2.9.4}$$

Moreover of $S(X,Y)$ is given by equation (2.9.4), then from (2.9.3), we get

$$\overline{S}(Y,X) + \overline{S}(X,Y) = 0,$$

This completes the proof of the theorem.

2.10 Curvature tensor of quarter symmetric non-metric connection

Theorem 2.10.1 *In an LP-Sasakian manifold if the curvature tensor with respect to quarter symmetric non-metric connection ∇ is given by*

$$\overline{R}(X,Y)Z = g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi, \tag{2.10.1}$$

then the manifold is projectively flat.

Proof: By the virtue of (2.1.9), (2.10.1) becomes

$$R(X,Y)Z = \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y. \tag{2.10.2}$$

Contracting (2.10.2) with respect to X we get

$$S(Y,Z) = (n-1)\ \eta(Y)\eta(Z). \tag{2.10.3}$$

From above equation we get

$$Q(Y) \ = \ (n-1)\eta(Y)\xi. \tag{2.10.4}$$

Contracting the equation (2.10.4) with respect to Y and using (1.16.2), we get

$$r \ = \ -(n-1). \tag{2.10.5}$$

The Projective curvature tensor is given by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\Big[S(Y,Z)X - S(X,Z)Y\Big]. \tag{2.10.6}$$

Taking account of (2.10.2) and (2.10.3), the above equation yields

$$P(X,Y)Z \ = \ 0.$$

This completes the proof.

Theorem 2.10.2 *For the associative curvature tensor of the quarter symmetric non-metric connection, we have*

$$\begin{aligned}
&(i) \quad {}'\overline{R}(X,Y,Z,U) + {}'\overline{R}(Y,X,Z,U) = 0,\\
&(ii) \quad {}'\overline{R}(X,Y,Z,U) + {}'\overline{R}(X,Y,U,Z) = 2\ [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\eta(U)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad + 2\ [g(Y,U)\eta(X) - g(X,U)\eta(Y)]\eta(Z),\\
&(iii) \quad {}'\overline{R}(X,Y,Z,U) + {}'\overline{R}(Y,Z,X,U) + {}'\overline{R}(Z,X,Y,U) = 0,
\end{aligned}$$

where ${}'\overline{R}(X,Y,Z,U) = g(\overline{R}(X,Y)Z,U)$.

Proof: (i) By the virtue of the equations (2.1.1) and (2.1.9). The associative curvature tensor ${}'\overline{R}$ with respect to quarter symmetric non-metric connection ∇ is given as

$$\begin{aligned}
{}'\overline{R}(X,Y,Z,U) \ &= \ g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\\
&+ \ g(Y,Z)\eta(X)\eta(U) - g(X,Z)\eta(Y)\eta(U)\\
&+ \ g(Y,U)\eta(X)\eta(Z) - g(X,U)\eta(Y)\eta(Z).
\end{aligned} \tag{2.10.7}$$

Interchanging X and Y in (2.10.7) we obtain

$$\begin{aligned} {}'\overline{R}(Y,X,Z,U) &= g(X,Z)g(Y,U) - g(Y,Z)g(X,U) \\ &+ g(X,Z)\eta(Y)\eta(U) - g(X,Z)\eta(X)\eta(U) \\ &+ g(X,U)\eta(Y)\eta(Z) - g(Y,U)\eta(X)\eta(Z). \end{aligned} \tag{2.10.8}$$

Adding (2.10.7) and (2.10.8), we get

$$'\overline{R}(X,Y,Z,U) + {}'\overline{R}(Y,X,Z,U) = 0.$$

(ii) Again, interchanging Z and U in the equation (2.10.7), we have

$$\begin{aligned} {}'\overline{R}(X,Y,U,Z) &= g(Y,U)g(X,Z) - g(X,U)g(Y,Z) \\ &+ g(Y,U)\eta(X)\eta(Z) - g(X,U)\eta(Y)\eta(Z) \\ &+ g(Y,Z)\eta(X)\eta(U) - g(X,Z)\eta(Y)\eta(U). \end{aligned} \tag{2.10.9}$$

Adding (2.10.7) and (2.10.9),we get

$$\begin{aligned} {}'\overline{R}(X,Y,Z,U) + {}'\overline{R}(X,Y,U,Z) &= 2\,[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\eta(U) \\ &+ 2\,[g(Y,U)\eta(X) - g(X,U)\eta(Y)]\eta(Z). \end{aligned}$$

(iii) It is obvious that

$$'\overline{R}(X,Y,Z,U) + {}'\overline{R}(Y,Z,X,U) + {}'\overline{R}(Z,X,Y,U) = 0.$$

This completes the proposition.

2.11 Some more results

Let D be Riemannian connection and ∇ be a quarter symmetric non-metric connection in an LP-Sasakian manifold M^n. Let

$$\nabla_X Y = D_X Y + H(X,Y), \tag{2.11.1}$$

where H is a tensor of type (1,2). Using (2.1.7) and (2.11.1), we have

$$H(X,Y) = -\eta(X)\varphi Y. \tag{2.11.2}$$

The torsion tensor T of M^n with respect to ∇ is defined as

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{2.11.3}$$

By use of (2.11.1) and (2.11.2), we get

$$T(X,Y) = \eta(Y)\varphi X - \eta(X)\varphi Y. \tag{2.11.4}$$

The nijenhuis tensor N is defined as

$$\begin{aligned} N(X,Y) &= (D_{\varphi X}\varphi)(Y) - (D_{\varphi Y}\varphi)(X) \\ &- \varphi(D_X\varphi)(Y) + \varphi(D_Y\varphi)(X), \end{aligned} \tag{2.11.5}$$

and hence

$$\begin{aligned} {}'N(X,Y)Z &= (D_{\varphi X}\Phi)(Y,Z) - (D_{\varphi Y}\Phi)(X,Z) \\ &- \varphi(D_X\Phi)(Y,Z) + \varphi(D_Y\Phi)(X,Z), \end{aligned} \tag{2.11.6}$$

where ${}'N(X,Y,Z) \Rightarrow g(N(X,Y)Z)$ and $\Phi(X,Y) = g(\varphi X, Y)$.

Theorem 2.11.1 *In an LP-Sasakian manifold with a quarter symmetric non-metric connection, the following relations hold:*

$$\begin{aligned}
&(i) && {}'H(X,Y,Z) = {}'H(X,Z,Y),\\
&(ii) && {}'T(X,Y,Z) + {}'T(Y,Z,X) + {}'T(Z,X,Y) = 0,\\
&(iii) && {}'H(X,Y,Z) - {}'H(Y,X,Z) = {}'T(X,Y,Z) = 0,\\
&(iv) && {}'T(X,Y,Z) + {}'T(Y,X,Z) = 0,\\
&(v) && {}'H(\varphi X,Y,Z) + {}'H(\varphi Y,X,Z) + {}'H(\varphi Z,X,Y) = 0,\\
&(vi) && {}'H(\varphi^2 X,Y,Z) + {}'H(\varphi X,Y,Z) = 0,\\
&(vii) && -{}'H(X,Y,\varphi^2 Z) = {}'H(\varphi X,\varphi Y,Z) - {}'H(X,Y,Z),\\
&(viii) && {}'H(\varphi X,\varphi Y,Z) + {}'H(\varphi X,Y,Z) = {}'H(\varphi^2 X,Y,\varphi^2 Z),\\
&(ix) && {}'H(\varphi X,Y,\varphi Z) = {}'T(\varphi X,\varphi Y,Z) = 0,\\
&(x) && {}'H(\varphi^2 X,Y,\varphi^2 Z) = {}'T(\varphi^2 X,\varphi Y,Z) = 0,\\
&(xi) && \eta(H(\varphi X,\varphi Y)) - \eta(H(X,Y)) = 0,\\
&(xii) && {}'T(\varphi X,Y,Z) = {}'T(\varphi Z,Y,X),
\end{aligned} \tag{2.11.7}$$

where $'H(X,Y,Z) = g(H(X,Y)Z)$ *and* $'T(X,Y,Z) = g(T(X,Y)Z)$.

Proof: (i) By the virtue of the equation (2.11.2), we get

$$\begin{aligned} 'H(X,Y,Z) - {}'H(X,Z,Y) &= g(H(X,Y)Z) + g(H(X,Z)Y) \\ &= g(-\eta(X)\varphi Y, Z) + g(-\eta(X)\varphi Z, Y) \\ &= -\eta(X)g(\varphi Y, Z) + \eta(X)g(\varphi Z, Y) \\ &= 0, \end{aligned} \tag{2.11.8}$$

From the equation (2.11.4), we have

$$\begin{aligned} 'T(X,Y,Z) &+ {}'T(Y,Z,X) + {}'T(Z,X,Y) \\ &= g(T(X,Y)Z) + g(T(Y,Z)X) + g(T(Z,X)Y) \\ &= g[\eta(Y)\varphi X - \eta(X)\varphi Y, Z] + g[\eta(Z)\varphi Y \\ &- \eta(Y)\varphi Z, X] + g[\eta(X)\varphi Z - \eta(Z)\varphi X, Y] \\ &= \eta(Y)g(\varphi X, Z) - \eta(X)g(\varphi Y, Z) + \eta(Z)g(\varphi Y, X) \\ &- \eta(Y)g(\varphi Z, X) + \eta(X)g(\varphi Z, Y) - \eta(Z)g(\varphi X, Y), \\ &= \eta(Y)[g(\varphi X, Z) - g(\varphi Z, X)] + \eta(Z)[g(\varphi Y, X) \\ &- g(\varphi X, Y)] + \eta(X)[g(\varphi Z, Y) - g(\varphi Y, Z)]. \\ &= 0. \end{aligned} \tag{2.11.9}$$

Similarly all the above relations follows by simple calculation and using the properties previously obtained.

Theorem 2.11.2 *Let* ∇ *be a quarter symmetric non-metric connection in an LP-Sasakian manifold* M^n *with a Riemannian connection* D*. then*

$$(\nabla_X \Phi)(Y,Z) = (D_X \varphi)(Y,Z) + 2\ \eta(X)g(\varphi Y, \varphi Z). \tag{2.11.10}$$

Proof: We know that

$$X \cdot \Phi(Y,Z) = (D_X \Phi)(Y,Z) + \Phi(D_X Y, Z) + \Phi(Y, D_X Z), \tag{2.11.11}$$

and

$$X \cdot \Phi(Y,Z) = (\nabla_X \Phi)(Y,Z) + \Phi(\nabla_X Y, Z) + \Phi(Y, \nabla_X Z). \tag{2.11.12}$$

Subtracting (2.11.11) from (2.11.12) and using the equations (2.11.1) and (2.11.2), we have

$$\begin{aligned}(D_X\Phi)(Y,Z)-(\nabla_X\Phi)(Y,Z) &= \Phi[(\nabla_XY-D_XY),Z]+\Phi[Y,(\nabla_XZ-D_XZ)]\\ &= \Phi[H(X,Y),Z]+\Phi[Y,H(X,Z)]\\ &= g[H(X,Y),\varphi Z]+g[\varphi Y,H(X,Z)]\\ &= g[-\eta(X)\varphi Y,\varphi Z]+g[\varphi Y,-\eta(X)\varphi Z]\\ &= \eta(X)g(\varphi Y,\varphi Z)-\eta(X)g(\varphi Y,\varphi Z)\\ &= -2\,\eta(X)g(\varphi Y,\varphi Z).\end{aligned}$$

Hence we get (2.11.10).

Theorem 2.11.3 *In an LP-Sasakian manifold with a quarter symmetric non-metric connection ∇, the Nijenhuis tensor satisfies the following relations:*

$$\begin{aligned}N(X,Y) &= (\nabla_{\varphi X}\varphi)(Y)-(\nabla_{\varphi Y}\varphi)(X)-\varphi(\nabla_X\varphi Y)\\ &+ \varphi(\nabla_Y\varphi X)+\nabla_XY-\nabla_YX\\ &+ \eta[\nabla_XY-\nabla_YX].\end{aligned}\tag{2.11.13}$$

Proof: In consequence of (2.1.7) and the fact that

$$(D_X\varphi)(Y)=D_X\varphi Y-\varphi(D_XY),\tag{2.11.14}$$

the equation (2.11.5) assumes the form

$$\begin{aligned}N(X,Y) &= D_{\varphi_X}\varphi Y-\varphi(D_{\varphi_X}Y)-D_{\varphi_Y}\varphi X\\ &+ \varphi(D_{\varphi Y}X)-\varphi(D_X\varphi Y)+\varphi^2(D_XY)\\ &+ \varphi(D_Y\varphi X)-\varphi^2(D_YX).\end{aligned}$$

Using (1.16.1), in the above equation we have

$$\begin{aligned}N(X,Y) &= (\nabla_{\varphi X}\varphi)(Y)-(\nabla_{\varphi Y}\varphi)(X)-\varphi(\nabla_X\varphi Y)\\ &+ \varphi(\nabla_Y\varphi X)+\nabla_XY-\nabla_YX\\ &+ \eta[\nabla_XY-\nabla_YX].\end{aligned}$$

Hence we get (2.11.13).

Chapter 3

Certain Curvature Conditions on LP-Sasakian Manifolds

3.1 Introduction

In an n-dimensional LP-Sasakian manifold with structure (φ, ξ, η, g) defined in (1.16.1-1.16.7) also hold the following relations (De *et al.*, 1999)

$$R(X, Y, \xi) = \eta(Y)X - \eta(X)Y, \tag{3.1.1}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{3.1.2}$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \tag{3.1.3}$$

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{3.1.4}$$

$$S(\varphi X, \varphi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{3.1.5}$$

$$(\nabla_X \eta)(Y) = g(X, \varphi Y) = g(\varphi X, Y), \tag{3.1.6}$$

$$S(X, \xi) = (n-1)\ \eta(X), \tag{3.1.7}$$

for any vector fields X, Y, Z, where R is the curvature tensor, S is the Ricci tensor.
An LP-Sasakian manifold M^n is said to be Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = kg(X,Y), \tag{3.1.8}$$

where $k = n-1$.
An LP-Sasakian manifold M^n is said to be η-Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = \alpha g(X,Y) + \beta\ \eta(X)\eta(Y), \tag{3.1.9}$$

for arbitrary vector fields X and Y, where α and β are smooth functions on M^n (Yano and Kon 1984; Blair, 1976).
The notion of the quasi conformal curvature tensor $\overline{C}$ was introduced by (Yano and Sawaki, 1968). They defined the quasi conformal curvature tensor by

$$\begin{aligned} \overline{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &- \frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)\ [g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3.1.10}$$

where a and b are constants such that $a, b \neq 0$, R is the Riemannian curvature tensor, S is the Ricci tensor, Q is the Ricci operator and r is the scalar curvature of the manifold.

3.2 Quasi conformal curvature tensor of LP-Sasakian manifolds

Theorem 3.2.1 *If an LP-Sasakian manifold M^n satisfies the condition $\overline{C}(\xi, X)\cdot R = 0$, then*

$$S(QX,Y) = \frac{1}{b}\Big[\{b(n-1) - A\}S(X,Y) + A(n-1)g(X,Y)\Big],$$

where,

$$A = \Big[a + b(n-1) - \frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)\Big]$$

.

Proof: Putting $X = \xi$ in the equation (3.1.10) and using the equations (1.16.2), (1.16.4), (3.1.2), (3.1.5) and (3.1.7), we get

$$\begin{aligned}\overline{C}(\xi, Y)Z &= \Big[a + b(n-1) - \frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)\Big]\Big[g(Y,Z)\xi - \eta(Z)Y\Big] \\ &+ b\Big[S(Y,Z)\xi - \eta(Z)QY\Big]. \qquad (3.2.1)\end{aligned}$$

Again, putting $Z = \xi$ in equation (3.1.10) and using (1.16.2), (3.1.1) and (3.1.7), we get

$$\begin{aligned}\overline{C}(X, Y)\xi &= \Big[a + b(n-1) - \frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)\Big]\Big[\eta(Y)X - \eta(X)Y\Big] \\ &+ b\Big[\eta(Y)QX - \eta(X)QY\Big]. \qquad (3.2.2)\end{aligned}$$

Now, taking inner product of the equations (3.1.10), (3.2.1) and (3.2.2) with ξ and using the equations (1.16.4), (1.16.2) and (3.1.7) we get

$$\begin{aligned}\eta(\overline{C}(X, Y)Z) &= \Big[a + b(n-1) - \frac{r}{n}\,(\frac{a}{n-1} + 2b)\Big]\Big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\Big] \\ &+ b\Big[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\Big], \qquad (3.2.3)\end{aligned}$$

$$\begin{aligned}\eta(\overline{C}(\xi, Y)Z) &= -\eta(\overline{C}(Y,\xi)Z) \\ &= \Big[a + b(n-1) - \frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)\Big]\Big[-g(Y,Z) - \eta(Y)\eta(Z)\Big] \\ &+ b\Big[-S(Y,Z) - \eta(QY)\eta(Z)\Big], \qquad (3.2.4)\end{aligned}$$

and

$$\eta(\overline{C}(X, Y)\xi) = 0, \qquad (3.2.5)$$

respectively.

Let $\overline{C}(\xi, X)\cdot R(Y,Z)U = 0$. Then we have

$$\begin{aligned}&\overline{C}(\xi, X)R(Y,Z)U - R(\overline{C}(\xi, X)Y, Z)U \\ &-R(Y, \overline{C}(\xi, X)Z)U - R(Y,Z)\overline{C}(\xi, X)U = 0, \qquad (3.2.6)\end{aligned}$$

which on using the equation (3.2.1), gives

$$
\begin{aligned}
&\Big[a+b(n-1)-\frac{r}{n}\Big(\frac{a}{n-1}+2b\Big)\Big]\Big[g(X,R(Y,Z)U)\xi-\eta(R(Y,Z)U)X\\
&-g(X,Y)R(\xi,Z)U+\eta(Y)R(X,Z)U-g(X,Z)R(Y,\xi)U+\eta(Z)R(Y,X)U\\
&-g(X,U)R(Y,Z)\xi+\eta(U)R(Y,Z)X\Big]+b\Big[S(X,R(Y,Z)U)\xi-\eta(R(Y,Z)U)QX\\
&-S(X,Y)R(\xi,Z)U+\eta(Y)R(QX,Z)U-S(X,Z)R(Y,\xi)U+\eta(Z)R(Y,QX)U\\
&-S(X,U)R(Y,Z)\xi+\eta(U)R(Y,Z)QX\Big]=0. \qquad (3.2.7)
\end{aligned}
$$

Taking inner product of the above equation with ξ and using the equations (1.16.2) and (1.16.4), we get

$$
\begin{aligned}
&\Big[a+b(n-1)-\frac{r}{n}\Big(\frac{a}{n-1}+2b\Big)\Big]\Big[-g(X,R(Y,Z)U)-\eta(R(Y,Z)U)\eta(X)\\
&-g(X,Y)\eta(R(\xi,Z)U)+\eta(Y)\eta(R(X,Z)U)-g(X,Z)\eta(R(Y,\xi)U)\\
&+\eta(Z)\eta(R(Y,X)U)-g(X,U)\eta(R(Y,Z)\xi)+\eta(U)\eta(R(Y,Z)X)\Big]\\
&+b\Big[-\ 'R(Y,Z,U,QX)-\eta(R(Y,Z)U)\eta(QX)-S(X,Z)\eta(R(Y,\xi)U)\\
&+\eta(Z)\eta(R(Y,QX)U)-S(X,U)\eta(R(Y,Z)\xi)\\
&+\eta(U)\eta(R(Y,Z)QX)\Big]=0. \qquad (3.2.8)
\end{aligned}
$$

Using the equations (3.1.1), (3.1.2), (3.1.3) and (3.1.4) in above equation, we get

$$
\begin{aligned}
&\Big[a+b(n-1)-\frac{r}{n}\Big(\frac{a}{n-1}+2b\Big)\Big]\Big[-'R(Y,Z,U,X)+g(Z,U)g(X,Y)\\
&-g(Y,U)g(X,Z)\Big]+b\Big[-'R(Y,Z,U,QX)+S(X,Y)g(Z,U)\\
&-S(X,Z)g(Y,U)\Big]=0. \qquad (3.2.9)
\end{aligned}
$$

Putting $Z=U=e_i$ in above equation and taking summation over i, $1\leq i\leq n$, we get

$$
S(QX,Y)=\frac{1}{b}\Big[\{b(n-1)-A\}S(X,Y)+A(n-1)g(X,Y)\Big], \qquad (3.2.10)
$$

where,

$$
A=\Big[a+b(n-1)-\frac{r}{n}\Big(\frac{a}{n-1}+2b\Big)\Big].
$$

This completes the proof.

Lemma 3.2.1 *(Deszcz and Yapark, 1994) Let ρ be a symmetric (0,2)-tensor at point x of an LP-Sasakian manifold (M^n,g), $n>2$, and let $T=g\,\overline{\wedge}\,\rho$ be the Kulkarni-Nomizu* *product of g and ρ. Then, the relation*

$$T.T = \alpha\ Q(g,T), \qquad\qquad \alpha\epsilon\mathbb{R},$$

is satisfied at x if and only if the condition

$$\rho^2 = \alpha\rho + \lambda g, \qquad\qquad \lambda\epsilon\mathbb{R}$$

holds at x.

From theorem (3.2.1) and lemma (3.2.1), we get the followings

Corollary 3.2.1 *Let an LP-Sasakian manifold M^n satisfies the condition $\overline{C}(\xi, X)\cdot R = 0$, then $T.T = \alpha\ Q(g,T)$, where $T = g \barwedge S$ and $\alpha = \Big[a + b(n-1) - \frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)\Big]$.*

Remark 3.2.1 *Here S is a symmetric (0,2) tensor at point x of an LP-Sasakian manifold M^n and $T = g \barwedge \rho$ is the Kulkarni-Nomizu product of g and S. Hence*

$$S^2 = \alpha\ S + \lambda\ g, \qquad \alpha \in \mathbb{R},$$

where $\alpha = \Big[a + b(n-1) - \frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)\Big]$ and $\lambda = \Big[a + b(n-1) - \frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)\Big](n-1)$.

3.3 An Einstein LP-Sasakian manifold satisfying $\overline{C}(X,Y)Z = 0$

Theorem 3.3.1 *The scalar curvature r of quasi conformal flat LP-Sasakian manifold M^n is constant, given by $r = n(n-1)$, provided $a + 2(n-1)b \neq 0$.*

Proof: We assume that $\overline{C}(X,Y)Z = 0$, then from (3.1.10), we get

$$\begin{aligned} a\,{}'R(X,Y,Z,W) &= -b\Big[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &+ g(Y,Z)g(QX,W) - g(X,Z)g(QY,W)\Big] \\ &+ \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\Big[g(Y,Z)X - g(X,Z)Y\Big], \end{aligned} \tag{3.3.1}$$

where

$${}'R(X,Y,Z,W) = g(R(X,Y)Z,W)$$

. Putting $X = W = \xi$ in (3.3.1), we get

$$\begin{aligned} a\,'R(\xi, Y, Z, \xi) &= -b\big[-S(Y, Z) - S(\xi, Z)\eta(Y) + g(Y, Z)S(\xi, \xi) - \eta(Z)S(Y, \xi)\big] \\ &+ \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\big[-g(Y, Z) - \eta(Y)\eta(Z)\big]. \end{aligned} \tag{3.3.2}$$

In view of equation (3.1.8), (3.1.9) and (3.3.2), we get

$$[a + 2b(n-1)][n(n-1) - r]g(\varphi Y, \varphi Z) = 0. \tag{3.3.3}$$

Since $g(\varphi Y, \varphi Z) \neq 0$. Hence from (3.3.3), we get $r = n(n-1)$, provided $a + 2b(n-1) \neq 0$. Hence, prove the theorem.

Theorem 3.3.2 *If in an LP-Sasakian manifold the relation* $(P_1^1\overline{C})(Y, Z) = 0$ *hold, then* M^n *is an Einstein manifold with scalar curvature* $r = n(n-1)$, *provided* $a + (n-2)b \neq 0$.

Proof: Contracting (3.1.10) with respect to X, we get

$$\begin{aligned} (P_1^1\overline{C})(Y, Z) &= aS(Y, Z) + b[(n-1)S(Y, Z) + r\, g(Y, Z) - S(Y, Z)] \\ &- \frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)\Big[(n-1)g(Y, Z)\Big] \end{aligned}$$

$$(P_1^1\overline{C})(Y, Z) = \Big(a + (n-2)b\Big)\ \Big[S(Y, Z) - \frac{r}{n}g(Y, Z)\Big] \tag{3.3.4}$$

where contraction of $\overline{C}(X, Y)Z$ with respect to X is defined by $(P_1^1\overline{C})(Y, Z)$. Let us assume that in an LP-Sasakian manifold

$$(P_1^1\overline{C})(Y, Z) = 0. \tag{3.3.5}$$

From (3.3.4) and (3.3.5), we get

$$\Big(a + (n-2)b\Big)\ \Big[S(Y, Z) - \frac{r}{n}\ g(Y, Z)\Big] = 0. \tag{3.3.6}$$

If $a + (n-2)b \neq 0$, then from (3.3.6), we get

$$S(Y, Z) = \frac{r}{n}\ g(Y, Z), \tag{3.3.7}$$

which shows that M^n is an Einstein manifold.
Putting $Z=\xi$ in (3.3.6), we get

$$\Big(a+(n-2)b\Big)\Big(n(n-1)-r\Big)\eta(Y)=0. \tag{3.3.8}$$

Since $\eta(Y)\neq 0$, so, we get $r=n(n-1)$, provided $a+(n-2)b\neq 0$.
Hence, proves the theorem.

3.4 An Einstein LP-Sasakian manifold satisfying $R(X,Y)\cdot\overline{C}=0$

Theorem 3.4.1 *If in an Einstein LP-Sasakian manifold, the relation $R(X,Y)\cdot\overline{C}=0$ holds, then it is of constant curvature.*

Proof: From equation (3.1.10), we have

$$\begin{aligned}'\overline{C}(X,Y,Z,W) &= a\,'R(X,Y,Z,W)+\ b\Big[S(Y,Z)g(X,W)-S(X,Z)g(Y,W)\\ &+\ g(Y,Z)S(X,W)-g(X,Z)S(Y,W)\Big]-\frac{r}{n}\left(\frac{a}{n-1}+2b\right)\\ &\Big[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\Big],\end{aligned} \tag{3.4.1}$$

where $'\overline{C}(X,Y,Z,W)=g(\overline{C}(X,Y)Z,W)$ and $'R(X,Y,Z,W)=g(R(X,Y)Z,W)$.
Let the Riemannian manifold M^n be an Einstein manifold, then (3.4.1) gives

$$\begin{aligned}'\overline{C}(X,Y,Z,W) &= a\,'R(X,Y,Z,W)Z+\Big\{2\ bk-\frac{r}{n}\Big(\frac{a}{n-1}+2b\Big)\Big\}\\ &\Big[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\Big].\end{aligned} \tag{3.4.2}$$

Using (3.1.4) in (3.4.2), we get

$$\eta(\overline{C}(X,Y)Z)=\Big\{a+\ 2bk-\frac{r}{n}\Big(\frac{a}{n-1}+2b\Big)\Big\}\Big[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\Big] \tag{3.4.3}$$

and

$$\eta(\overline{C}(X,Y)\xi)=0. \tag{3.4.4}$$

Now,

$$\begin{aligned} R(X,Y)\cdot\overline{C}(U,V)W &= R(X,Y)\overline{C}(U,V)W-\overline{C}(R(X,Y)U,V)W \\ &- \overline{C}(U,R(X,Y)V)W-\overline{C}(U,V)R(X,Y)W. \end{aligned} \tag{3.4.5}$$

We assume that

$$R(X,Y)\cdot\overline{C}(U,V)W=0,$$

then from (3.4.5), we have

$$\begin{aligned} &R(X,Y)\overline{C}(U,V)W-\overline{C}(R(X,Y)U,V)W \\ &-\overline{C}(U,R(X,Y)V)W-\overline{C}(U,V)R(X,Y)W=0. \end{aligned} \tag{3.4.6}$$

Therefore,

$$\begin{aligned} &g(R(\xi,Y)\overline{C}(U,V)W,\xi)-g(\overline{C}(R(\xi,Y)U,V)W,\xi) \\ &-g(\overline{C}(U,R(\xi,Y)V)W)-g(\overline{C}(U,V)R(\xi,Y)W,\xi)=0. \end{aligned} \tag{3.4.7}$$

From this it follows that

$$\begin{aligned} &-'\overline{C}(U,V,W,Y)-\eta(Y)\eta(\overline{C}(U,V)W) \\ &+\eta(U)\eta(\overline{C}(Y,V)W)+\eta(V)\eta(\overline{C}(U,Y)W) \\ &+\eta(W)\eta(\overline{C}(U,V)Y)-g(Y,U)\eta(\overline{C}(\xi,V)W) \\ &-g(Y,V)\eta(\overline{C}(U,\xi)W)-g(Y,W)\eta(\overline{C}(U,V)\xi)=0. \end{aligned} \tag{3.4.8}$$

Putting $Y=U$ in (3.4.8), we get

$$\begin{aligned} &-'\overline{C}(U,V,W,U)+\eta(V)\eta(\overline{C}(U,U)W) \\ &+\eta(W)\eta(\overline{C}(U,V)U)-g(U,U)\eta(\overline{C}(\xi,V)W) \\ &-g(U,V)\eta(\overline{C}(U,\xi)W)-g(U,W)\eta(\overline{C}(U,V)\xi)=0. \end{aligned} \tag{3.4.9}$$

Let $\{e_i\}$, $i=1,2,3,...n$ be an orthonormal basis of the tangent space at any point. Then the sum for $1\le i\le n$ of the relation (3.4.9), for $U=e_i$, gives

$$
\begin{aligned}
\eta(\overline{C}(\xi,V)W) &= \Big\{a+2bk-\frac{r}{n}\Big(\frac{a}{n-1}+2b\Big)\Big\}\eta(V)\eta(W) \\
&+ \frac{1}{n}\Big[-aS(V,W)+(n-1)\Big\{a+2bk-\frac{r}{n}\Big(\frac{a}{n-1}+2b\Big)\Big\} \\
&\quad g(V,W)\Big]. \qquad (3.4.10)
\end{aligned}
$$

Using (3.4.2) and (3.4.10), it follows from (3.4.9) that

$$
\begin{aligned}
\overline{C}(U,V,W,Y) &= -\frac{1}{n}a\Big[S(V,W)g(Y,U)-S(U,W)g(Y,V)\Big] \\
&+ \frac{1}{n}\Big[a+(n-1)\Big\{2bk-\frac{r}{n}\Big(\frac{a}{n-1}+2b\Big)\Big\}\Big] \\
&\quad \Big[g(V,W)g(Y,U)-g(V,Y)g(U,W)\Big]. \qquad (3.4.11)
\end{aligned}
$$

Making use of (3.1.8), the equation (3.4.11) yields

$$
\begin{aligned}
\overline{C}(U,V,W,Y) &= (n-1)\Big[-\frac{a}{n}+\ 2bk-\frac{r}{n}\Big(\frac{a}{n-1}+2b\Big)\Big] \\
&\quad \Big[g(V,W)g(Y,U)-g(V,Y)g(U,W)\Big]. \qquad (3.4.12)
\end{aligned}
$$

From (3.4.2) and (3.4.12), we get

$$
a\,'R(U,V,W,Y)=\Big(\frac{n-1}{n}\Big)\,a\Big[g(V,W)g(Y,U)-g(V,Y)g(U,W)\Big],
$$

which gives
$'R(U,V,W,Y)=\frac{n-1}{n}\Big[g(V,W)g(Y,U)-g(V,Y)g(U,W)\Big]$, provided that $a\neq 0$.
This proves that it is of constant curvature tensor.

3.5 An Einstein LP-Sasakian manifold satisfying $(div\overline{C})(X,Y)Z=0$

Theorem 3.5.1 *An Einstein LP-Sasakian manifold $(M^n,g)(n>2)$ is quasi conformally conservative if and only if the scalar curvature is constant, provided $[b(n-4)(n-1)-2a]\neq 0$.*

Proof: A manifold $(M^n, g)(n > 2)$ is called quasi conformally conservative if $div\overline{C} = 0$ (Hicks, 1969). In this section we assume that

$$div\overline{C} = 0, \tag{3.5.1}$$

where div denotes divergence. Now differentiating the equation (3.1.10) covariantly, we get

$$\begin{aligned} (D_U\overline{C})(X,Y)Z &= a(D_UR)(X,Y)Z \\ &+ b\Big[(D_US)(Y,Z)X - (D_US)(X,Z)Y \\ &+ g(Y,Z)(D_UQ)(X) - g(X,Z)(D_UQ)(Y)\Big] \\ &- \frac{(D_Ur)}{n}\Big(\frac{a}{n-1} + 2b\Big)\Big[g(Y,Z)X - g(X,Z)Y\Big]. \end{aligned} \tag{3.5.2}$$

Contraction of the equation (3.5.2) gives

$$\begin{aligned} (div\overline{C})(X,Y)Z &= a(divR)(X,Y)Z \\ &+ b\Big[(D_XS)(Y,Z) - (D_YS)(X,Z) \\ &+ g(Y,Z)(divQ)(X) - g(X,Z)(divQ)(Y)\Big] \\ &- \frac{1}{n}\Big(\frac{a}{n-1} + 2b\Big)\Big[g(Y,Z)dr(X) - g(X,Z)dr(Y)\Big]. \end{aligned} \tag{3.5.3}$$

But from (Eisenhart, 1926), we have

$$(divR)(X,Y)Z = (D_XS)(Y,Z) - (D_YS)(X,Z). \tag{3.5.4}$$

If LP-Sasakian manifold is an Einstein manifolds, then from the equation (3.1.8) and (3.5.4), we get

$$(divR)(X,Y)Z = (D_XS)(Y,Z) - (D_YS)(X,Z) = 0. \tag{3.5.5}$$

From equation (3.5.3) and (3.5.5), we get

$$(div\overline{C})(X,Y)Z = \frac{1}{n}\Big[\frac{b(n-4)}{2} - \frac{a}{n-1}\Big]\Big[g(Y,Z)dr(X) - g(X,Z)dr(Y)\Big]. \tag{3.5.6}$$

From equation (3.5.1) and (3.5.6), we get

$$g(Y,Z)dr(X) - g(X,Z)dr(Y) = 0, \tag{3.5.7}$$

provided $[b(n-4)(n-1)-2a] \neq 0$ which shows that r is constant. Again if r is constant then from equation (3.5.6) , we get

$$(div\overline{C})(X,Y)Z = 0. \tag{3.5.8}$$

This completes the proof.

3.6 φ-Quasi conformally flat LP-Sasakian manifolds

Theorem 3.6.1 *Let M^n be an n-dimensional $(n > 2)$ φ-quasi conformally flat LP-Sasakian manifold, then M^n is an η-Einstein manifold.*

Proof: A differentiable manifold $(M^n, g)(n > 2)$ satisfying the condition (Cabrerizo *et al.*, 1999)

$$\varphi^2\overline{C}(\varphi X, \varphi Y)\varphi Z = 0, \tag{3.6.1}$$

is called φ-quasi conformal flat LP-Sasakian manifold.
Suppose that $(M^n, g)(n > 2)$ is a φ quasi conformally flat LP-Sasakian manifold. It is easy to see that $\varphi^2\overline{C}(\varphi X, \varphi Y)\varphi Z = 0$, holds if and only if $g(\overline{C}(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0$, for any vector fields X, Y, Z, W.
By the use of equation (3.1.10), φ-quasi conformal flat LP-Sasakian manifold (3.6.1) gives

$$\begin{aligned} a'R(\varphi X, \varphi Y, \varphi Z, \varphi W) &= -b\Big[S(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - S(\varphi X, \varphi Z)g(\varphi X, \varphi W) \\ &+ g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) - g(\varphi X, \varphi Z)S(\varphi Y, \varphi W)\Big] \\ &+ \frac{1}{n}\Big(\frac{a}{n-1} + 2b\Big)\Big[g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) \\ &- g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)\Big]. \end{aligned} \tag{3.6.2}$$

where

$$'R(X, Y, Z, W) = g(R(X, Y)Z, W).$$

Let $e_1, e_2, ..., e_{n-1}, \xi$ be a local orthonormal basis of vector fields in M^n by using the fact that $\varphi e_1, \varphi e_2, ..., \varphi e_{n-1}, \xi$ is also a local orthonormal basis, if we put $X = W = e_i$ in equation (3.6.2) and sum up with respect to i, then we have

$$\begin{aligned}\sum_{i=1}^{n-1} a\,'R(\varphi e_i, \varphi Y, \varphi Z, \varphi e_i) &= -b\sum_{i=1}^{n-1}\Big[S(\varphi Y, \varphi Z)g(\varphi e_i, \varphi e_i) - S(\varphi e_i, \varphi Z)g(\varphi Y, \varphi e_i)\\
&+ g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i) - g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i)\Big]\\
&+ \frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)\Big[g(\varphi Y, \varphi Z)g(\varphi e_i, \varphi e_i)\\
&- g(\varphi e_i, \varphi Z)g(\varphi Y, \varphi e_i)\Big]. \qquad (3.6.3)\end{aligned}$$

On an LP-Sasakian manifold, we have (Özgür, 2003)

$$\sum_{i=1}^{n-1} {}'R(\varphi e_i, \varphi Y, \varphi Z, \varphi e_i) = S(\varphi Y, \varphi Z) + g(\varphi Y, \varphi Z), \qquad (3.6.4)$$

$$\sum_{i=1}^{n-1} S(\varphi e_i, \varphi e_i) = r + (n-1), \qquad (3.6.5)$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i) = S(\varphi Y, \varphi Z), \qquad (3.6.6)$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi e_i) = n + 1, \qquad (3.6.7)$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi Z)g(\varphi Y, \varphi e_i) = g(\varphi Y, \varphi Z). \qquad (3.6.8)$$

So by virtue of the equations (3.6.4)-(3.6.8), the equation (3.6.3) takes the form as

$$S(\varphi Y, \varphi Z) = \Big(\frac{r}{n-1} - 1\Big)g(\varphi Y, \varphi Z). \qquad (3.6.9)$$

By making the use of (1.16.3) and (3.1.5) in the equation (3.6.10), we get

$$S(Y, Z) = \Big(\frac{r}{n-1} - 1\Big)g(Y, Z) + \Big(\frac{r}{n-1} - n\Big)\eta(Y)\eta(Z), \qquad (3.6.10)$$

which shows that, M^n is an η-Einstein manifold.
Hence, proves the theorem.

3.7 M-projective Curvature Tensor of LP-Sasakian manifolds

(Pokhariyal and Mishra, 1971) defined a tensor field W^* on a Riemannian manifold M^n as

$$\begin{aligned} W^*(X,Y)Z &= R(X,Y)Z - \frac{1}{2(n-1)}\Big[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\Big], \end{aligned} \tag{3.7.1}$$

for vector fields X, Y and Z, where S is the Ricci tensor of type (0,2), Q is the Ricci operator and $'W^*(X,Y,Z,U) = g(W^*(X,Y)Z,U)$.
Putting $X = \xi$ in equation (3.7.1) and using equations (1.16.2), (1.16.4), (3.1.2) and (3.1.7) we get

$$\begin{aligned} W^*(\xi,Y)Z &= -W^*(Y,\xi)Z \\ &= \frac{1}{2}\Big[g(Y,Z)\xi - \eta(Z)Y\Big] \\ &- \frac{1}{2(n-1)}\Big[S(Y,Z)\xi - \eta(Z)QY\Big]. \end{aligned} \tag{3.7.2}$$

Again, putting Z=ξ in equation (3.7.1) and using equations (1.16.4), (3.1.1) and (3.1.7), we get

$$\begin{aligned} W^*(X,Y)\xi &= \frac{1}{2}\Big[\eta(Y)X - \eta(X)Y\Big] \\ &- \frac{1}{2(n-1)}\Big[\eta(Y)QX - \eta(X)QY\Big]. \end{aligned} \tag{3.7.3}$$

Now, taking the inner product of equations (3.7.1), (3.7.2) and (3.7.3) with ξ and using equations (1.16.2), (1.16.4) and (3.1.7), we get

$$\begin{aligned} \eta(W^*(X,Y)Z) &= \frac{1}{2}\Big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\Big] \\ &- \frac{1}{2(n-1)}\Big[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\Big], \end{aligned} \tag{3.7.4}$$

$$\begin{aligned}\eta(W^*(\xi,Y)Z) &= -\eta(W^*(Y,\xi)Z) \\ &= -\frac{1}{2}g(Y,Z) + \frac{1}{2(n-1)}S(Y,Z), \end{aligned} \tag{3.7.5}$$

and

$$\eta(W^*(X,Y)\xi) = 0, \tag{3.7.6}$$

respectively.

Theorem 3.7.1 *An LP- Sasakian manifold M^n satisfying the condition $R(\xi, X) \cdot W^* = 0$, is an Einstein manifold.*

Proof: Let $(R(\xi, X) \cdot W^*)(Y, Z)U = 0$. Then we have

$$\begin{aligned} &R(\xi,X)W^*(Y,Z)U - W^*(R(\xi,X)Y,Z)U \\ &-W^*(Y,R(\xi,X)Z)U - W^*(Y,Z)R(\xi,X)U = 0, \end{aligned} \tag{3.7.7}$$

which on using equation (3.1.2), gives

$$\begin{aligned} &g(X,W^*(Y,Z)U)\xi - \eta(W^*(Y,Z)U)X - g(X,Y)W^*(\xi,Z)U \\ &-g(X,Z)W^*(Y,\xi)U - g(X,U)W^*(Y,Z)\xi + \eta(Y)W^*(X,Z)U \\ &+\eta(Z)W^*(Y,X)U + \eta(U)W^*(Y,Z)X = 0. \end{aligned} \tag{3.7.8}$$

Now, taking the inner product of above equation with ξ and using equations (1.16.2), (1.16.4), (3.1.2), (3.7.1), (3.7.4), (3.7.5) and (3.7.6), we obtain

$$\begin{aligned} {}'R(Y,Z,U,X) &= g(X,Y)g(Z,U) - g(X,Z)g(Y,U) \\ &+ \frac{1}{2}\Big[g(X,Z)\eta(Y)\eta(U) - g(X,Y)\eta(Z)\eta(U)\Big] \\ &+ \frac{1}{2(n-1)}\Big[S(X,Y)\eta(Z)\eta(U) - S(X,Z)\eta(Y)\eta(U)\Big]. \end{aligned} \tag{3.7.9}$$

Taking a frame field and contraction over Z and U, we get

$$S(X,Y) = (n-1)g(X,Y).$$

This shows that M^n is an Einstein manifold.

Theorem 3.7.2 *If an LP-Sasakian manifold M^n satisfies the condition $W^*(\xi, X) \cdot R = 0$,* *then*

$$S(QX,Y) = (n-1)^2\ g(X,Y).$$

Proof: Let $(W^*(\xi,X)\cdot R)(Y,Z)U = 0$. Then, we have

$$\begin{aligned} &W^*(\xi,X)R(Y,Z)U - R(W^*(\xi,X)Y,Z)U \\ &-R(Y,W^*(\xi,X)Z)U - R(Y,Z)W^*(\xi,X)U = 0, \end{aligned} \tag{3.7.10}$$

which on using equation (3.7.2), gives

$$\begin{aligned} &g(X,R(Y,Z)U)\xi - \eta(R(Y,Z)U)X - g(X,Y)R(\xi,Z)U \\ &+\eta(Y)R(X,Z)U - g(X,Z)R(Y,\xi)U + \eta(Z)R(Y,X)U \\ &-g(X,U)R(Y,Z)\xi + \eta(U)R(Y,Z)X - \frac{1}{n-1}\Big[S(X,R(Y,Z)U)\xi \\ &-\eta(R(Y,Z)U)QX - S(X,Y)R(\xi,Z)U + \eta(Y)R(QX,Z)U \\ &-S(X,Z)R(Y,\xi)U + \eta(Z)R(Y,QX)U - S(X,U)R(Y,Z)\xi \\ &+\eta(U)R(Y,Z)QX\Big] = 0. \end{aligned} \tag{3.7.11}$$

Now, taking the inner product of above equation with ξ and using equations (1.16.2), (1.16.4), (3.1.2), (3.1.1) and (3.1.7), we obtain

$$\begin{aligned} &g(X,R(Y,Z)U) - g(X,Y)\eta(R(\xi,Z)U) + \eta(Y)\eta(R(X,Z)U) \\ &-g(X,Z)\eta(R(Y,\xi)U) + \eta(Z)\eta(R(Y,X)U) - g(X,U)\eta(R(Y,Z)\xi) \\ &+\eta(U)\eta(R(Y,Z)X) - \frac{1}{n-1}\Big[\,'R(Y,Z,U,QX) - S(X,Y)\eta(R(\xi,Z)U) \\ &+\eta(Y)\eta(R(QX,Z)U) - S(X,Z)\eta(R(Y,\xi)U) + \eta(Z)\eta(R(Y,QX)U) \\ &-S(X,U)\eta(R(Y,Z)\xi) + \eta(U)\eta(R(Y,Z)QX)\Big] = 0. \end{aligned} \tag{3.7.12}$$

Taking a frame field and contraction over Z and U, we get

$$S(QX,Y) = (n-1)^2\ g(X,Y).$$

This completes the proof.

Theorem 3.7.3 *If an LP-Sasakian manifold M^n satisfies the condition $W^*(\xi,X)\cdot S = 0$, then*

$$S(QX,Y) = -(n-1)^2 g(X,Y) + 2(n-1)S(X,Y).$$

Proof: Let $W^*(\xi, X) \cdot S(Y, Z) = 0$. Then, we have

$$S(W^*(\xi, X)Y, Z) + S(Y, W^*(\xi, X)Z) = 0, \tag{3.7.13}$$

which on using equation (3.7.2), gives

$$(n-1)\left[g(X,Y)\eta(Z) + g(X,Z)\eta(Y)\right] - S(X,Z)\eta(Y) - S(X,Y)\eta(Z)$$
$$+\frac{1}{(n-1)}\Big[S(QX,Y)\eta(Z) - S(QX,Z)\eta(Y)\Big] = 0. \tag{3.7.14}$$

Now, putting $Z = \xi$ in above equation and using equations (1.16.2), (1.16.4) and (3.1.7), we get

$$S(QX, Y) = -(n-1)^2 g(X, Y) + 2\ (n-1)S(X, Y).$$

This completes the proof.

3.8 LP-Sasakian manifolds satisfying $P(\xi, X) \cdot W^* = 0$ and $W^*(\xi, X) \cdot P = 0$

Projective curvature tensor P of the manifold M^n is given by (Mishra, 1984)

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\Big[S(Y,Z)X - S(X,Z)Y\Big]. \tag{3.8.1}$$

Putting X=ξ in above equation and using equation (3.1.2)and (3.1.7), we get

$$P(\xi, Y)Z = -P(Y, \xi)Z = g(Y,Z)\xi - \frac{1}{n-1}\ S(Y,Z)\xi. \tag{3.8.2}$$

Again, Putting Z=ξ in equation (3.8.1) and using equations (3.1.1) and (3.1.7), we get

$$P(X,Y)\xi = 0. \tag{3.8.3}$$

Now, taking the inner product of equations (3.8.1), (3.8.2) and (3.8.3) with ξ, we get

$$\begin{aligned} \eta(P(X,Y)Z &= g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \\ &- \frac{1}{n-1}\Big[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\Big], \end{aligned} \tag{3.8.4}$$

$$\eta(P(\xi,Y)Z) \;=\; -\eta(P(Y,\xi)Z) = -g(Y,Z) + \frac{1}{n-1}\; S(Y,Z), \tag{3.8.5}$$

and

$$\eta(P(X,Y)\xi = 0 \tag{3.8.6}$$

respectively.

Theorem 3.8.1 *If an LP-Sasakian manifold M^n satisfies the condition $P(\xi,X)\cdot W^* = 0$ then*

$$S(QX,Y) = 2(n-1)\;[S(X,Y) - (n-1)g(X,Y)].$$

Proof: Let $(P(\xi,X)\cdot W^*)(Y,Z)U = 0$. Then, we have

$$\begin{aligned} &P(\xi,X)W^*(Y,Z)U - W^*(P(\xi,X)Y,Z)U \\ &-W^*(Y,P(\xi,X)Z)U - W^*(Y,Z)P(\xi,X)U = 0, \end{aligned} \tag{3.8.7}$$

which on using equation (3.8.2), gives

$$\begin{aligned} &g(X,W^*(Y,Z)U)\xi - g(X,Y)W^*(\xi,Z)U - g(X,Z)W^*(Y,\xi)U \\ &-g(X,U)W^*(Y,Z)\xi - \frac{1}{n-1}\Big[S(X,W^*(Y,Z)U)\xi - S(X,Y)W^*(\xi,Z)U \\ &-S(X,Z)W^*(Y,\xi)U - S(X,U)W^*(Y,Z)\xi\Big] = 0. \end{aligned} \tag{3.8.8}$$

Now, taking the inner product of above equation with ξ and using equation (1.16.2), (1.16.4), (3.7.1), (3.7.4), (3.7.5) and (3.7.6), we obtain

$$\begin{aligned} \frac{1}{(n-1)}\;'R(Y,Z,U,QX) \;&=\; 'R(Y,Z,U,X) + \frac{1}{2}\Big[g(X,Z)g(Y,U) - g(X,Y)g(Z,U)\Big] \\ &- \;\frac{1}{2(n-1)^2}\Big[g(Z,U)S(QX,Y) - g(Y,U)S(QX,Z)\Big]. \end{aligned} \tag{3.8.9}$$

Taking a frame field and contraction over Z and U, we get

$$S(QX,Y) = 2(n-1)\;[S(X,Y) - (n-1)g(X,Y)].$$

This completes the proof.

Theorem 3.8.2 *If an LP-Sasakian manifold M^n satisfies the condition $W^*(\xi,X)\cdot P = 0$*

then

$$S(QX,Y) = -\frac{n(n-1)^2}{n-2}g(X,Y) + \frac{2n(n-1)}{n-2}S(X,Y).$$

Proof: Let $(W^*(\xi,X)\cdot P)(Y,Z)U = 0$. Then, we have

$$\begin{aligned} &W^*(\xi,X)P(Y,Z)U - P(W^*(\xi,X)Y,Z)U \\ &-P(Y,W^*(\xi,X)Z)U - P(Y,Z)W^*(\xi,X)U = 0, \end{aligned} \tag{3.8.10}$$

which on using equation (3.7.2), gives

$$\begin{aligned} &g(X,P(Y,Z)U)\xi - \eta(P(Y,Z)U)X - g(X,Y)P(\xi,Z)U + \eta(Y)P(X,Z)U \\ &-g(X,Z)P(Y,\xi)U + \eta(Z)P(Y,X)U - g(X,U)P(Y,Z)\xi + \eta(U)P(Y,Z)X \\ &-\frac{1}{n-1}\Big[S(X,P(Y,Z)U)\xi - \eta(P(Y,Z)U)QX - S(X,Y)P(\xi,Z)U \\ &+\eta(Y)P(QX,Z)U - S(X,Z)P(Y,\xi)U + \eta(Z)P(Y,QX)U \\ &-S(X,U)P(Y,Z)\xi + \eta(U)P(Y,Z)QX\Big] = 0. \end{aligned} \tag{3.8.11}$$

Now, taking the inner product of above equation with ξ and using equations (1.16.2), (1.16.4), (3.8.1) , (3.8.4), (3.8.5) and (3.8.6), we obtain

$$\begin{aligned} &-{}'R(Y,Z,U,X) + g(X,Y)g(Z,U) - g(X,Z)g(Y,U) + g(X,Z)\eta(Y)\eta(U) \\ &-g(X,Y)\eta(Z)\eta(U) - \frac{1}{(n-1)}\Big[\,{}'R(Y,Z,U,QX) + 2S(X,Y)\eta(Z)\eta(U) \\ &-2S(X,Z)\eta(Y)\eta(U) + S(X,Z)g(Y,U) - S(X,Y)g(Z,U)\Big] \\ &+\frac{1}{(n-1)^2}\Big[S(QX,Z)\eta(Y)\eta(U) - S(QX,Y)\eta(Z)\eta(U)\Big] = 0. \end{aligned} \tag{3.8.12}$$

Put $Z = U = e_i$ in above equation and taking summation over i, $1 \le i \le n$, we get

$$S(QX,Y) = -\frac{n(n-1)^2}{n-2}\; g(X,Y) + \frac{2n(n-1)}{n-2}\; S(X,Y).$$

This completes the proof.

3.9 LP-Sasakian manifolds Satisfying $C(\xi, X) \cdot W^* = 0$ and $W^*(\xi, X) \cdot C = 0$

Conformal curvature tensor C of the manifold M^n is given by (Mihai *et al.*, 1999)

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{(n-2)}\Big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY\Big] + \frac{r}{(n-1)(n-2)}\Big[g(Y,Z)X - g(X,Z)Y\Big]. \end{aligned} \tag{3.9.1}$$

Putting $X = \xi$ in above equation and using equation (3.1.2) and (3.1.7), we get

$$\begin{aligned} C(\xi,Y)Z &= -C(Y,\xi)Z \\ &= \frac{1+r-n}{(n-1)(n-2)}\Big[g(Y,Z)\xi - \eta(Z)Y\Big] \\ &- \frac{1}{n-2}\Big[S(Y,Z)\xi - \eta(Z)QY\Big]. \end{aligned} \tag{3.9.2}$$

Again, Putting $Z = \xi$ in equation (3.9.1) and using equations (3.1.1) and (3.1.7), we get

$$\begin{aligned} C(X,Y)\xi &= \frac{1+r-n}{(n-1)(n-2)}\Big[\eta(Y)X - \eta(X)Y\Big] \\ &- \frac{1}{n-2}\Big[\eta(Y)QX - \eta(X)QY\Big]. \end{aligned} \tag{3.9.3}$$

Now, taking the inner product of equations (3.9.1), (3.9.2) and (3.9.3) with ξ, we get

$$\begin{aligned} \eta(C(X,Y)Z &= \frac{1+r-n}{(n-1)(n-2)}\Big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\Big] \\ &- \frac{1}{n-2}\Big[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\Big], \end{aligned} \tag{3.9.4}$$

$$\begin{aligned} \eta(C(\xi,Y)Z) &= -\eta(C(Y,\xi)Z) \\ &= \frac{1+r-n}{(n-1)(n-2)}\Big[-g(Y,Z) - \eta(Y)\eta(Z)\Big] \\ &- \frac{1}{n-2}\Big[-S(Y,Z) - \eta(Z)\eta(QY)\Big] \end{aligned} \tag{3.9.5}$$

and

$$\eta(C(X,Y)\xi = 0 \tag{3.9.6}$$

respectively.

Theorem 3.9.1 *If an LP-Sasakian manifold M^n satisfies the condition $W^*(\xi, X)\cdot C = 0$ then the manifold is an Einstein manifold.*

Proof: Let $(W^*(\xi, X)\cdot C)(Y, Z)U = 0$. Then, we have

$$\begin{aligned} &W^*(\xi, X)C(Y, Z)U - C(W^*(\xi, X)Y, Z)U \\ &-C(Y, W^*(\xi, X)Z)U - C(Y, Z)W^*(\xi, X)U = 0, \end{aligned} \tag{3.9.7}$$

which on using equation (3.7.2), gives

$$\begin{aligned} &g(X, C(Y, Z)U) - \eta(C(Y, Z)U)X - g(X, Y)C(\xi, Z)U \\ &+\eta(Y)C(X, Z)U - g(X, Z)C(Y, \xi)U + \eta(Z)C(Y, X)U \\ &-g(X, U)C(Y, Z)\xi + \eta(U)C(Y, Z)X - \frac{1}{n-1}\Big[S(X, C(Y, Z)U)\xi \\ &-\eta(C(Y, Z)U)QX - S(X, Y)C(\xi, Z)U + \eta(Y)C(QX, Z)U \\ &-S(X, Z)C(Y, \xi)U + \eta(Z)C(Y, QX)U - S(X, U)C(Y, Z)\xi \\ &+\eta(U)C(Y, Z)QX\Big] = 0. \end{aligned} \tag{3.9.8}$$

Now, taking the inner product of above equation with ξ and using equations (1.16.2), (1.16.4), (3.9.1), (3.9.4), (3.9.5) and (3.9.6), we obtain

$$\begin{aligned} &-\frac{1}{(n-1)}\ 'R(Y, Z, U, QX) = -\ 'R(Y, Z, U, X) + \frac{1}{(n-2)}\Big[S(X, Y)g(Z, U) \\ &-S(X, Z)g(Y, U) + (n-1)\{g(X, Z)\eta(Y)\eta(U) - g(X, Y)\eta(Z)\eta(U)\} \\ &+2\{S(X, Y)\eta(Z)\eta(U) - S(X, Z)\eta(Y)\eta(U)\} + \frac{1}{(n-1)}\ \{S(QX, Z)g(Y, U) \\ &-S(QX, Y)g(Z, U) + S(QX, Z)\eta(Y)\eta(U) - S(QX, Y)\eta(Z)\eta(U)\}\Big] \\ &-\frac{r}{(n-1)(n-2)}\Big[g(Z, U)g(X, Y) - g(Y, U)g(X, Z) + \frac{1}{(n-1)}\{g(Z, U)S(X, Y) \\ &-g(Y, U)S(X, Z)\}\Big] + \frac{1+r-n}{(n-1)(n-2)}\Big[g(Z, U)g(X, Y) - g(Y, U)g(X, Z) \\ &+\frac{1}{(n-1)}\{S(X, Z)g(Y, U) - S(X, Y)g(Z, U)\}\Big]. \end{aligned} \tag{3.9.9}$$

Put $Z = U = e_i$ in above equation and taking summation over i, $1 \leq i \leq n$, we get

$$S(X, Y) = 2r\ g(X, Y).$$

This shows that the manifold is an Einstein manifold.

Theorem 3.9.2 *If an LP-Sasakian manifold M^n satisfies the condition $C(\xi, X)\cdot W^* = 0$ then*

$$S(QX, Y) = \Big(\frac{n^2 - 3n + r + 2}{n-1}\Big)\ S(X, Y) + (1 + r - n)\ g(X, Y).$$

Proof: Let $(C(\xi, X) \cdot W^*)(Y, Z)U = 0$. Then, we have

$$\begin{aligned} &C(\xi, X)W^*(Y, Z)U - W^*(C(\xi, X)Y, Z)U \\ &-W^*(Y, C(\xi, X)Z)U - W^*(Y, Z)C(\xi, X)U = 0, \end{aligned} \tag{3.9.10}$$

which on using equation (3.9.2), gives

$$\begin{aligned} &\frac{1 + r - n}{(n-1)(n-2)}\Big[g(X, W^*(Y, Z)U)\xi - \eta(W^*(Y, Z)U)X - g(X, Y)W^*(\xi, Z)U \\ &+\eta(Y)W^*(X, Z)U - g(X, Z)W^*(Y, \xi)U + \eta(Z)W^*(Y, X)U \\ &-g(X, U)W^*(Y, Z)\xi + \eta(U)W^*(Y, Z)X\Big] - \frac{1}{n-2}\Big[S(X, W^*(Y, Z)U)\xi \\ &-\eta(W^*(Y, Z)U)QX - S(X, Y)W^*(\xi, Z)U + \eta(Y)W^*(QX, Z)U \\ &-S(X, Z)W^*(Y, \xi)U + \eta(Z)W^*(Y, QX)U - S(X, U)W^*(Y, Z)\xi \\ &+\eta(U)W^*(Y, Z)QX\Big] = 0. \end{aligned} \tag{3.9.11}$$

Now, taking the inner product of above equation with ξ and using equations (1.16.2) and (1.16.4), we get

$$\begin{aligned} &\frac{1 + r - n}{(n-1)(n-2)}\Big[- g(X, W^*(Y, Z)U) - \eta(W^*(Y, Z)U)\eta(X) - g(X, Y)\eta(W^*(\xi, Z)U) \\ &+\eta(Y)\eta(W^*(X, Z)U) - g(X, Z)\eta(W^*(Y, \xi)U) + \eta(Z)\eta(W^*(Y, X)U) \\ &-g(X, U)\eta(W^*(Y, Z)\xi) + \eta(U)\eta(W^*(Y, Z)X)\Big] - \frac{1}{n-2}\Big[- S(X, W^*(Y, Z)U) \\ &-\eta(W^*(Y, Z)U)\eta(QX) - S(X, Y)\eta(W^*(\xi, Z)U) + \eta(Y)\eta(W^*(QX, Z)U) \\ &-S(X, Z)\eta(W^*(Y, \xi)U) + \eta(Z)\eta(W^*(Y, QX)U) - S(X, U)\eta(W^*(Y, Z)\xi) \\ &+\eta(U)\eta(W^*(Y, Z)QX)\Big] = 0. \end{aligned} \tag{3.9.12}$$

Using the equations (3.7.1), (3.7.4), (3.7.5) and (3.7.6) in above equation, we obtain

$$\frac{1 + r - n}{(n-1)(n-2)}\Big[- \ 'R(Y, Z, U, X) + \frac{1}{2(n-1)}\{g(Z, U)S(X, Y) - g(Y, U)S(X, Z)\}$$

$$+\frac{1}{2}\{g(X,Y)g(Z,U)-g(X,Z)g(Y,U)+g(X,Z)\eta(Y)\eta(U)-g(X,Y)\eta(Z)\eta(U)\}$$
$$+\frac{1}{2(n-1)}\{S(X,Y)\eta(Z)\eta(U)-S(Z,X)\eta(Y)\eta(U)\}\Big]$$
$$-\frac{1}{(n-2)}\Big[-\ 'R(Y,Z,U,QX)+\frac{1}{2(n-1)}\{g(Z,U)S(QX,Y)-g(Y,U)S(QX,Z)\}$$
$$+\frac{1}{2}\{S(X,Y)g(Z,U)-S(X,Z)g(Y,U)+S(X,Z)\eta(Y)\eta(U)-S(X,Y)\eta(Z)\eta(U)\}$$
$$+\frac{1}{2(n-1)}\{S(QX,Y)\eta(Z)\eta(U)-S(QX,Z)\eta(Y)\eta(U)\}\Big]=0. \tag{3.9.13}$$

Put $Z=U=e_i$ in above equation and taking summation over i, $1\leq i\leq n$, we get

$$S(QX,Y)=\Big(\frac{n^2-3n+r+2}{n-1}\Big)S(X,Y)+(1-n+r)\ g(X,Y).$$

This completes the proof.

3.10 LP-Sasakian manifolds Satisfying $\overline{C}(\xi,X).W^*=0$

The notion of the quasi-conformal curvature tensor $\overline{C}$ was introduced by (Yano and Sawaki, 1968). They defined the quasi-conformal curvature tensor by

$$\begin{aligned}\overline{C}(X,Y)Z &= aR(X,Y)Z+\ b\Big[S(Y,Z)X-S(X,Z)Y\\ &+\ g(Y,Z)QX-g(X,Z)QY\Big]\\ &-\ \frac{r}{n}\Big(\frac{a}{n-1}+2b\Big)\Big[g(Y,Z)X-g(X,Z)Y\Big],\end{aligned} \tag{3.10.1}$$

where a and b are constants such that $ab\neq 0$. If a=1 and $b=-\frac{1}{n-2}$, then above equation reduces to conformal curvature tensor given by (3.9.1). Thus the conformal curvature tensor C is a particular case of the Quasi-conformal curvatue tensor $\overline{C}$.
Putting $X=\xi$ in equation (3.10.1) and using equations (3.1.2) and (3.1.7), we get

$$\begin{aligned}\overline{C}(\xi,Y)Z &= -\overline{C}(Y,\xi)Z)=\Big\{a+b(n-1)-\frac{r}{n}(\frac{a}{n-1}+2b)\Big\}\\ &\Big[g(Y,Z)\xi-\eta(Z)Y\Big]+b\Big[S(Y,Z)\xi-\eta(Z)QY\Big].\end{aligned} \tag{3.10.2}$$

Again, Putting $Z = \xi$ in equation (3.10.1) and using equations (3.1.1) and (3.1.7), we get

$$\begin{aligned}\overline{C}(X,Y)\xi &= \Big\{a + b(n-1) - \frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)\Big\}\ \Big[\eta(Y)X - \eta(X)Y\Big] \\ &+ b\Big[\eta(Y)QX - \eta(X)QY\Big]. \end{aligned} \tag{3.10.3}$$

Now, taking the inner product of equations (3.10.1), (3.10.2) and (3.10.3) with ξ, we get

$$\begin{aligned}\eta(\overline{C}(X,Y)Z) &= \Big\{a + b(n-1) - \frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)\Big\}\ \Big[g(Y,Z)\eta(X) \\ &- g(X,Z)\eta(Y)\Big] + b\Big[\eta(Y)QX - \eta(X)QY\Big], \end{aligned} \tag{3.10.4}$$

$$\begin{aligned}\eta(\overline{C}(\xi,Y)Z) &= -\eta(C(Y,\xi)Z) \\ &= \Big\{a + b(n-1) - \frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)\Big\}\Big[-g(Y,Z) - \eta(Y)\eta(Z)\Big] \\ &+ b\,[-S(Y,Z) - \eta(Z)\eta(QY)] \end{aligned} \tag{3.10.5}$$

and

$$\eta(\overline{C}(X,Y)\xi) = 0 \tag{3.10.6}$$

respectively.

Theorem 3.10.1 *If an LP-Sasakian manifold M^n satisfies the condition $\overline{C}(\xi,X)\cdot W^* = 0$ then*

$$\begin{aligned} S(QX,Y) &= \Big[(n-1) - \frac{A}{b}\Big]\ S(X,Y) - \Big[\frac{2(n-1)+r}{n}\Big]\frac{A}{b}\ \eta(X)\eta(Y) \\ &+ \Big[\frac{n(n-1)-r}{n}\Big]\frac{A}{b}\ g(X,Y), \end{aligned}$$

where $A = \Big[a + b(n-1) - \frac{r}{n}(\frac{a}{n-1} + 2b)\Big]$.

Proof: Let $(\overline{C}(\xi,X)\cdot W^*)(Y,Z)U = 0$. Then, we have

$$\begin{aligned}&\overline{C}(\xi,X)W^*(Y,Z)U - W^*(\overline{C}(\xi,X)Y,Z)U \\ &-W^*(Y,\overline{C}(\xi,X)Z)U - W^*(Y,Z)\overline{C}(\xi,X)U = 0,\end{aligned} \tag{3.10.7}$$

which on using equation (3.10.2), gives

$$A\Big[g(X,W^*(Y,Z)U)\xi - \eta(W^*(Y,Z)U)X - g(X,Y)W^*(\xi,Z)U$$

$$
\begin{aligned}
&+\eta(Y)W^*(X,Z)U - g(X,Z)W^*(Y,\xi)U + \eta(Z)W^*(Y,X)U \\
&-g(X,U)W^*(Y,Z)\xi + \eta(U)W^*(Y,Z)X\Big] \\
&+b\Big[S(X,W^*(Y,Z)U)\xi + \eta(W^*(Y,Z)U)X - S(X,Y)W^*(\xi,Z)U \\
&+\eta(Y)W^*(QX,Z)U - S(X,Z)W^*(Y,\xi)U + \eta(Z)W^*(Y,QX)U \\
&-S(X,U)W^*(Y,Z)\xi + \eta(U)W^*(Y,Z)QX\Big] = 0, \qquad (3.10.8)
\end{aligned}
$$

Now, taking the inner product of above equation with ξ and using equations (1.16.2) and (1.16.4), we get

$$
\begin{aligned}
&A\Big[-g(X,W^*(Y,Z)U) - \eta(W^*(Y,Z)U)\eta(X) - g(X,Y)\eta(W^*(\xi,Z)U) \\
&+\eta(Y)\eta(W^*(X,Z)U) - g(X,Z)\eta(W^*(Y,\xi)U) + \eta(Z)\eta(W^*(Y,X)U) \\
&-g(X,U)\eta(W^*(Y,Z)\xi) + \eta(U)\eta(W^*(Y,Z)X)\Big] \\
&+b\Big[-S(X,W^*(Y,Z)U) + \eta(W^*(Y,Z)U)\eta(X) - S(X,Y)\eta(W^*(\xi,Z)U) \\
&+\eta(Y)\eta(W^*(QX,Z)U) - S(X,Z)\eta(W^*(Y,\xi)U) + \eta(Z)\eta(W^*(Y,QX)U) \\
&-S(X,U)\eta(W^*(Y,Z)\xi) + \eta(U)\eta(W^*(Y,Z)QX)\Big] = 0, \qquad (3.10.9)
\end{aligned}
$$

using the equations (3.7.1), (3.7.4), (3.7.5) and (3.7.6) in above equation, we obtain

$$
\begin{aligned}
&A\Big[-\ 'R(Y,Z,U,X) - \frac{1}{2(n-1)}\ \{g(Z,U)S(X,Y) - g(Y,U)S(X,Z) \\
&-2\ S(Y,U)\eta(X)\eta(Z) - S(Z,U)g(X,Y) - S(Z,U)\eta(X)\eta(Y) - S(X,Z)\eta(Y)\eta(U) \\
&+S(X,Y)\eta(Z)\eta(U)\} + \frac{1}{2}\{g(Z,U)g(X,Y) - g(X,Z)g(Y,U) \\
&+g(X,Z)\eta(Y)\eta(U) - g(X,Y)\eta(Z)\eta(U)\}\Big] \\
&+b\Big[-\ 'R(Y,Z,U,QX) + \frac{1}{2(n-1)}\{g(Z,U)S(X,QY) - g(Y,U)S(X,QZ) \\
&+S(QX,Y)\eta(Z)\eta(U) - S(QX,Z)\eta(Y)\eta(U)\} + \frac{1}{2}\{g(Z,U)S(X,Y) \\
&-g(Y,U)S(X,Z) + S(X,Z)\eta(Y)\eta(U) - S(X,Y)\eta(Z)\eta(U)\}\Big]. \qquad (3.10.10)
\end{aligned}
$$

Put $Z = U = e_i$ in above equation and taking summation over i, $1 \leq i \leq n$, we get

$$
\begin{aligned}
S(QX,Y) &= \Big[(n-1) - \frac{A}{b}\Big]\ S(X,Y) - \Big[\frac{2(n-1)+r}{n}\Big]\frac{A}{b}\ \eta(X)\eta(Y) \\
&+ \Big[\frac{n(n-1)-r}{n}\Big]\frac{A}{b}\ g(X,Y).
\end{aligned}
$$

Chapter 4

Concircularly and M-Projectively Semi-Generalized Recurrent Manifolds

4.1 Introduction

A Riemannian manifold (M^n, g) is called a semi-generalized recurrent manifold (Prasad, 2000) if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y, Z)W = A(X)R(Y, Z)W + B(X)g(Z, W)Y, \tag{4.1.1}$$

where A and B are two 1-forms, B is non zero, P_1 and P_2 are two vector fields such that

$$g(X, P_1) = A(X) \qquad g(X, P_2) = B(X), \tag{4.1.2}$$

for any vector field X and ∇ denotes the operator of covariant differentiation with respect to the metric g. Such a manifold is called a semi-generalized recurrent manifold and the 1-form B may be called its associated 1-form. An n-dimensional semi-generalized recurrent manifold shall be denoted by $(SGK)_n$. If the 1-form B in (4.1.1) becomes zero, then the manifold reduces to a recurrent manifold (Walker, 1951).
In an n-dimensional differentiable manifold M^n a Para-Sasakian (briefly P-Sasakian) manifold with structure (φ, ξ, η, g) defined in (1.17.1-1.17.4) also satisfy the following relations (Sato, 1976; Aditi, 1977).

$$(\nabla_X \varphi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{4.1.3}$$

$$\nabla_X \xi = \varphi X, \tag{4.1.4}$$

$$(\nabla_X \eta)(Y) = g(\varphi X, Y) = g(\varphi Y, X), \tag{4.1.5}$$

$$rank(\varphi) = (n-1), \tag{4.1.6}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{4.1.7}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{4.1.8}$$

$$R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi, \tag{4.1.9}$$

$$R(\xi, X)\xi = X - \eta(X)\xi, \tag{4.1.10}$$

$$Q\xi = -(n-1)\xi, \tag{4.1.11}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{4.1.12}$$

$$S(\varphi X, \varphi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{4.1.13}$$

for all vector fields X, Y, Z, where R and S are the Riemannian curvature tensor of the manifold respectively.

4.2 Semi-generalized Ricci recurrent P-Sasakian manifolds

Definition 4.2.1 *A Riemannian manifold (M^n, g) is semi-generalized Ricci recurrent manifold (De and Guha, 1991; Blair, 1976) if*

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + nB(X)g(Y,Z). \tag{4.2.1}$$

Theorem 4.2.1 *The scalar curvature r of a semi-generalized recurrent P-Sasakian manifold is related in terms of contact forms $\eta(P_1)$ and $\eta(P_2)$ as given by*

$$r = -\frac{1}{\eta(P_1)}\Big[(n^2+2)\eta(P_2) + 2(n-1)\eta(P_1)\Big].$$

Proof: Permutting equation (4.1.1) twice with respect to X, Y, Z; adding the three equations and using Bianchi's second identity, we have

$$\begin{aligned} &A(X)R(Y,Z)W + B(X)g(Z,W)Y + A(Y)R(Z,X)W \\ &+B(Y)g(X,W)Z + A(Z)R(X,Y)W + B(Z)g(Y,W)X = 0. \end{aligned} \tag{4.2.2}$$

Contracting (4.2.2) with respect to Y, we get

$$\begin{aligned} &A(X)S(Z,W) + nB(X)g(Z,W) - g(R(Z,X)P_1, W) \\ &+B(Z)g(X,W) - A(Z)S(X,W) + B(Z)g(X,W) = 0. \end{aligned} \tag{4.2.3}$$

In view of $S(Y,Z) = g(QY,Z)$, the equation (4.2.3) reduces to

$$\begin{aligned} &A(X)g(QZ,W) + nB(X)g(Z,W) - g(R(Z,X)P_1), W) \\ &+B(Z)g(X,W) - A(Z)g(QX,W) + B(Z)g(X,W) = 0. \end{aligned} \tag{4.2.4}$$

Factoring off W, we get from (4.2.4)

$$\begin{aligned} &A(X)QZ + nB(X)Z - R(Z,X)P_1 \\ &+B(Z)X - A(Z)QX + B(Z)X = 0. \end{aligned} \tag{4.2.5}$$

Contracting (4.2.5) with respect to Z, we get

$$A(X)r + (n^2+2)B(X) - 2S(X,P_1) = 0. \tag{4.2.6}$$

Putting $X = \xi$ in the equation (4.2.6) and using the equations (4.1.2) and (4.1.12), we get

$$r = -\frac{1}{\eta(P_1)}\Big[(n^2+2)\eta(P_2) + 2(n-1)\eta(P_1)\Big].$$

This completes the proof.

Theorem 4.2.2 *In a semi-generalized Ricci-recurrent P-Sasakian manifold, the 1-forms*

A and B are related as

$$-(n-1)A(X) + nB(X) = 0.$$

Proof: Taking $Z = \xi$ in (4.2.1), we have

$$(\nabla_X S)(Y, \xi) = A(X)S(Y, \xi) + nB(X)g(Y, \xi). \tag{4.2.7}$$

The left hand side of (4.2.7) clearly can be written in the form

$$(\nabla_X S)(Y, \xi) = \nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi)$$

which in view of (4.1.4), (4.1.5) and (4.1.12) gives

$$-(n-1)g(Y, \varphi X) - S(Y, \varphi X).$$

While the right hand side of (4.2.7) equals

$$A(X)S(Y, \xi) + nB(X)g(Y, \xi) = -(n-1)A(X)\eta(Y) + nB(X)\eta(Y).$$

Hence,

$$-(n-1)g(Y, \varphi X) - S(Y, \varphi X) = -(n-1)A(X)\eta(Y) + nB(X)\eta(Y). \tag{4.2.8}$$

Putting $Y = \xi$ in (4.2.8) and then using (1.17.2), (1.17.3) and (4.1.12), we get

$$-(n-1)\eta(\varphi X) + (n-1)\eta(\varphi X) = -(n-1)A(X) + nB(X),$$

or,

$$-(n-1)A(X) + nB(X) = 0. \tag{4.2.9}$$

This completes the proof.

Theorem 4.2.3 *If a semi-generalized Ricci-recurrent P-Sasakian manifold is an Einstein manifold then 1-forms A and B are related as* $\lambda A(Y) + nB(Y) = 0$.

Proof: For an Einstein manifold, we have $S(Y, Z) = \lambda g(Y, Z)$ which gives $(\nabla_U S) = 0$, where λ is constant.

Hence from (4.2.1) we have

$$\begin{aligned}[\lambda A(X)+nB(X)]\,g(Y,Z) &+ [\lambda A(Y)+nB(Y)]\,g(Z,X)\\ &+ [\lambda A(Z)+nB(Z)]\,g(X,Y)=0. \qquad (4.2.10)\end{aligned}$$

Replacing Z by ξ in (4.2.10) and using (4.1.2), (1.17.2) and (1.17.3) we get

$$\begin{aligned}[\lambda A(X)+nB(X)]\,\eta(Y) &+ [\lambda A(Y)+nB(Y)]\,\eta(X)\\ &+ [\lambda\eta(P_1)+n\eta(P_2)]\,g(X,Y)=0. \qquad (4.2.11)\end{aligned}$$

Again, taking $X=Y=\xi$ in (4.2.11) and using (4.1.2), (1.17.2) and (1.17.3), we have

$$\lambda\eta(P_1)+n\eta(P_2)=0. \qquad (4.2.12)$$

Using (4.1.2), (1.17.2) and (1.17.3) in the above relation, it follows that

$$\lambda A(Y)+nB(Y)=0.$$

Hence, proves the theorem.

4.3 Semi-generalized Ricci recurrent Lorentzian α-Sasakian manifolds

Lorentzian α-Sasakian manifold is defined by the equations (1.16.12-1.16.15) which further satisfies the relations given in (1.16.16-1.16.24).

Theorem 4.3.1 *The scalar curvature r of a semi-generalized recurrent Lorentzian α-Sasakian manifolds is related in terms of contact forms $\eta(P_1)$ and $\eta(P_2)$ as given by*

$$r=\frac{1}{\eta(P_1)}\Big[2(n-1)\alpha^2\eta(P_1)-(n^2+2)\eta(P_2)\Big].$$

Proof: Permutting the equation (4.1.1) twice with respect to X,Y,Z; adding the three equations and using Bianchi's second identity, we have

$$\begin{aligned}&A(X)R(Y,Z)W+B(X)g(Z,W)Y+A(Y)R(Z,X)W\\ &+B(Y)g(X,W)Z+A(Z)R(X,Y)W+B(Z)g(Y,W)X=0. \qquad (4.3.1)\end{aligned}$$

Contracting (4.3.1) with respect to Y, we get

$$\begin{aligned}&A(X)S(Z,W) + nB(X)g(Z,W) - g(R(Z,X)P_1,W)\\&+B(Z)g(X,W) - A(Z)S(X,W) + B(Z)g(X,W) = 0.\end{aligned} \tag{4.3.2}$$

In view of $S(Y,Z) = g(QY,Z)$, the equation (4.3.2) reduces to

$$\begin{aligned}&A(X)g(QZ,W) + nB(X)g(Z,W) - g(R(Z,X)P_1,W)\\&+B(Z)g(X,W) - A(Z)g(QX,W) + B(Z)g(X,W) = 0.\end{aligned} \tag{4.3.3}$$

Factoring off W, we get from (4.3.3)

$$\begin{aligned}&A(X)QZ + nB(X)Z - R(Z,X)P_1\\&+B(Z)X - A(Z)QX + B(Z)X = 0.\end{aligned} \tag{4.3.4}$$

Contracting (4.3.4) with respect to Z, we get

$$A(X)r + (n^2+2)B(X) - 2S(X,P_1) = 0. \tag{4.3.5}$$

Putting $X = \xi$ in the equation (4.3.5) and using the equation (4.1.2) and (1.16.22), we get

$$r = \frac{1}{\eta(P_1)}\Big[2(n-1)\alpha^2\eta(P_1) - (n^2+2)\eta(P_2)\Big].$$

completes the proof of the theorem.

Theorem 4.3.2 *In a semi-generalized Ricci-recurrent Lorentzian α-Sasakian manifolds, the 1-forms A and B are related as*

$$(n-1)\alpha^2 A(X) + nB(X) = 0.$$

Proof: Taking $Z = \xi$ in (4.2.1), we have

$$(\nabla_X S)(Y,\xi) = A(X)S(Y,\xi) + nB(X)g(Y,\xi). \tag{4.3.6}$$

The left hand side of (4.3.6), clearly can be written in the form

$$(\nabla_X S)(Y,\xi) = \nabla_X S(Y,\xi) - S(\nabla_X Y,\xi) - S(Y,\nabla_X\xi),$$

which in view of (1.16.16), (1.16.17) and (1.16.22) gives

$$-(n-1)\alpha^3 g(Y, \varphi X) - \alpha S(Y, \varphi X).$$

While the right hand side of (4.3.6) equals

$$A(X)S(Y, \xi) + nB(X)g(Y, \xi) = (n-1)\alpha^2 A(X)\eta(Y) + nB(X)\eta(Y).$$

Hence,

$$(n-1)\alpha^3 g(Y, \varphi X) - \alpha S(Y, \varphi X) = (n-1)\alpha^2 A(X)\eta(Y) + nB(X)\eta(Y). \tag{4.3.7}$$

Putting $Y = \xi$ in (4.3.7) and then using (1.16.12), (1.16.15) and (1.16.22) we get

$$(n-1)\alpha^3 \eta(\varphi X) - (n-1)\alpha^2 \eta(\varphi X) = -(n-1)\alpha^2 A(X) - nB(X),$$

or

$$(n-1)\alpha^2 A(X) + nB(X) = 0.$$

Hence, completes the proof.

Theorem 4.3.3 *If a semi-generalized Ricci-recurrent Lorentzian α-Sasakian manifolds is an Einstein manifold then 1-forms A and B are related as*

$$\lambda A(Y) + nB(Y) = 0.$$

Proof: For an Einstein manifold, we have $S(Y, Z) = \lambda g(Y, Z)$ which gives $(\nabla_U S) = 0$, where λ is constant.
Hence from (4.2.1) we have

$$\begin{aligned}[\lambda A(X) + nB(X)]\, g(Y, Z) &+ [\lambda A(Y) + nB(Y)]\, g(Z, X) \\ &+ [\lambda A(Z) + nB(Z)]\, g(X, Y) = 0. \end{aligned} \tag{4.3.8}$$

Replacing Z by ξ in (4.3.8) and using (4.1.2) and (1.16.15), we have

$$\begin{aligned}[\lambda A(X) + nB(X)]\, \eta(Y) &+ [\lambda A(Y) + nB(Y)]\, \eta(X) \\ &+ [\lambda \eta(P_1) + n\eta(P_2)]\, g(X, Y) = 0. \end{aligned} \tag{4.3.9}$$

Again, taking $X = Y = \xi$ in (4.3.9) and using (4.1.2), (1.16.12) and (1.16.15), we get

$$\lambda\eta(P_1) + n\eta(P_2) = 0. \tag{4.3.10}$$

Using (4.1.2) and (1.16.15) in the above equation, it follows that

$$\lambda A(Y) + nB(Y) = 0.$$

This completes the proof.

4.4 Semi-generalized φ-recurrent P-Sasakian manifolds

Definition 4.4.1 *A P-Sasakian manifold (M^n, g) is called semi-generalized φ recurrent if its curvature tensor R satisfies the condition*

$$\varphi^2\big((\nabla_W R)(X,Y)Z\big) = A(W)R(X,Y)Z + B(W)g(Y,Z)X, \tag{4.4.1}$$

where A and B are two 1-forms, B is non-zero and these are defined by

$$A(W) = g(W, P_1), \quad B(W) = g(W, P_2) \tag{4.4.2}$$

and P_1 and P_2 are vector fields associated with 1-forms A and B, respectively.

Theorem 4.4.1 *A semi generalized φ-recurrent P-Sasakian manifold (M^n, g) is an Einstein manifold and moreover; the 1-forms A and B are related as*

$$(n-1)A(W) = nB(W).$$

Proof: Let us consider a semi-generalized φ-recurrent P-Sasakian manifold. Then by virtue of (1.17.1) and (4.4.1) we have

$$\begin{aligned}(\nabla_W R)(X,Y)Z - \eta\big((\nabla_W R)(X,Y)Z\big)\xi \\ = A(W)R(X,Y)Z + B(W)g(Y,Z)X.\end{aligned} \tag{4.4.3}$$

From which it follows that

$$g((\nabla_W R)(X,Y)Z,U) - \eta((\nabla_W R)(X,Y)Z)\eta(U)$$
$$= A(W)g(R(X,Y)Z,U) + B(W)g(Y,Z)g(X,U). \tag{4.4.4}$$

Let $\{e_i\}, i = 1, 2, ...n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (4.4.4) and taking summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned}(\nabla_W S)(Y,Z) \quad &- \quad \sum_{i=1}^{n} \eta\big((\nabla_W R)(e_i,Y)Z\big)\eta(e_i) \\ &= \quad A(W)S(Y,Z) + nB(W)g(Y,Z).\end{aligned} \tag{4.4.5}$$

Putting $Z = \xi$ in the above equation and using (1.17.3) and (4.1.12) we get

$$\begin{aligned}(\nabla_W S)(Y,\xi) \quad &- \quad \sum_{i=1}^{n} g((\nabla_W R)(e_i,Y)\xi,\xi)\eta(e_i) \\ &= \quad -(n-1)A(W)\eta(Y) + nB(W)\eta(Y).\end{aligned} \tag{4.4.6}$$

We know that

$$\begin{aligned}g((\nabla_W R)(e_i,Y)\xi,\xi) \quad &= \quad g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(\nabla_W e_i,Y)\xi,\xi) \\ &= \quad -g(R(e_i,\nabla_W Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi).\end{aligned} \tag{4.4.7}$$

at $p \in M^n$. Since $\{e_i\}$ is an orthonormal basis, $\nabla_X e_i = 0$ at p. Using (4.1.8) we find

$$\begin{aligned}g(R(e_i,\nabla_W Y)\xi,\xi) \quad &= \quad g\Big(\eta(e_i)\nabla_W Y - \eta(\nabla_W Y)e_i,\xi\Big) \\ &= \quad \eta(e_i)g(\nabla_W Y,\xi) - \eta(\nabla_W Y)g(e_i,\xi) \\ &= \quad 0.\end{aligned} \tag{4.4.8}$$

Using (4.4.8) in (4.4.7) we have

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi). \tag{4.4.9}$$

Since

$$g(R(e_i,Y)\xi,\xi) = -g(R(\xi,\xi)Y,e_i) = 0,$$

we get

$$g(\nabla_W R(e_i, Y)\xi, \xi) + g(R(e_i, Y)\xi, \nabla_W \xi) = 0. \tag{4.4.10}$$

In consequence of (4.4.10), the equation (4.4.9) becomes

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = -g(R(e_i, Y)\xi, \nabla_W \xi) - g(R(e_i, Y)\nabla_W \xi, \xi). \tag{4.4.11}$$

Using (4.1.4) and (4.1.7) in the above equation, we get

$$\begin{aligned} g((\nabla_W R)(e_i, Y)\xi, \xi) &= -g(R(e_i, Y)\xi, \varphi W) - g(R(e_i, Y)\varphi W, \xi) \\ &= -\eta(e_i)g(Y, \varphi W) + \eta(Y)g(e_i, \varphi W) \\ &- \eta(Y)g(e_i, \varphi W) + g(Y, \varphi W)\eta(e_i) \\ &= 0, \end{aligned} \tag{4.4.12}$$

i.e.,

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = 0. \tag{4.4.13}$$

By using (4.4.13) in the equation (4.4.6) we get

$$(\nabla_W S)(Y, \xi) = -(n-1)A(W)\eta(Y) + nB(W)\eta(Y). \tag{4.4.14}$$

Further, we know that

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi).$$

Using (4.1.4), (4.1.5) and (4.1.13) in the above relation, it follows

$$(\nabla_W S)(Y, \xi) = -(n-1)g(\varphi W, Y) - S(\varphi W, Y). \tag{4.4.15}$$

In consequence of (4.4.14) and (4.4.15) we obtain

$$-(n-1)g(\varphi W, Y) - S(\varphi W, Y) = -(n-1)A(W)\eta(Y) + nB(W)\eta(Y). \tag{4.4.16}$$

Replacing $Y = \xi$ in (4.4.16) then using (1.17.2) and (1.17.3), we get

$$(n-1)A(W) = nB(W). \tag{4.4.17}$$

Using (4.4.17) in (4.4.16), we obtain

$$S(Y, \varphi W) = -(n-1)g(Y, \varphi W). \tag{4.4.18}$$

Again, replacing Y by φY both sides in the above equation (4.4.18) and using the equations (1.17.4) and (4.1.13), we obtain

$$S(Y, W) = -(n-1)g(Y, W),$$

i.e., the manifold is an Einstein manifold.

4.5 Semi-generalized φ-recurrent Lorentzian α-Sasakian manifolds

Definition 4.5.1 *A Lorentzian α-Sasakian manifolds (M^n, g) is called semi-generalized φ recurrent if its curvature tensor R satisfies the condition*

$$\varphi^2\big((\nabla_W R)(X, Y)Z\big) = A(W)R(X, Y)Z + B(W)g(Y, Z)X, \tag{4.5.1}$$

where A and B are two 1-forms, B is non-zero and these are defined by

$$A(W) = g(W, P_1), \quad B(W) = g(W, P_2) \tag{4.5.2}$$

and P_1 and P_2 are vector fields associated with 1-forms A and B, respectively.

Theorem 4.5.1 *A semi generalized ϕ-recurrent Lorentzian α-Sasakian manifolds (M^n, g) is an Einstein manifold and moreover the 1-forms A and B are related as*

$$\left[\alpha^2(n-1)\right] A(W) = nB(W).$$

Proof: Let us consider a semi-generalized φ-recurrent Lorentzian α-Sasakian manifolds. Then by virtue of (1.16.13) and (4.5.1) we have

$$\begin{aligned}&(\nabla_W R)(X, Y)Z + \eta((\nabla_W R)(X, Y)Z)\xi \\ &= A(W)R(X, Y)Z + B(W)g(Y, Z)X,\end{aligned} \tag{4.5.3}$$

from which it follows that

$$
\begin{aligned}
& g((\nabla_W R)(X,Y)Z,U) + \eta((\nabla_W R)(X,Y)Z)\eta(U) \\
& = A(W)g(R(X,Y)Z,U) + B(W)g(Y,Z)g(X,U). \qquad (4.5.4)
\end{aligned}
$$

Let $\{e_i\}, i = 1, 2, ...n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (4.5.4) and taking summation over i, $1 \leq i \leq n$, we get

$$
\begin{aligned}
(\nabla_W S)(Y,Z) \quad + & \quad \sum_{i=1}^{n} \eta\big((\nabla_W R)(e_i,Y)Z\big)\eta(e_i) \\
= & \quad A(W)S(Y,Z) + nB(W)g(Y,Z). \qquad (4.5.5)
\end{aligned}
$$

The second term of left hand side of (4.5.5) by putting Z=ξ takes the form $g((\nabla_W R)(e_i,Y)\xi,\xi)$ which is zero in this case. So, by replacing Z by ξ in (4.5.5) and using (1.16.22), we get

$$
(\nabla_W S)(Y,\xi) = (n-1)\alpha^2 A(W)\eta(Y) + nB(W)\eta(Y). \qquad (4.5.6)
$$

We know that

$$
(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi).
$$

Using (1.16.16), (1.16.17) and (1.16.22) in the above relation, it follows

$$
(\nabla_W S)(Y,\xi) = (n-1)\alpha^3 g(\varphi W,Y) - \alpha S(\varphi W,Y). \qquad (4.5.7)
$$

From (4.5.6) and (4.5.7) we obtain

$$
\begin{aligned}
(n-1)\alpha^3 g(\varphi W,Y) - \alpha S(\varphi W,Y) \quad = & \quad \alpha^2(n-1)A(W)\eta(Y) \\
+ & \quad nB(W)\eta(Y). \qquad (4.5.8)
\end{aligned}
$$

Replacing $Y = \xi$ in (4.5.8) and using (1.16.15) and (1.16.22), we get

$$
-\alpha^2(n-1)A(W) = nB(W). \qquad (4.5.9)
$$

Using (4.5.9) in (4.5.8) we obtain

$$S(Y, \varphi W) = (n-1)\alpha^2 g(Y, \varphi W). \tag{4.5.10}$$

Again, replacing Y by φY both sides in (4.5.10) and using (1.16.14) and (1.16.24), we obtain

$$S(Y, W) = (n-1)\alpha^2 g(Y, W),$$

i.e., the manifold is an Einstein manifold.

4.6 Semi-generalized concircular φ-recurrent P-Sasakian manifolds

Definition 4.6.1 *A P-Sasakian manifold (M^n, g) is called semi-generalized concircular φ-recurrent if its concircular curvature tensor defined (Maralabhavi and Rathnamma, 1999)*

$$L(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\Big[g(Y,Z)X - g(X,Z)Y\Big] \tag{4.6.1}$$

satisfies the condition

$$\varphi^2\big((\nabla_W L)(X,Y)Z\big) = A(W)L(X,Y)Z + B(W)g(Y,Z)X \tag{4.6.2}$$

where A and B are defined as (4.4.2) and r is the scalar curvature of the manifold (M^n, g).

Theorem 4.6.1 *Let (M^n, g) be a semi-generalized concircular φ-recurrent P-Sasakian manifold then*

$$\left[-(n-1) - \frac{r}{n}\right] A(W) + nB(W) = 0.$$

Proof: Let us consider a semi-generalized φ-recurrent P-Sasakian manifold. Then by virtue of (1.17.1) and (4.6.2), we have

$$\begin{aligned}&(\nabla_W L)(X,Y)Z - \eta((\nabla_W L)(X,Y)Z)\xi \\ &= A(W)L(X,Y)Z + B(W)g(Y,Z)X,\end{aligned} \tag{4.6.3}$$

from which it follows that

$$g((\nabla_W L)(X,Y)Z,U) - \eta((\nabla_W L)(X,Y)Z)\eta(U)$$
$$= A(W)g(L(X,Y)Z,U) + B(W)g(Y,Z)g(X,U). \tag{4.6.4}$$

Let $\{e_i\}, i = 1, 2, ...n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $Y = Z = e_i$ in (4.6.4) and taking summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned}(\nabla_W S)(X,U) &= \frac{W(r)}{n} g(X,U) - \frac{W(r)}{n}\eta(X)\eta(U) \\ &+ (\nabla_W S)(X,\xi)\eta(U) + nB(W)g(X,U) \\ &+ \left[S(X,U) - \frac{r}{n}g(X,U)\right] A(W).\end{aligned} \tag{4.6.5}$$

Replacing U by ξ in (4.6.5) and using (1.17.4), (4.1.12) and (1.17.3), we have

$$\left[-(n-1) - \frac{r}{n}\right] A(W)\eta(X) + nB(W)\eta(X) = 0. \tag{4.6.6}$$

Putting $X = \xi$ in (4.6.6), we obtain

$$\left[-(n-1) - \frac{r}{n}\right] A(W) + nB(W) = 0.$$

This completes the proof.

Theorem 4.6.2 *A semi-generalized concircular φ-recurrent P-Sasakian manifold is an Einstein manifold.*

Proof: Putting $X = U = e_i$ in (4.6.4) and taking summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned}(\nabla_W S)(Y,Z) &= \sum_{i=1}^{n} g((\nabla_W R)(e_i,Y)Z,\xi)g(e_i,\xi) \\ &+ \frac{W(r)}{n} g(Y,Z) - \frac{W(r)}{n(n-1)}\left[g(Y,Z) - \eta(Y)\eta(Z)\right] \\ &+ \left[S(Y,Z) - \frac{r}{n}g(Y,Z)\right] A(W) + nB(W)g(Y,Z).\end{aligned} \tag{4.6.7}$$

Replacing Z by ξ in (4.6.7) and using (4.6.6), we have

$$(\nabla_W S)(Y,Z) = \frac{W(r)}{n}\eta(Y). \tag{4.6.8}$$

We know that

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi).$$

Using (4.1.4), (4.1.5) and (4.1.12) in the above relation, it follows that

$$(\nabla_W S)(Y,\xi) = -(n-1)g(Y,\varphi W) - S(Y,\varphi W). \tag{4.6.9}$$

In view of (4.6.8) and (4.6.9), we obtain

$$S(Y,\varphi W) = -(n-1)g(Y,\varphi W) - \frac{W(r)}{n}\,\eta(Y). \tag{4.6.10}$$

Replacing Y by φY in (4.6.10) then using (1.17.4), (1.17.3) and (4.1.13), we obtain

$$S(Y,W) = -(n-1)g(Y,W).$$

4.7 Semi-generalized concircular φ-recurrent Lorentzian α-Sasakian manifolds

Definition 4.7.1 *A Lorentzian α-Sasakian manifolds (M^n, g) is called semi-generalized concircular φ-recurrent if its concircular curvature tensor (Venkatesha and Bagewadi, 2006) defined in (4.6.1) satisfies the condition*

$$\varphi^2\big((\nabla_W L)(X,Y)Z\big) = A(W)L(X,Y)Z + B(W)g(Y,Z)X \tag{4.7.1}$$

where A and B are defined as (4.4.2) and r is the scalar curvature of the manifold (M^n, g).

Theorem 4.7.1 *Let (M^n, g) be a semi-generalized concircular φ-recurrent Lorentzian α-Sasakian manifolds then*

$$\Big[(n-1)\alpha^2 - \frac{r}{n}\Big]\,A(W) + n\;B(W) = 0.$$

Proof: Let us consider a semi-generalized φ-recurrent Lorentzian α-Sasakian manifolds. Then by virtue of (1.16.13) and (4.7.1), we have

$$\begin{aligned}&(\nabla_W L)(X,Y)Z + \eta((\nabla_W L)(X,Y)Z)\xi\\ &= A(W)L(X,Y)Z + B(W)g(Y,Z)X,\end{aligned} \tag{4.7.2}$$

from which it follows that

$$\begin{aligned} & g((\nabla_W L)(X,Y)Z,U) + \eta((\nabla_W L)(X,Y)Z)\eta(U) \\ &= A(W)g(L(X,Y)Z,U) + B(W)g(Y,Z)g(X,U). \end{aligned} \tag{4.7.3}$$

Let $\{e_i\}, i = 1, 2, ...n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $Y = Z = e_i$ in (4.7.3) and taking summation over i, $1 \le i \le n$, we get

$$\begin{aligned} (\nabla_W S)(X,U) &= \frac{W(r)}{n} g(X,U) + \frac{W(r)}{n}\eta(X)\eta(U) \\ &- (\nabla_W S)(X,\xi)\eta(U) + nB(W)g(X,U) \\ &+ \left[S(X,U - \frac{r}{n}g(X,U)\right] A(W). \end{aligned} \tag{4.7.4}$$

Replacing U by ξ in (4.7.4) and using (1.16.12), (1.16.15) and (1.16.22), we have

$$\left[(n-1)\alpha^2 - \frac{r}{n}\right] A(W)\eta(X) + nB(W)\eta(X) = 0. \tag{4.7.5}$$

Putting $X = \xi$ in (4.7.5), we obtain

$$\left[(n-1)\alpha^2 - \frac{r}{n}\right] A(W) + nB(W) = 0.$$

Hence, completes the proof.

Theorem 4.7.2 *A semi-generalized concircular φ-recurrent Lorentzian α-Sasakian manifolds is an Einstein manifold.*

Proof: Putting $X = U = e_i$ in (4.7.3) and taking summation over i, $1 \le i \le n$, we get

$$\begin{aligned} (\nabla_W S)(Y,Z) &= \sum_{i=1}^{n} g((\nabla_W R)(e_i,Y)Z,\xi)g(e_i,\xi) \\ &+ \frac{W(r)}{n} g(Y,Z) + \frac{W(r)}{n(n-1)}\left[g(Y,Z) - \eta(Y)\eta(Z)\right] \\ &+ \left[S(Y,Z) - \frac{r}{n}g(Y,Z)\right] A(W) + nB(W)g(Y,Z). \end{aligned} \tag{4.7.6}$$

Replacing Z by ξ in (4.7.6) and using (4.7.5), we have

$$(\nabla_W S)(Y,\xi) = \frac{n+1}{n(n-1)} W(r)\eta(Y). \tag{4.7.7}$$

We know that

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi).$$

Using (1.16.16), (1.16.17) and (1.16.22) in the above relation, it follows that

$$(\nabla_W S)(Y,\xi) = (n-1)\alpha^3 g(Y,\varphi W) - \alpha S(Y,\varphi W). \tag{4.7.8}$$

In view of (4.7.7) and (4.7.8), we obtain

$$S(Y,\varphi W) = \frac{1}{\alpha}\left[(n-1)\alpha^3 g(Y,\varphi W) - \frac{n+1}{n(n-1)}\, W(r)\eta(Y)\right]. \tag{4.7.9}$$

Replacing Y by φY in (4.7.9) then using (1.16.3), (1.16.12) and (1.16.22), we obtain

$$S(Y,W) = (n-1)\alpha^2 g(Y,W).$$

This completes the proof.

4.8 Semi-generalized M-Projective φ-recurrent P-Sasakian manifolds

Definition 4.8.1 *A P-Sasakian manifold (M^n,g) is called semi-generalized M-projective φ-recurrent if M-projective curvature defined in (Prasad, 2000) satisfies the condition*

$$\varphi^2((\nabla_V W^*)(X,Y)Z) = A(V)W^*(X,Y)Z + B(V)g(Y,Z)X, \tag{4.8.1}$$

where A and B are defined as (4.4.2).

Theorem 4.8.1 *Let (M^n,g) be a semi-generalized M-Projective φ-recurrent P-Sasakian manifold then*

$$-\left[\frac{n^2-n+r}{2(n-1)}\right] A(V) + nB(V) = 0.$$

Proof: Let us consider a semi-generalized φ-recurrent P-Sasakian manifold. Then by virtue of (1.17.1) and (4.8.1), we have

$$\begin{aligned}(\nabla_V W^*)(X,Y)Z &- \eta((\nabla_V W^*)(X,Y)Z)\xi \\ &= A(V)W^*(X,Y)Z + B(V)g(Y,Z)X,\end{aligned} \tag{4.8.2}$$

from which it follows that

$$
\begin{aligned}
&g((\nabla_V W^*)(X,Y)Z,U) - \eta((\nabla_V W^*)(X,Y)Z))\eta(U) \\
&\quad = A(V)g(W^*(X,Y)Z,U) + B(V)g(Y,Z)g(X,U). \qquad (4.8.3)
\end{aligned}
$$

Let $\{e_i\}, i = 1, 2, ...n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $Y = Z = e_i$ in (4.8.3) and taking summation over i, $1 \leq i \leq n$, we get

$$
\begin{aligned}
& \frac{n}{2(n-1)}\,(\nabla_V S)(X,U) - \frac{V(r)}{2(n-1)}g(X,U) \\
- & \frac{n}{2(n-1)}\,(\nabla_V S)(X,\xi)\eta(U) + \frac{V(r)}{2(n-1)}\eta(X)\eta(U) \\
= & \left[\frac{n}{2(n-1)}S(X,U) - \frac{r}{2(n-1)}g(X,U)\right]A(V) \\
+ & \; nB(V)g(X,U). \qquad (4.8.4)
\end{aligned}
$$

Replacing U by ξ in (4.8.4) and using (1.17.2) and (1.17.3), we have

$$
-\left[\frac{n^2-n+r}{2(n-1)}\right]A(V)\eta(X) + nB(V)\eta(X) = 0. \qquad (4.8.5)
$$

Putting $X = \xi$ in (4.8.5), we obtain

$$
-\left[\frac{n^2-n+r}{2(n-1)}\right]A(V) + nB(V) = 0.
$$

This completes the proof.

Theorem 4.8.2 *A semi-generalized M-Projective φ-recurrent P-Sasakian manifold is an Einstein manifold.*

Proof: Putting $X = U = e_i$ in (4.8.3) and taking summation over i, $1 \leq i \leq n$, we get

$$
\begin{aligned}
\frac{n}{2(n-1)}\,(\nabla_V S)(Y,Z) &= \sum_{i=1}^{n} g((\nabla_V R)(e_i,Y)Z,\xi)g(e_i,\xi) \\
&+ \frac{V(r)}{2(n-1)}g(Y,Z) \\
&- \frac{1}{2(n-1)}\Big[(\nabla_V S)(Y,Z)g(\xi,\xi) - (\nabla_V S)(\xi,Z)\eta(Y) \\
&+ g(Y,Z)(\nabla_V S)(\xi,\xi) - (\nabla_V S)(Y,\xi)\eta(Z)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \left[\frac{n}{2(n-1)}S(Y,Z) - \frac{r}{2(n-1)}g(Y,Z)\right]A(V) \\
&+ nB(V)g(Y,Z). \qquad (4.8.6)
\end{aligned}
$$

Replacing Z by ξ in (4.8.6) and using (1.17.3) and (4.1.12), we have

$$(\nabla_V S)(Y,\xi) = -\frac{V(r)}{n}\eta(Y). \qquad (4.8.7)$$

We know that

$$(\nabla_V S)(Y,\xi) = \nabla_V S(Y,\xi) - S(\nabla_V Y,\xi) - S(Y,\nabla_V\xi).$$

Using (4.1.4), (4.1.5) and (4.1.12) in above the relation, it follows that

$$(\nabla_V S)(Y,\xi) = -(n-1)g(Y,\varphi V) - S(Y,\varphi V). \qquad (4.8.8)$$

In view of (4.8.7) and (4.8.8)

$$S(Y,\varphi V) = -(n-1)g(Y,\varphi V) + \frac{V(r)}{n}\eta(Y). \qquad (4.8.9)$$

Replacing Y by φY in (4.8.9) then using (1.17.4) and (4.1.13), we get

$$S(Y,V) = -(n-1)g(Y,V).$$

This completes the proof.

4.9 Semi-generalized M-Projective φ-recurrent Lorentzian α-Sasakian manifolds

Definition 4.9.1 *A Lorentzian α-Sasakian manifolds (M^n, g) is called semi-generalized M-projective φ-recurrent if M-Projective curvature tensor defined in (Prasad, 2000) satisfies the condition*

$$\varphi^2((\nabla_V W^*)(X,Y)Z) = A(V)W^*(X,Y)Z + B(V)g(Y,Z)X, \qquad (4.9.1)$$

where A and B are defined as (4.4.2).

Theorem 4.9.1 *Let (M^n, g) be a semi-generalized M-Projective φ-recurrent Lorentzian*

α-Sasakian manifolds then

$$\left[\frac{(n\alpha^2(n-1)-r)}{2(n-1)}\right] A(V)+nB(V)=0.$$

Proof: Let us consider a semi-generalized φ-recurrent Lorentzian α-Sasakian manifolds. Then by virtue of (1.16.13) and (4.9.1), we have

$$\begin{aligned}
&(\nabla_V W^*)(X,Y)Z+\eta((\nabla_V W^*)(X,Y)Z)\xi \\
&= A(V)W^*(X,Y)Z+B(V)g(Y,Z)X,
\end{aligned} \tag{4.9.2}$$

from which it follows that

$$\begin{aligned}
&g((\nabla_V W^*)(X,Y)Z,U)+\eta((\nabla_V W^*)(X,Y)Z))\eta(U) \\
&= A(V)g(W^*(X,Y)Z,U)+B(V)g(Y,Z)g(X,U).
\end{aligned} \tag{4.9.3}$$

Let $\{e_i\}, i=1,2,...n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $Y=Z=e_i$ in (4.9.3) and taking summation over i, $1\leq i\leq n$, we get

$$\begin{aligned}
&- \frac{n}{2(n-1)}(\nabla_V S)(X,U)-\frac{V(r)}{2(n-1)}g(X,U) \\
&- \frac{n}{2(n-1)}(\nabla_V S)(X,\xi)\eta(U)-\frac{V(r)}{2(n-1)}\eta(X)\eta(U) \\
&= \left[\frac{n}{2(n-1)}S(X,U)-\frac{r}{2(n-1)}g(X,U)\right] A(V) \\
&+ nB(V)g(X,U).
\end{aligned} \tag{4.9.4}$$

Replacing U by ξ in (4.9.4) and using (1.16.12), (1.16.15) and (1.16.22), we have

$$A(V)\left[\frac{(n\alpha^2(n-1)-r)}{2(n-1)}\right]\eta(X)+nB(V)\eta(X)=0. \tag{4.9.5}$$

Putting $X=\xi$ in (4.9.5), we obtain

$$\left[\frac{(n\alpha^2(n-1)-r)}{2(n-1)}\right] A(V)+nB(V)=0.$$

This completes the proof.

Theorem 4.9.2 *A semi-generalized M-Projective φ-recurrent Lorentzian α-Sasakian manifold is an Einstein manifold.*

Proof: Putting $X = U = e_i$ in (4.9.3) and taking summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned}(\nabla_V S)(Y,Z) &= \frac{2(n-1)}{n}\Big[-\sum_{i=1}^{n} g\big((\nabla_V R)(e_i,Y)Z,\xi\big)g(e_i,\xi)\\ &+ \frac{V(r)}{2(n-1)}g(Y,Z)\\ &+ \frac{1}{2(n-1)}\Big\{-(\nabla_V S)(Y,Z)-(\nabla_V S)(\xi,Z)\eta(Y)\\ &+ g(Y,Z)(\nabla_V S)(\xi,\xi)-(\nabla_V S)(Y,\xi)\eta(Z)\Big\}\\ &+ \Big\{\frac{n}{2(n-1)}S(Y,Z)-\frac{r}{2(n-1)}g(Y,Z)\Big\}A(V)\\ &+ nB(V)g(Y,Z)\Big]. \end{aligned} \tag{4.9.6}$$

Replacing Z by ξ in (4.9.6) and using (1.16.12), (1.16.15) and (1.16.22), we have

$$(\nabla_V S)(Y,\xi) = \frac{V(r)}{n}\eta(Y). \tag{4.9.7}$$

We know that

$$(\nabla_V S)(Y,\xi) = \nabla_V S(Y,\xi) - S(\nabla_V Y,\xi) - S(Y,\nabla_V \xi).$$

Using (1.16.16), (1.16.17) and (1.16.22) in above relation, it follows that

$$(\nabla_V S)(Y,\xi) = (n-1)\alpha^3 g(Y,\varphi V) - \alpha S(Y,\varphi V). \tag{4.9.8}$$

In view of (4.9.7) and (4.9.8)

$$S(Y,\varphi V) = \frac{1}{\alpha}\left[-\frac{V(r)}{n}\eta(Y) + (n-1)\alpha^3 g(Y,\varphi V)\right]. \tag{4.9.9}$$

Replacing Y by φY in (4.9.9) then using (1.16.14), (1.16.12) and (1.16.24), we get

$$S(Y,V) = (n-1)\alpha^2 g(Y,V).$$

This completes the proof.

4.10 Three dimensional locally semi-generalized φ-recurrent P-Sasakian manifolds

Theorem 4.10.1 *The curvature tensor of three dimensional semi-generalized φ-recurrent P-Sasakian manifold is given by*

$$R(X,Y,Z) = \left[\frac{dr(e_i)}{2A(e_i)} - \frac{B(e_i)}{A(e_i)}\right] g(Y,Z)X - \left[\frac{dr(e_i)}{2A(e_i)}\right] g(X,Z)Y.$$

Proof: In a three-dimensional Riemannian manifold (M^3, g), we have

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &- S(X,Z)Y + \frac{r}{2}\Big[g(X,Z)Y - g(Y,Z)X\Big], \end{aligned} \tag{4.10.1}$$

where Q is the Ricci operator, i.e., $S(X,Y) = g(QX,Y)$ and r is the scalar curvature of the manifold. In 3-dimensional P-Sasakian manifolds the equations (4.1.11) and (4.1.12) assume the following form

$$Q\,\xi = -2\,\xi, \tag{4.10.2}$$

and

$$S(X,\xi) = -2\,\eta(X), \tag{4.10.3}$$

respectively.

Now putting $Z = \xi$ in (4.10.1) and using the equations (1.17.3) and (4.10.3), we get

$$\begin{aligned} R(X,Y)\xi &= \eta(Y)QX - \eta(X)QY + 2\,[\eta(X)Y - \eta(Y)X] \\ &+ \frac{r}{2}\Big[\eta(X)Y - \eta(Y)X\Big]. \end{aligned} \tag{4.10.4}$$

Using (4.1.8) in (4.10.4), we have

$$\Big\{1 + \frac{r}{2}\Big\}\Big[\eta(X)Y - \eta(Y)X\Big] = \eta(X)QY - \eta(Y)QX. \tag{4.10.5}$$

Putting $Y = \xi$ in the equation (4.10.5) and using the equations (1.17.2) and (4.10.2), we get

$$QX = -\Big\{3 + \frac{r}{2}\Big\}\eta(X)\xi + \Big\{1 + \frac{r}{2}\Big\}X. \tag{4.10.6}$$

Therefore, it follows from (4.10.6) that

$$S(X,Y) = -\Big\{3 + \frac{r}{2}\Big\}\eta(X)\eta(Y) + \Big\{1 + \frac{r}{2}\Big\}g(X,Y). \tag{4.10.7}$$

Thus from (4.10.1), (4.10.6) and (4.10.7), we get

$$\begin{aligned} R(X,Y)Z &= -\Big\{2 + \frac{r}{2}\Big\}\Big[g(X,Z)Y - g(Y,Z)X\Big] \\ &- \Big\{3 + \frac{r}{2}\Big\}\Big[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\Big]. \end{aligned} \tag{4.10.8}$$

Taking the covariant differentiation to the both sides of the equation (4.10.8), we get

$$\begin{aligned} (\nabla_W R)(X,Y)Z &= \frac{dr(W)}{2}\Big[g(X,Z)Y - g(Y,Z)X + g(Y,Z)\eta(X)\xi \\ &- g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\Big] \\ &- \Big\{3 + \frac{r}{2}\Big\}\Big[g(Y,Z)\eta)(X) - g(X,Z)\eta)(Y)\Big](\nabla_W \xi) \\ &- \Big\{3 + \frac{r}{2}\Big\}\Big[g(Y,Z)\xi - \eta(Z)Y\Big](\nabla_W \eta)(X) \\ &- \Big\{3 + \frac{r}{2}\Big\}\Big[\eta(Y)X - \eta(X)Y\Big](\nabla_W \eta)(Z) \\ &+ \Big\{3 + \frac{r}{2}\Big\}\Big[g(X,Z)\xi - \eta(Z)X\Big](\nabla_W \eta)(Y). \end{aligned} \tag{4.10.9}$$

Noting, that we may assume that all vector fields X, Y, Z, W are orthogonal to ξ and using (1.17.1), we get

$$\begin{aligned} (\nabla_W R)(X,Y)Z &= \frac{dr(W)}{2}\Big[g(X,Z)Y - g(Y,Z)X\Big] \\ &- \Big\{3 + \frac{r}{2}\Big\}\ \Big[g(Y,Z)(\nabla_W \eta)(X) \\ &- g(X,Z)(\nabla_W \eta)(Y)\Big]\xi. \end{aligned} \tag{4.10.10}$$

Applying φ^2 to the both side of (4.10.10) and using (1.17.1) and (4.1.4), we get

$$\varphi^2\Big((\nabla_W R)(X,Y)Z\Big) = \frac{dr(W)}{2}\Big[g(Y,Z)X - g(X,Z)Y\Big]. \tag{4.10.11}$$

By (4.4.1), the equation (4.10.11) reduces to

$$A(W)R(X,Y)Z = \Big[\frac{dr(W)}{2} - B(W)\Big]g(Y,Z)X - \frac{dr(W)}{2}g(X,Z)Y.$$

Putting $W = \{e_i\}$, where $i = 1, 2, 3$ is an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1 \leq i \leq 3$, we obtain

$$R(X,Y)Z = \left[\frac{dr(e_i)}{2A(e_i)} - \frac{B(e_i)}{A(e_i)}\right] g(Y,Z)X - \Big[\frac{dr(e_i)}{2A(e_i)}\Big]g(X,Z)Y.$$

Hence, prove the theorem.

4.11 Three dimensional locally semi-generalized φ-recurrent Lorentzian α-Sasakian manifolds

Theorem 4.11.1 *The curvature tensor of three dimensional semi-generalized φ-recurrent Lorentzian α-Sasakian manifold is given by*

$$R(X,Y,Z) = \left[\frac{dr(e_i)}{2A(e_i)} - \frac{B(e_i)}{A(e_i)}\right] g(Y,Z)X - \Big[\frac{dr(e_i)}{2A(e_i)}\Big]g(X,Z)Y.$$

Proof: In a three-dimensional Riemannian manifold (M^3, g), we have

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &- S(X,Z)Y + \frac{r}{2}\Big[g(X,Z)Y - g(Y,Z)X\Big], \end{aligned} \tag{4.11.1}$$

where Q is the Ricci operator, i.e., $S(X,Y) = g(QX,Y)$ and r is the scalar curvature of the manifold. In 3-dimensional Lorentzian α-Sasakian manifolds the equation (1.16.23) and (1.16.22) assume the following form

$$Q\,\xi = 2\alpha^2\,\xi, \tag{4.11.2}$$

$$S(X,\xi) = 2\alpha^2\,\eta(X), \tag{4.11.3}$$

respectively.

Now putting $Z = \xi$ in (4.11.1) and using the equations (1.16.15) and (4.11.3), we get

$$R(X,Y)\xi = \eta(Y)QX - \eta(X)QY + \Big\{2\,\alpha^2 - \frac{r}{2}\Big\}\Big[\eta(Y)X - \eta(X)Y\Big]. \tag{4.11.4}$$

Using (1.16.20) in (4.11.4), we have

$$\Big\{-\alpha^2 + \frac{r}{2}\Big\}\Big[\eta(Y)X - \eta(X)Y\Big] = \eta(Y)QX - \eta(X)QY. \tag{4.11.5}$$

Putting $Y = \xi$ in (4.11.5) and using the equations (1.16.12) and (4.11.2), we get

$$QX = \Big\{-\alpha^2 + \frac{r}{2}\Big\}X + \Big\{-3\,\alpha^2 + \frac{r}{2}\Big\}\eta(X)\xi. \tag{4.11.6}$$

Therefore, it follows from (4.11.6) that

$$S(X,Y) = \Big\{-\alpha^2 + \frac{r}{2}\Big\}g(X,Y) + \Big\{-3\alpha^2 + \frac{r}{2}\Big\}\eta(X)\eta(Y). \tag{4.11.7}$$

Thus from (4.11.1), (4.11.6) and (4.11.7), we get

$$\begin{aligned} R(X,Y)Z &= \Big\{-2\alpha^2 + \frac{r}{2}\Big\}\Big[g(Y,Z)X - g(X,Z)Y\Big] \\ &+ \Big\{-3\alpha^2 + \frac{r}{2}\Big\}\Big[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\Big]. \end{aligned} \tag{4.11.8}$$

Taking the covariant differentiation to the both sides of the equation (4.11.8), we get

$$\begin{aligned} (\nabla_W R)(X,Y)Z &= \frac{dr(W)}{2}\Big[g(Y,Z)X - g(X,Z)Y + g(Y,Z)\eta(X)\xi \\ &- g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\Big] \\ &+ \Big\{-3\alpha^2 + \frac{r}{2}\Big\}\Big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\Big](\nabla_W\xi) \\ &+ \Big\{-3\alpha^2 + \frac{r}{2}\Big\}\Big[g(Y,Z)\xi - \eta(Z)Y\Big](\nabla_W\eta)(X) \\ &+ \Big\{-3\alpha^2 + \frac{r}{2}\Big\}\Big[\eta(Y)X - \eta(X)Y\Big](\nabla_W\eta)(Z) \\ &- \Big\{-3\alpha^2 + \frac{r}{2}\Big\}\Big[g(X,Z)\xi - \eta(Z)X\Big](\nabla_W\eta)(Y). \end{aligned} \tag{4.11.9}$$

Noting, that we may assume that all vector fields X, Y, Z, W are orthogonal to ξ and using (1.16.12), we get

$$\begin{aligned}(\nabla_W R)(X,Y)Z &= \frac{dr(W)}{2}\Big[g(Y,Z)X - g(X,Z)Y\Big] \\ &+ \Big\{-3\alpha^2 + \frac{r}{2}\Big\}\Big[g(Y,Z)(\nabla_W\eta)(X) \\ &- g(X,Z)(\nabla_W\eta)(Y)\Big]\xi. \end{aligned} \tag{4.11.10}$$

Applying φ^2 to the both side of (4.11.10) and using (1.16.12) and (1.16.13), we get

$$\varphi^2\Big((\nabla_W R)(X,Y)Z\Big) = \frac{dr(W)}{2}\Big[g(Y,Z)X - g(X,Z)Y\Big]. \tag{4.11.11}$$

By (4.4.1), the equation (4.11.11) reduces to

$$A(W)R(X,Y)Z = \left[\frac{dr(W)}{2} - B(W)\right] g(Y,Z)X - \frac{dr(W)}{2} g(X,Z)Y.$$

Putting $W = \{e_i\}$, where $i = 1,2,3$ is an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1 \leq i \leq 3$, we obtain

$$R(X,Y,Z) = \left[\frac{dr(e_i)}{2A(e_i)} - \frac{B(e_i)}{A(e_i)}\right] g(Y,Z)X - \left[\frac{dr(e_i)}{2A(e_i)}\right] g(X,Z)Y.$$

Hence, prove the theorem.

Chapter 5

On The Almost r-paracontact Submanifold

5.1 Introduction

Let M^n be an n-dimensional Riemannian manifold with a positive definite metric g. If on M^n there exist a tensor field φ of type (1,1), r-vector fields $\xi_1, \xi_2, ...\xi_r (r < n)$, r 1-form $\eta^1, \eta^2, ...\eta^r$ such that

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \quad \alpha, \beta \in (r) = 1, 2, ..., r \tag{5.1.1}$$

$$\varphi^2(X) = X - \eta^\alpha(X)\xi_\alpha, \tag{5.1.2}$$

$$\eta^\alpha(X) = g(X, \xi_\alpha), \quad \alpha \in (r), \tag{5.1.3}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y), \tag{5.1.4}$$

where X and Y are vector fields on M and $a^\alpha b_\alpha \Rightarrow \sum_\alpha a^\alpha b_\alpha$, then the structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha\epsilon(r)}$ is said to be an almost r-paracontact Riemannian structure on M^n and M^n is an almost r-paracontact Riemannian manifold (Bucki and Miernowski, 1985).

From the definitions of almost r-paracontact Riemannian structure, it follows that

$$\varphi(\xi_\alpha) = 0, \quad \alpha \in (r), \tag{5.1.5}$$

and

$$\eta^\alpha \circ \varphi = 0, \quad \alpha \in (r). \tag{5.1.6}$$

If

$$\Phi(X, Y) \Rightarrow g(\varphi X, Y) = g(X, \varphi Y), \tag{5.1.7}$$

then the tensor field Φ is symmetric

$$\Phi(X, Y) = \Phi(Y, X), \tag{5.1.8}$$

and

$$\Phi(X, Y) = (D_X \eta^\alpha)(Y), \tag{5.1.9}$$

5.2 Almost r-paracontact submanifold

Let M^{n-1} be a submanifold of M^{n+1} with the inclusion $b : M^{n-1} \longrightarrow M^{n+1}$ map such that $p \in M^{n-1}$ to $bp \in M^{n+1}$, the map b induces a linear transformation (Jacobian map) B such that

$$B : T^{n-1} \longrightarrow T^{n+1},$$

where T^{n-1} is a tangent space to M^{n-1} at a point p and T^{n+1} is the tangent space to M^{n+1} such that (X in M^{n-1} at p) $\longrightarrow$ (BX in M^{n+1} at bp).
Agreement: In what follows the equation containing X, Y, Z hold for arbitrary vector field X, Y, Z in M^{n-1}. Suppose that M^{n+1} is an almost r-paracontact Riemannian manifold with metric tensor $\widetilde{g}$. Then the submanifold M^{n-1} is also an almost r-paracontact Riemannian manifold with metric tensor g such that

$$g(X, Y) = \widetilde{g}(BX, BY). \tag{5.2.1}$$

Let $\dot{\nabla}$ be the connection induced on the submanifold M^{n-1} from the connection $\widetilde{\dot{\nabla}}$ on the enveloping manifold with respect to unit normal vectors N_1 and N_2, then we have the Gauss Equation (Mishra, 1972).

$$\widetilde{\dot{\nabla}}_{BX} BY = B(\dot{\nabla}_X Y) + h(X, Y) N_1 + k(X, Y) N_2, \tag{5.2.2}$$

for arbitrary vector fields X and Y in M^{n-1}, where h and k are the second fundamental tensors of M^{n-1}.
The Weingarten equations are given by (Nivas and Verma, 2005)

$$\dot{\widetilde{\nabla}}_{BX} N_1 = -BHX + l(X)N_2, \tag{5.2.3}$$

$$\dot{\widetilde{\nabla}}_{BX} N_2 = -BKX - l(X)N_1, \tag{5.2.4}$$

where H and K are tensor of type $(1,1)$ such that

$$g(H(X), Y) = h(X, Y),$$

$$g(K(X), Y) = k(X, Y).$$

Theorem 5.2.1 *The necessary and sufficient conditions that M^{n-1} be an almost r-paracontact metric submanifold with the structure $(\varphi, \xi_\alpha, \eta^\alpha)$ in an almost r-paracontact metric manifold M^{n+1} are*

$$\begin{aligned}&\widetilde{\eta}^\alpha(BX) \circ b = \eta^\alpha(X), \qquad r(X) = s(X) = 0, \qquad \rho = \sigma = 0,\\ &\widetilde{\varphi}(BX) = B\overline{X}, \qquad \widetilde{\xi}_\alpha = B\xi_\alpha,\end{aligned}$$

where $\overline{X} \overset{def}{=} \varphi X$.

Proof: Let us consider

$$\widetilde{\varphi}(BX) = B\overline{X} + r(X)N_1 + s(X)N_2, \tag{5.2.5}$$

$$\widetilde{\xi}_\alpha = B\xi_\alpha + \rho N_1 + \sigma N_2, \tag{5.2.6}$$

$$\widetilde{\varphi}(N_1) = -Bp + \theta N_2, \tag{5.2.7}$$

$$\widetilde{\varphi}(N_2) = -Bq + \theta N_1. \tag{5.2.8}$$

On pre-multiplying in (5.2.5) by $\widetilde{\varphi}$ and using (5.1.2), (5.2.6), (5.2.7) and (5.2.8), we obtained

$$\begin{aligned} BX - \widetilde{\eta}_\alpha(BX)B\xi_\alpha + \rho N_1 + \sigma N_2 &= B\overline{\overline{X}} - B_p\ \alpha(X) - B_q\ r(X) \\ &+ N_1\left\{r(\overline{X}) + \theta s(X)\right\} \\ &+ N_2\left\{s(\overline{X}) + \Theta r(X)\right\}. \end{aligned} \tag{5.2.9}$$

Substituting from (5.2.5) in

$$\widetilde{g}\ (\widetilde{\varphi}BX, \widetilde{\varphi}BY) = \widetilde{g}(BX, BY) - \sum_\alpha \widetilde{\eta}^\alpha(BX)\ \widetilde{\eta}^\alpha(BY),$$

and using (5.2.1), we obtain

$$\begin{aligned} g(\overline{X}, \overline{Y}) &= g(X,Y) - \sum_\alpha \{\eta^\alpha(BX)\}\ \{\eta^\alpha(BY)\} \\ &- r(X)r(Y) - s(X)s(Y). \end{aligned} \tag{5.2.10}$$

Equation (5.2.9) and (5.2.10) imply

$$\overline{\overline{X}} = X - \eta^\alpha(BX)\xi_\alpha,$$

$$g(\overline{X}, \overline{Y}) = g(X,Y) - \Sigma_\alpha \eta^\alpha(BX)\eta^\alpha(BY),$$

if and only if

$$\begin{aligned} &\widetilde{\eta}^\alpha(BX) \circ b = \eta^\alpha(X), \\ &r(X)r(Y) + s(X)s(Y) = 0, \\ &pr(X) + qs(X) = 0, \\ &r(\overline{X}) + \theta s(X) + \rho\eta^\alpha(X) = 0, \\ &s(\overline{X}) + \Theta r(X) + \sigma\eta^\alpha(X) = 0. \end{aligned} \tag{5.2.11}$$

The above equations are consistent if and only if

$$\begin{aligned} &\widetilde{\eta}^\alpha(BX) \circ b = \eta^\alpha(X), \quad r(X) = s(X) = 0, \\ &and \quad \rho = \sigma = 0. \end{aligned} \tag{5.2.12}$$

Substituting (5.2.4) and (5.2.12) in (5.2.5) and (5.2.6), we obtain

$$\widetilde{\varphi}(BX) = B\overline{X}, \qquad \widetilde{\xi}_\alpha = B\xi_\alpha. \tag{5.2.13}$$

Thus (5.2.12) and (5.2.13) are necessary and sufficient condition.

Theorem 5.2.2 *When M^{n-1} is an almost r-paracontact manifold then the following relations hold*

$$(a) \quad \widetilde{\eta}^\alpha(N_1) = 0, \quad (b) \qquad \widetilde{\eta}^\alpha(N_2) = 0, \tag{5.2.14}$$

$$(a) \qquad \widetilde{\varphi}(N_1) = N_2, \quad (b) \qquad \widetilde{\varphi}(N_2) = N_1. \tag{5.2.15}$$

Proof: We have

$$\widetilde{g}\ (\widetilde{\varphi}N_1, BX) - \widetilde{g}(\widetilde{\varphi}BX, N_1) = 0. \tag{5.2.16}$$

By virtue of (5.2.7) and (5.2.12), the equation (5.2.16) assume the form

$$g(p, X) = 0 \Rightarrow p = 0. \tag{5.2.17}$$

Similarly

$$q = 0. \tag{5.2.18}$$

Pre-multiplying (5.2.7) by $\widetilde{\varphi}$ and using (5.1.2) and (5.2.8), we get

$$N_1 - \widetilde{\eta}^\alpha(N_1)\widetilde{\xi}_\alpha = \theta\Theta N_1,$$

which yield (5.2.14(a)) and

$$\Theta\theta = 1. \tag{5.2.19}$$

The equation (5.2.14(b)) follows similarly. From (5.1.4), (5.2.14(a)) and (5.2.14(b)), we have

$$\widetilde{g}(\widetilde{\varphi}N_1, \widetilde{\varphi}N_1) = \widetilde{g}(N_1, N_1) = 1.$$

Substituting from (5.2.7) and using (5.2.17) in above equation, we get

$$\Theta^2 = 1.$$

On putting $\Theta = 1$ in (5.2.19), we get

$$\theta = 1.$$

Hence we have (5.2.15).

Theorem 5.2.3 *If ν and $\mathcal{N}$ be Nijenhuis tensor in the almost r-paracontact submanifold M^{n-1} and almost r-paracontact manifold M^{n+1} respectively then*

$$\mathcal{N}(BX, BY) = B\nu(X, Y). \tag{5.2.20}$$

Proof: In consequence of (5.2.2) and (5.2.12), we have

$$\begin{aligned}
\widetilde{\varphi}(\widetilde{\varphi}\,[BX, BY]) &= \widetilde{\varphi}\left(\widetilde{\varphi}\left(\widetilde{\dot{\nabla}}_{BX}BY - \widetilde{\dot{\nabla}}_{BY}BX\right)\right) \\
&= \widetilde{\varphi}\left(\widetilde{\varphi}\left(B\widetilde{\dot{\nabla}}_{BX}BY - B\widetilde{\dot{\nabla}}_{BY}BX\right)\right) \\
&= B\overline{\overline{[X, Y]}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{N}(BX, BY) &= B\left(\overline{\overline{[X, Y]}} + [\overline{X}, \overline{Y}] - \overline{[\overline{X}, \overline{Y}]} - \overline{[X, \overline{Y}]}\right) \\
&= B\nu(X, Y).
\end{aligned}$$

Hence prove the theorem.

Definition 5.2.1 *An almost r-paracontact manifold is said to be normal if and only if (Bucki, 1998)*

$$\mathcal{N}(X, Y) - 2d\widetilde{\eta}^{\alpha}(X, Y)\widetilde{\xi}_{\alpha} = 0.$$

Theorem 5.2.4 *If an almost r-paracontact manifold M^{n+1} is normal then almost r-paracontact submanifold M^{n-1} is also normal.*

Proof: Let almost r-paracontact manifold M^{n+1} is normal then

$$\begin{aligned} &\mathcal{N}(BX, BY) - 2d\widetilde{\eta}^{\alpha}(BX, BY)\widetilde{\xi}_{\alpha} = 0 \\ \Rightarrow\ &\mathcal{N}(BX, BY) - 2\left[(\widetilde{\dot{\nabla}}_{BX}\eta^{\alpha})(BY) - (\widetilde{\dot{\nabla}}_{BY}\eta^{\alpha})(BX)\right]\widetilde{\xi}_{\alpha} = 0. \end{aligned} \tag{5.2.21}$$

On using (5.2.2), (5.2.12), (5.2.13), (5.2.14) and (5.2.20) in (5.2.21), we get

$$\begin{aligned} &\nu(X, Y) - 2\left[(\dot{\nabla}_X\eta^{\alpha})(Y) - (\dot{\nabla}_Y\eta^{\alpha})(X)\right]\xi_{\alpha} = 0, \\ \Rightarrow\ &\nu(X, Y) - 2d\eta^{\alpha}(X, Y)\xi_{\alpha} = 0. \end{aligned}$$

This shows that almost r-paracontact submanifold M^{n-1} is also normal.

Definition 5.2.2 *If the second fundamental form h and k of M^{n-1} are of the form $h(X, Y) = \mu_1\widetilde{g}(X, Y)$ and $k(X, Y) = \mu_2\widetilde{g}(X, Y)$ where $\mu_1, \mu_2 = \frac{TrB}{n^1}$ then M^{n-1} is called totally umbilical. In our case, we take $\mu_1 = \mu_2 = \mu$. If the second fundamental form vanishes identically then M^{n-1} is said to be totally geodesic (Yano and Kon, 1977).*

Theorem 5.2.5 *Let M^{n-1} be a submanifold tangent to the structure vector field $\widetilde{\xi}_{\alpha}$ of an almost r-paracontact metric manifold M^{n+1}. If M^{n-1} is totally umbilical then M^{n-1} is totally geodesic.*

Proof: From Gauss equation, we have

$$\widetilde{\dot{\nabla}}_{BX}\widetilde{\xi}_{\alpha} = B\dot{\nabla}_X\widetilde{\xi}_{\alpha} + h(X, \widetilde{\xi}_{\alpha})N_1 + k(X, \widetilde{\xi}_{\alpha})N_2,$$

or

$$B\widetilde{\varphi}X = B\dot{\nabla}_X\widetilde{\xi}_{\alpha} + h(X, \widetilde{\xi}_{\alpha})N_1 + k(X, \widetilde{\xi}_{\alpha})N_2.$$

Equating tangential and normal parts, we get

$$\widetilde{\varphi}X = \dot{\nabla}_X\widetilde{\xi}_{\alpha},$$

and

$$h(X, \widetilde{\xi}_{\alpha}) = 0, \qquad k(X, \widetilde{\xi}_{\alpha}) = 0.$$

Thus

$$h(\widetilde{\xi}_{\alpha}, \widetilde{\xi}_{\alpha}) = 0, and \qquad k(\widetilde{\xi}_{\alpha}, \widetilde{\xi}_{\alpha}) = 0.$$

If M^{n-1} is totally umblical, then $h(X,Y) = \mu g(X,Y) = k(X,Y)$.
Using $\widetilde{\xi}_\alpha$ for both X and Y, we get

$$\begin{aligned} & h(\widetilde{\xi}_\alpha, \widetilde{\xi}_\alpha) = k(\widetilde{\xi}_\alpha \widetilde{\xi}_\alpha) = 0, \\ \Rightarrow \quad & \mu g(\widetilde{\xi}_\alpha, \widetilde{\xi}_\alpha) = 0, \\ \Rightarrow \quad & \mu = 0, \end{aligned}$$

which implies that $h(X,Y) = k(X,Y) = 0$. Thus M^{n-1} is totally geodesic.
If M^{n-1} is totally geodesic then $h(X,\xi)$=0, that is $\widetilde{\varphi}X$ is tangent to M^{n-1} and hence M^{n-1} is an invariant submanifold.

Theorem 5.2.6 *Let M^{n-1} be an almost r-paracontact submanifold in the almost r-paracontact metric manifold M^{n+1}. Let there exist affine connection in M^{n-1} and M^{n+1} be respectively $\dot{\nabla}$ and $\widetilde{\dot{\nabla}}$ such that*

$$\begin{aligned} & (a) \quad \widetilde{\dot{\nabla}}_\lambda \ \widetilde{\xi}_\alpha = \widetilde{\varphi} \ \lambda, \\ & (b) \quad \widetilde{\dot{\nabla}}_\lambda(\widetilde{\varphi} \ \mu) = \widetilde{\dot{\nabla}}_\mu(\widetilde{\varphi} \ \lambda) + \widetilde{\varphi} \ [\lambda, \mu] + \lambda(\widetilde{\eta}^\alpha(\mu)) - \mu(\widetilde{\eta}^\alpha(\lambda)), \\ & (c) \quad (\widetilde{\dot{\nabla}}_\lambda \eta^\alpha)(\mu) - (\widetilde{\dot{\nabla}}_\mu \eta^\alpha)(\lambda) = 0. \end{aligned} \tag{5.2.22}$$

Then the condition that $\dot{\nabla}$ also satisfyies similar equations

$$\begin{aligned} & (a) \quad \dot{\nabla}_X \ \eta = \overline{X}, \\ & (b) \quad \dot{\nabla}_X \overline{Y} = \nabla_Y \overline{X} + \overline{[X,Y]} + X\eta(Y) - Y\eta(X), \\ & (c) \quad (\dot{\nabla}_X \ \eta)(Y) - (\dot{\nabla}_Y \ \eta)(X) = 0, \end{aligned} \tag{5.2.23}$$

are

$$h(\overline{X}, \overline{Y}) - h(X,Y) = 0, \quad \textit{and} \ \ k(\overline{X}, \overline{Y}) - k(X,Y) = 0$$

.

Proof: From (5.2.22(a)), we have

$$\begin{aligned} & \widetilde{\dot{\nabla}}_{BX} B\widetilde{\xi}_\alpha = \widetilde{\varphi} BX \\ \Rightarrow \quad & B(\dot{\nabla}_X \ \xi_\alpha) + h(X, \xi_\alpha)N_1 + k(X, \xi_\alpha)N_2 = B\overline{X}. \end{aligned}$$

On equating tangential and normal part, we get

$$
\begin{aligned}
&(a) \quad \dot{\nabla}_X \xi_\alpha = \overline{X} \\
&(b) \quad h(X, \xi_\alpha) = k(X, \xi_\alpha) = 0. \qquad (5.2.24)
\end{aligned}
$$

From (5.2.22(b)), we have

$$
\begin{aligned}
\tilde{\dot{\nabla}}_{BX}(\widetilde{\varphi}BY) &= \tilde{\dot{\nabla}}_{BY}(\widetilde{\varphi}BX) + \widetilde{\varphi}[BX, BY] \\
&+ BX(\widetilde{\eta}^\alpha(BY)) - BY(\widetilde{\eta}^\alpha(BX)). \qquad (5.2.25)
\end{aligned}
$$

On using (5.2.13(a)) and (5.2.12(a)) in (5.2.25), we get

$$
\begin{aligned}
B(\dot{\nabla}_X \overline{Y}) + h(X, \overline{Y})N_1 + k(X, \overline{Y})N_2 &= B\nabla_Y \overline{X} + h(Y, \overline{X})N_1 + k(Y, \overline{X})N_2 \\
&+ B\overline{[X, Y]} + BX(\eta^\alpha(Y)) - BY(\eta^\alpha(X)).
\end{aligned}
$$

This gives

$$
\dot{\nabla}_X \overline{Y} = \dot{\nabla}_Y \overline{X} + \overline{[X, Y]} + X\eta^\alpha(Y) - Y\eta^\alpha(X),
$$

if and only if

$$
h(X, \overline{Y}) = h(Y, \overline{X}) \qquad and \qquad k(X, \overline{Y}) = k(Y, \overline{X}),
$$

or

$$
h(\overline{X}, \overline{Y}) - h(X, Y) = 0 \qquad and \qquad k(\overline{X}, \overline{Y}) - k(X, Y) = 0.
$$

In consequence of (5.2.22(c)) and (5.2.12(a)), we have

$$
\begin{aligned}
&(\tilde{\dot{\nabla}}_\lambda \widetilde{\eta}^\alpha)\ (\mu) - (\tilde{\dot{\nabla}}_\mu \widetilde{\eta}^\alpha)(\lambda) = 0, \\
\Rightarrow\ &(\dot{\nabla}_X\ \eta^\alpha)(Y) - (\dot{\nabla}_Y\ \eta^\alpha)(X) = 0.
\end{aligned}
$$

Hence prove the theorem.

Theorem 5.2.7 *Let M^{n-1} be an almost r-paracontact submanifold in the almost r-paracontact manifold M^{n+1}. Let there exist affine connection $\tilde{\dot{\nabla}}$ in M^{n+1} be such that*

$$
(\tilde{\dot{\nabla}}_\lambda\ \widetilde{\varphi})\mu = 0 \qquad (5.2.26)
$$

then the condition that induced connection $\dot{\nabla}$ in M^{n-1} also satisfies a similar condition

$$(\dot{\nabla}_X\ \varphi)(Y) = 0 \tag{5.2.27}$$

are

$$h(X,\overline{Y}) = k(X,Y) \quad \textit{and} \quad k(X,\overline{Y}) = h(X,Y).$$

Proof: From (5.2.26), we have

$$(\widetilde{\dot{\nabla}}_{BX}\ \widetilde{\varphi})(BY) = 0,$$

$$\Rightarrow \widetilde{\dot{\nabla}}_{BX}\ \widetilde{\varphi}(BY) = \widetilde{\varphi}(\widetilde{\dot{\nabla}}_{BX}BY),$$

$$\Rightarrow \widetilde{\dot{\nabla}}_{BX}B\overline{Y} = \widetilde{\varphi}(\widetilde{\dot{\nabla}}_{BX}BY).$$

Using (5.2.2), (5.2.13(a)) and (5.2.15) in above equation, we get

$$B\dot{\nabla}_X\overline{Y} + h(X,\overline{Y})N_1 + k(X,\overline{Y})N_2 = B\overline{\dot{\nabla}_X Y} + h(X,Y)N_2 + k(X,Y)N_1.$$

This implies that

$$\begin{aligned} & \dot{\nabla}_X\overline{Y} - \overline{\dot{\nabla}_X Y} = 0, \\ \Rightarrow \quad & (\dot{\nabla}_X\varphi)(Y) = 0, \end{aligned}$$

if and only if

$$h(X,\overline{Y}) = k(X,Y) \quad \textit{and} \quad k(X,\overline{Y}) = h(X,Y).$$

Hence prove the theorem.

Theorem 5.2.8 *Let M^{n-1} be an almost r-paracontact metric submanifold in an almost r-paracontact metric manifold M^{n+1}. Let there exist affine connection $\widetilde{\dot{\nabla}}$ in M^{n+1} be such that*

$$\widetilde{\eta}^\alpha(\mu)\widetilde{\dot{\nabla}}_\lambda\ \widetilde{\xi}_\alpha + \widetilde{\xi}_\alpha(\widetilde{\dot{\nabla}}_\lambda\ \widetilde{\eta}^\alpha)(\mu) = 0 \tag{5.2.28}$$

then the condition that induced connection $\dot{\nabla}$ in M^{n-1} also satisfies a similar condition

$$\eta^\alpha(Y)\ \dot{\nabla}_X\ \xi_\alpha + \xi_\alpha(\dot{\nabla}_X\ \eta^\alpha)(Y) = 0, \tag{5.2.29}$$

if and only if

$$h(X, \xi_\alpha) = k(X, \xi_\alpha).$$

Proof: From (5.2.28), we have

$$\widetilde{\eta}^\alpha(BY)\ \widetilde{\dot{\nabla}}_{BX}\ \widetilde{\xi}_\alpha + \widetilde{\xi}_\alpha(\widetilde{\dot{\nabla}}_{BX}\ \widetilde{\eta}^\alpha)(BY) = 0,$$

which by virtue of (5.2.13), (5.2.2), (5.2.12) and (5.2.14) implies that

$$\eta^\alpha(Y)\ [B\dot{\nabla}_X\ \xi_\alpha + h(X, \xi_\alpha)N_1 + k(X, \xi_\alpha)N_2] + B\ \xi_\alpha\ (\dot{\nabla}_X\ \eta^\alpha)(Y) = 0.$$

This gives

$$\eta^\alpha(Y)\ \dot{\nabla}_X\ \xi_\alpha + \xi_\alpha(\dot{\nabla}_X\ \eta^\alpha)(Y) = 0,$$

and

$$h(X, \xi_\alpha) = k(X, \xi_\alpha) = 0.$$

Hence prove the theorem.

Chapter 6

Summary and Conclusion

In Chapter 1, we have defined Differentiable manifold, Tangent vector, Tangent space and Vector field, Tensor, Lie-bracket, Covarient derivatives, Lie derivative and Exterior derivatives, Connection, Riemannian manifolds, Torsion tensor, Ricci tensor, Curvature tensors on Riemannian manifold, Almost contact manifold, Almost paracontact metric manifold, Lorentzian paracontact manifold, Lorentzian α-Sasakian manifold, Recurrent manifold, Submanifold are also defined in introduction.

The chapter 2 is about the study of some properties of quarter symmetric non-metric connection on an LP-Sasakian manifold. We prove that an n-dimensional LP-Sasakian manifold M^n is Locally $\overline{W^*} - \varphi$-symmetric with respect to the quarter symmetric non-metric connection ∇ if and only if it is so with respect to the Riemannian connection D. We show that an n-dimensional LP-Sasakian manifold is $\xi - \overline{W^*}$-projectively flat with respect to the quarter symmetric non- metric connection if and only if the manifold is $\xi - W^*$-projectively flat with respect to the Riemannian connection provided that the vector fields X and Y are orthogonal to ξ. Next, we prove that an n-dimensional φ-conharmonicaly flat LP-Sasakian manifolds admitting a quarter-symmetric non-metric connection, then M^n is an η-Einstein manifold with the scalar curvature $r = -\frac{2(n-1)}{n-2}$ with respect to the Riemannian connection. We also prove that a LP-Sasakian manifolds admitting a quarter symmetric non-metric connection satisfying $\overline{P} \cdot \overline{S} = 0$, $\overline{R} \cdot \overline{S} = 0$ and $\overline{L} \cdot \overline{S} = 0$ is an η-Einstein manifold. Finally, have shown that in an LP-Sasakian manifold if the curvature tensor with respect to quarter symmetric non-metric connection ∇ is given by $\overline{R}(X,Y)Z = g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi$, then the manifold is projectively flat.

The chapter 3 is devoted to the study of certain curvature conditions of Quasi conformal curvature tensor and M-projective curvature on LP-Sasakian manifolds. we have shown that if in an LP-Sasakian manifold the relation $(P_1^1\overline{C})(Y,Z) = 0$ hold, then M^n is an Einstein manifold with scalar curvature $r = n(n-1)$, provided $a + (n-2)b \neq 0$. We prove that an Einstein LP-Sasakian manifold $(M^n, g)(n > 2)$ is quasi conformally conservative if and only if the scalar curvature is constant, provided $[b(n-4)\ (n-1) - 2a] \neq 0$. Then we show that an n-dimensional $(n > 2)$ φ-quasi conformally flat LP-Sasakian manifold is an η-Einstein manifold. Next, it is proved that an LP-Sasakian manifold M^n satisfying the condition $R(\xi, X) \cdot W^* = 0$ and $W^*(\xi, X) \cdot C = 0$, is an Einstein manifold. Finally some curvature properties of M-projective curvature tensor are obtained in an LP-Sasakian manifold.

In Chapter 4, we have discussed semi-generalized concircular recurrent manifolds and semi-generalized M-projectively recurrent manifolds and obtained some interesting results. Semi-generalized recurrent Lorentzian α-Sasakian manifolds, semi-generalized recurrent P-Sasakian manifolds, semi-generalized φ-recurrent Lorentzian α-Sasakian manifolds and semi-generalized φ-recurrent P-Sasakian manifolds are discussed in this chapter. We have shown that a semi-generalized M-Projective φ-recurrent P-Sasakian manifold is an Einstein manifold. Then we show that a semi generalized ϕ-recurrent Lorentzian α-Sasakian manifolds (M^n, g) is an Einstein manifold and moreover the 1-forms A and B are related as $-\left[\alpha^2(n-1)\right] A(W) = nB(W)$. Finally, three dimensional locally semi-generalized φ-recurrent P-Sasakian manifolds and three dimensional locally semi-generalized φ-recurrent Lorentzian α-Sasakian manifolds are considered and the expression for the curvature tensors are obtained.

Chapter 5 is devoted to the study of the almost r-paracontact submanifold. It is shown that If ν and $\mathcal{N}$ be Nijenhuis tensor in the almost r-paracontact submanifold M^{n-1} and almost r-paracontact manifold M^{n+1} respectively then $\mathcal{N}(BX, BY) = B\nu(X, Y)$ and If an almost r-paracontact manifold M^{n+1} is normal then almost r-paracontact submanifold M^{n-1} is also normal. Finally, we prove that if a submanifold M^{n-1} tangent to the structure vector field $\widetilde{\xi}_\alpha$ of an almost r-paracontact metric manifold M^{n+1}. If M^{n-1} is totally umbilical then M^{n-1} is totally geodesic.

Finally we conclude that whole work of this thesis gives the properties and geometrical structure of the vectors related with quarter symmetric non-metric connection equipped with LP-Sasakian manifold, certain curvature conditions with LP-Sasakian manifolds, semi-generalized recurrent Lorentzian α-Sasakian manifolds, semi-generalized recurrent P-Sasakian manifolds and on the almost r-paracontact submanifold.

Future Scope of the Study:

Now a days the study of semi symmetric metric and non-metric connection is very popular in Indian modern differential geometry. The semi symmetric metric connection has important physical application such as the displacement on the earth surface following a fixed point is metric and semi-symmetric. Golab (1975) introduced and studied quarter symmetric connection in a Riemannian manifold with an affine connection which generalizes the idea of semi symmetric connection.

In the present study we consider a quarter symmetric non-metric connection on an LP-Sasakian manifold. This connection can also be study in other differentiable manifolds such as generalized Sasakian space form, Quasi Sasakian manifolds etc. The notion of quarter symmetric non-metric connection further can be extended to the study of hypersurfaces and submanifolds of almost contact manifolds.

Semi-generalized recurrent manifold defined by Prasad (2000) can also be study in generalized Sasakian space form, Quasi Sasakian manifolds, β-kenmotsu manifolds etc. One can consider the curvature tensors like pseudo projective curvature tensor, Quasi conformal curvature tensor, Conformal curvature tensor, W_2-curvature tensor to obtain certain geometrical properties of curvature tensors with respect to semi-generalized recurrent manifolds.

Bibliography

Adati, T. and Matsumoto, K. (1977). On conformally recurrent and conformally symmetric P-Sasakian manifolds, *T.R.U Math.* **13**, 25-32.

Adati, T. and Miyazawa, T. (1979). On P-Sasakian manifolds satisfying certain conditions, *Tensor (N.S.)* **33**, 173-178.

Agashe, N.S. and Chafle, M.R. (1992). A semi symmetric non-metric connection in a Riemannian manifold, *Indian J. Pure Appl. Math.* **23**, 399-409.

Ahmad, M., Jun, J. B. and Haseeb, A. (2011). Of an almost r-paracontact Riemannian manifold endowed with a quarter-symmetric non-metric connection, *J. Chunggheong Math. Soc.* **24(1)**, 91-103.

Ahmad, M., Haseeb, A., Jun, Jae-Bok. and Rahman, S. (2010). On almost r-paracontact Riemannian manifold with a certain, *Comm. Korean. Math. Soc.* **25(2)**, 235-243.

Al, S. and Nivas, R. (2000). On submanifolds of a manifold with quarter-symmetric connection, *Riv. Mat. Uni. Parma.* **6(3)**, 11-23.

Berman, Ajit (2013). Semi-Symmetric non-metric Connection in a P-Sasakian manifold, *Novi Sad J. Math.* **43(2)**, 117-124.

Berman, Ajit (2014). On Lorentzian α-Sasakian manifolds admitting a type of semi-symmetric metric connection, *Novi Sad J. Math.* **44(2)**, 77-88.

Bhattacharya, A. and Patra, C. (2014). Semi-symmetric metric connections on pseudo-symmetric Lorentzian α-Sasakian manifolds, *ACTA ET Commentations Universitatis Tartunesis De Mathematica* **18**, 197-210.

Biswas, S.C. and De, U.C. (1997). Quarter symmetric metric connection in an *LP*-Sasakian manifold, *Commun. Fac. Sci. Univ. Ank. Series* **46**, 49-56.

Blair, D.E. (1976). Contact manifold in Riemannian Geometry, Lecture notes in math., *Springer Verlag*, Berlin-Heidelberg, New-York, 509.

Bucki, A. (1974). Almost r-paracontact submanifolds, *Tensor (N.S.)* **28**.

Bucki, A. (1985). Almost r-paracontact structures of P-Sasakian type, *Tensor (N.S.)* **42**, 42-54.

Bucki, A. and Miernowski, A. (1985). Almost r-paracontact structures, *Ann. Univ. Mariae Curie-Sklodowska, Sect. A.* **39**, 13-26.

Bucki, A. (1998). Product submanifolds of almost r-paracontact Riemannian manifolds of P-Sasakian type, *Soochow J. of Mathematics* **24(4)**, 255-259.

Cabrerizo, J. L., Fernandez, L. M., Fernandez, M. and Zhen, G. (1999). The structure of a class of k-contact manifolds, *Acta Math. Hungar* **82**, 331-340.

Chaki, M.C. and Saha, S.K. (1994). On Pseudo projective Ricci-symmetric manifolds, *Bulg. Journ. Phys.* 24.

Chaturvedi, B.B. and Pandey, P.N. (2008). Semi-symmetric non-metric connection on a Kahler manifold, *Differential Geometry-Dynamical Systems* **10**, 86-90.

Chaubey, S.K. (2011). Almost contact metric manifolds admitting semi-symmetric non-metric connection, *Bull. Math. Anal. and Appl.* **3(2)**, 252-260.

Chaubey, S.K. (2012). On Weakely M-Projectively Symmetric Manifolds, *Novi Sad J. Math.* **42(1)**, 67-79.

De, U.C. and De, K. (2011). On three dimensional Kenmotsu manifolds admitting a quarter-symmetric metric connection, *Azerbaijan J. Math.* **1**, 132-142.

De, U.C. and Guha, N. (1991). On generalized recurrent manifold, *J. Nat. Acad. Math.* **9**, 85-92.

De, U.C. and Kamilya, D. (1994). On generalized concircular recurrent manifolds, *Bull. Cal. Math. Soc.* **86**, 69-72.

De,U.C., Guha, N. and Kamilya, D.(1995). On generalized Ricci-recurrent manifolds, *Tensor (N.S.)* **56**, 312-317.

De, U.C., Matsumoto, K. and Shaikh, A.A. (1999). On Lorentzian para-Sasakian manifolds, *Rensicontidel Seminario Matematico di Messina, Serie II, Supplemento al.* **3**, 149-158.

De, U.C. and Pathak, G. (1994). On P-Sasakian manifolds satisfying certain conditions. *J. Indian Acad. Math.* **16**, 27-50.

De, U.C. and Pathak, G. (2004). On 3-dimensional Kenmotsu manifolds, *Indian J. Pure Appl. Math.* **35(2)**, 159-165.

De, U.C. and Sarkar, A. (2009). On a type of P-Sasakian manifolds, *Math. Reports* **11(61)**, 2, 139-144.

De, U.C. and Sengupta, J. (2001). On a type of semi-symmetric metric connection on an almost contact metric manifold, *Facta Universitatis, Ser. Math. Inform.* **16**, 87-96.

De, U.C., Shaikh, A.A. and Sengupta, A. (2002). On LP-Sasakian manifold with coefficient α, *Kyungpook Math. J.* **42**, 177-186.

De, U.C. and Tripathi, M.M.(2003). Ricci tensor on 3-dimensional trans-Sasakian manifolds, *Kyungpook Math. J.* **43**, 247-255.

Debnath, D. and Bhattacharya, A. (2013). On generalized φ-recurrent trans-Sasakian manifolds, *Acta Universitatis Apulensis* **36**, 253-266.

Deszcz, L. Verstraelen and Yapark, S. (1994). Warped products realizing a certain condition of pseudo symmetry type imposed on the Weyl curvature tensor, *Chin. J. Math.* **22(2)**, 139-157.

Devi, Saroja M. and Singh, J.P.(2015). On a type of M-projective curvature tensor on Kenmotsu manifolds, *Int. J. Math. Sci. & Engg. Appls.* **9**, No. III, 37-49.

Dhruwa Narain, Prakash Amit and Prasad, B. (2009). A Pseudo projective curvature tensor On a Lorentzian Para-Sasakian manifold, *Ale Universitat "AL.I. CUZA" IASI (S.N.) Matematica, Tomul LV*, 275-284.

Eisenhart, L.P. (1926). Riemannian Geometry, *Princeton University Press, N.J.*, 82-91.

Friedmann, A. and Schouten, J.A. (1924). Uber die geometric der halbsymmetrischen ubertragung, *Math. Zeitschr* **21**, 211-223.

Golab, S. (1975). On semi-symmetric and quarter-symmetric linear connections, *Tensor (N.S.)* **29**, 249-254.

Guha, N. (2000). On generalized Ricci-recurrent Sasakian manifolds, *Bull. Cal. Math. Soc.* **92(5)**, 361-364.

Hatakeyama, Y. (1963). Some notes on differentiable manifolds with almost contact structure, *Tohoku Math. J.* **15** 176-181.

Hatakeyama, Y., Ogawa, Y. and Tanano, S. (1963). Some properties of manifolds with contact metric structures, *Tohoku Math. J.* **15**, 42-48.

Hayden, H.A. (1932). Of a space with torsion tensor, *Proc. London Math. Soc.* **34**, 27-50.

Hicks, N.J.(1969). *Notes On Differential Geometry*, Affliated East West Press Pvt. Ltd., 95.

Jaiswal, J.P. and Ojha, R.H.(2009). On generalized φ-recurrent LP-Sasakian manifolds, *Kyungpook Math. J.* **49**, 779-788.

Kenmotsu, K. (1992). A class of almost contact Riemannian manifolds, *Tohoku Math. J.* **24**, 93-103.

Khan, Quddus (2004). On generalized recurrent Sasakian manifolds, *Kyungpook Math. J.* **44**, 167-172.

Kim, J.S., Prasad, R. and Tripathi, M.M. (2002). On generalized Ricci-recurrent trans-sasakian manifolds, *J. Korean Math. Soc.* **39(6)**, 953-961.

Kobayashi, S. and Nomizu, K. (1963). *Foundations of differential geometry*, Interscience Publishers, New York **1**, 574.

Kumar, Rajesh (2010). Hypersurfaces of almost paracontact manifolds, *Journal of Progressive Science* **1(1)**, 58-62.

Kumar, R., Singh, J.P. and Chowdhury, J. (2014). On semi-generalized recurrent LP-Sasakian manifolds, *Int. J. of Math. and Statistics Inv.* **2(11)**, 63-66.

Kumar, R., Singh, J.P. and Chowdhury, J. (2015). On Generalized Ricci-Recurrent LP-Sasakian manifolds, *J. Math. and Comp. Sci.* **14**, 205-210.

Lokesh, S., Bagewadi, C.S. and Pradeep Kumar, K.T. (2012). On Lorentzian α-Sasakian manifolds, *Adv. Pure math.* **2**, 177-182.

Maralabhavi, Y.B. and Rathnamma, M. (1999). Generalized recurrent and concircular recurrent manifolds, *Ind. J. Pure Appl. Math.* **30**, 1167-1171.

Matsumoto, K. (1977). Killing Vector fields in a P-Sasakian manifolds, *J. Korean Math. Soc.* **14(1)**, 135-142.

Matsumoto, K. (1989). On Lorentzian para contact manifolds, *Bull. of Yamagata Univ. Nat. Sci* **12**, 151-156.

Matsumoto, K., Ianus S. and Mihai, I. (1986). On a P-Sasakian manifolds which admit certain tensor fields, *Publ. Math. Debrecen* **33**, 61-65.

Matsumoto, K. and Mihai, I. (1988). On a certain transformations in Lorentzian para Sasakian manifold, *Tensor (N.S.)* **47**, 189-197.

Mihai, I., De, U.C. and shaikh, A.A (1999). On Lorentzian Para-Sasakian manifolds. *Korean J. Math. Sci.* **6**, 1-13.

Mihai, I. and Rosca, R. (1992). On Lirentzian P-Sasakian manifolds, Classical Analysis (Kazimierz Dolny, 1991), *World Sci. Publ., River Edge, NJ*, 155-169.

Mishra, R.S. (1972). Almost contact manifold with a specified affine connection, *Tensor (N.S.)* **23**, 41-45.

Mishra, R.S. (1981). On P-Sasakian manifolds, *Prog. of math.* **15 (1-2)**, 55-67.

Mishra, R.S. (1984). *Structures on a differentiable manifold and their aplications*, Chandrama Prakashan, Allahabad, 396.

Mishra, R.S. (1991). Almost contact metric manifold, *Tensor Soc. India*, Lucknow, Monograph(1).

Mishra, R.S. and Pandey, S.N. (1980). On quarter symmetric metric F-connections, *Tensor (N.S.)* **34**, 1-7.

Miyazawa, T. and Yamaguchi, S.(1966). Some theorems on K-contact metric manifolds and Sasakian manifolds, *T.R.U. Math. Japan* **2**, 46-52.

Mondal, A.K. and De, U.C. (2009). Some properties of a quarter-symmetric metric connection on a Sasakian manifold, *Bull. Math. Anal. and Appl.* **1(3)**, 99-108.

Mukhopadhya, S. and Barua, B. (1990). A note on the projective curvature tensor of a semi-symmetric recurrent metric connection on a Riemannian manifold, *Ale Universitat, "AL. I Cuza" Iasi, Tomul XXXVI, Mathematica*, 415-417.

Mukhopadhya, A. K. and Barua, B. (1991). Some properties of a quarter symmetric metric connection on a Riemannian manifold, *Soochow J. Math.* **17**, 205-211.

Nagaraja, H.G. (2011). ϕ-recurrent trans-Sasakia manifolds, Matematnykn Bechnk **63(2)**, 79-86.

Nivas, R. and Verma, G. (2005). On quarter-symmetric non-metric connection in a Riemannian manifold, *J. Rajasthan Acad. Phys. Sci.* **4(1)**, 57-68.

Ojha, R.H. (1974). almost contact submanifolds, *Tensor (N.S.)* 28.

Ojha, R.H. (1975). A note on the M-projective curvature tensor, *Ind. J. Pure Appl. Math.* **8(12)**, 1531-1534.

Ojha, R.H. (1986). M-projectively flat Sasakian manifolds, *Ind. J. Pure Appl. Math.* **17(4)**, 481-484.

Ojha, R.H. and Prasad B.(1994). On a semi-symmetric non-metric connection in almost Grayan manifolds, *Bull. Cal. Math. Soc.* **86**, 213-220.

Okumura, M. (1962). On infinitesimal conformal and projective transformations of normal contact spaces, *Tohoku Math. J.* **14**, 398-412.

Oubina, J. (1985). New classes of almost contact metric structures, *Publications Mathematicae* **32**, 187-193.

Özgür, C. (2003). ϕ-conformally flat LP-Sasakian manifolds, *Radovi matematici* **12**, 99-106.

Özgür, C. (2007). On generalized recurrent Kenmotsu manifolds, *World Appl. Sci. J.* **2(1)**, 9-33.

Özgür, C. and Tripathi (2007). On P-Sasakian manifolds satisfying certain conditions on the concircular curvature tensor, *Turk J Math.* **31**, 171-179.

Pandey, S.N. and Mishra, R.S. (1980). On quarter-symmetric metric F-connections, *Tensor (N.S.)* **34**, 1-7.

Pandey, L.K. and Ojha, R.H. (2001). Semi-symmetric metric and non-metric connections in Lorentzian Para contact manifold, *Bull. Cal. Math. Soc.* **93(6)**, 497-504.

Patil, D.A., Prakasha, D.G. and Bagewadi, C.S. (2009). On generalized φ-recurrent Sasakian manifolds, *Bull. Math. Anal. and Apll.* **1**, 42-48.

Perktas, S.Y., and Tripathi, M.M. (2010). On a semi-symmetric metric connection in a Lorentzian para-sasakian manifold, *Differential Geometry-Dynamical Systems* **13**, 299-310.

Pokhariyal, G.P. (1982). Study of a new curvature tensor in a Sasakian manifold, *Tensor N.S.* **36**, 222-225.

Pokhariyal, G.P. and Mishra, R.S. (1970). Curvature tensor and their relativistic significance I., *Yokohama Math. J.* **18**, 105-108.

Pokhariyal, G.P. and Mishra, R.S. (1971). Curvature tensor and their relativistic significance II., *Yokohama Math. J.* **19**, 97-103.

Prakash, D.G. (2010). On Ricci η-recurrent $(LCS)_N$-manifolds, *Acta Univ. Apulensis* **24**, 109-118.

Prakash, A. and Dhruwa, N. (2011). On a quarter symmetric non-metric connection in an Lorentzian para-Sasakian manifolds, *Int. Electronic J. Geom.* **4(1)**, 129-137.

Prakash, A. (2012). On Concircularlly φ-recurrent Para-Sasakian manifolds, *Novi. Sad. J. Math.* **42(1)**, 81-87.

Prakasha, D.G. and Yildiz, A. (2010). Generalized ϕ-Recurrent Lorentzian α-Sasakian manifold, *comm. Fac. Sci. Univ. Ank. Series Al.* **59(1)**, 53-62.

Prasad, B. (1994). On a semi-symmetric non-metric connection in an SP-Sasakian manifold, *Instanbul Univ. Fen Fak. Mat. Der.*, 77-80.

Prasad, B. (2000). On semi-generalized recurrent manifold, *Mathematica Balkanica, New series* **14**, 77-82.

Prasad, B. and Kumar, R. (2002). A semi-symmetric non-metric SP-connection in an SP-Sasakian manifold, *J. Nat. Acad. Math.* **16**, 37-45.

Prasad, V.S. and Bagewadi, C.S. (1999). Note on Kenmotsu manifolds, *Bull. Cal. Math. Soc.* **91(5)**, 379-384.

Rastogi, S.C. (1987). On quarter symmetric metric connection, *Tensor (N.S.)* **44(2)**, 133-141.

Ruse, H.S. (1951). A classification of K^*-spaces, *Proc. London Math. Soc.* **53**, 212-229.

Sasaki, S. (1965, 67, 68). Almost contact manifold, I, A Lecture Note, *Tohoku University.*

Sasaki, S. (1965, 67, 68). Almost contact manifold, II, A Lecture Note, *Tohoku University.*

Sasaki, S. (1965, 67, 68). Almost contact manifold, III, A Lecture Note, *Tohoku University.*

Sasaki, S. (1960). On differentiable manifolds with certain structures which are closely related to almost contact structure, *Tohoku Math. J.*, **12**, 459-476.

Sasaki, S. (1975). Lecture notes on almost contact manifolds, part 1, *Tohoku University.*

Sasaki, S. and Hatakeyama, Y. (1961). On differentiable manifolds with certain structures which are closely related to almost contact structure, *Tohoku Math. J.* **13**, 281-294.

Sato, I. (1976). On a structure similar to the almost contact structure, Tensor (N.S.) **30**, 219-224.

Sato, I. and Matsumoto, K. (1976). On P-Sasakian manifolds satisfying certain conditions, *Tensor (N.S.)* **33**, 173.

Sengupta, J., De, U.C. and Binh, T.Q. (2000). On a type of semi-symmetric non-metric connection on a Riemannian manifold, *Indian J. Pure Appl. Math.* **31(1-2)**, 1659-1670.

Shaikh, A.A. and Baishya, K.K. (2005). On φ-symmetric Lorentzian para-Sasakian manifolds, *Yokohama Math. J.* **52**, 97-112.

Shaikh A. A. and Biswas, S. (2004). On LP-sasakian manifolds, *Bull. Malaysian Math. Sci. Soc.* **27**, 17-26.

Shaikh, A.A. and De, U.C. (2000). On 3-dimensional LP-Sasakian manifolds, *Soochow J. Math.* **26(4)**, 359-368.

Sharfuddin, A. and Husain, S.I. (1976). Semi-symmetric metric connections in almost contact manifolds, *Tensor (N.S.)* **30**, 133-139.

Shukla, S.S. and Shukla, M.K. (2010). On φ-Symmetric Para-Sasakian manifolds, *Int. J. Math. Anal.* **4(16)**, 761-769.

Sinha, B.B. (1978). On almost contact metric manifold with semi-symmetric metric connection, *Math Rive. Amer. Math. Soc.* **12(1-2)**.

Singh, H. and Khan, Q. (1999). On symmetric Riemannian manifolds, *Novi Sad J. Math.* **29(3)**, 301-308.

Singh, H. and Khan, Q. (2000). On generalized recurrent Riemannian manifolds, *Publ. Math. Debrecen* **56(1-2)**, 87-95.

Singh, J.P. (2009). On an Einstein M-projective P-Sasakian manifolds, *Bull. Cal. Math. Soc.* **101(2)**, 175-180.

Singh, J.P. (2012). On M-projective recurrent Riemannian manifold, *Int. J. Math. Anal.* **6(24)**, 1173-1178.

Singh, J.P. (2013). Some properties of LP-Sasakian manifolds admitting a Quarter symmetric non metric connection, *Tamsui Oxford J. of Info. Math. Sci.* **29(4)**, 505-517.

Singh, J.P. (2014). On an Einstein Quarter symmetric metric F-connection, *Int. J. of advances in Math.* **1(1)**, 1-8.

Singh, J.P. (2014). On Generalized φ-recurrent and Generalized Concircularly φ-recurrent P-Sasakian manifolds, *Novi Sad J. Math.* **44(1)**, 153-163.

Singh, J.P. (2015). On a type of LP-Sasakian manifolds admitting a quarter symmetric non-metric connection, *The Mathematical Student* **84(1-2)**, 57-67.

Singh, J.P. (2016). Generalized Sasakian space forms with M-projective curvature tensor, *Acta Math. Univ. Comeniannce*, **LXXXV (1)**, 135-146.

Singh, R.N. and Pandey, Giteshwari (2013). On the W_2-Curvature tensors of the semi-symmetric non-metric connection in a Kenmotsu manifold, *Novi Sad J. Math.* **43(2)**, 91-105.

Sinha, B.B. (1978). On almost contact metric manifold with semi-symmetric metric connection, *Math Rive. Amer. Math. Soc.* **12(1-2)**.

Sreenivasa, G.T., Venkatesha, Bagewadi, C.S. and Naganagoud, K. (2009). On φ-recurrent Lorentzian β-Kenmotsu manifolds, *SDU J. Sci.(E-Journal)* **4(1)**, 75-79.

Szabo, Z.I. (1982). Structure theorems on Riemannian spaces satisfying $R(X,Y).R = 0$, I, The local version, *J. Diff. Geome.* **17**, 531-582.

Takahashi, T. (1977). Sasakian φ-symmetric spaces, *Tohoku Math. J.* **29**, 91-113.

Takano, K. (1971). On a space with birecurrent curvature, *Tensor (N.S.)* **22**, 329-32.

Tamassy, L. and Binh, T.Q. (1993). On weak symmetries of Einstein and Sasakian manifolds, *Tensor (N.S.)* **3**, 140-148.

Taleshian, A. and Asghari, N. (2010). On LP-Sasakian manifolds satisfying certain conditions on the concircular curvature tensor, *Differential Geometry-Dynamical Systems* **12**, 228-232.

Taleshian, A. and Asghari, N. (2012). On Lorentzian α-Sasakian manifolds, *J. Math. and Comp. sci.* **4**, 295- 300.

Tripathi, M.M. and Kakar, N. (2001). On a semi-symmetric non-metric connection on a Kahler manifold, *Differential Geometry-Dynamical Systems* **4**, 86-90.

Tripathi, M.M. and Kakar, N. (2001). On a semi-symmetric non-metric connection on a Kenmotsu manifold, *Bull. Cal. Math. Soc.* **16(4)**, 323-330.

Venkatesha and Bagewadi, C.S. (2006). On pseudo projective φ-recurrent Kenmotsu manifoldss, *Soochow J. Math.* **32(3)**, 433-439.

Venkatesha and Bagewadi, C.S. (2008). On concircular φ-recurrent LP-Sasakian manifold, *Balkan Soc. Geometers* **10**, 312-319.

Venkatesha, Bagewadi, C.S. and Kumar Pradeep, K.T. (2011). Some results on Lorentzian Para-Sasakian manifolds, *ISRN Geometry*, ID 161523, **2011**, 1-9.

Walker, A.G. (1951). On Ruse's spaces of recurrent curvature, *Proc. London Math. Soc.* **52**, 36-64.

Yadav, S. and Suthar, D.L. (2012). Certain derivation on Lorentzin α-Sasakian manifolds, *GJSFR: Math. and Decision Sci.* **12(2)**, 101-106.

Yano, K. (1970). On semi symmetric connection, *Rev. Roum. Math. Pures, Appliqu*, 1579-1586.

Yano, K. and Imai, T. (1982). Quarter symmetric metric connection and their curvature tensors, *Tensor (N.S.)* **38**, 13-18.

Yano, K. and Kon, M. (1977). Anti invariant submanifold of Sasakian space forms, *I. Tohku Math. J.*, 23-29.

Yano, K. and Kon, M. (1984). *Structures on Manifolds*, Series in Pure Math.**3**, World Sci.

Yano, K. and Sawaki, S. (1968). Riemannian manifold admitting a conformal transformation group, *J. Differ. Geome.* **2**, 161-184.

Yildiz, A. and Murathan, C. (2005). On Lorentizian α-Sasakian Manifolds, *Kyungpook Math. J.* **45(1)**, 95-103.

Yildiz, A. and Turan, M. (2009). A Class of Lorentzian α-Sasakian manifolds, *Kyungpook Math. J.* **49**, 789-799.

Yildiz, A., Turan, M. and Acet, B.E. (2009). On three-dimensional Lorentzian α-Sasakian manifolds, *Bull. Math. Anal. Appl.* **1**, 90-98.

Appendices

(A) LIST OF RESEARCH PUBLICATIONS

(1) J.P. Singh, A.Singh (2014): Quarter symmetric non-metric connection on an LP-Sasakian manifold, International Journal of applied mathematical Sciences, (7)2, 113-121. ISSN: 0973-0176

(2) J.P. Singh, Archana Singh and Rajesh Kumar (2015): On a type of semi-generalized recurrent lorentzian α-Sasakian manifolds, Bull. Cal. Math. Soc. 107(5), 357-370. ISSN: 0008-0659

(3) Archana Singh, J.P. Singh and Rajesh Kumar(2016): On a type of semi-generalized recurrent P-Sasakian manifolds, Facta Universitatis (NIŠ), Ser. Math. Inform. 31(1), 213-225. ISSN: 0352-9665

(4) J.P. Singh, Archana Singh and Rajesh Kumar(2015): Some curvature conditions on LP-Sasakian manifolds admitting a quarter symmetric non-metric connection, J. Statistics Maths. Engg. 1(3), 1-14.

(5) J.P. Singh, Archana Singh and Rajesh Kumar(2014): Some curvature properties of LP-Sasakian manifolds, Jour. Pure Math., Vol. 31, 13-28. ISSN: 2277-355X

(B) CONFERENCES/ SEMINAR/ WORKSHOPS

(1) Presented a paper "Some Results On Semi-Generalized Recurrent α-Cosymplectic manifolds" Second Mizoram Mathematics Congress organized by Mizoram Mathematical Society (MMS) in Collaboration with Department of Mathematics (UG & PG), Mizoram University, Aizawl-796 004, Mizoram, $13-14^{th}$ August, 2015.

(2) Presented a paper "Some Curvature conditions on LP-Sasakian manifolds admitting a quarter symmetric non-metric connection" in National Conference on Application of Mathematics organized by Department of Mathematics and Computer Science, Mizoram University, Aizawl-796 004, Mizoram, $25-26^{th}$ February, 2016.