# A STUDY OF CERTAIN STRUCTURES ON ALMOST CONTACT MANIFOLDS

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# A STUDY OF CERTAIN STRUCTURES ON ALMOST CONTACT MANIFOLDS

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# CERTIFICATE

This is to certify that the thesis entitled "A Study of Certain Structures on Almost Contact Manifolds" submitted by Ms. K. Lalnunsiami (Registration No: MZU/Ph. D./1078 of 07.11.2017) for the degree of Doctor of Philosophy (Ph. D.) of the Mizoram University, embodies the record of original investigation carried out by her under my supervision. She has been duly registered and the thesis presented is worthy of being considered for the award of the Ph. D. degree. This work has not been submitted for any degree of any other universities.

(Dr. JAY PRAKASH SINGH)

Supervisor

**MIZORAM UNIVERSITY**

**TANHRIL**

**Month: August** **Year: 2021**

## DECLARATION

I, K. Lalnunsiami, hereby declare that the subject matter of this thesis entitled "A Study of Certain Structures on Almost Contact Manifolds" is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in other University/Institute.

This is being submitted to the Mizoram University for the degree of Doctor of Philosophy (Ph. D.) in Mathematics.

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## PREFACE

The present thesis entitled **"A Study of Certain Structures on Almost Contact Manifolds"** is an outcome of the research carried out by the author under the supervision of Dr. Jay Prakash Singh, Associate Professor, Department of Mathematics and Computer Science, Mizoram University, Aizawl, Mizoram.

This thesis has been divided into six chapters and each chapter is subdivided into smaller sections. The first chapter is the general introduction which includes the basic definitions of differential geometry such as topological manifold, differentiable manifolds, tangent vector, tangent space, vector field, Lie bracket, Lie derivative, connection, covariant derivative, contraction, Riemannian manifold, Riemannian connection, torsion tensor, semi-symmetric and quarter symmetric connection, different curvature tensors, almost contact metric manifolds, almost contact para-contact metric manifolds, recurrent manifolds and symmetric manifolds. Finally, the review of literature is given.

The second chapter is dedicated to the study of a semi-symmetric metric connection in weakly symmetric almost contact manifolds. We studied weakly symmetric Kenmotsu manifolds with respect to a semi-symmetric metric connection. We considered weakly Ricci symmetric, weakly concircular symmetric, weakly concircular Ricci symmetric and weakly $m$-projectively symmetric Kenmotsu manifolds with respect to such a connection. Weakly symmetric and weakly Ricci symmetric Para-Sasakian manifolds admitting a semi-symmetric metric connection are considered. An example of a 3-dimensional weakly symmetric and weakly Ricci symmetric Para-Sasakian manifold admitting a semi-symmetric metric connection is given.

The third chapter is related to the study of semi-generalized recurrent almost contact manifolds. We studied semi-generalized $W_3$ recurrent manifolds and obtained a necessary and sufficient condition for the scalar curvature to be constant in such a manifold. Ricci symmetric and decomposable semi-generalized $W_3$ recurrent manifolds are studied. Finally, we constructed two examples of a semi-generalized $W_3$

recurrent manifold.

In the fourth chapter, we study $N(k)$-quasi Einstein manifolds. $W^*$-Ricci pseudosymmetric, $W_2$-pseudosymmetric and $Z$-generalized pseudosymmetric $N(k)$-quasi Einstein manifolds are studied. We considered the curvature properties of the pseudo projective, $W_2$ and conharmonic curvature tensors in an $N(k)$-quasi Einstein manifold. Also, we have given examples to support the results.

In the fifth chapter, we considered the weak symmetry of the $Z$-tensor in almost contact manifolds. Weakly $Z$-symmetric manifolds with Codazzi type and cyclic parallel $Z$ tensor, Einstein weakly $Z$-symmetric manifolds and conformally flat weakly $Z$-symmetric manifolds are studied. A totally umbilical hypersurface of a conformally flat weakly $Z$-symmetric manifold is considered. We investigate decomposable weakly $Z$-symmetric manifolds and we construct examples to support our results.

In Chapter 6, we gave the summary and the conclusion. The references of the mentioned papers have been given with the surname of the author and the years of the publication, which are decoded in chronological order in the Bibliography.

# Contents

# Chapter 1

# Introduction

## 1.1 Topological Manifold

A topological space $M^n$ is a locally Euclidean space such that every point $p$ of $M^n$ has a neighborhood which is homeomorphic to an open subset $U$ of the Euclidean space $\mathbb{R}^n$. If $\phi$ is a homeomorphism from $U \subset M^n$ onto $U$, then $U$ is called a coordinate neighborhood; $\phi$ is called a coordinate map; the functions $x^i = t^i \cdot \phi$, where $t^i$ denotes the $i^{th}$ canonical coordinate function on $\mathbb{R}^n$ are called the coordinate functions and the pair $(U, \phi)$ is called a coordinate system or a chart.

A topological manifold of dimension $n$ is a Hausdorff, second countable, locally Euclidean space of dimension $n$.

**Definition 1.1** *The charts $(U, \phi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ are said to be $C^\infty$-compatible if $\phi \cdot \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ and $\psi \cdot \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ are $C^\infty$-mappings.*

**Definition 1.2** *An atlas $\mathfrak{M}$ of class $C^\infty$ on a locally Euclidean space $M^n$ is a collection of coordinate systems $\{(U_\alpha, \phi_\alpha) : \alpha \in \mathfrak{M}\}$ that cover $M^n$, i. e., such that $\cup_{\alpha \in \mathfrak{M}} U_\alpha = M^n$.*

**Definition 1.3** *A differentiable structure (or maximal atlas) on a locally Euclidean space $M^n$ is an atlas $\mathfrak{M} = \{(U_\alpha, \phi_\alpha) : \alpha \in \mathfrak{M}\}$ of class $C^\infty$, such that it is not* *contained in a larger atlas, i. e., if $\mathfrak{U}$ is any atlas containing $\mathfrak{M}$, then $\mathfrak{M} = \mathfrak{U}$.*

## 1.2 Differentiable Manifold

**Definition 1.4** *A topological manifold $M^n$ together with a maximal atlas $\mathfrak{M}$ is called a differentiable manifold of class $C^\infty$ of dimension $n$ (or simply differentiable manifold of dimension $n$ or $C^\infty$ manifold or $n$ dimensional manifold).*

A manifold is said to have dimension $n$ if all of its connected components have dimension $n$. A 1-dimensional manifold is also called a curve, a 2-dimensional manifold a surface.

## 1.3 Tangent Vectors and Tangent Spaces

Let $M^n$ be a differentiable manifold and $p \in M^n$, $C^\infty(p)$ be the set of all real-valued $C^\infty$-functions, each defined on some neighborhood $\cup$ of $p$. Let us consider a vector $X$ at $p$ such that

(i) $X \in M^n,\ f \in C^\infty(p)$, then $Xf \in C^\infty(p)$,

(ii) $X(f+g) = Xf + Xg, \quad f, g \in C^\infty(p)$ ,

(iii) $X(fg) = f(Xg) + (Xf)g$,

(iv) $X(af) = a(Xf), \quad a \in \mathbb{R}$,

then $X$ is called a tangent vector to $M^n$ at $p$.

The set of all tangent vectors at $p$ forms a vector space over $\mathbb{R}$ and is called the tangent space of $M^n$ at $p$ and is denoted by $T_p(M^n)$.

## 1.4 Vector Field

A vector field $X$ on $M^n$ is a linear mapping $X : C^\infty(M^n) \to C^\infty(M^n)$ such that the map $f \to Xf$ satisfies

(i) $X(f+g) = Xf + Xg$,

(ii) $X(af) = aXf$,

(iii) $X(fg) = (Xf)g + f(Xg)$

$\forall f, g \in C^\infty(M^n), a \in \mathbb{R}$. Thus to each point $p \in M^n$ such a derivation assigns a linear map $X_p : C^\infty(M^n) \to \mathbb{R}$ defined by $X_{(p)}f = (Xf)(p)$ for each $f \in C^\infty(M^n)$ and hence the map $p \in X_p$ assigns a field of tangent vectors.

## 1.5 Lie Bracket

If $X, Y$ are $C^\infty$ vector fields, then we define a $C^\infty$-mapping called the Lie bracket (or Poisson Bracket) $[\ ,\ ] : M^n \times M^n \to M^n$ as

$$[X, Y]f = X(Yf) - Y(Xf),$$

where $f \in C^\infty(M^n)$.

The Lie bracket satisfy the following properties:

(i) $[X, Y](f+g) = [X, Y]f + [X, Y]g$,

(ii) $[X, Y](fg) = f[X, Y]g + g[X, Y]f$,

(iii) $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$,

(iv) $[X, Y] = -[Y, X]$, $\quad$ *(skew symmetry)*

(v) $[X, aY + bZ] = a[X, Y] + b[X, Z]$, $\quad$ *(bilinear)*

(vi) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$, $\quad$ *(Jacobi identity)*

where $f, g \in C^\infty(M^n)$, $X, Y, Z \in C^\infty$ and $a, b$ are scalars.

## 1.6 Lie Derivative

Let $X$ be a $C^\infty$ vector field on an open set of $M^n$. Then the Lie derivative along $X$ is a type preserving linear mapping

$$\mathcal{L}_X : T^r_s \to T^r_s,$$

such that

(i) $\mathcal{L}_X f = Xf$,

(ii) $\mathcal{L}_X Y = [X, Y]$,

(iii) $(\mathcal{L}_X A)(Y) = X(A(Y)) - A([X, Y]), \qquad A \ \ is\ a\ 1 - form$

(iv) $\mathcal{L}_X a = 0, \qquad a \in \mathbb{R}$

(v)
$$\begin{aligned}(\mathcal{L}_X P)(A_1, ..., A_r, X_1, ..., X_s) &= X(P(A_1, ..., A_r, X_1, ..., X_s)) \\ &- P(\mathcal{L}_X A_1, ..., A_r, X_1, ..., X_s)... \\ &- P(A_1, ..., A_r, [X, X_1], X_2, ..., X_s)... \\ &- P(A_1, ..., X_{s-1}, [X, X_s]), \quad P \in T^r_s,\end{aligned}$$

where $f$ is a $C^\infty$ function, $X_1, X_2, ......, X_s$ are vector fields and $A_1, A_2, ......., A_r$ are 1-forms.

## 1.7 Connection

Let $p \in M^n$ be a point of $M^n$, $T_p(M^n)$ be a tangent space to $M^n$ at $p$ and $T^r_s$ be a vector space whose elements are the tensors of type $(r, s)$. A connection $\nabla$ is a type preserving mapping $\nabla : T_p \otimes T^r_s \to T^r_s$ that assigns to each pair of $C^\infty$-vector fields $(X, P)$, $X \in T_p, P \in T^r_s$, a $C^\infty$ vector field $\nabla_X P$ such that

(i) $\nabla_X f = Xf$,

(ii) $\nabla_X a = 0, \ a \in \mathbb{R}$

(iii) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z,$

(iv) $\nabla_X(fY) = (Xf)Y + f(\nabla_X Y),$

(v) $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z,$

(vi) $\nabla_{fX} Z = f\nabla_X Z,$

(vii) $(\nabla_X \nu)(Y) = X(\nu(Y)) - \nu(\nabla_X Y),$

(viii) $$\begin{aligned}(\nabla_X P)(\nu_1, ..., \nu_r, X_1, ..., X_s) &= X(P(\nu_1, ..., \nu_r, X_1, ..., X_s)) \\ &\quad - P(\nabla_X \nu_1, ..., \nu_r, X_1, ..., X_s)... \\ &\quad - P(\nu_1 ..., \nu_r, X_1, ..., \nabla_X X_s),\end{aligned}$$

where $f$ is a $C^\infty$-function.

## 1.8 Covariant Derivative

A linear affine connection on $M^n$ is a function

$$\nabla : \chi(M^n) \times \chi(M^n) \to \chi(M^n)$$

such that

(i) $\nabla_X f = Xf,$

(ii) $\nabla_{fX+gY} Z = f(\nabla_X Z) + g(\nabla_Y Z),$

(iii) $\nabla_X(fY + gZ) = f(\nabla_X Y) + g(\nabla_X Z) + (Xf)Y + (Xg)Z,$

where $X, Y, Z$ are arbitrary vector fields and $f, g \in C^\infty(M^n)$. $\nabla_X$ is a smooth function called the covariant derivative and $\nabla_X Y$ is called the covariant derivative of $Y$ with respect to $X$.

The covariant derivative of a 1-form $v$ is given by

$$(\nabla_X v)(Y) = X(v(Y)) - v(\nabla_X Y).$$

## 1.9 Contraction

The linear mapping

$$C_h^k : T_s^r \to T_{s-1}^{r-1}; \quad (i \leq h \leq r), \quad (i \leq k \leq s),$$

such that

$$\begin{aligned} C_h^k(\nu_1 \otimes \nu_2 \otimes ... \otimes \nu_r \otimes \beta_1 \otimes \beta_2 \otimes ... \otimes \beta_s) \quad &= \quad \beta_k(\nu_1 \otimes ... \otimes \nu_{h-1} \otimes \nu_{h+1} ... \otimes \nu_r \\ &\quad \otimes \beta_1 \otimes ... \otimes \beta_{k-1} \otimes \beta_{k+1} ... \otimes \beta_s), \end{aligned}$$

where $\nu_1, \nu_2, ..., \nu_r, \beta_1, \beta_2, ..., \beta_s \in T_p(M^n)$ and $\otimes$ denote the tensor product is called contraction with respect to the $h^{th}$ contravariant and $k^{th}$ covariant places.

## 1.10 Riemannian Manifold

Consider an $n$-dimensional $C^\infty$ with the tangent space $T_p$ at $p \in M^n$. A real valued, bilinear symmetric, non-singular positive definite function $g$ on the ordered pair $(X, Y)$ of tangent vectors $T_p$ at each point $p$, such that

(i) $g(X, Y)$ is a real number,

(ii) $g$ is symmetric $\Rightarrow g(X, Y) = g(Y, X)$,

(iii) $g$ is non-singular i. e., $g(X, Y) = 0$, for all $Y \neq 0 \Rightarrow X = 0$,

(iv) $g$ is positive definite i. e., $g(X, X) \geqslant 0$, for all $X \in C^\infty$ and $g(X, X) = 0$ if and only if $X = 0$,

(v) $g(aX + bY, Z) = ag(X, Z) + bg(Y, Z)$; $a, b \in \mathbb{R}$,

is called the Riemannian metric tensor or the fundamental tensor of type (0,2). The manifold $M^n$ with a Riemannian metric $g$ is called a Riemannian manifold and its geometry is called a Riemannian geometry denoted by $(M^n, g)$ or $(M, g)$ or simply by $M$.

## 1.11 Riemannian Connection

Let $(M^n, g)$ be an $n$-dimensional manifold and $\nabla$ be an affine connection on $M^n$. Then $\nabla$ is said to be a Riemannian connection (or Levi-Civita connection) if

(i) $\nabla$ is symmetric or torsion free i. e.,

$$\nabla_X Y - \nabla_Y X = [X, Y], \tag{1.1}$$

and

(ii) $\nabla$ is a metric compatible or metric connection i. e.,

$$(\nabla_X g)(Y, Z) = 0. \tag{1.2}$$

Thus a Riemannian connection on a Riemannian manifold is a linear connection which is torsion free and metric compatible.

## 1.12 Torsion Tensor

The mapping $T : \chi(M^n) \otimes \chi(M^n) \to \chi(M^n)$ defined by

$$T(X, Y) \stackrel{def}{=} \nabla_X Y - \nabla_Y X - [X, Y]. \tag{1.3}$$

is called a torsion tensor of the connection $\nabla$ for all $X, Y \in \chi(M^n)$.

The torsion tensor is vector valued, skew-symmetric, bilinear function $T$ of type $(1, 2)$. A tensor is said to be symmetric or torsion free, if

$$T(X, Y) = 0,$$

and $\nabla = 0$ .

## 1.13 Semi Symmetric and Quarter Symmetric Connection

A linear connection $\tilde{\nabla}$ on an $n$-dimensional Riemannian manifold $(M^n, g)$ is called a quarter symmetric connection (Golab, 1975) if its torsion tensor $T$ defined by

$$T(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y], \tag{1.4}$$

satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.5}$$

where $\eta$ is 1-form and $\phi$ is a $(1,1)$ tensor field. In particular, if $\phi(X) = X$, then the quarter symmetric connection reduces to a semi-symmetric connection. Thus the notion of quarter symmetric connection generalizes the notion of semi symmetric connection.

Moreover, if a quarter symmetric connection $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_X g)(Y,Z) = 0, \tag{1.6}$$

for all $X, Y, Z \in T_p(M^n)$, where $T_p(M^n)$ is the Lie algebra of vector fields of the manifold $M^n$, then $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection, otherwise it is called a quarter symmetric non-metric connection.

## 1.14 Curvature Tensor

The curvature tensor $R$ of type $(1,3)$ with respect to the Riemannian connection $\nabla$ is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.7}$$

for all $X, Y, Z \in T_p(M^n)$.

Let $R'$ be the associative curvature tensor of the type $(0,4)$ of the curvature tensor $R$. Then

$$R'(X,Y,Z,U) = g(R(X,Y)Z,U), \tag{1.8}$$

$R'$ is called the Riemannian-Christoffel curvature tensor of first kind.

The associative curvature tensor $R'$ satisfies the following properties:

$R'$ is skew-symmetric in first two slot

$$i.e., \quad R'(X,Y,Z,U) = -\ R'(Y,X,Z,U). \tag{1.9}$$

$R'$ is skew-symmetric in last two slot

$$i.e., \quad R'(X,Y,Z,U) = -\ R'(X,Y,U,Z). \tag{1.10}$$

$R'$ is symmetric in two pair of slot

$$i.e., \quad R'(X,Y,Z,U) = \quad R'(Z,U,X,Y). \tag{1.11}$$

$R'$ satisfies Bianchi's first identities

$$i.e., \quad R'(X,Y,Z,U) + \ R'(Y,Z,X,U) + \ R'(Z,X,Y,U) = 0, \tag{1.12}$$

and $R'$ satisfies Bianchi's second identities

$$i.e., \ (\nabla_X \ R')(Y,Z,U,V) + (\nabla_Y \ R')(Z,X,U,V) + (\nabla_Z \ R')(X,Y,U,V) = 0. \tag{1.13}$$

## 1.15 Ricci-Tensor

The tensor of type (0, 2) defined by

$$S(Y,Z) \stackrel{def}{=} (C_1^1 R)(Y,Z) = -(C_1^2 R)(Z,Y), \tag{1.14}$$

is called the Ricci tensor where $C_1^1$ and $C_1^2$ are the respective contractions. It is a symmetric tensor,

$$i.e., \quad S(X, Y) = S(Y, X).$$

The linear map $Q$ of type (1, 1) given by

$$g(QX, Y) \stackrel{def}{=} S(X, Y), \tag{1.15}$$

is called a Ricci map. It is self adjoint,

$$i.e., \qquad g(QX, Y) = g(X, QY). \tag{1.16}$$

The scalar curvature $r$ of $M^n$ at the point $p$ is defined as

$$r \stackrel{def}{=} (C_1^1 S). \tag{1.17}$$

A Riemannian manifold $M^n$ is said to be Einstein if

$$S(X, Y) = \frac{r}{n} g(X, Y). \tag{1.18}$$

A Riemannian manifold $M^n$ is said to be a flat manifold if

$$R(X, Y)Z = 0. \tag{1.19}$$

## 1.16 $Z$-tensor

A generalized (0, 2) tensor defined by

$$Z(X, Y) = S(X, Y) + \phi g(X, Y), \tag{1.20}$$

where $\phi$ is a smooth function and $S$ is the Ricci tensor is called the $Z$-tensor (Mantica and Suh, 2012).

## 1.17 Certain Curvature Tensors on a Riemannian manifold

**(A) Concircular curvature tensor**:

The concircular curvature tensor $C'$ of type $(0,4)$, is given by (Yano, 1940)

$$\begin{aligned} C'(X,Y,Z,U) &= R'(X,Y,Z,U) - \frac{r}{n(n-1)}\big[g(Y,Z)g(X,U) \\ &- g(X,Z)g(Y,U)\big]. \end{aligned} \tag{1.21}$$

It satisfies the following algebraic properties

(i) $C'(X,Y,Z,U) = -C'(Y,X,Z,U)$,

(ii) $C'(X,Y,Z,U) = -C'(X,Y,U,Z)$,

(iii) $C'(X,Y,Z,U) = C'(Z,U,X,Y)$,

(iv) $C'(X,Y,Z,U) + C'(Y,Z,X,U) + C'(Z,X,Y,U) = 0$,

where

$$C'(X,Y,Z,U) = g(C(X,Y)Z,U).$$

**(B) Conharmonic curvature tensor**:

The conharmonic curvature tensor $H'$ is defined as (Ishii, 1957)

$$\begin{aligned} H'(X,Y,Z,U) &= R'(X,Y,Z,U) - \frac{1}{(n-2)}\big[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \\ &+ S(X,U)g(Y,Z) - S(Y,U)g(X,Z)\big]. \end{aligned} \tag{1.22}$$

It satisfies the following properties

$$(i)\ H'(X,Y,Z,U) = -H'(Y,X,Z,U),$$
$$(ii)\ H'(X,Y,Z,U) = -H'(X,Y,U,Z),$$
$$(iii)\ H'(X,Y,Z,U) = H'(Z,U,X,Y),$$
$$(iv)\ H'(X,Y,Z,U) + H'(Y,Z,X,U) + H'(Z,X,Y,U) = 0,$$

where

$$H'(X,Y,Z,U) = g(H(X,Y)Z,U).$$

**(C) Projective curvature tensor**:

The projective curvature tensor $P'$ of type $(0,4)$ is defined by (Yano and Bochner, 1953)

$$\begin{aligned} P'(X,Y,Z,U) &= R'(X,Y,Z,U) - \frac{1}{(n-1)}\big[S(Y,Z)g(X,U) \\ &- S(X,Z)g(Y,U)\big]. \end{aligned} \tag{1.23}$$

The projective curvature tensor $P'$ satisfies

$$(i)P'(X,Y,Z,U) = -P'(Y,X,Z,U),$$
$$(ii)C_1^1P = C_2^1P = C_3^1P = 0,$$
$$(iii)P'(X,Y,Z,U) + P'(Y,Z,X,U) + P'(Z,X,Y,U) = 0,$$

where

$$P'(X,Y,Z,U) = g(P(X,Y)Z,U).$$

**(D) Pseudo-projective curvature tensor:** The pseudo-projective curvature tensor $\bar{P}$ is defined by (Prasad, 2002)

$$\begin{aligned} \bar{P}(X,Y)Z &= \alpha R(X,Y)Z + \beta\big[S(Y,Z)X - S(X,U)Z\big] \\ &- \frac{r}{n}\Big(\frac{\alpha}{n-1} + \beta\Big)\big[g(Y,Z)X - g(X,Z)Y\big], \end{aligned} \tag{1.24}$$

where $\alpha$ and $\beta$ are non zero constants.

It satisfies the following properties

$$(i)\bar{P}'(X,Y,Z,U) = -\bar{P}'(Y,X,Z,U),$$

$$(ii)\bar{P}'(X,Y,Z,U) = \bar{P}'(Y,X,U,Z),$$

where

$$\bar{P}'(X,Y,Z,U) = g(\bar{P}(X,Y)Z,U).$$

**(E)** $m$**-projective curvature tensor**:

Pokhariyal and Mishra (1971) defined the $m$-projective curvature tensor $W^{*'}$ of the type $(0,4)$ by

$$\begin{aligned} W^{*'}(X,Y,Z,U) &= R'(X,Y,Z,U) - \frac{1}{2(n-1)}\big[g(X,U)S(Y,Z) - g(Y,U)S(X,Z) \\ &+ S(X,U)g(Y,Z) - S(Y,U)g(X,Z)\big]. \end{aligned} \tag{1.25}$$

It satisfies the following algebraic properties

$$(i)\ W^{*'}(X,Y,Z,U) = W^{*'}(Z,U,X,Y),$$

$$(ii)\ W^{*'}(X,Y,Z,U) = -W^{*'}(Y,X,Z,U),$$

$$(iii)\ W^{*'}(X,Y,Z,U) = -\ W^{*'}(X,Y,U,Z),$$

$$(iv)\ W^{*'}(X,Y,Z,U) + W^{*'}(Y,Z,X,U) + W^{*'}(Z,X,Y,U) = 0,$$

where

$$W^{*'}(X,Y,Z,U) = g(W^{*}(X,Y)Z,U).$$

**(F)** $W_2$ **curvature tensor:**

Pokhariyal and Mishra (1971) also defined the $W_2$ curvature tensor of the type $(0,4)$ as

$$\begin{aligned} W_2'(X,Y,Z,U) &= R'(X,Y,Z,U) - \frac{1}{(n-1)}\big[g(Y,Z)S(X,U) \\ &- g(X,Z)S(Y,U)\big]. \end{aligned} \tag{1.26}$$

It satisfies the following properties

$$(i)\ W_2'(X,Y,Z,U) = \ -W_2'(Y,X,Z,U),$$

$$(ii)\ W_2'(X,Y,Z,U) + \ W_2'(Y,Z,X,U) + \ W_2'(Z,X,Y,U) = 0,$$

where

$$W_2'(X,Y,Z,U) = g(W_2(X,Y)Z,U).$$

**(G) Conformal curvature tensor:** The Weyl conformal curvature tensor $\bar{C}'$ of type (0, 4) is defined as (Yano and Kon, 1984)

$$\begin{aligned}\bar{C}'(X,Y,Z,U) &= R'(X,Y,Z,U) - \frac{1}{(n-2)}\big[g(Y,Z)S(X,U) - g(X,Z)S(Y,U) \\ &+ S(Y,Z)g(X,U) - S(X,Z)g(Y,U)\big] \\ &+ \frac{r}{(n-1)(n-2)}\big[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\big]. \qquad (1.27)\end{aligned}$$

## 1.18 Almost Contact Metric Manifold

Let $M^n(n = 2m + 1)$ be an odd-dimensional differentiable manifold. Let $\phi$ be a tensor field of type (1,1), $\zeta$ a vector field, $\eta$ a 1-form on $M^n$ satisfying for arbitrary vectors $X, Y, Z$

$$\phi^2 X = -X + \eta(X)\zeta, \qquad (1.28)$$

$$\eta(\zeta) = 1, \qquad (1.29)$$

$$\phi(\zeta) = 0, \qquad (1.30)$$

$$\eta(\phi X) = 0, \qquad (1.31)$$

and

$$rank(\phi) = n - 1, \tag{1.32}$$

then $M^n$ is called an almost contact manifold (Sasaki, 1965) and the structure $(\phi, \eta, \zeta)$ is called an almost contact structure (Sasaki, 1960; Sasaki and Hatakeyama, 1961; Hatakeyama et al., 1963).

An almost contact manifold $M^n$ on which $\exists$ a Riemannian metric tensor $g$ satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.33}$$

and

$$g(X, \zeta) = \eta(X), \tag{1.34}$$

is called an almost contact metric manifold and the structure $(\phi, \zeta, \eta, g)$ is called an almost contact metric structure (Sasaki, 1960).

The fundamental 2-form $F'$ of an almost contact metric manifold $M^n$ is defined by

$$F'(X, Y) = g(\phi X, Y). \tag{1.35}$$

We have

$$F'(X, Y) = -F'(Y, X). \tag{1.36}$$

If in an almost contact metric manifold

$$2F'(X, Y) = (\nabla_X \eta)(Y) - (\nabla_Y \eta)(X), \tag{1.37}$$

then $M^n$ is called an almost Sasakian manifold.

An almost contact metric manifold is called a Kenmotsu manifold (Kenmotsu,

1972) if

$$\nabla_X \zeta = X - \eta(X)\zeta, \tag{1.38}$$

$$(\nabla_X \phi)(Y) = g(\phi X, Y)\zeta - \eta(Y)\phi X, \tag{1.39}$$

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y). \tag{1.40}$$

In a Kenmotsu manifold, we have

$$R(X, Y)\zeta = \eta(X)Y - \eta(Y)X, \tag{1.41}$$

$$R(X, \zeta)Y = g(X, Y)\zeta - \eta(Y)X, \tag{1.42}$$

$$R(\zeta, X)\zeta = X - \eta(X)\zeta, \tag{1.43}$$

$$S(X, \zeta) = -(n-1)\eta(X), \tag{1.44}$$

$$S(\zeta, \zeta) = -(n-1). \tag{1.45}$$

## 1.19 Almost Para-Contact Metric Manifold

Let $M^n$ be an $n$-dimensional $C^\infty$-manifold. If there exist a tensor field $\phi$ of the type $(1,1)$, a vector field $\zeta$ and a 1-form $\eta$ in $M^n$ satisfying

$$\phi^2 X = X - \eta(X)\zeta, \tag{1.46}$$

$$\phi(\zeta) = 0, \qquad \eta(\zeta) = 1, \tag{1.47}$$

then $M^n$ is called an almost Para-contact manifold.

Let $g$ be a Riemannian metric satisfying

$$\eta(X) = g(X, \zeta), \qquad \eta(\phi X) = 0, \tag{1.48}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.49}$$

then the structure $(\phi, \zeta, \eta, g)$ satisfying (1.46) - (1.49) is called an almost Para-contact Riemannian structure. The manifold with such structure is called an almost Para-contact Riemannian manifold (Sato and Matsumoto, 1976).

If we define $F'(X, Y) = g(\phi X, Y)$, then the following relations are satisfied:

$$F'(X, Y) = F'(Y, X), \tag{1.50}$$

and

$$F'(\phi X, \phi Y) = F'(X, Y). \tag{1.51}$$

An almost Para-contact metric manifold $M^n$ is said to be Para-Sasakian or $P$-Sasakian if (Adati and Matsumoto, 1977)

$$d\eta = 0, \quad i.e., \eta \text{ is closed}, \tag{1.52}$$

$$(\nabla_X \phi)(Y) = -g(X, Y)\zeta - \eta(Y)X + 2\eta(X)\eta(Y)\zeta, \tag{1.53}$$

$$\nabla_X \zeta = \phi X, \tag{1.54}$$

$$rank(\phi) = (n - 1), \tag{1.55}$$

$$(\nabla_X \eta)(Y) = g(\phi X, Y) = g(\phi Y, X), \tag{1.56}$$

for any vector fields $X, Y$ where $\nabla$ denotes the covariant differentiation with respect to $g$.

Also, in a Para-Sasakian manifold, the following relations hold (Sato, 1976; Adati and Matsumoto, 1977):

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{1.57}$$

$$R(X,\zeta)Y = g(X,Y)\zeta - \eta(Y)X, \tag{1.58}$$

$$R(\zeta,X)\zeta = X - \eta(X)\zeta, \tag{1.59}$$

$$R(X,Y)\zeta = \eta(X)Y - \eta(Y)X, \tag{1.60}$$

$$S(X,\zeta) = -(n-1)\eta(X), \tag{1.61}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{1.62}$$

where $R$ and $S$ are the curvature and the Ricci tensors of the manifold respectively.

## 1.20 Recurrent Manifold

Let $M^n$ be an $n$-dimensional smooth Riemannian manifold and $\chi(M^n)$ denotes the set of all differentiable vector fields on $M^n$. Let $X, Y \in \chi(M^n)$; $\nabla_X Y$ denotes the covariant derivative of $Y$ with respect to $X$ and $R$ be the Riemannian curvature tensor of type $(1,3)$. Then, $M^n$ is said to be recurrent (Kobayashi and Nomizu, 1963) if

$$(\nabla_U R)(X,Y)Z = \alpha(U)R(X,Y)Z, \tag{1.63}$$

where $X, Y, Z \in \chi(M^n)$ and $\alpha$ is a non-zero 1-form known as recurrence parameter. If the 1-form $\alpha$ is zero in (1.63), then the manifold reduces to a symmetric manifold (Singh and Khan, 1999).

A Riemannian manifold $(M^n, g)$ is said to be Ricci-recurrent (Patterson, 1952) if it satisfies the relation

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z), \tag{1.64}$$

for all $X, Y, Z \in \chi(M^n)$, where $\nabla$ denotes the Levi-Civita connection and $A$ is a 1-form on $M^n$. If the 1-form $A$ vanishes identically on $M^n$, then a Ricci-recurrent manifold becomes a Ricci-symmetric manifold.

A Riemannian manifold $(M^n, g)$ is called a generalized recurrent manifold (De and Guha, 1991) if its curvature tensor $R$ satisfies the following condition:

$$(\nabla_X R)(Y, Z)U = A(X)R(Y, Z)U + B(X)[g(Z, U)Y - g(Y, U)Z], \tag{1.65}$$

where $A$ and $B$ are 1-forms, $B$ is non-zero and defined by

$$A(X) = g(X, \rho_1), \qquad B(X) = g(X, \rho_2), \tag{1.66}$$

$\rho_1$ and $\rho_2$ are vector fields associated with 1-forms $A$ and $B$ respectively.

A Riemannian manifold $(M^n, g)$ is said to be $\phi$-recurrent (De et al., 2003) if there exists a non-zero 1-form $A$ such that

$$\phi^2((\nabla_W R)(Y, Z)U) = A(W)R(Y, Z)U, \tag{1.67}$$

for arbitrary vector fields $Y, Z, U, W$.

A Riemannian manifold $(M^n, g)$ is called generalized $\phi$-recurrent (Shaikh and Ahmad, 2011) if its curvature tensor $R$ satisfies

$$\begin{aligned} \phi^2((\nabla_W R)(Y, Z)U) &= A(W)R(Y, Z)U \\ &+ B(W)[g(Z, U)Y - g(Y, U)Z], \end{aligned} \tag{1.68}$$

where $A$ and $B$ are 1-forms and $B$ is non-zero.

## 1.21 Weakly Symmetric Manifold

A non-flat Riemannian manifold $(M^n, g), (n > 2)$ is called a weakly symmetric manifold (Tamassy and Binh, 1989) if the curvature tensor $R$ of type (1,3) satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z)U &= A(X)R(Y,Z)U + B(Y)R(X,Z)U + C(Z)R(Y,X)U \\ &+ D(U)R(Y,Z)X + g(R(Y,Z)U, P), \qquad (1.69)\end{aligned}$$

for all $X, Y, Z, U \in \chi(M^n)$, where $\nabla$ denotes the Levi-Civita connection on $(M^n, g)$ and $A, B, C, D$ and $P$ are 1-forms and a vector field respectively which are non-zero simultaneously.

A non-flat Riemannian manifold $(M^n, g), (n > 2)$ is said to be weakly Ricci symmetric (Tamassy and Binh, 1993) if the Ricci tensor $S$ satisfies the condition

$$(\nabla_X S)(Y,Z) = \alpha(X)S(Y,Z) + \beta(Y)S(X,Z) + \gamma(Z)S(Y,X), \qquad (1.70)$$

where $\alpha, \beta, \gamma$ are simultaneously non-zero.

## 1.22 Semi-symmetric Manifold

A Riemannian manifold $(M^n, g)$ is known as a semi-symmetric manifold (Cartan, 1946) if it satisfies the relation

$$R(X,Y) \cdot R(U,V)W = 0, \qquad (1.71)$$

for all $X, Y, Z, U, V, W \in \chi(M^n)$.

A Riemannian manifold $M^n$ is said to be Ricci semi-symmetric (Cartan, 1946) if the Ricci tensor $S$ satisfies

$$R(X,Y) \cdot S(U,V) = 0. \qquad (1.72)$$

## 1.23 Pseudosymmetric Manifold

An $n$-dimensional Riemannian manifold $M^n, (n > 2)$ is called a pseudosymmetric manifold (Deszcz, 1992) if $R \cdot R$ and $Q(g, R)$ are linearly dependent, i. e.,

$$R \cdot R = L_R Q(g, R) \tag{1.73}$$

holds on the set $U_R = \{x \in M^n : Q(g, R) \neq 0 \text{ at } x\}$, where $L_R$ is some function on $U_R$.

Also, $M^n$ is called Ricci pseudosymmetric and Ricci-generalized pseudosymmetric manifold if

$$R \cdot S = L_S Q(g, S), \tag{1.74}$$

and

$$R \cdot R = L_G Q(S, R) \tag{1.75}$$

holds on the set $U_S = \{x \in M^n : Q(g, S) \neq 0 \;\text{ at }\; x\}$ and $U_G = \{x \in M^n : Q(S, R) \neq 0 \;\text{ at }\; x\}$ respectively, where $L_S$ and $L_G$ are some functions on $U_S$ and $U_G$.

## 1.24 Methodology

Differentiable manifold was defined on the basis of differential calculus, topology and real analysis. With the help of differentiable manifold, we can study curves and surfaces in $n$-dimensional Euclidean space. Riemannian manifold is a part of differentiable manifold which we study by index free notation and tensor notation. The fundamental theorem of Riemannian Geometry, Lie algebra, Ricci Identity, Jacobi Identity, Bianchi first Identity, Bianchi second Identity, Contraction method, Koszul's formula and Levi-Civita connection are used in our study. The details of some of the above mentioned methods are given as:

(i) **Ricci identity**:

For a tensor field $R$ of type $(0,1)$ on a Riemannian manifold $(M^n, g)$,

$([\nabla_X \nabla_Y]u - \nabla_{[X,Y]}u)(Z) = -u(R(X,Y)Z)$,

$([\nabla_X, \nabla_Y]P - \nabla_{[X,Y]}P)Z = R(X,Y)(P(Z)) - P(R(X,Y)Z)$.

(ii) **Jacobi identity**:

$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$, for $X, Y, Z \in \chi(M^n)$.

(iii) **Bianchi's First identity**:

For a tensor field $R$ of type $(0,1)$ on a Riemannian manifold $(M^n, g)$,

$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$,

where $X, Y, Z$ are vector fields.

(iv) **Bianchi's Second identity**:

For a Riemannian connection $\nabla$, we have

$(\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W + (\nabla_Z R)(Y,X)W = 0$,

where $R$ is the curvature tensor.

(v) **Koszul's Formula**:

$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)$

for all $X, Y, Z \in \chi(M^n)$.

(vi) **Fundamental Theorem of Riemannian Geometry**:

Every Riemannian manifold $(M^n, g)$ of dimension $n$ admits a unique torsion free connection.

## 1.25 Review of Literature

The idea of semi-symmetric linear connection on a differentiable manifold was introduced by Friedman and Schouten (1924). Hayden (1932) defined a metric connection with torsion on a Riemannian manifold. Yano (1970) studied some curvature and derivational conditions for semi-symmetric connections in Riemannian manifolds. As a generalization of this, Golab (1975) introduced quarter symmetric connection on a differentiable manifold.

A semi-symmetric metric connection was defined in an almost contact manifold by Sharfuddin and Hussain (1976). Manifolds admitting semi-symmetric metric connection have been studied by Amur and Pujar (1978), Szabo (1982, 87), De and Biswas (1997), De and Sengupta (2000), Murathan and Ozgur (2008), Yilmaz et al. (2011) and other geometers. De and Sengupta (2000) investigated the curvature tensor of an almost contact metric manifold admitting a type of semi-symmetric metric connection and studied the curvature properties of the conformal curvature tensor and the projective curvature tensor. This was also studied by many geometers like Sasaki and Hatakeyama (1961), Hatakeyama (1963), Hatakeyama et al. (1963), Oubina (1985). Agashe and Chafle (1992) introduced a semi symmetric non-metric connection on a Riemannian manifold and this was further studied by De and Kamilya (1994), Pandey and Ojha (2001), Prasad and Kumar (2002), Chaturvedi and Pandey (2008), Chaubey (2011) and others. Verma (2020) studied the properties of trans-Sasakian manifolds admitting a semi-symmetric metric connection.

The notion of Para-Sasakian manifolds was first defined by Adati and Masumoto in 1927. Kenmotsu (1972) initiated the study of Kenmotsu manifolds. Tamassy and Binh (1989, 1993) introduced the notion of weakly symmetric manifolds and weakly Ricci symmetric manifolds. In 2005, De and Ghosh defined the weakly concircular Ricci symmetric manifolds. Shaikh and Hui initiated the notion of weakly concircular symmetric manifolds in 2009. In 2015, Singh studied some properties of $LP$-Sasakian manifolds with respect to a quarter symmetric non-metric connection. Prakasha and Vikas (2015) studied some properties of weakly symmetric Kenmotsu manifolds admitting a quarter-symmetric metric connection. $\phi$-symmetric $LP$-Sasakian manifolds admitting semi-symmetric metric connection were studied by Shaikh and Hui in 2015. In 2018, Hui and Lemence studied generalized $\phi$-recurrent Kenmotsu manifolds with respect to quarter-symmetric metric connection. Some curvature properties of trans Sasakian manifolds with respect to a quarter symmetric non-metric connection have been studied by Lalmalsawma and Singh (2019). In 2020, Yadav et al. investigated the properties of Kenmotsu manifolds with respect to a semi-symmetric metric connection.

The idea of recurrent manifolds was introduced by Walker (1950). Several authors have generalized the notion of recurrent manifolds such as 2-recurrent manifolds by Lichnerowicz (1950), Ricci recurrent manifolds by Patterson (1952), projective 2-recurrent manifolds by Ghosh (1970) and others. A tensor field of type $(0, p)$ is said to be recurrent if

$$\begin{aligned}&(\nabla_X T)(Y_1, Y_2, ...., Y_p)T(Z_1, Z_2, ...., Z_p)\\&-T(Y_1, Y_2, ...., Y_p)(\nabla_X T)(Z_1, Z_2, ...., Z_p) = 0,\end{aligned}$$

holds on $(M^n, g)$.

De and Guha (1991) studied generalized recurrent manifold with the non-zero 1-form $A$ and another non-zero associated 1-form $B$. Such a manifold has been denoted by $GK_n$. If $B$ becomes zero then the manifold $GK_n$ reduces to recurrent manifold introduced by Ruse (1951) denoted by $K_n$. Khan (2004) introduced the notion of generalized recurrent Sasakian manifolds to generalize the notion of recurrency. Generalized recurrent and generalized Ricci recurrent manifolds have been studied by several authors such as Özgür (2007), Arslan et al. (2009), Mallick et al. (2013) and many others. Rajesh Kumar et al. (2010) extended the study of semi-generalized recurrent manifolds to LP- Sasakian manifolds and obtained some interesting results. Archana Singh et al. (2016) extended this study to Para- Sasakian manifolds.

Prasad (2000) initiated the notion of semi-generalized recurrent manifold. A Riemannian manifold $(M^n, g)$ is called a semi-generalized recurrent manifold if its curvature tensor $R$ satisfies the condition

$$(\nabla_X R)(Y, Z)W = A(X)R(Y, Z)W + B(X)g(Y, Z)W \tag{1.76}$$

where $A, B$ are two 1-forms, $B$ is non zero, $P_1$ and $P_2$ are two vector fields such that

$$g(X, P_1) = A(X), \quad g(X, P_2) = B(X)$$

for any vector field $X$ and $\nabla$ denote the operator of covariant differentiation with respect to $g$. Singh and Khan (2000) studied generalized recurrent and generalized conformally recurrent manifolds. Generalized concircularly recurrent manifolds have been studied by De and Gazi (2009). Generalized $\phi$-recurrent and generalized concircular $\phi$-recurrent $P$-Sasakian manifold were studied by Singh (2014a). Singh (2014b) studied $m$-projective recurrent Riemannian manifold. In 2014, De and Pal studied some geometric properties of generalized $m$-projectively recurrent manifolds. Jaiswal and Yadav (2016) studied generalized $m$-projective $\phi$-recurrent trans-Sasakian manifolds.

An $n$-dimensional Riemannian or semi-Riemannian manifold $M(n > 2)$ is said to be an Einstein manifold if it satisfies

$$S = \frac{r}{n} g, \tag{1.77}$$

where $S$ and $r$ are the Ricci tensor and the scalar curvature respectively. Equation (1.77) is called the Einstein metric condition (Besse, 1987). The notion of a quasi Einstein manifold was introduced during the study of exact solutions to the Einstein field equations and consideration of quasi-umbilical hypersurfaces (Chaki and Maity, 2000). A non-flat Riemannian manifold $(M, g)$ is said to be quasi Einstein if its Ricci tensor $S$ satisfies

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{1.78}$$

$\forall X, Y \in T(M)$ where $a$ and $b$ are smooth functions, $b \neq 0$ called the associated scalars, $\eta$ is a non zero 1-form defined by

$$g(X, \zeta) = \eta(X), \; g(\zeta, \zeta) = 1, \tag{1.79}$$

called the associated 1-form and the unit vector field $\zeta$ is called the generator of the manifold. The study of quasi Einstein manifolds was continued by Chaki (2001), Guha (2003), De and Ghosh (2004b) and several other geometers. The notion of quasi Einstein manifolds have been extended to generalized quasi Einstein manifolds (De and Ghosh, 2004a), mixed generalized quasi Einstein manifolds (Bhattacharya and Debnath, 2004), generalized Einstein manifolds (Bejan and Binh, 2008) and others. $\ddot{O}$zg$\ddot{u}$r also studied generalized quasi Einstein manifolds (2006) and super quasi Einstein manifolds (2009). The $k$-nullity distribution of a Riemannian manifold $M$ is defined as

$$N(k) : p \to N_p(k) = \{Z \in T_p(M) : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\} \quad (1.80)$$

for some smooth function (Tanno, 1988). If the generator $\zeta$ in a quasi Einstein manifold $M$ belongs to some $k$-nullity distribution, then $M$ is called an $N(k)$-quasi Einstein manifold (Tripathi and Kim, 2007).

In 2016, Mallick and De studied the derivation conditions $R(\zeta, X) \cdot Z = 0$ and $P(\zeta, X) \cdot Z = 0$ in an $N(k)$-quasi Einstein manifold, where $P$ is the projective curvature tensor. $N(k)$-quasi Einstein manifolds satisfying $C(\zeta, X) \cdot R = 0$, $R(\zeta, X) \cdot W^* = 0$ and $W^*(\zeta, X) \cdot S = 0$, where $C$ is the conformal curvature tensor have been studied by De et al. (2016). Also, in 2019, Chaubey studied $W^*$-pseudosymmetric and $Z$-recurrent $N(k)$-quasi Einstein manifolds. $N(k)$-quasi Einstein manifolds satisfying certain curvature conditions have been studied by Tripathi and Kim (2007), Hosseinzadeh and Taleshian (2012), Hui and Lemence (2013), De et al. (2016), Chaubey (2017) and so on. In 2020, $\ddot{U}$nal studied $N(k)$-quasi Einstein manifolds with respect to a type of semi-symmetric metric connection.

In 2012, Mantica and Suh defined a (0, 2) tensor known as the $Z$-tensor as

$$Z(X,Y) = S(X,Y) + \phi g(X,Y),$$

where $\phi$ is a scalar function. Mantica and Molinari (2012) generalized the notion of weakly Ricci symmetric manifolds to weakly $Z$-symmetric manifolds and studied several geometric properties. The study of the $Z$-tensor was continued by Mantica and Suh (2014), Mallick and De (2016) and other geometers.

In 1926, Cartan studied Riemannian symmetric spaces obtaining a classification. A Riemannian manifold $(M^n, g)$ is called a locally symmetric manifold if $\nabla R = 0$, where $R$ is the Riemannian curvature tensor and $\nabla$ is the Levi Civita connection. The notion of locally symmetric manifolds have been extended to conformally symmetric manifolds (Chaki and Gupta, 1963), pseudo symmetric manifolds (Chaki, 1987), weakly symmetric manifolds (Tamassy and Binh, 1989) and so on. Prvanovic (1995) introduced the notion of weakly symmetric manifolds. A non-flat Riemannian manifold $(M^n, g)$ is said to be weakly symmetric (Tamassy and Binh, 1989) if the curvature tensor $R$ satisfies equation (1.69). We denote such a manifold by $(WS)_n$. In a $(WS)_n$, we have $B = C$ and $D = E$ (De and Bandhyopadhyay, 1999).

A Riemannian manifold $(M^n, g)(n > 2)$ is said to be weakly $Z$-symmetric (Mantica and Molinari, 2012) if the $Z$ tensor satisfies

$$(\nabla_X Z)(V, W) = A(U)Z(V, W) + B(V)Z(W, U) + C(W)Z(U, V)$$

where $A, B, C$ are the associated 1-forms. Various properties of the $Z$-tensor were pointed out by Mantica and Suh (2012). This notion was further generalized by De et al. (2015) to weakly cyclic $Z$-symmetric manifolds. The concept of $Z$-recurrent form embraces both pseudo $Z$-symmetric and weakly $Z$-symmetric manifolds.

The study of cyclic parallel and Codazzi type Ricci tensor were introduced by Gray (1978). A Riemannian manifold is said to have cyclic parallel Ricci tensor if $S$ is non-zero and

$$(\nabla_X S)(U, V) + (\nabla_U S)(X, V) + (\nabla_V S)(U, X) = 0. \tag{1.81}$$

The Ricci tensor $S$ in a Riemannian manifold is said to be of Codazzi type if $S$ is not zero and satisfies

$$(\nabla_X S)(U, V) = (\nabla_V S)(U, X). \tag{1.82}$$

In 2014, De and Prajjwal studied almost pseudo $Z$-symmetric manifolds with cyclic parallel $Z$-tensor and proved that an almost pseudo $Z$-symmetric manifold with Codazzi type $Z$ tensor is quasi-Einstein. Recently, De et al. (2015) obtained a condition for a conformally flat weakly cyclic $Z$-symmetric manifold to be of quasi constant curvature. Weakly cyclic generalized $Z$-symmetric manifolds were studied by Pandey (2020).

# Chapter 2

# Properties of Semi-symmetric Metric Connection

In this chapter we studied weakly symmetric, weakly Ricci symmetric, weakly concircular symmetric and weakly concircular Ricci symmetric properties of a Kenmotsu manifolds with respect to a semi-symmetric metric connection. Weakly $m$-projectively symmetric Kenmotsu manifold with respect to such a connection are considered. Also, we studied weakly symmetric Para-Sasakian manifolds with respect to a semi-symmetric metric connection.

## 2.1 Introduction

**Definition 2.1** *A semi-symmetric connection in a Riemmanian manifold is defined by Friedman and Schouten (1924) as a connection $\nabla$ whose torsion tensor $T$ satisfies*

$$T(X,Y) = \eta(Y)X - \eta(X)Y \tag{2.1}$$

*where*

$$g(X,\zeta) = \eta(X) \tag{2.2}$$

Singh, J. P. and Lalnunsiami, K. (2019). Some results on weakly symmetric Kenmotsu manifolds, *Science and Technology Journal*, **7 (1)**, 13-21.

*is a 1-form and* $X, Y \in \chi(M^n)$ *where* $\chi(M^n)$ *is the set of all differentiable vector fields in* $M^n$*. In addition, if* $\nabla g = 0$ *then* $\nabla$ *is known as a semi-symmetric metric connection.*

Weakly symmetric and weakly Ricci symmetric manifolds have been defined by Tamassy and Binh (1989, 1993).

**Definition 2.2** *A Riemannian manifold* $M^n(n > 2)$ *is said to be a weakly concircular symmetric manifold (Shaikh and Hui, 2009) if* $\exists$ *1-forms* $B_1, B_2, B_3, B_4, B_5$ *such that*

$$\begin{aligned}(\nabla_X C')(Y,Z,U,V) &= B_1(X)C'(Y,Z,U,V) + B_2(Y)C'(X,Z,U,V) \\ &+ B_3(Z)C'(Y,X,U,V) + B_4(U)C'(Y,Z,X,V) \\ &+ B_5(V)C'(Y,Z,U,X) \end{aligned} \tag{2.3}$$

*where* $B_1, B_2, B_3, B_4, B_5$ *are not simultaneously zero and the concircular curvature tensor* $C'$ *is defined by* (1.21).

**Definition 2.3** *A Riemmanian manifold* $M^n(n > 2)$ *is called a weakly concircular Ricci symmetric manifold (De and Ghosh, 2005) if* $\exists$ *1-forms* $A_1, A_2, A_3$*, not simultaneously zero such that the concircular Ricci tensor* $P$ *given by*

$$P(X,Y) = \sum_{i=1}^{n} C'(e_i, X, Y, e_i) = S(X,Y) - \frac{r}{n} g(X,Y) \tag{2.4}$$

*is not identically zero and satisfies*

$$(\nabla_X P)(Y,Z) = A_1(X)P(Y,Z) + A_2(Y)P(X,Z) + A_3(Z)P(Y,X) \tag{2.5}$$

*for all* $X, Y, Z \in \chi(M^n)$.

**Definition 2.4** *A weakly m-projectively symmetric manifold is a non m-projectively flat manifold* $M^n(n > 2)$ *where*

$$\begin{aligned}(\nabla_X W^*)(Y,Z,U,V) &= B_1(X)W^*(Y,Z,U,V) + B_2(Y)W^*(X,Z,U,V) \\ &+ B_3(Z)W^*(Y,X,U,V) + B_4(U)W^*(Y,Z,X,V) \\ &+ B_5(V)W^*(Y,Z,U,X) \end{aligned} \tag{2.6}$$

*where $W^*$ is the m-projective curvature tensor defined by* (1.25).

An $n$-dimensional differentiable manifold $M^n (n > 2)$ is called a Kenmotsu manifold (Kenmotsu, 1972) if $\exists$ an almost contact structure $(\phi, \zeta, \eta, g)$ satisfying equations (1.28)-(1.32) and (1.38)-(1.40) respectively.

An $n$-dimensional differentiable manifold $M^n (n > 3)$ is called a Para-Sasakian manifold (Adati and Matsumoto, 1977) if it admits an almost paracontact structure $(\phi, \zeta, \eta, g)$ satisfying (1.46)-(1.49) and (1.53)-(1.56) respectively.

## 2.2 Semi-symmetric metric connection in a Kenmotsu manifold

A semi-symmetric metric connection in a Kenmotsu manifold is given by Yano (1970) as

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\zeta. \tag{2.7}$$

We obtain a relation between the Riemmanian curvature tensor $R$ with respect to the Levi-Civita connection $\nabla$ and the curvature tensor $\tilde{R}$ with respect to the semi-symmetric metric connection $\tilde{\nabla}$ as (Prakasha and Vikas, 2013)

$$\begin{aligned} \tilde{R}(X,Y)Z &= R(X,Y)Z - 3[g(Y,Z)X - g(X,Z)Y] + 2[\eta(Y)X \\ &- \eta(X)Y]\eta(Z) - 2[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\zeta. \end{aligned} \tag{2.8}$$

On contracting equation (2.8), we get

$$\tilde{S}(X,Y) = S(X,Y) - (3n-5)g(X,Y) + 2(n-2)\eta(X)\eta(Y), \tag{2.9}$$

where $\tilde{S}$ and $S$ are the Ricci tensors with respect to $\tilde{\nabla}$ and $\nabla$ respectively.

Again by contraction, equation (2.9) reduces to

$$\tilde{r} = r - 2(n-1), \tag{2.10}$$

where $\tilde{r}$ and $r$ are the scalar curvatures with respect to $\tilde{\nabla}$ and $\nabla$ respectively.

Making use of equations (1.28)-(1.31), (1.41)-(1.45), we get

$$\tilde{R}(X,Y)\zeta = 2\big[\eta(X)Y - \eta(Y)X\big], \tag{2.11}$$

$$\tilde{R}(X,\zeta)Y = 2\big[g(X,Y)\zeta - \eta(Y)X\big], \tag{2.12}$$

$$\tilde{R}(\zeta,X)\zeta = 2\big[X - \eta(X)\big], \tag{2.13}$$

$$\tilde{S}(X,\zeta) = -2(n-1)\eta(X), \tag{2.14}$$

$$\tilde{S}(\zeta,\zeta) = -2(n-1). \tag{2.15}$$

## 2.3 Weakly concircular symmetric Kenmotsu manifolds with respect to a semi-symmetric metric connection

Consider a Kenmotsu manifold which is weakly concircular symmetric with respect to $\tilde{\nabla}$. Then,

$$\begin{aligned}(\tilde{\nabla}_X\tilde{C'})(Y,Z,U,V) &= B_1(X)\tilde{C'}(Y,Z,U,V) + B_2(Y)\tilde{C'}(X,Z,U,V)\\ &+ B_3(Z)\tilde{C'}(Y,X,U,V) + B_4(U)\tilde{C'}(Y,Z,X,V)\\ &+ B_5(V)\tilde{C'}(Y,Z,U,X),\end{aligned} \tag{2.16}$$

where $\tilde{C'}$ is the concircular curvature tensor with respect to $\tilde{\nabla}$ which is not identically zero.

In a weakly concircular symmetric Kenmotsu manifold with respect to a semi-symmetric metric connection, $B_2 = B_3$ and $B_4 = B_5$. So equation (2.16) can be written as

$$
\begin{aligned}
(\tilde{\nabla}_X\tilde{C}')(Y,Z,U,V) &= B_1(X)\tilde{C}'(Y,Z,U,V)+B_2(Y)\tilde{C}'(X,Z,U,V)\\
&+ B_2(Z)\tilde{C}'(Y,X,U,V)+B_4(U)\tilde{C}'(Y,Z,X,V)\\
&+ B_4(V)\tilde{C}'(Y,Z,U,X). \qquad (2.17)
\end{aligned}
$$

Substituting $Y=V=\zeta$ in equation (2.17) and taking summation over $i, 1\leq i\leq n$, we get

$$
\begin{aligned}
(\tilde{\nabla}_X\tilde{S})(Z,U)-\frac{d\tilde{r}(X)}{n}g(Z,U) &= B_1(X)\big[\tilde{S}(Z,U)-\frac{\tilde{r}}{n}g(Z,U)\big]\\
&+ B_2(Z)\big[\tilde{S}(X,U)-\frac{\tilde{r}}{n}g(X,U)\big]\\
&+ B_4(U)\big[\tilde{S}(Z,X)-\frac{\tilde{r}}{n}g(Z,X)\big]\\
&+ B_2(\tilde{R}(X,Z)U)+B_4(\tilde{R}(X,U)Z)\\
&- \frac{\tilde{r}}{n(n-1)}\Bigg[g(Z,U)\{B_2(X)+B_4(X)\}\\
&- B_2(Z)g(X,U)-B_4(U)g(X,Z)\Bigg]. \qquad (2.18)
\end{aligned}
$$

Taking $X=Z=U=\zeta$, equation (2.18) reduces to

$$B_1(\zeta)+B_2(\zeta)+B_4(\zeta)=\frac{d\tilde{r}(\zeta)}{\tilde{r}+2n(n-1)}, \qquad (2.19)$$

provided $\tilde{r}+2n(n-1)\neq 0$.

Substituting $X, Z$ by $\zeta$ in equation (2.18) and using equation (2.19), we have

$$B_4(U)=B_4(\zeta)\eta(U). \qquad (2.20)$$

Similarly, on substitution of $X, U$ by $\zeta$ in equation (2.18) and using equation (2.19), we obtain

$$B_2(Z)=B_2(\zeta)\eta(Z). \qquad (2.21)$$

Taking $Z=U=\zeta$ in equation (2.18) and using equations (2.11), (2.19), (2.20) and (2.21) we get

$$B_1(X) = \frac{d\tilde{r}(X)}{\tilde{r}+2n(n-1)} - \big[B_2(\zeta)+B_4(\zeta)\big]\eta(X). \tag{2.22}$$

This leads to the theorem:

**Theorem 2.1** *In a weakly concircular symmetric Kenmotsu manifold admitting a semi-symmetric metric connection $\tilde{\nabla}$, the relation between the associated 1-forms $B_1, B_2, B_4$ is given by equation* (2.22)*.*

## 2.4 On weakly concircular Ricci symmetric Kenmotsu manifolds admitting a semi-symmetric metric connection

Consider a weakly concircular Ricci symmetric Kenmotsu manifold admitting a semi-symmetric metric connection $\tilde{\nabla}$. We have,

$$(\tilde{\nabla}_X\tilde{P})(Y,Z) = A_1(X)\tilde{P}(Y,Z)+A_2(Y)\tilde{P}(X,Z)+A_3(Z)\tilde{P}(Y,X), \tag{2.23}$$

where

$$\begin{aligned}\tilde{P}(X,Y) &= \sum_{i=1}^{n}\tilde{C'}(e_i,X,Y,e_i)\\ &= \tilde{S}(X,Y)-\frac{\tilde{r}}{n}g(X,Y)\end{aligned} \tag{2.24}$$

is not identically zero for all $X,Y,Z\in\chi(M^n)$.

Suppose equation (2.23) holds. Then, we have

$$\begin{aligned}(\tilde{\nabla}_X\tilde{S})(Y,Z)-\frac{d\tilde{r}(X)}{n}g(Y,Z) &= A_1(X)\big[\tilde{S}(Y,Z)-\frac{\tilde{r}}{n}g(Y,Z)\big]\\ &+ A_2(Y)\big[\tilde{S}(X,Z)-\frac{\tilde{r}}{n}g(X,Z)\big]\\ &+ A_3(Z)\big[\tilde{S}(Y,X)-\frac{\tilde{r}}{n}g(Y,X)\big].\end{aligned} \tag{2.25}$$

Replacing $X, Y, Z$ by $\zeta$ in equation (2.25), we obtain

$$A_1(\zeta) + A_2(\zeta) + A_3(\zeta) = \frac{d\tilde{r}(\zeta)}{\tilde{r} + 2n(n-1)}, \tag{2.26}$$

provided $\tilde{r} + 2n(n-1) \neq 0$.

Taking $X = Y = \zeta$ in equation (2.25) and using (2.26), we get

$$A_3(Z) = A_3(\zeta)\eta(Z). \tag{2.27}$$

Similarly, replacing $X, Z$ by $\zeta$ and using equation (2.26), (2.25) becomes

$$A_2(Y) = A_2(\zeta)\eta(Y), \tag{2.28}$$

and replacing $Y, Z$ by $\zeta$ in equation (2.25) and using (2.26), we get

$$A_1(X) = \frac{d\tilde{r}(X)}{\tilde{r} + 2n(n-1)} + \big[A_1(\zeta) - \frac{d\tilde{r}(\zeta)}{\tilde{r} + 2n(n-1)}\big]\eta(X), \tag{2.29}$$

provided $\tilde{r} + 2n(n-1) \neq 0$.

Adding equations (2.27), (2.28) and (2.29), we have

$$A_1(X) + A_2(X) + A_3(X) = \frac{d\tilde{r}(X)}{\tilde{r} + 2n(n-1)}, \tag{2.30}$$

provided $\tilde{r} + 2n(n-1) \neq 0$.

**Theorem 2.2** *The sum of the associated 1-forms $A_1, A_2, A_3$ in a weakly concircular Ricci symmetric Kenmotsu manifold which admits a semi-symmetric metric connection $\tilde{\nabla}$ is given by equation* (2.30).

## 2.5 Weakly $m$-projectively symmetric Kenmotsu manifolds admitting a semi-symmetric metric connection

Suppose a Kenmotsu manifold is weakly $m$-projectively symmetric with respect to $\tilde{\nabla}$. Then, we have

$$\begin{aligned}(\tilde{\nabla}_X \tilde{W}^*)(Y,Z,U,V) &= A_1(X)\tilde{W}^*(Y,Z,U,V) + A_2(Y)\tilde{W}^*(X,Z,U,V) \\ &+ A_3(Z)\tilde{W}^*(Y,X,U,V) + A_4(U)\tilde{W}^*(Y,Z,X,V) \\ &+ A_5(V)\tilde{W}^*(Y,Z,U,X), \qquad (2.31)\end{aligned}$$

where

$$\begin{aligned}\tilde{W}^{*'}(X,Y,Z,U) &= \tilde{R}'(X,Y,Z,U) - \frac{1}{2(n-1)}\big[g(X,U)\tilde{S}(Y,Z) - g(Y,U)\tilde{S}(X,Z) \\ &+ \tilde{S}(X,U)g(Y,Z) - \tilde{S}(Y,U)g(X,Z)\big], \qquad (2.32)\end{aligned}$$

for all vector fields $X,Y,Z,U,V \in \chi(M^n)$ and $A_1, A_2, A_3, A_4, A_5$ are 1-forms, not simultaneously zero.

In a weakly $m$-projectively symmetric Kenmotsu manifold admitting a semi-symmetric metric connection, $A_2 = A_3, A_4 = A_5$. So equation (2.31) can be written as

$$\begin{aligned}(\tilde{\nabla}_X \tilde{W}^*)(Y,Z,U,V) &= A_1(X)\tilde{W}^*(Y,Z,U,V) + A_2(Y)\tilde{W}^*(X,Z,U,V) \\ &+ A_2(Z)\tilde{W}^*(Y,X,U,V) + A_4(U)\tilde{W}^*(Y,Z,X,V) \\ &+ A_4(V)\tilde{W}^*(Y,Z,U,X). \qquad (2.33)\end{aligned}$$

From equation (2.32), we can obtain

$$\sum_{i=1}^{n} \tilde{W}^*(e_i,Y,Z,e_i) = \frac{n}{2(n-1)}\big[\tilde{S}(Y,Z) - \frac{\tilde{r}}{n}g(Y,Z)\big]. \qquad (2.34)$$

Also,

$$\sum_{i=1}^{n} \tilde{W}^*(X, Y, e_i, e_i) = 0, \tag{2.35}$$

$$\sum_{i=1}^{n} \tilde{W}^*(e_i, e_i, e_i, e_i) = 0. \tag{2.36}$$

The $m$-projective curvature tensor $\tilde{W}^*$ with respect to $\tilde{\nabla}$ satisfies

$$\tilde{W}^*(X, Y, Z, U) + \tilde{W}^*(Y, Z, X, U) + \tilde{W}^*(Z, X, Y, U) = 0, \tag{2.37}$$

$$\tilde{W}^*(X, Y, U, Z) + \tilde{W}^*(Y, Z, U, X) + \tilde{W}^*(Z, X, U, Y) = 0. \tag{2.38}$$

**Nature of the scalar curvature with respect to semi-symmetric metric connection.**

Let $\tilde{Q}$ be the Ricci operator with respect to $\tilde{\nabla}$ defined by

$$g(\tilde{Q}X, Y) = \tilde{S}(X, Y).$$

On covariant differentiation of equation (2.32) along $X$ and using Bianchi identity, we obtain

$$\begin{aligned}
(\tilde{\nabla}_X \tilde{W}^*)(Y, Z, U, V) &+ (\tilde{\nabla}_Y \tilde{W}^*)(Z, X, U, V) + (\tilde{\nabla}_Z \tilde{W}^*)(X, Y, U, V) \\
&= -\frac{1}{2(n-1)}\Big[\{(\tilde{\nabla}_X \tilde{S})(Z, U) - (\tilde{\nabla}_Z \tilde{S})(X, U)\}g(Y, V) \\
&+ \{(\tilde{\nabla}_Y \tilde{S})(X, U) - (\tilde{\nabla}_X \tilde{S})(Y, U)\}g(Z, V) \\
&+ \{(\tilde{\nabla}_Z \tilde{S})(Y, U) - (\tilde{\nabla}_Y \tilde{S})(Z, U)\}g(X, V) \\
&+ \{(\tilde{\nabla}_X \tilde{S})(Y, V) - (\tilde{\nabla}_Y \tilde{S})(X, V)\}g(Z, U) \\
&+ \{(\tilde{\nabla}_Y \tilde{S})(Z, V) - (\tilde{\nabla}_Z \tilde{S})(Y, V)\}g(X, U)\Big].
\end{aligned} \tag{2.39}$$

Suppose $\tilde{S}$ is a Codazzi tensor, then

$$(\tilde{\nabla}_X \tilde{S})(Y, Z) = (\tilde{\nabla}_Y \tilde{S})(X, Z). \tag{2.40}$$

Using (2.40) in (2.39), we get

$$(\tilde{\nabla}_X \tilde{W}^*)(Y,Z,U,V) + (\tilde{\nabla}_Y \tilde{W}^*)(Z,X,U,V) + (\tilde{\nabla}_Z \tilde{W}^*)(X,Y,U,V) = 0. \quad (2.41)$$

Suppose equation (2.41) holds. Then, clearly the Ricci tensor $\tilde{S}$ is of Codazzi type. Thus, we can state the theorem:

**Theorem 2.3** *The necessary and sufficient condition for the Ricci tensor $\tilde{S}$ in a weakly m-projectively symmetric Kenmotsu manifold with respect to the semi-symmetric metric connection $\tilde{\nabla}$ to be of Codazzi type is that the relation* (2.41) *holds.*

Suppose $\tilde{S}$ is Codazzi. Then, equation (2.41) holds. Using (2.33) in equation (2.41), we get

$$\lambda(X)\tilde{W}^*(Y,Z,U,V) + \lambda(Y)\tilde{W}^*(Z,X,U,V) + \lambda(Z)\tilde{W}^*(X,Y,U,V) = 0, \quad (2.42)$$

where $\lambda(X) = A_1(X) - 2A_2(X)$, for all $X, Y, Z, U, V \in \chi(M^n)$.

Putting $Y = V = e_i$ and taking summation over $i, 1 \leq i \leq n$, (2.42) reduces to

$$\frac{n}{2(n-1)}\left[\lambda(X)\{\tilde{S}(Z,U) - \frac{\tilde{r}}{n}g(Z,U)\} - \lambda(Z)\{\tilde{S}(X,U) - \frac{\tilde{r}}{n}g(X,V)\}\right] = 0. \quad (2.43)$$

Again, substituting $X = U = \zeta$ in (2.43) and summing over $i, 1 \leq i \leq n$, we get $\lambda(\tilde{Q}Z) = \frac{\tilde{r}}{n}\lambda(Z)$, which implies that

$$\tilde{S}(Z,T) = \frac{\tilde{r}}{n}g(Z,T). \quad (2.44)$$

This leads to the theorem:

**Theorem 2.4** *If the Ricci tensor $\tilde{S}$ in a weakly m-projectively symmetric Kenmotsu manifold admitting a semi-symmetric metric connection $\tilde{\nabla}$ is of Codazzi type, then $\frac{\tilde{r}}{n}$ is an eigenvalue of $\tilde{S}$ corresponding to the eigenvector $T$ defined by $g(X,T) = \lambda(X)$.*

## 2.6 Semi-symmetric metric connection in a Para-Sasakian manifold

A semi-symmetric metric connection $\tilde{\nabla}$ in a Para-Sasakian manifold is given by equation (2.7). A relation between the curvature tensor $\tilde{R}$ with respect to the semi-symmetric metric connection $\tilde{\nabla}$ and the curvature tensor $R$ with respect to the Levi-Civita connection $\nabla$ in such a manifold is obtained as (Barman, 2014)

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + g(X,\phi Z)Y - \eta(X)\eta(Z)Y \\ &+ \eta(Y)\eta(Z)X - g(Y,\phi Z)X + g(X,Z)Y \\ &- g(Y,Z)X + g(X,Z)\phi Y - g(Y,Z)\phi X \\ &- g(X,Z)\eta(Y)\zeta + g(Y,Z)\eta(X)\zeta. \end{aligned} \tag{2.45}$$

By suitable contraction of (2.45), we get

$$\begin{aligned}\tilde{S}(Y,Z) &= S(Y,Z) - (n-2)g(Y,\phi Z) + (n-2)\eta(Y)\eta(Z) \\ &- (n-2+\psi)g(Y,Z), \end{aligned} \tag{2.46}$$

where $\tilde{S}$ and $S$ are the Ricci tensors of $\tilde{\nabla}$ and $\nabla$ respectively and $\psi = trace\ of\ \phi = \sum_{i=1}^{n} g(e_i, \phi e_i)$. Also, by contraction of (2.46) we obtain,

$$\tilde{r} = r - (2n-1)\psi - (n-1)(n-2), \tag{2.47}$$

where $\tilde{r}$ is the scalar curvature of the manifold with respect to $\tilde{\nabla}$.

From equation (2.46) we can show that $\tilde{S}$ is symmetric. By doing suitable calculations and using the relations (1.46) - (1.48), (1.57) - (1.61), (2.45) and (2.46), it follows that

$$\tilde{S}(Y,\zeta) = -(n-1+\psi)\eta(Y), \tag{2.48}$$

$$\tilde{R}(X,Y)\zeta = \eta(X)(Y+\phi Y) - \eta(Y)(X+\phi X), \tag{2.49}$$

$$\tilde{R}(X,\zeta)Y = g(X, Y + \phi Y)\zeta - \eta(Y)(X + \phi X). \tag{2.50}$$

## 2.7 Weakly symmetric Para-Sasakian manifolds with respect to semi-symmetric metric connection

Let us consider a weakly symmetric Para-Sasakian manifold admitting a semi-symmetric metric connection $\tilde{\nabla}$. Then

$$\begin{aligned}(\tilde{\nabla}_X \tilde{R})(Y,Z)V &= A(X)\tilde{R}(Y,Z)V + B(Y)\tilde{R}(X,Z)V + C(Z)\tilde{R}(Y,X)V \\ &+ D(V)\tilde{R}(Y,Z)X + g(\tilde{R}(Y,Z)V,X)P \end{aligned} \tag{2.51}$$

$\forall X, Y, Z, V \in \chi(M^n)$.

By suitable contraction of (2.51), we obtain

$$\begin{aligned}(\tilde{\nabla}_X \tilde{S})(Z,V) &= A(X)\tilde{S}(Z,V) + B(\tilde{R}(X,Z)V) + C(Z)\tilde{S}(X,V) \\ &+ D(V)\tilde{S}(X,Z) + E(\tilde{R}(X,V)Z), \end{aligned} \tag{2.52}$$

where $E(X) = g(X,P)$. Putting $V = \zeta$ in (2.52) and using (2.47), (2.49), (2.50) and (2.51) we get

$$\begin{aligned}(\tilde{\nabla}_X \tilde{S})(Z,\zeta) &= -(n-1+\psi)[A(X)\eta(Z) + C(Z)\eta(X)] \\ &+ \eta(X)[B(Z) + B(\phi Z)] - \eta(Z)[B(X) + B(\phi X)] \\ &+ D(\zeta)\tilde{S}(X,Z) + E(\zeta)g(X, Z + \phi Z) \\ &- \eta(Z)[E(X) + E(\phi X)]. \end{aligned} \tag{2.53}$$

We know that

$$(\tilde{\nabla}_X \tilde{S})(Z,\zeta) = \tilde{\nabla}_X \tilde{S}(Z,\zeta) - \tilde{S}(\tilde{\nabla}_X Z,\zeta) - \tilde{S}(Z,\tilde{\nabla}_X \zeta). \tag{2.54}$$

Using equations (1.46) - (1.49), (1.52), (1.54), (2.45) and (2.49) in (2.54), we obtain

$$(\tilde{\nabla}_X \tilde{S})(Z, \zeta) = -\tilde{S}(Z, X + \phi X) - (n - 1 + \psi) g(Z, X + \phi X). \tag{2.55}$$

Comparing (2.53) and (2.55), we have

$$\begin{aligned}
&-\tilde{S}(Z, X + \phi X) - (n - 1 + \psi) g(Z, X + \phi X) = -(n - 1 + \psi) \\
&[A(X)\eta(Z) + C(Z)\eta(X)] + \eta(X)[B(Z) + B(\phi Z)] \\
&-\eta(Z)[B(X) + B(\phi X)] + D(\zeta)\tilde{S}(X, Z) + E(\zeta) g(X, Z + \phi Z) \\
&-\eta(Z)[E(X) + E(\phi X)].
\end{aligned} \tag{2.56}$$

Putting $X = Z = \zeta$ in (2.56) and using (1.46) - (1.49) and (2.49), we obtain

$$-(n - 1 + \psi)[A(\zeta) + C(\zeta) + D(\zeta)] = 0. \tag{2.57}$$

Since $n > 3$, this implies

$$A(\zeta) + C(\zeta) + D(\zeta) = 0. \tag{2.58}$$

Also by replacing $Z$ with $\zeta$ in (2.52) and doing suitable calculations, we obtain

$$\begin{aligned}
&-\tilde{S}(V, X + \phi X) - (n - 1 + \psi) g(V, X + \phi X) = -(n - 1 + \psi) \\
&[A(X)\eta(V) + D(V)\eta(X)] + C(\zeta)\tilde{S}(X, V) + B(\zeta) g(X, V + \phi V) \\
&-[B(X) + B(\phi X)]\eta(V) + [E(V) + E(\phi V)]\eta(X) \\
&-[E(X) + E(\phi X)]\eta(V).
\end{aligned} \tag{2.59}$$

Substituting $V = \zeta$ in (2.59) and using (1.46) - (1.49) and (2.49), we get

$$\begin{aligned}
&-(n - 1 + \psi)[A(X)\eta(V) + C(\zeta)\eta(X) + D(\zeta)\eta(X)] + \eta(X)[B(\zeta) \\
&+E(\zeta)] - [B(X) + B(\phi X)] - [E(X) + E(\phi X)] = 0.
\end{aligned} \tag{2.60}$$

Similarly putting $X = \zeta$ in (2.60), we have

$$-(n-1+\psi)[A(\zeta)\eta(V) + C(\zeta)\eta(V) + D(V)]$$
$$+[E(V) + E(\phi V)] - E(\zeta)\eta(V) = 0. \tag{2.61}$$

Substituting $V = X$ in (2.61), we obtain

$$-(n-1+\psi)[A(\zeta)\eta(X) + C(\zeta)\eta(X) + D(X)]$$
$$+[E(X) + E(\phi X)] - E(\zeta)\eta(X) = 0. \tag{2.62}$$

Adding (2.61) and (2.62) and using (2.59), we get

$$-(n-1+\psi)[A(X) + D(X)] - (n-1+\psi)C(\zeta)\eta(X)$$
$$+B(\zeta)\eta(X) - [B(X) + B(\phi X)] = 0. \tag{2.63}$$

Substituting $X = \zeta$ in (2.57) and then using (1.46) - (1.49) and (2.49), we get

$$-(n-1+\psi)[A(\zeta)\eta(Z) + D(\zeta)]\eta(Z) - (n-1+\psi)C(Z)$$
$$+[B(Z) + B(\phi Z)] - \eta(Z)B(\zeta) = 0. \tag{2.64}$$

Substituting $Z$ by $X$ in (2.64), we have

$$-(n-1+\psi)[A(\zeta)\eta(X) + D(\zeta)\eta(X) + C(X)]$$
$$+[B(X) + B(\phi X)] - \eta(X)B(\zeta) = 0. \tag{2.65}$$

Taking the sum of (2.63) and (2.65) and using (2.58), we get

$$-(n-1+\psi)[A(X) + C(X) + D(X)] = 0. \tag{2.66}$$

Since $n > 3$, this implies

$$A(X) + C(X) + D(X) = 0,$$

for any $X$ in $M^n$. This leads to the following theorem:

**Theorem 2.5** *In a weakly symmetric Para-Sasakian manifold $M^n(n > 3)$ admitting* *a semi-symmetric metric connection $\tilde{\nabla}$, the sum of the associated 1- forms $A, C$ and $D$ vanishes everywhere.*

## 2.8 Weakly Ricci symmetric Para-Sasakian manifolds with respect to semi-symmetric metric connection

Let $M^n$ be a weakly Ricci symmetric Para-Sasakian manifold with respect to $\tilde{\nabla}$. Then,

$$(\tilde{\nabla}_X \tilde{S})(Y, Z) = \alpha(X)\tilde{S}(Y, Z) + \beta(Y)\tilde{S}(X, Z) + \gamma(Z)\tilde{S}(X, Y), \tag{2.67}$$

$\forall X, Y, Z \in \chi(M^n)$.

Putting $Z = \zeta$ in (2.67), we get

$$(\tilde{\nabla}_X \tilde{S})(Y, \zeta) = \alpha(X)\tilde{S}(Y, \zeta) + \beta(Y)\tilde{S}(X, \zeta) + \gamma(\zeta)\tilde{S}(X, Y). \tag{2.68}$$

By (2.55) and (2.68), we have

$$\begin{aligned} -\tilde{S}(Y, X + \phi X) \quad &- \quad (n - 1 + \psi)g(X + \phi X, Y) = \alpha(X)\tilde{S}(Y, \zeta) \\ &+ \quad \beta(Y)\tilde{S}(X, \zeta) + \gamma(\zeta)\tilde{S}(X, Y). \end{aligned} \tag{2.69}$$

Putting $X = Y = \zeta$ in (2.69) and by using (1.46) - (1.49) and (2.48), we get

$$-(n - 1 + \psi)[\alpha(\zeta) + \beta(\zeta) + \gamma(\zeta)] = 0.$$

Since $n > 3$, we have

$$\alpha(\zeta) + \beta(\zeta) + \gamma(\zeta) = 0. \tag{2.70}$$

Putting $Y = \zeta$ in (2.69) and using (1.46) - (1.49) and (2.49), we get

$$\alpha(X) = \alpha(\zeta)\eta(X). \tag{2.71}$$

Similarly,

$$\beta(X) = \beta(\zeta)\eta(X), \tag{2.72}$$

$$\gamma(X) = \gamma(\zeta)\eta(X). \tag{2.73}$$

Adding equations (2.71), (2.72) and (2.73), we get

$$\alpha(X) + \beta(X) + \gamma(X) = 0.$$

for all vector fields $X \in M^n$. This gives the following theorem:

**Theorem 2.6** *There does not exist a weakly Ricci symmetric Para-Sasakian manifold $M^n(n > 3)$ admitting a semi-symmetric metric connection unless the sum of the associated 1-forms $\alpha, \beta$ and $\gamma$ is zero everywhere.*

Suppose a weakly Ricci-symmetric Para-Sasakian manifold admitting a semi-symmetric metric connection $\tilde{\nabla}$ is Ricci-recurrent. This implies

$$(\tilde{\nabla}_X \tilde{S})(Y, Z) = \alpha(X)\tilde{S}(Y, Z). \tag{2.74}$$

From (2.74), we have

$$\beta(Y)\tilde{S}(X, Z) + \gamma(Z)\tilde{S}(X, Y) = 0. \tag{2.75}$$

Putting $X = Y = Z = \zeta$ in (2.75), we get

$$\beta(\zeta) + \gamma(\zeta) = 0, \quad (\text{since } n > 3). \tag{2.76}$$

Putting $X = Y = \zeta$ in (2.75), we have

$$\gamma(Z) = -\eta(Z)\gamma(\zeta). \tag{2.77}$$

Similarly,

$$\beta(Z) = -\eta(Z)\beta(\zeta). \tag{2.78}$$

Therefore, by adding (2.77) and (2.78), we get

$$\gamma(Z) + \beta(Z) = 0, \tag{2.79}$$

for any vector field $Z \in M^n$. This yields the following theorem:

**Theorem 2.7** *In a weakly Ricci symmetric Para-Sasakian manifold $M^n (n > 3)$ admitting a semi-symmetric metric connection $\tilde{\nabla}$ where the connection $\tilde{\nabla}$ is Ricci-recurrent, the 1-forms $\beta$ and $\gamma$ are in the opposite direction.*

Consider a weakly concircular Ricci symmetric Para-Sasakian manifold admitting a semi-symmetric metric connection $\tilde{\nabla}$. We have,

$$(\tilde{\nabla}_X \tilde{P})(Y, Z) = \alpha(X)\tilde{P}(Y, Z) + \beta(Y)\tilde{P}(X, Z) + \gamma(Z)\tilde{P}(Y, X). \tag{2.80}$$

Then by the definition and equation (2.80), we have

$$\begin{aligned}(\tilde{\nabla}_X \tilde{S})(Y, Z) \quad &- \quad \frac{d\tilde{r}}{n} g(Y, Z) = \alpha(X)\big[\tilde{S}(Y, Z) - \frac{\tilde{r}}{n} g(Y, Z)\big] \\ &+ \quad \beta(Y)\big[\tilde{S}(X, Z) - \frac{\tilde{r}}{n} g(X, Z)\big] \\ &+ \quad \gamma(Z)\big[\tilde{S}(Y, X) - \frac{\tilde{r}}{n} g(Y, X)\big].\end{aligned} \tag{2.81}$$

Setting $X = Y = Z = \zeta$ in (2.81), we have

$$\alpha(\zeta) + \beta(\zeta) + \gamma(\zeta) = \frac{d\tilde{r}}{\tilde{r} + n(n - 1 + \psi)}. \tag{2.82}$$

Substituting $X = Y = \zeta$ in (2.81), we have

$$\gamma(Z) = \gamma(\zeta)\eta(Z), \tag{2.83}$$

provided $\tilde{r} + n(n - 1 + \psi) \neq 0$.

Similarly,

$$\beta(Z) = \beta(\zeta)\eta(Z), \tag{2.84}$$

provided $\tilde{r} + n(n - 1 + \psi) \neq 0$.

Put $Y = Z = \zeta$ in (2.81), we have

$$\alpha(X) = \frac{d\tilde{r}(X)}{\tilde{r} + n(n-1+\psi)} + \left[\alpha(\zeta) - \frac{d\tilde{r}(\zeta)}{\tilde{r} + n(n-1+\psi)}\right]\eta(X). \tag{2.85}$$

By adding (2.83), (2.84) and (2.85), we get

$$\begin{aligned} \alpha(X) + \beta(X) + \gamma(X) &= \frac{d\tilde{r}(X)}{\tilde{r} + n(n-1+\psi)} \\ &= \frac{dr(X)}{r - (n-1)(\psi + 2(n-1))}, \end{aligned} \tag{2.86}$$

for any $X \in M^n$. This leads to the following:

**Theorem 2.8** *The sum of the associated 1-forms in a weakly concircular Ricci symmetric Para-Sasakian manifold $M^n (n > 3)$ admitting a semi-symmetric metric connection $\tilde{\nabla}$ is zero if the scalar curvature is constant and $\tilde{r} + n(n-1+\psi) \neq 0$.*

## 2.9 Example of a 3-dimensional weakly symmetric and weakly Ricci symmetric Para-Sasakian manifold admitting a semi-symmetric metric connection

In this section, we construct an example of a 3-dimensional Para-Sasakian manifold admitting a semi-symmetric metric connection which supports Theorems 2.5 and 2.6.

Consider a 3 - dimensional manifold $M = \{(x, y, z) : (x, y, z) \in \mathbb{R}^3\}$. We choose the vector fields

$$e_1 = e^x \frac{\partial}{\partial y}, \qquad e_2 = e^x (\frac{\partial}{\partial y} + \frac{\partial}{\partial z}), \qquad e_3 = -\frac{\partial}{\partial x},$$

which are linearly independent at each point of M. Let $g$ be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1,$$
$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0.$$

Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any vector field $Z$ on $M$. Define the (1, 1) tensor field $\phi$ as $\phi(e_1) = e_1, \phi(e_2) = e_2, \phi(e_3) = 0$. By linearity property of $\phi$ and $g$, we have

$$\eta(e_3) = 1, \quad \phi^2 X = X - \eta(X)e_3,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields $X, Y$ on $M$. Thus for $e_3 = \zeta, (\phi, \zeta, \eta, g)$ is an almost paracontact structure on $M$.

Let $\nabla$ be the Levi- Civita connection with respect to $g$. Then, we have

$$[e_1, e_2] = 0, [e_2, e_3] = e_2, [e_1, e_3] = e_1.$$

The Riemannian connection $\nabla$ of the metric $g$ is given by Koszul's formula,

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &\quad + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y), \end{aligned}$$

which yields

$$\nabla_{e_1} e_2 = 0, \ \nabla_{e_1} e_3 = e_1, \ \nabla_{e_1} e_1 = -e_3,$$
$$\nabla_{e_2} e_3 = e_2, \ \nabla_{e_2} e_2 = -e_3, \ \nabla_{e_2} e_1 = 0,$$
$$\nabla_{e_3} e_3 = 0, \ \nabla_{e_3} e_2 = 0, \ \nabla_{e_3} e_1 = 0.$$

From the above, it can be easily seen that $(\phi, \zeta, \eta, g)$ is a Para-Sasakian structure on $M$. Hence, $(M, \phi, \zeta, \eta, g)$ is a 3-dimensional Para-Sasakian manifold. By using the above results we can easily obtain,

$$R(e_1,e_2)e_2 = -e_1,\ R(e_1,e_3)e_3 = -e_1,\ R(e_2,e_1)e_1 = -e_2,$$
$$R(e_2,e_3)e_3 = -e_2,\ R(e_3,e_1)e_1 = -e_3,\ R(e_3,e_2)e_2 = -e_3,$$
$$R(e_1,e_2)e_3 = 0,\ R(e_3,e_2)e_1 = 0,\ R(e_3,e_1)e_2 = 0.$$

The definition of Ricci tensor in a 3 - dimensional manifold implies that

$$S(X,Y) = \sum_{i=1}^{3} g(R(e_i,X)Y,e_i). \tag{2.87}$$

Using the components of the curvature tensor in (2.87) we get

$$S(e_1,e_1) = -2,\ S(e_2,e_2) = -2,\ S(e_3,e_3) = -2,$$
$$S(e_1,e_2) = 0,\ S(e_2,e_3) = 0,\ S(e_3,e_1) = 0.$$

The semi-symmetric metric connection $\tilde{\nabla}$ is given by (2.45) which yields,

$$\tilde{\nabla}_{e_1}e_2 = 0,\ \tilde{\nabla}_{e_1}e_3 = 2e_1,\ \tilde{\nabla}_{e_1}e_1 = -2e_3,$$
$$\tilde{\nabla}_{e_2}e_3 = 2e_2,\ \tilde{\nabla}_{e_2}e_2 = -2e_3,\ \tilde{\nabla}_{e_2}e_1 = 0,$$
$$\tilde{\nabla}_{e_3}e_3 = 0,\ \tilde{\nabla}_{e_3}e_2 = 0,\ \tilde{\nabla}_{e_3}e_3 = 0.$$

By using (2.46), we have

$$\tilde{R}(e_1,e_2)e_2 = -4e_1,\ \tilde{R}(e_1,e_3)e_3 = -2e_1,\ \tilde{R}(e_2,e_1)e_1 = -4e_2,$$
$$\tilde{R}(e_2,e_3)e_3 = -2e_2,\ \tilde{R}(e_3,e_1)e_1 = -2e_1,\ \tilde{R}(e_3,e_2)e_2 = -2e_2,$$
$$\tilde{R}(e_1,e_2)e_3 = 0,\ \tilde{R}(e_3,e_2)e_3 = 0,\ \tilde{R}(e_3,e_1)e_2 = 0.$$

Using the components of the curvature tensor, we can easily calculate the components of the Ricci tensor with respect to the Levi-Civita connection and the semi-symmetric metric connection, respectively as,

$$\tilde{S}(e_1,e_1) = -6,\ \tilde{S}(e_2,e_2) = -6,\ \tilde{S}(e_3,e_3) = -4,$$
$$\tilde{S}(e_1,e_2) = 0,\ \tilde{S}(e_2,e_3) = 0,\ \tilde{S}(e_3,e_1) = 0.$$

Using the above components of the curvature tensor with respect to the semi-symmetric metric connection and equation (2.51), we get

$$A(e_i) + C(e_i) + D(e_i) = 0, \quad \forall i = 1, 2, 3.$$

Also, using the above components of the Ricci tensor with respect to the semi-symmetric metric connection and equation (2.67), we get

$$\alpha(e_i) + \beta(e_i) + \gamma(e_i) = 0, \quad \forall i = 1, 2, 3.$$

Thus, this is an example of a 3-dimensional Para-Sasakian manifold admitting a semi-symmetric metric connection which is weakly symmetric and weakly Ricci-symmetric.

# Chapter 3

# Semi-generalized $W_3$ Recurrent Manifolds

In this chapter we considered semi-generalized $W_3$ recurrent manifolds. We obtained a necessary and sufficient condition for the scalar curvature to be constant in such a manifold. Later Ricci symmetric and decomposable semi-generalized $W_3$ recurrent manifolds are studied. Also, we obtained a sufficient condition for such a manifold to be quasi Einstein. Finally, we constructed two examples of a semi-generalized $W_3$ recurrent manifold.

## 3.1 Introduction

A Riemannian manifold $(M^n, g)$ is called a semi-generalized recurrent manifold if its curvature tensor $R$ satisfies equation (1.76) where $A, B$ are two 1-forms, $B$ is non zero, $P_1$ and $P_2$ are two vector fields defined by

$$g(X, P_1) = A(X), \quad g(X, P_2) = B(X)$$

for any vector field $X$ and $\nabla$ denote the operator of covariant differentiation with respect to $g$. We considered a non-flat Riemannian manifold $(M^n, g)(n > 2)$ whose

$W_3$ curvature tensor satisfies

$$(\nabla_X W_3)(Y, Z, U, V) = \alpha(X) W_3(Y, Z, U, V) + \beta(X) g(Z, U) g(Y, V), \tag{3.1}$$

where $\alpha, \beta$ are 1-forms, $\beta$ is non-zero and $W_3$ is the $W_3$ curvature tensor defined by (Pokhariyal, 1973)

$$\begin{aligned} W_3(Y, Z, U, V) &= R(Y, Z, U, V) + \frac{1}{(n-1)}\big[g(Z, U) S(Y, V) \\ &\quad - g(Z, V) S(Y, U)\big]. \end{aligned} \tag{3.2}$$

Such a manifold is called a semi-generalized $W_3$ recurrent manifold.

Let $Q$ denote the symmetric endomorphism of the tangent space at each point of $M^n$ corresponding to the Ricci tensor $S$ such that

$$S(X, Y) = g(QX, Y) \tag{3.3}$$

for every vector fields $X$ and $Y$.

Chaki and Maity (2000) introduced the notion of quasi Einstein manifold. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called a quasi Einstein manifold if $S$ is not identically zero and satisfies

$$S(X, Y) = a g(X, Y) + b \eta(X) \eta(Y), \tag{3.4}$$

where $a, b (b \neq 0)$ are scalars and $\eta$ is a non-zero 1-form defined by

$$\eta(X) = g(X, \rho),$$

where $\rho$ is a unit vector field.

## 3.2 Necessary and sufficient condition for the scalar curvature to be constant in a semi-generalized $W_3$ recurrent manifold

From equations (3.1) and (3.2), we have

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= \alpha(X)R(Y,Z,U,V) + \frac{1}{(n-1)}\Big[\alpha(X)\big\{g(Z,U)S(Y,V)\\ &\quad -g(Z,V)S(Y,U)\big\} - \big\{g(Z,U)(\nabla_X S)(Y,V)\\ &\quad -g(Z,V)(\nabla_X S)(Y,U)\big\}\Big] + \beta(X)g(Z,U)g(Y,V). \qquad (3.5)\end{aligned}$$

Using Bianchi's second identity and (3.5), we have

$$\begin{aligned}&\alpha(X)R(Y,Z,U,V) + \alpha(Y)R(Z,X,U,V) + \alpha(Z)R(X,Y,U,V)\\ &+\beta(X)g(Z,U)g(Y,V) + \beta(Y)g(X,U)g(Z,V) + \beta(Z)g(Y,U)g(X,V)\\ &+\frac{1}{(n-1)}\Big[\alpha(X)\big\{g(Z,U)S(Y,V) - g(Z,V)S(Y,U)\big\}\\ &+\alpha(Y)\big\{g(X,U)S(Z,V) - g(X,V)S(Z,U)\big\}\\ &+\alpha(Z)\big\{g(Y,U)S(X,V) - g(Y,V)S(X,U)\big\}\\ &-\big\{g(Z,U)(\nabla_X S)(Y,V) - g(Z,V)(\nabla_X S)(Y,U)\big\}\\ &-\big\{g(X,U)(\nabla_Y S)(Z,V) - g(X,V)(\nabla_Y S)(Z,U)\big\}\\ &-\big\{g(Y,U)(\nabla_Z S)(X,V) - g(Y,V)(\nabla_Z S)(X,U)\big\}\Big] = 0. \qquad (3.6)\end{aligned}$$

Putting $Y = V = e_i$ in (3.6) and summing over $i, 1 \leq i \leq n$, we get

$$\begin{aligned}&\alpha(X)S(Z,U) + \alpha(R(Z,X)U) - \alpha(Z)S(X,U)\\ &+n\beta(X)g(Z,U) + 2\beta(Z)g(X,U)\\ &+\frac{1}{(n-1)}\Big[\alpha(X)\{rg(Z,U) - 2S(Z,U)\} + \alpha(QZ)g(X,U)\\ &-(n-1)\alpha(Z)S(X,U) + 2(\nabla_Z S)(X,U)\\ &-\big\{g(Z,U)dr(X) + g(X,U)\frac{dr(Z)}{2}\}\Big] = 0. \qquad (3.7)\end{aligned}$$

Contracting (3.7) with $Z$ and $U$, we obtain

$$\left(\frac{2n-3}{n-1}\right)r\alpha(X)-\left(\frac{3n-4}{n-1}\right)\alpha(QX)+(n^2+2)\beta(X)-\frac{1}{(n-1)}dr(X)=0, \quad (3.8)$$

which can be written as

$$r\alpha(X)=\left(\frac{3n-4}{2n-3}\right)\alpha(QX)-\frac{(n^2+2)(n-1)}{(2n-3)}\beta(X)+\frac{1}{(2n-3)}dr(X). \quad (3.9)$$

Thus, we can state the following:

**Theorem 3.1** *In a semi-generalized $W_3$ recurrent manifold, the scalar curvature $r$ is constant if and only if*

$$r\alpha(X)=\left(\frac{3n-4}{2n-3}\right)\alpha(QX)-\frac{(n^2+2)(n-1)}{(2n-3)}\beta(X),$$

*for all vector fields $X$.*

Suppose $r$ is constant in a semi-generalized $W_3$ recurrent manifold, i. e., $dr=0$. Then, equation (3.9) becomes

$$r\alpha(X)=\left(\frac{3n-4}{2n-3}\right)\alpha(QX)-\frac{(n^2+2)(n-1)}{(2n-3)}\beta(X). \quad (3.10)$$

Contraction of equation (3.5) yields

$$\begin{aligned}(\nabla_X S)(Z,U) &= \alpha(X)S(Z,U)+n\beta(X)g(Z,U)+\frac{1}{(n-1)}\Big[\alpha(X)\{rg(Z,U)\\ &\quad -S(Z,U)\}-\{dr(X)g(Z,U)-(\nabla_X S)(Z,U)\}\Big]. \quad (3.11)\end{aligned}$$

Making use of (3.10) and $dr=0$ in (3.11), we obtain

$$\begin{aligned}(\nabla_X S)(Z,U) &= \alpha(X)S(Z,U)+\frac{(n-1)}{(n-2)(2n-3)}\Big[(n^2-3n-2)\beta(X)\\ &\quad +(3n-4)\alpha(QX)\Big]g(Z,U)\end{aligned}$$

which can be written as

$$(\nabla_X S)(Z,U)=\alpha(X)S(Z,U)+n\gamma(X)g(Z,U),$$

where

$$\gamma(X) = \frac{(n-1)}{n(n-2)(2n-3)}\Big[(n^2-3n-2)\beta(X) + (3n-4)\alpha(QX)\Big].$$

This leads to the theorem:

**Theorem 3.2** *A semi-generalized $W_3$ recurrent manifold with constant scalar curvature is semi-generalized Ricci recurrent.*

## 3.3 Ricci symmetric semi-generalized $W_3$ recurrent manifold

Assume that the semi-generalized $W_3$ recurrent manifold is Ricci symmetric. Then, $\nabla S = 0$, i. e., $\nabla Q = 0$. This implies that $r$ is constant and $dr = 0$. Then, from equation (3.11), we have

$$\Big(\frac{n-2}{n-1}\Big)\alpha(X)S(Z,U) + \Big[\frac{r}{(n-1)}\alpha(X) + n\beta(X)\Big]g(Z,U) = 0. \tag{3.12}$$

Since $r$ is constant, equation (3.10) holds. Substituting the value of $\beta(X)$ from equation (3.10) in (3.12), we have

$$S(Z,U) = \frac{n}{(n-2)(n^2+2)}\Big[r(n^2-3n-2) - (3n-4)\frac{\alpha(QX)}{\alpha(X)}\Big]g(Z,U),$$

which can be written as

$$S(Z,U) = \lambda g(Z,U),$$

where $\lambda = \frac{n}{(n-2)(n^2+2)}\Big[r(n^2-3n-2) - (3n-4)\frac{\alpha(QX)}{\alpha(X)}\Big]$. Thus, we have:

**Theorem 3.3** *A Ricci symmetric semi-generalized $W_3$ recurrent manifold is an Einstein manifold.*

## 3.4 Sufficient condition for a semi-generalized $W_3$ recurrent manifold to be a quasi Einstein manifold

Equation (3.11) yields

$$\begin{aligned}(\nabla_X S)(Y,Z) &= \alpha(X)S(Y,Z) + \frac{n}{(n-1)}\Big[(n-1)\beta(X) \\ &+ \frac{r}{n}\alpha(X) - \frac{dr(X)}{n}\Big]g(Y,Z). \end{aligned} \tag{3.13}$$

A vector field $P$ defined by $g(X,P) = \alpha(X)$ is said to be a concircular vector field if

$$(\nabla_X \alpha)(Y) = \nu g(X,Y) + \omega(X)\alpha(Y), \tag{3.14}$$

where $\nu$ is a non-zero scalar and $\omega$ is a closed 1-form. If $P$ is unit, then (3.14) can be written as

$$(\nabla_X \alpha)(Y) = \nu[g(X,Y) - \alpha(X)\alpha(Y)]. \tag{3.15}$$

Suppose a semi-generalized $W_3$ recurrent manifold admits a unit concircular vector field $P$. Using Ricci identity in (3.15), we have

$$\alpha(R(X,Y)Z) = -\nu^2[g(X,Z)\alpha(Y) - g(Y,Z)\alpha(X)]. \tag{3.16}$$

Contraction of equation (3.16) with respect to $Y$ and $Z$ gives

$$\alpha(QX) = (n-1)\nu^2\alpha(X), \tag{3.17}$$

where $Q$ is the Ricci operator given by (3.3).

This implies

$$S(X,P) = (n-1)\nu^2\alpha(X). \tag{3.18}$$

We know that

$$(\nabla_X S)(Y,P) = \nabla_X S(Y,P) - S(\nabla_X Y,P) - S(Y,\nabla_X P). \tag{3.19}$$

Using equation (3.18) in (3.19), we have

$$(\nabla_X S)(Y,P) = (n-1)\nu^2\nabla_X \alpha(Y) - (n-1)\nu^2\alpha(\nabla_X Y) - S(Y,\nabla_X P),$$

or

$$(\nabla_X S)(Y,P) = (n-1)\nu^2(\nabla_X \alpha)(Y) - S(Y,\nabla_X P).$$

Applying (3.15) in the above equation, we get

$$(\nabla_X S)(Y,P) = (n-1)\nu^3[g(X,Y) - \alpha(X)\alpha(Y)] - S(Y,\nabla_X P). \tag{3.20}$$

Now,

$$\begin{aligned}(\nabla_X \alpha)(Y) &= \nabla_X \alpha(Y) - \alpha(\nabla_X Y) = \nabla_X g(Y,P) - g(\nabla_X Y,P)\\ &= g(Y,\nabla_X P), \quad \text{since} \quad (\nabla_X g)(Y,P) = 0.\end{aligned}$$

By virtue of (3.15), this implies

$$\nu[g(X,Y) - \alpha(X)\alpha(Y)] = g(Y,\nabla_X P),$$

$$\Rightarrow g(\nu X,Y) - g(\nu\alpha(X)P,Y) = g(\nabla_X P,Y),$$

$$\text{or,}\ \ \nabla_X P = \nu[X - \alpha(X)P].$$

Therefore,

$$S(Y,\nabla_X P) = S(Y,\nu X) - S(Y,\nu\alpha(X)P),$$

$$\text{which implies}\ \ S(Y,\nabla_X P) = \nu[S(X,Y) - \alpha(X)S(Y,P)]. \tag{3.21}$$

Making use of equation (3.21) in (3.20), we have

$$\begin{aligned}(\nabla_X S)(Y,P) &= (n-1)\nu^3[g(X,Y) - \alpha(X)\alpha(Y)]\\ &\quad -\nu[S(X,Y) - \alpha(X)S(Y,P)].\end{aligned} \tag{3.22}$$

Applying (3.18) in (3.22), we obtain

$$(\nabla_X S)(Y,P) = (n-1)\nu^3 g(X,Y) - \nu S(X,Y). \tag{3.23}$$

From (3.23), we have

$$\begin{aligned}(\nabla_X S)(Y,P) &= \Big(\frac{n-2}{n}\Big)\alpha(X)S(Y,P) + (n-1)\Big[\beta(X) \\ &\quad -\frac{r}{n(n-2)}\alpha(X) + \frac{dr(X)}{n(n-1)}\Big]g(Y,P).\end{aligned}$$

Using equations (3.20) and (3.23), the above equation becomes

$$\begin{aligned}(n-1)\nu^3 g(X,Y) - \nu S(X,Y) &= \frac{(n-2)}{n(n-1)}\nu^2\alpha(X)\alpha(Y) \\ +(n-1)\Big[\beta(X) - \frac{r}{n(n-2)}\alpha(X) &+ \frac{dr(X)}{n(n-1)}\Big]\alpha(Y).\end{aligned} \tag{3.24}$$

If the scalar curvature is constant, then $dr = 0$. From (3.10), we have $r\alpha(X) = \left(\frac{3n-4}{2n-3}\right)\alpha(QX) - \frac{(n^2+2)(n-1)}{(2n-3)}\beta(X)$, which can be written as

$$\beta(X) = \frac{1}{(n^2+2)}\Big[(3n-4)\nu^2 - \left(\frac{2n-3}{n-1}\right)r\Big]\alpha(X). \tag{3.25}$$

Making use of equation (3.25) and $dr = 0$ in (3.24), we get

$$\begin{aligned}(n-1)\nu^3 g(X,Y) - \nu S(X,Y) &= \Big[(n-1)\nu^2 \\ +\frac{1}{n(n-2)(n^2+2)}\big\{(n-1)(3n-4)\nu^2 &- r(n^2-3n-2)\big\}\Big]\alpha(X)\alpha(Y),\end{aligned}$$

$$\begin{aligned}\text{i. e.,}\ (n-1)\nu^3 g(X,Y) - \nu S(X,Y) &= \frac{(n-1)}{(n-2)(n^2+2)}\Big[\big\{(n^3+n^2-2n-4)\nu^2 \\ &- r(n^2-3n-2)\big\}\Big]\alpha(X)\alpha(Y).\end{aligned}$$

Thus, we get

$$\begin{aligned}S(X,Y) &= (n-1)\nu^2 g(X,Y) + \frac{(n-1)}{(n-2)(n^2+2)}\Big[\Big\{\frac{r}{\nu}(n^2-3n-2) \\ &- (n^3+n^2-2n-4)\nu\Big\}\Big]\alpha(X)\alpha(Y),\end{aligned}$$

$$\text{or,} \quad S(X,Y) = ag(X,Y) + b\alpha(X)\alpha(Y),$$

where $a = (n-1)\nu^2$ and $b = \dfrac{(n-1)}{(n-2)(n^2+2)}\left[\left\{\dfrac{r}{\nu}(n^2-3n-2)-(n^3+n^2-2n-4)\nu\right\}\right]$ are two non-zero constants. Hence, the manifold is a quasi Einstein manifold. Thus, we have the theorem:

**Theorem 3.4** *A semi-generalized $W_3$ recurrent manifold which admits a unit concircular vector field and whose associated scalar is a non-zero constant is a quasi Einstein manifold.*

## 3.5 Decomposable semi-generalized $W_3$ recurrent manifold

**Definition 3.1** *A Riemannian manifold $(M^n, g)(n > 2)$ is said to be a decomposable Riemannian manifold (Schouten, 1954) if it can be expressed in the form $M^n = M_1^p \times M_2^{n-p}$ for some $p$, $2 \leq p \leq (n-2)$, i. e., in some coordinate neighbourhood of $M^n$ , the metric $g$ can be written as*

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ab}dx^a dx^b + g^*_{\alpha\beta}dx^\alpha dx^\beta, \tag{3.26}$$

*where $\bar{g}_{ab}$ are functions of $x^1, x^2, ....., x^p$ denoted by $\bar{x}$, $g^*_{\alpha\beta}$ are functions of $x^{p+1}, x^{p+2}, ....., x^n$ denoted by $x^*$, $a, b, c......$ runs from 1 to $p$ and $\alpha, \beta, \gamma, .....$ runs from $p+1$ to $n$. $M_1^p$ and $M_2^{n-p}$ are called the components of $M^n$.*

Suppose a semi-generalized $W_3$ recurrent manifold $(M^n, g)(n > 2)$ is decomposable. Then, $M^n = M_1^p \times M_2^{n-p}$ for some $p$, $2 \leq p \leq (n-2)$, Let $\bar{X}, \bar{Y}, \bar{Z}, \bar{U}, \bar{V} \in \chi(M_1^p)$, $X^*, Y^*, Z^*, U^*, V^* \in \chi(M_2^{n-p})$. Since $M^n$ is decomposable, we have

$$S(\bar{X}, \bar{Y}) = \bar{S}(\bar{X}, \bar{Y}),$$

$$S(X^*, Y^*) = S^*(X^*, Y^*),$$

$$(\nabla_{\bar{X}} S)(\bar{Y}, \bar{Z}) = (\bar{\nabla}_{\bar{X}} S)(\bar{Y}, \bar{Z}),$$

$$(\nabla_{X^*}S)(Y^*, Z^*) = (\nabla^*_{X^*}S)(Y^*, Z^*)$$

and $r = \bar{r} + r^*$.

From (3.1), we have

$$W_3(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) = \bar{W}_3(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}), \tag{3.27}$$

$$W_3(X^*, Y^*, Z^*, U^*) = W^*_3(X^*, Y^*, Z^*, U^*),$$

$$W_3(Y^*, \bar{Z}, \bar{U}, \bar{V}) = 0 = W_3(\bar{Y}, Z^*, U^*, V^*) = W_3(\bar{Y}, Z^*, \bar{U}, \bar{V}) = W_3(\bar{Y}, \bar{Z}, U^*, \bar{V}),$$

$$W_3(\bar{Y}, Z^*, U^*, \bar{V}) = \frac{1}{(n-1)} g(Z^*, U^*) S(\bar{Y}, \bar{V}), \tag{3.28}$$

$$W_3(Y^*, \bar{Z}, \bar{U}, V^*) = \frac{1}{(n-1)} g(\bar{Z}, \bar{U}) S(Y^*, V^*), \tag{3.29}$$

$$W_3(Y^*, \bar{Z}, U^*, \bar{V}) = -\frac{1}{(n-1)} g(\bar{Z}, \bar{V}) S(Y^*, U^*),$$

$$W_3(\bar{Y}, Z^*, \bar{U}, V^*) = -\frac{1}{(n-1)} g(Z^*, V^*) S(\bar{Y}, \bar{U}),$$

$$(\nabla_{X^*}W_3)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = 0 = (\nabla_{\bar{X}}W_3)(Y^*, Z^*, U^*, V^*).$$

From (3.2), we get

$$(\nabla_{\bar{X}}W_3)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = \alpha(\bar{X}) W_3(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) + \beta(\bar{X}) g(\bar{Z}, \bar{U}) g(\bar{Y}, \bar{V}),$$

$$\text{and } \alpha(X^*) W_3(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) + \beta(X^*) g(\bar{Z}, \bar{U}) g(\bar{Y}, \bar{V}) = 0. \tag{3.30}$$

Also,

$$\beta_{(\bar{p}, p^*)}(0 \oplus v) = 0,$$

where $\bar{p} \in M_1, p^* \in M_2$ and $v \in T_{p^*}(M_2)$. Also, for every $(\bar{p}, p^*) \in M^n$, we have from (3.1),

$$(\nabla_{X^*}W_3)_{(\bar{p}, p^*)}(Y^*, Z^*, U^*, V^*) = (\nabla^*_{X^*}W_3)_{p^*}(Y^*, Z^*, U^*, V^*) \tag{3.31}$$

and the R. H. S does not depend on $\bar{p} \in M_1$.

Suppose $\beta(X^*) = 0$, $\forall X^* \in \chi(M_2)$, then (3.30) yields

$$\alpha(X^*)W_3(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = 0, \tag{3.32}$$

$$\text{and} \ \ \alpha(X^*)\bar{W}_3(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = 0. \tag{3.33}$$

If $M_1$ is not $W_3$ flat, i. e., $(\bar{W}_3)_{\bar{p}_0} \neq 0$, for some $\bar{p}_0 \in M_1$, then equations (3.32) and (3.33) gives

$$\alpha_{(\bar{p},p^*)}(0 \oplus v) = 0. \tag{3.34}$$

Then, (3.2) yields

$$(\nabla_{X^*} W_3)_{(\bar{p},p^*)}(Y^*, Z^*, U^*, V^*) = 0,$$

for every $\bar{p} \in M_1, p^* \in M_2$ and $v \in T_{p^*}(M_2)$. It follows that if $M_1$ is not $W_3$ flat, then

$$\alpha_{(\bar{p},p^*)}(W_3^*)_{p^*}(Y^*, Z^*, U^*, V^*) = 0. \tag{3.35}$$

for all $\bar{p} \in M_1, p^* \in M_2$.

Assume that

$$(\nabla_X W_3)(Y, Z, U, V) = \bar{\alpha}(X)W_3(Y, Z, U, V) + \bar{\beta}(X)g(Z, U)g(Y, V), \tag{3.36}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are 1-forms.

Using (3.36) in (3.2), we obtain

$$[\alpha(X) - \bar{\alpha}(X)]W_3(Y, Z, U, V) + [\beta(X) - \bar{\beta}(X)]g(Z, U)g(Y, V) = 0. \tag{3.37}$$

Contraction of (3.37) over $Y$ and $V$ gives

$$\begin{aligned}[]
[\alpha(X) - \bar{\alpha}(X)]\Big[S(Z, U) \ &- \ \frac{1}{(n-1)}\{rg(Z, U) - S(Z, U)\}\Big] \\
&+ \ [\beta(X) - \bar{\beta}(X)]g(Z, U) = 0.
\end{aligned} \tag{3.38}$$

Again contracting (3.38) over $Z$ and $U$, we have

$$\beta(X) = \bar{\beta}(X),$$

which implies, from (3.37)

$$\alpha(X) = \bar{\alpha}(X),$$

for all $X \in M^n$ provided $W_3 \neq 0$, i. e., the manifold is not $W_3$ flat. Thus, the 1-forms $\alpha$ and $\beta$ are uniquely determined provided that the manifold is not $W_3$ flat. So, from equation (3.34), we have

$$\alpha_{(\bar{p},p^*)}(X^*) = 0, \tag{3.39}$$

for all $\bar{p} \in M_1, p^* \in M_2$.

Hence, from equation (3.32), we can conclude that either

(1) $\alpha(X^*) = 0$, or

(2) $M_1$ is $W_3$ flat.

Also, from (3.2), we get

$$\begin{aligned}(\nabla_{X^*}W_3)(Y^*, \bar{Z}, \bar{U}, V^*) &= \alpha(X^*)W_3(Y^*, \bar{Z}, \bar{U}, V^*) \\ &\quad +\beta(X^*)g(\bar{Z}, \bar{U})g(Y^*, V^*). \end{aligned} \tag{3.40}$$

Consider case (1). From (3.40), we have

$$(\nabla_{X^*}W_3)(Y^*, \bar{Z}, \bar{U}, V^*) = 0,$$

which by virtue of (3.28) gives

$$(\nabla_{X^*}S)(Y^*, V^*) = 0, \tag{3.41}$$

i. e., the component $M_2$ is Ricci symmetric. Using equations (3.32), (3.34), (3.37), (3.38) and (3.39), and $\alpha(X^*) = 0$, $\beta(X^*) = 0$, for all $X^* \in M_2$, we have from (3.2),

$$(\nabla_{X^*}W_3)(Y^*, Z^*, U^*, V^*) = 0,$$

and hence

$$\begin{aligned}(\nabla_{X^*}R)(Y^*,Z^*,U^*,V^*) &+ \frac{1}{(n-1)}\big[g(Z^*,U^*)(\nabla_{X^*}S)(Y^*,V^*) \\ &- g(Z^*,V^*)(\nabla_{X^*}S)(Y^*,U^*)\big] = 0,\end{aligned}$$

which by virtue of (3.41) yields

$$(\nabla_{X^*}R)(Y^*,Z^*,U^*,V^*) = 0.$$

Hence, $M_2$ is locally symmetric. Similarly, we can prove for $M_1$. Thus, we can state the theorem:

**Theorem 3.5** *Let $M^n$ be a decomposable semi-generalized $W_3$ recurrent manifold which is not $W_3$ flat such that $M^n = M_1^p \times M_2^{n-p}$, $2 \leq p \leq (n-2)$. If $\beta(X^*) = 0$ for all $X^* \in M_2$, (respectively $\beta(\bar{X}) = 0$, for all $\bar{X} \in M_1$), then either (1) or (2) holds.*

*(1) $\alpha(X^*) = 0, \quad \forall X^* \in \chi(M_2)$,(respectively $\alpha(\bar{X}) = 0, \quad \forall \bar{X} \in \chi(M_1)$), and hence $M_2$(respectively $M_1$) is Ricci symmetric as well as locally symmetric.*

*(2) $M_2$(respectively $M_1$) is $W_3$ flat.*

Also, from (3.2), we have

$$\begin{aligned}(\nabla_{\bar{X}}W_3)(\bar{Y},Z^*,U^*,\bar{V}) &= \alpha(\bar{X})W_3(\bar{Y},Z^*,U^*,\bar{V}) \\ &\quad +\beta(\bar{X})g(Z^*,U^*)g(\bar{Y},\bar{V}).\end{aligned} \tag{3.42}$$

Using equation (3.31) in (3.42), we get

$$\begin{aligned}\frac{1}{(n-1)}g(Z^*,U^*)(\nabla_{\bar{X}}S)(\bar{Y},\bar{V}) &= \frac{\alpha(\bar{X})}{(n-1)}g(Z^*,U^*)S(\bar{Y},\bar{V}) \\ &\quad +\beta(\bar{X})g(Z^*,U^*)g(\bar{Y},\bar{V}).\end{aligned} \tag{3.43}$$

Assume $g(Z^*,U^*) \neq 0$, then (3.43) becomes

$$(\nabla_{\bar{X}}S)(\bar{Y},\bar{V}) = \alpha(\bar{X})S(\bar{Y},\bar{V}) + (n-1)\beta(\bar{X})g(\bar{Y},\bar{V}),$$

$$\Rightarrow (\nabla_{\bar{X}}S)(\bar{Y},\bar{V}) = A(\bar{X})S(\bar{Y},\bar{V}) + nB(\bar{X})g(\bar{Y},\bar{V}),$$

where $A(\bar{X}) = \alpha(\bar{X})$ and $B(\bar{X}) = \frac{(n-1)}{n}\beta(\bar{X})$ are two non-zero 1-forms. This leads to the theorem:

**Theorem 3.6** *Let $M^n$ be a decomposable semi-generalized $W_3$ recurrent manifold which is not $W_3$ flat such that $M^n = M_1^p \times M_2^{n-p}$, $2 \leq p \leq (n-2)$. Then $M_1$(respectively $M_2$) is semi-generalized Ricci recurrent.*

## 3.6 Example of a semi-generalized $W_3$ recurrent manifold

### Example 1:

Consider $\mathbb{R}^4$ with the Riemannian metric defined by

$$ds^2 = g_{ij}dx^i dx^j = (1-4q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2], \tag{3.44}$$

where $q = \frac{e^{x^1}}{k^2}$, for a non-zero constant $k$ and $x^1 \neq 0$. The non-vanishing components of the Christoffel's symbols, the Riemannian curvature tensors and the Ricci tensors are

$$\Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = \frac{2q}{1-4q},$$
$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = -\frac{2q}{1-4q},$$
$$R_{1221} = R_{1331} = R_{1441} = -\frac{2q}{1-4q},$$
$$S_{11} = \frac{6q}{(1-4q)^2}, \quad S_{22} = S_{33} = \frac{2q}{(1-4q)^2}$$

and the components which can be obtained by symmetry properties. Using

$$r = g^{ij}S_{ij}, \tag{3.45}$$

we get $r = \dfrac{12q}{(1-4q)^3}$, which is non-zero. By virtue of (3.2), we get the non-zero components of the $W_3$ curvature tensor as

$$(W_3)_{1221} = -\frac{4q}{1-4q}, \quad (W_3)_{1331} = (W_3)_{1441} = -\frac{8q}{3(1-4q)},$$

whose non-zero covariant derivatives are

$$(W_3)_{1221,1} = -\frac{4q}{(1-4q)^2}, \quad (W_3)_{1331,1} = (W_3)_{1441,1} = -\frac{8q}{3(1-4q)^2}$$

and their symmetric components. Here "," denotes the operator of covariant differentiation with respect to the metric $g$. To show that $(\mathbb{R}^4, g)$ is a semi-generalized $W_3$ recurrent manifold, we choose the 1-forms $\alpha$ and $\beta$ as

$$\alpha_i = \begin{cases} \dfrac{1}{1-4q}, & i = 1, \\ 0, & otherwise, \end{cases}$$

$$\beta_i = 0, \text{for } i = 1, 2, 3, 4.$$

Then, equation (3.1) reduces to

$$(W_3)_{1221,1} = \alpha_1 (W_3)_{1221}, \tag{3.46}$$

$$(W_3)_{1331,1} = \alpha_1 (W_3)_{1331}, \tag{3.47}$$

$$(W_3)_{1441,1} = \alpha_1 (W_3)_{1441} \tag{3.48}$$

and the other cases hold trivially.

$$\begin{aligned} \text{R. H. S of (3.46)} &= \alpha_1 (W_3)_{1221} \\ &= \Big(\frac{1}{1-4q}\Big).\Big(-\frac{4q}{1-4q}\Big) \\ &= -\frac{4q}{(1-4q)^2} = (W_3)_{1221,1} \\ &= \text{L. H. S of (3.46).} \end{aligned}$$

Similarly, equations (3.47) and (3.48) can be proved. Therefore, $(\mathbb{R}^4, g)$ is a semi-generalized $W_3$ recurrent manifold.

## Example 2:

Define a Riemannian metric $g$ on $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = (x^1)^{\frac{1}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2, \text{ where } x^1 \neq 0. \quad (3.49)$$

We obtain the non-vanishing components of the Christoffel's symbols, the curvature tensors and the Ricci tensors as

$$\Gamma^1_{22} = \Gamma^1_{33} = \frac{1}{6x^1},$$

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = -\frac{1}{6x^1},$$

$$R_{1221} = -\frac{5}{36(x^1)^{\frac{5}{3}}} = R_{1331}, \quad R_{2332} = -\frac{1}{36(x^1)^{\frac{5}{3}}},$$

$$S_{11} = -\frac{5}{18(x^1)^2}, \quad S_{22} = S_{33} = \frac{1}{9(x^1)^2}$$

and their symmetric components.

Using (3.45), we get $r = \frac{1}{2(x^1)^{\frac{7}{3}}}$, which is non-zero and non constant. From equation (3.2), we obtain

$$(W_3)_{1221} = -\frac{5}{108(x^1)^{\frac{5}{3}}} = (W_3)_{1331}, \ (W_3)_{2332} = -\frac{1}{36(x^1)^{\frac{5}{3}}},$$

and the components obtained by symmetric properties. Using these, we get the covariant derivatives of the $W_3$ curvature tensors as

$$(W_3)_{1221,1} = \frac{25}{324(x^1)^{\frac{8}{3}}} = (W_3)_{1331,1}, \quad (W_3)_{2332,1} = \frac{5}{108(x^1)^{\frac{8}{3}}}.$$

To show that the manifold under consideration is semi-generalized $W_3$ recurrent, we choose the 1-forms $\alpha$ and $\beta$ as

$$\alpha_i = \begin{cases} -\dfrac{5}{3(x^1)}, & i = 1, \\ 0, & otherwise, \end{cases}$$

$$\beta_i = 0, \text{for } i = 1, 2, 3, 4.$$

From equation (3.1), we have

$$(W_3)_{1221,1} = \alpha_1 (W_3)_{1221}, \tag{3.50}$$

$$(W_3)_{1331,1} = \alpha_1 (W_3)_{1331}, \tag{3.51}$$

$$(W_3)_{2332,1} = \alpha_1 (W_3)_{2332}, \tag{3.52}$$

and all other cases hold trivially. Now,

$$\begin{aligned} \text{R. H. S of (3.50)} &= \alpha_1 (W_3)_{1221} \\ &= \Big(-\frac{5}{3(x^1)}\Big).\Big(-\frac{5}{108(x^1)^{\frac{5}{3}}}\Big) \\ &= \frac{25}{324(x^1)^{\frac{8}{3}}} = (W_3)_{1221,1} \\ &= \text{L. H. S of (3.50)} \end{aligned}$$

and equations (3.51) and (3.52) can be proved in a similar manner. Therefore, $\mathbb{R}^4$ with the given metric is a semi-generalized $W_3$ recurrent manifold.

# Chapter 4

# Curvature Properties of $N(k)$-quasi Einstein Manifolds

In this chapter we considered $N(k)$-quasi Einstein manifolds satisfying certain curvature conditions. $W^*$-Ricci pseudosymmetric, $W_2$-pseudosymmetric and $Z$-generalized pseudosymmetric $N(k)$-quasi Einstein manifolds are studied. We considered $N(k)$-quasi Einstein manifolds satisfying the curvature conditions $\bar{P}(\zeta, X) \cdot W_2 = 0$ and $\bar{P}(\zeta, X) \cdot H = 0$, where $\bar{P}$, $W_2$ and $H$ are the pseudo projective, $W_2$ and conharmonic curvature tensors respectively. We studied pseudo projectively symmetric $N(k)$-quasi Einstein manifolds and showed that there does not exist a pseudo projectively semisymmetric $N(k)$-quasi Einstein manifold. Also, we constructed some examples to support the existence of such manifolds.

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Singh, J. P. and Lalnunsiami, K. (2020). Certain Curvature Properties of $N(k)$-quasi Einstein Manifolds, *SUT Journal of Mathematics*, **56(1)**, 55-69.

Singh, J. P. and Lalnunsiami, K. (2020). On a type of $N(k)$-quasi Einstein manifolds, *Bulletin of the Transilvania University of Braşov Series III: Mathematics, Informatics, Physics,* **Vol 13(62), No. 1**, 219-236.

## 4.1 Introduction

A non-flat Riemannian manifold $(M, g)$ is said to be quasi Einstein if its Ricci tensor $S$ satisfies equation (1.78). In 2007, Tripathi and Kim defined $N(k)$-quasi Einstein manifolds. In an $N(k)$-quasi Einstein manifold, $k$ is not arbitrary as given by (Özgür and Tripathi, 2009):

**Lemma 4.1** *In an $n$-dimensional $N(k)$-quasi Einstein manifold,*

$$k = \frac{a+b}{n-1}. \tag{4.1}$$

Also, in an $N(k)$-quasi Einstein manifold, we have

$$R(X,Y)\zeta = k\big[\eta(Y)X - \eta(X)Y\big], \tag{4.2}$$

$$R(X,\zeta)Y = k\big[\eta(Y)X - g(X,Y)\zeta\big] = -R(\zeta,X)Y, \tag{4.3}$$

$$\eta(R(X,Y)Z) = k\big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big]. \tag{4.4}$$

In 1971, Pokhariyal and Mishra defined the $m$-projective and the $W_2$ curvature tensors given by equations (1.25) and (1.26) respectively. The $Z$ tensor in a Riemannian manifold is defined by equation (1.20) (Mantica and Molinari, 2012). The conharmonic curvature tensor (Ishii, 1957) and the pseudo projective curvature tensor (Prasad, 2002) are defined by equations (1.22) and (1.24) respectively.

Using equations (1.78) and (1.79), we obtain

$$S(X,\zeta) = (a+b)\eta(X), \tag{4.5}$$

$$r = na + b, \tag{4.6}$$

where $r$ is the scalar curvature of the manifold. In an $n$-dimensional $N(k)$-quasi

Einstein manifold, we have

$$W_2(X,Y)\zeta = \frac{b}{(n-1)}\big[\eta(Y)X - \eta(X)Y\big], \tag{4.7}$$

$$W_2(\zeta,X)Y = \frac{1}{(n-1)}\big[\eta(Y)QX - (a+b)\eta(Y)X\big], \tag{4.8}$$

$$\eta(W_2(X,Y)Z) = 0, \tag{4.9}$$

$$W^*(X,Y)\zeta = \frac{b}{2(n-1)}\big[\eta(Y)X - \eta(X)Y\big], \tag{4.10}$$

$$W^*(\zeta,X)Y = \frac{b}{2(n-1)}\big[g(X,Y)\zeta - \eta(Y)X\big], \tag{4.11}$$

$$\eta(W^*(X,Y)Z) = \frac{b}{2(n-1)}\big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big], \tag{4.12}$$

$$\bar{P}(X,Y)\zeta = \Big[\frac{\beta(n-1)+\alpha}{b}\Big]\Big[\eta(Y)X - \eta(X)Y\Big], \tag{4.13}$$

$$\begin{aligned}\bar{P}(\zeta,X)Y &= \frac{(\alpha-\beta)}{n}\Big[g(X,Y)\zeta - \eta(Y)X\Big] \\ &+ \beta b\big[\eta(Y)\eta(X) - \eta(Y)X\big],\end{aligned} \tag{4.14}$$

$$\eta(\bar{P}(X,Y)Z) = \frac{(\alpha-\beta)}{n}\Big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\Big], \tag{4.15}$$

$$H(X,Y)\zeta = \frac{(na+b)}{(n-1)(n-2)}\Big[\eta(Y)X - \eta(X)Y\Big], \tag{4.16}$$

$$H(\zeta,X)Y = \frac{(na+b)}{(n-1)(n-2)}\Big[g(X,Y)\zeta - \eta(Y)X\Big], \tag{4.17}$$

$$\begin{aligned}\eta(H(X,Y)Z) &= \frac{(na+b)}{(n-1)(n-2)}\Big[g(Y,Z)\eta(X) \\ &- g(X,Z)\eta(Y)\Big].\end{aligned} \tag{4.18}$$

The generalized $Z$-tensor in an $N(k)$-quasi Einstein manifold takes the form,

$$Z(X,Y) = (a+\phi)g(X,Y) + b\eta(X)\eta(Y), \tag{4.19}$$

which by contraction, reduces to

$$Z = (a+\phi)n + b. \tag{4.20}$$

Also,

$$Z(X,\zeta) = (a+b+\phi)\eta(X), \tag{4.21}$$

$$Z(\zeta,\zeta) = (a+b+\phi), \tag{4.22}$$

$\forall X, Y, Z \in M^n$.

## 4.2 $m$-projective curvature tensor in an $N(k)$-quasi Einstein manifold

Suppose an $N(k)$-quasi Einstein manifold satisfies

$$W^*(\zeta, X) \cdot W_2 = 0,$$

or,

$$\begin{aligned} &W^*(\zeta,X)W_2(U,V)Z - W_2(W^*(\zeta,X)U,V)Z \\ &-W_2(U,W^*(\zeta,X)V)Z - W_2(U,V)W^*(\zeta,X)Z = 0. \end{aligned} \tag{4.23}$$

Using (4.12), (4.23) it becomes

$$\frac{b}{2(n-1)}\Big[g(X,W_2(U,V)Z)\zeta-\eta(W_2(U,V)Z)X$$
$$-g(X,U)W_2(\zeta,V)Z+\eta(U)W_2(X,V)Z$$
$$-g(X,V)W2(U,\zeta)Z+\eta(V)W_2(U,X)Z$$
$$-g(X,Z)W_2(U,V)\zeta+\eta(Z)W_2(U,V)X\Big]=0. \tag{4.24}$$

Since $b\neq 0$ and $n>1$, we have

$$g(X,W_2(U,V)Z)\zeta-\eta(W_2(U,V)Z)X$$
$$-g(X,U)W_2(\zeta,V)Z+\eta(U)W_2(X,V)Z$$
$$-g(X,V)W_2(U,\zeta)Z+\eta(V)W_2(U,X)Z$$
$$-g(X,Z)W_2(U,V)\zeta+\eta(Z)W_2(U,V)X=0. \tag{4.25}$$

Taking inner product of (4.25) with respect to $\zeta$, we have

$$W_2'(U,V,Z,X)-\eta(W_2(U,V)Z)\eta(X)$$
$$-g(X,U)\eta(W_2(\zeta,V)Z)+\eta(U)\eta(W_2(X,V)Z)$$
$$-g(X,V)\eta(W_2(U,\zeta)Z)+\eta(V)\eta(W_2(U,X)Z)$$
$$-g(X,Z)\eta(W_2(U,V)\zeta)+\eta(Z)\eta(W_2(U,V)X)=0. \tag{4.26}$$

From equations (4.9) and (4.26), it follows that $W_2'(U,V,Z,X)=0$. Thus, we can state the following theorem:

**Theorem 4.1** *An $n$-dimensional $N(k)$-quasi Einstein manifold satisfies the condition $W^*(\zeta,X)\cdot W_2=0$ if and only if the manifold is $W_2$-flat.*

**Definition 4.1** *A Riemannian manifold is said to be semi-symmetric (Szabo; 1982, 1987) if*

$$R\cdot R=0, \tag{4.27}$$

*where $R$ is the Riemannian curvature tensor.*

Consider an $N(k)$-quasi Einstein manifold which is $W^*$-semisymmetric. Then, we have

$$(R(X,Y)\cdot W^*)(U,V)Z = 0,$$

which implies that

$$\begin{aligned}&R(X,Y)W^*(U,V)Z - W^*(R(X,Y)U,V)Z\\ &-W^*(U,R(X,Y)V)Z - W^*(U,V)R(X,Y)Z = 0. \qquad (4.28)\end{aligned}$$

Taking inner product of (4.28) with respect to $\zeta$, we have

$$\begin{aligned}&g(R(X,Y)W^*(U,V)Z,\zeta) - g(W^*(R(X,Y)U,V)Z,\zeta)\\ &-g(W^*(U,R(X,Y)V)Z,\zeta) - g(W^*(U,V)R(X,Y)Z,\zeta) = 0. \qquad (4.29)\end{aligned}$$

Substituting $X=\zeta$, (4.29) reduces to

$$\begin{aligned}&g(R(\zeta,Y)W^*(U,V)Z,\zeta) - g(W^*(R(\zeta,Y)U,V)Z,\zeta)\\ &-g(W^*(U,R(\zeta,Y)V)Z,\zeta) - g(W^*(U,V)R(\zeta,Y)Z,\zeta) = 0. \qquad (4.30)\end{aligned}$$

Using equations (4.3) and (4.11) in (4.30), we get

$$W^*(U,V,Z,X) - \frac{b}{2(n-1)}\Big[g(U,Y)g(V,Z) - g(V,Y)g(U,Z)\Big] = 0. \qquad (4.31)$$

Making use of (1.20) and (4.31), we obtain

$$\begin{aligned}&R'(U,V,Z,Y) - \frac{1}{2(n-1)}\Big[S(V,Z)g(U,Y) - S(U,Z)g(V,Y)\\ &+S(U,Y)g(V,Z) - S(V,Y)g(U,Z)\Big]\\ &-\frac{b}{2(n-1)}\Big[g(U,Y)g(V,Z) - g(V,Y)g(U,Z)\Big] = 0. \qquad (4.32)\end{aligned}$$

Contracting (4.32) with respect to $U$ and $Y$, we have

$$S(V,Z) = (a+b)g(V,Z),$$

which is a contradiction as the manifold is quasi Einstein. This leads to the theorem:

**Theorem 4.2** *There does not exist a $W^*$-semisymmetric $N(k)$-quasi Einstein manifold.*

**Definition 4.2** *A Riemannian manifold is said to be a symmetric manifold (Kobayashi and Nomizu, 1963; Desai and Amur, 1975) if*

$$(\nabla_X R)(Y, Z)V = 0, \tag{4.33}$$

*where $\nabla$ is the operator of covariant differentiation with respect to the metric $g$.*

Consider an $N(k)$-quasi Einstein manifold which is $W^*$-symmetric. Then, we can write

$$(\nabla_X W^*)(U, V, Z, Y) = 0.$$

Using equation (1.25), we have

$$\begin{aligned}(\nabla_X R')(U, V, Z, Y) &= \frac{1}{2(n-1)}\Big[(\nabla_X S)(V, Z)g(Y, U) - (\nabla_X S)(U, Z)g(V, Y) \\ &+ (\nabla_X S)(U, Y)g(V, Z) - (\nabla_X S)(V, Y)g(U, Z)\Big].\end{aligned} \tag{4.34}$$

Setting $U = Y = e_i$ and summing over $i, 1 \leq i \leq n$, we get

$$(\nabla_X S)(V, Z) = \frac{dr(X)}{n} g(V, Z). \tag{4.35}$$

Using equation (1.79) in (4.35), we obtain

$$\begin{aligned}&da(X)g(V, Z) + db(X)\eta(V)\eta(Z) + b\big[(\nabla_X \eta)(Z)\eta(V) \\ &+(\nabla_X \eta)(V)\eta(Z)\big] = \frac{dr(X)}{n} g(V, Z).\end{aligned} \tag{4.36}$$

Putting $Z = V = \zeta$, we get

$$dr(X) = n[da(X) + db(X)]. \tag{4.37}$$

Also, from (4.6), it follows that

$$dr(X) = nda(X) + db(X). \tag{4.38}$$

From equations (4.37) and (4.38), we get

$$db(X) = 0,$$

i. e., $b$ is constant. Therefore, we have the theorem:

**Theorem 4.3** *There exists no $W^*$-symmetric $N(k)$-quasi Einstein manifold unless the associated scalar $b$ is a non-zero constant.*

From equation (1.25), we can write

$$\begin{aligned}(div\ W^*)(X,Y)Z &= (div\ R)(X,Y)Z - \frac{1}{2(2n-3)}\big[dr(X)g(Y,Z) \\ &- dr(Y)g(X,Z)\big], \qquad (4.39)\end{aligned}$$

where "*div*" denotes the divergence.

We know that in a Riemannian manifold,

$$(div\ R)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z). \qquad (4.40)$$

Using equation (4.39) in (4.40), we get

$$\begin{aligned}(div\ W^*)(X,Y)Z &= (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \\ &- \frac{1}{2(2n-3)}\Big[dr(X)g(Y,Z) - dr(Y)g(X,Z)\Big]. \qquad (4.41)\end{aligned}$$

Suppose that an $N(k)$-quasi Einstein manifold is $W^*$-conservative. Then,

$$(div\ W^*)(X,Y)Z = 0,$$

or,

$$\begin{aligned}(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) &= \frac{1}{2(2n-3)}\Big[dr(X)g(Y,Z) \\ &- dr(Y)g(X,Z)\Big]. \qquad (4.42)\end{aligned}$$

Making use of equation (4.6) in (4.42), we obtain

$$\begin{aligned}
& da(X)g(Y,Z) + db(X)\eta(Y)\eta(Z) - da(Y)g(X,Z) - db(Y)\eta(X)\eta(Z) \\
& +b\big[(\nabla_X\eta)(Y)\eta(Z) + (\nabla_X\eta)(Z)\eta(Y) - (\nabla_Y\eta)(X)\eta(Z) - (\nabla_Y\eta)(Z)\eta(X)\big] \\
& = \frac{1}{2(2n-3)}\Big[dr(X)g(Y,Z) - dr(Y)g(X,Z)\Big]. \qquad (4.43)
\end{aligned}$$

Assume that the associated scalar $b$ is non-zero constant. Then $db(X) = 0$, from which it follows that $dr(X) = nda(X)$, $\forall X$. Therefore equation (4.43) becomes

$$\begin{aligned}
& \frac{3(n-2)}{2(2n-3)}\Big[da(X)g(Y,Z) - da(Y)g(X,Z)\Big] + b\Big[(\nabla_X\eta)(Y)\eta(Z) \\
& +(\nabla_X\eta)(Z)\eta(Y) - (\nabla_Y\eta)(X)\eta(Z) - (\nabla_Y\eta)(Z)\eta(X)\Big] = 0. \qquad (4.44)
\end{aligned}$$

Substituting $Y = Z = \zeta$ in equation (4.44), we obtain

$$b(\nabla_\zeta\eta)(X) = \frac{3(n-2)}{2(2n-3)}\Big[da(X) - da(\zeta)\eta(X)\Big]. \qquad (4.45)$$

Contracting equation (4.44) over $Y$ and $Z$, we have

$$b\big[(\nabla_\zeta\eta)(X) + \eta(X)\sum_{i=1}^{n}(\nabla_{e_i}\eta)(e_i)\big] - \frac{3(n-1)(n-2)}{2(2n-3)}da(X) = 0. \qquad (4.46)$$

From equations (4.45) and (4.46), it follows that

$$\begin{aligned}
b\eta(X)\sum_{i=1}^{n}(\nabla_{e_i}\eta)(e_i) &= \frac{3(n-1)(n-2)}{2(2n-3)}da(X) \\
&- \frac{3(n-2)}{2(2n-3)}\Big[da(X) - da(\zeta)\eta(X)\Big]. \qquad (4.47)
\end{aligned}$$

Taking $X = \zeta$, equation (4.47) becomes

$$\sum_{i=1}^{n}(\nabla_{e_i}\eta)(e_i) = \frac{3(n-1)(n-2)}{2(2n-3)}da(\zeta). \qquad (4.48)$$

Making use of equations (4.45) and (4.48), (4.46) becomes

$$da(X) = da(\zeta)\eta(X). \qquad (4.49)$$

Substituting $X = \zeta$ in equation (4.44) and using (4.49), we get

$$b[(\nabla_X\eta)(Y) - (\nabla_Y\eta)(X)] = 0,$$

or,

$$(\nabla_X\eta)(Y) - (\nabla_Y\eta)(X) = 0, \quad \text{(since } n \neq 0\text{)}$$

which implies that the 1-form $\eta$ is closed.

Setting $X = \zeta$, the above equation reduces to

$$(\nabla_\zeta\eta)(Y) = 0,$$

which implies that

$$\nabla_\zeta\zeta = 0.$$

Therefore, we can state the theorem:

**Theorem 4.4** *On an $(n > 3)$-dimensional $N(k)$-quasi Einstein manifold which is $W^*$-conservative and $b$ is non-zero constant, the associated 1-form $\eta$ is closed and the integral curves of the generator $\zeta$ are geodesics.*

## 4.3 $W^*$-Ricci pseudosymmetric $N(k)$-quasi Einstein manifold

**Definition 4.3** *A Riemannian manifold is said to be Ricci pseudosymmetric (Deszcz, 1992) if the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent at every point of $M^n$, i. e.,*

$$R \cdot S = L_S Q(g, S),$$

*where $L_S$ is a smooth function on $A_S = \{x \in \mathbb{R} : S \neq \frac{r}{n} g \text{ in } x\}$.*

Consider an $N(k)$-quasi Einstein manifold which is $W^*$-Ricci pseudosymmetric. Then the vectors $W^* \cdot S$ and $Q(g, S)$ are linearly dependent, i.e.,

$$(W^*(X, Y) \cdot S)(Z, U) = L_S Q(g, S)(Z, U; X, Y), \tag{4.50}$$

where $L_S$ is a function on $A_S = \{x \in \mathbb{R} : S \neq \frac{r}{n} g \text{ at } x\}$. Then,

$$\begin{aligned} S(W^*(X,Y)Z,U) + S(Z,W^*(X,Y)U) &= L_S\big[S((X \wedge Y)Z,U) \\ &+ S(Z,(X \wedge Y)U)\big]. \end{aligned} \tag{4.51}$$

Taking $X = \zeta$ in (4.51), we have

$$\begin{aligned} S(W^*(\zeta,Y)Z,U) + S(Z,W^*(\zeta,Y)U) &= L_S\big[S((\zeta \wedge Y)Z,U) \\ &+ S(Z,(\zeta \wedge Y)U)\big]. \end{aligned} \tag{4.52}$$

Using (4.11) and

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{4.53}$$

equation (4.52) becomes

$$\begin{aligned} &\Big[\frac{b}{2(n-1)} - L_S\Big]\big[S(U,X)g(Y,Z) - S(U,Y)g(X,Z) \\ &+g(U,Y)S(Z,X) - g(U,X)S(Z,Y)\big] = 0, \end{aligned} \tag{4.54}$$

which implies that either

$$L_S = \frac{b}{2(n-1)},$$

or,

$$\begin{aligned} &S(U,X)g(Y,Z) - S(U,Y)g(X,Z) \\ &+g(U,Y)S(Z,X) - g(U,X)S(Z,Y) = 0. \end{aligned} \tag{4.55}$$

Using equation (1.78), (4.55) can be written as

$$\begin{aligned} &a\big[g(U,X)g(Y,Z) - g(U,Y)g(X,Z) \\ &+g(U,Y)g(Z,X) - g(U,X)g(Z,Y)\big] \\ &+b\big[\eta(U)\eta(X)g(Y,Z) - \eta(U)\eta(Y)g(X,Z) \\ &+g(U,Y)\eta(Z)\eta(X) - g(U,X)\eta(Z)\eta(Y)\big] = 0. \end{aligned} \tag{4.56}$$

Contracting equation (4.56) with respect to $X$ and $U$, we get

$$g(Y, Z) = n\eta(Y)\eta(Z).$$

Substituting $Y = Z = \zeta$ in the above equation, we have

$$n = 1,$$

which is a contradiction. Therefore,

$$L_S = \frac{b}{2(n-1)}.$$

Thus, we can state:

**Theorem 4.5** *An $n$-dimensional $W^*$-Ricci pseudosymmetric $N(k)$-quasi Einstein manifold satisfies the relation $L_S = \frac{b}{2(n-1)}$.*

## 4.4 $W_2$-pseudosymmetric $N(k)$-quasi Einstein manifold

**Definition 4.4** *An $n$-dimensional Riemannian manifold is said to be pseudosymmetric (Deszcz, 1992) if*

$$R \cdot R = LQ(g, R),$$

*i.e., $R \cdot R$ and $Q(g, R)$ are linearly dependent and $L$ is a function on $B = \{x \in \mathbb{R} : Q(g, R) \neq 0$ at $x\}$.*

Suppose that an $N(k)$-quasi Einstein manifold is $W_2$-pseudosymmetric. Then,

$$(R(X, Y) \cdot W_2)(U, V)Z = L_{W_2}Q(g, W_2)(U, V, Z; X, Y), \tag{4.57}$$

where $L_{W_2}$ is a smooth function on $B_{W_2} = \{x \in \mathbb{R} : Q(g, W_2) \neq 0 \text{ at } x\}$.

From (4.57), we have

$$\begin{aligned}
&R(X,Y)W_2(U,V)Z - W_2(R(X,Y)U,V)) \\
&-W_2(U,R(X,Y)V)Z - W_2(U,V)R(X,Y)Z \\
&= L_{W_2}\big[(X \wedge_{W_2} Y)W_2(U,V)Z - W_2((X \wedge_{W_2} Y)U,V)Z \\
&-W_2(U,(X \wedge_{W_2} Y)V)Z - W_2(U,V)(X \wedge_{W_2} Y)Z\big]. \qquad (4.58)
\end{aligned}$$

Put $X = \zeta$ in the above equation, we have

$$\begin{aligned}
&R(\zeta,Y)W_2(U,V)Z - W_2(R(\zeta,Y)U,V)Z) \\
&-W_2(U,R(\zeta,Y)V)Z - W_2(U,V)R(\zeta,Y)Z \\
&= L_{W_2}\big[(\zeta \wedge_{W_2} Y)W_2(U,V)Z - W_2((\zeta \wedge_{W_2} Y)U,V)Z \\
&-W_2(U,(\zeta \wedge_{W_2} Y)V)Z - W_2(U,V)(\zeta \wedge_{W_2} Y)Z\big]. \qquad (4.59)
\end{aligned}$$

Using (4.3) and (4.53), we get

$$\begin{aligned}
&(k - L_{W_2})\big[W_2'(U,V,Z,Y)\zeta - \eta(W_2(U,V)Z)Y \\
&-g(Y,U)W_2(\zeta,V)Z + \eta(U)W_2(Y,V)Z \\
&-g(Y,V)W_2(U,\zeta)Z + \eta(V)W_2(U,Y)Z \\
&-g(Y,Z)W_2(U,V)\zeta + \eta(Z)W_2(U,V)X\big] = 0. \qquad (4.60)
\end{aligned}$$

Taking inner product of (4.60) with respect to $\zeta$, we get

$$\begin{aligned}
&(k - L_{W_2})\big[W_2'(U,V,Z,Y) - \eta(W_2(U,V)Z)\eta(Y) \\
&-g(Y,U)\eta(W_2(\zeta,V)Z) + \eta(U)\eta(W_2(Y,V)Z) \\
&-g(Y,V)\eta(W_2(U,\zeta)Z) + \eta(V)\eta(W_2(U,Y)Z) \\
&-g(Y,Z)\eta(W_2(U,V)\zeta) + \eta(Z)\eta(W_2(U,V)X)\big] = 0. \qquad (4.61)
\end{aligned}$$

By virtue of (4.9), (4.61) reduces to

$$(k - L_{W_2})W_2'(U,V,Z,Y) = 0.$$

Since $W_2 \neq 0$, we have

$$k - L_{W_2} = 0,$$

or,

$$k = L_{W_2}.$$

This leads to the theorem:

**Theorem 4.6** *An $N(k)$-quasi Einstein manifold is $W_2$-pseudosymmetric provided that $k = L_{W_2}$.*

## 4.5 $Z$-generalized pseudosymmetric $N(k)$-quasi Einstein manifold

**Definition 4.5** *A Riemannian manifold is said to be Ricci-generalized pseudosymmetric (Deszcz, 1992) if at every point of $M^n$, the tensors $R \cdot R$ and $Q(S, R)$ are linearly dependent, i. e.,*

$$R \cdot R = LQ(S, R),$$

*where $L$ is a function on $A = \{x \in \mathbb{R} : Q(S, R) \neq 0 \text{ at } x\}$.*

Consider an $N(k)$-quasi Einstein manifold which is $Z$-generalized pseudosymmetric. Then,

$$R \cdot R = L_Z Q(Z, R),$$

where $L_Z$ is a function on $A_Z = \{x \in \mathbb{R} : Q(Z, R) \neq 0 \text{ at } x\}$. Then,

$$\begin{aligned}&R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W\\&-R(U,V)R(X,Y)W = L_Z\big[(X \wedge_Z Y)R(U,V)W\\&-R((X \wedge_Z Y)U,V)W - R(U,(X \wedge_Z Y)V)W - R(U,V)(X \wedge_Z Y)W\big]. \quad (4.62)\end{aligned}$$

Taking $X = \zeta$ in (4.62), we have

$$R(\zeta, Y)R(U, V)W - R(R(\zeta, Y)U, V)W - R(U, R(\zeta, Y)V)W$$
$$-R(U, V)R(\zeta, Y)W = L_Z\big[(\zeta \wedge_Z Y)R(U, V)W$$
$$-R((\zeta \wedge_Z Y)U, V)W - R(U, (\zeta \wedge_Z Y)V)W - R(U, V)(\zeta \wedge_Z Y)W\big]. \quad (4.63)$$

Using (4.3) and

$$(X \wedge_Z Y)U = Z(Y, U)X - Z(X, U)Y,$$

in (4.63), we have

$$[k - L_Z(a + \phi)]\big[R'(U, V, W, Y)\zeta - \eta(R(U, V)W)Y$$
$$-g(Y, U)R(\zeta, V)W + \eta(U)R(Y, V)W$$
$$-g(Y, V)R(U, \zeta)W + \eta(V)R(U, Y)W$$
$$-g(Y, W)R(U, V)\zeta + \eta(W)R(U, V)Y\big]$$
$$= L_Z b\big[\eta(Y)\eta(R(U, V)W)\zeta - \eta(R(U, V)W)Y$$
$$-g(Y, U)R(\zeta, V)W + \eta(U)R(Y, V)W$$
$$-g(Y, V)R(U, \zeta)W + \eta(V)R(U, Y)W$$
$$-g(Y, W)R(U, V)\zeta + \eta(W)R(U, V)Y\big]. \quad (4.64)$$

Taking inner product of (4.64) with respect to $\zeta$, we have

$$
\begin{aligned}
&[k-L_Z(a+\phi)]\big[R'(U,V,W,Y)-\eta(R(U,V)W)\eta(Y)\\
&-g(Y,U)\eta(R(\zeta,V)W)+\eta(U)\eta(R(Y,V)W)\\
&-g(Y,V)\eta(R(U,\zeta)W)+\eta(V)\eta(R(U,Y)W)\\
&-g(Y,W)\eta(R(U,V)\zeta)+\eta(W)\eta(R(U,V)Y)\big]\\
&=L_Zb\big[\eta(Y)\eta(R(U,V)W)-\eta(R(U,V)W)\eta(Y)\\
&-g(Y,U)\eta(R(\zeta,V)W)+\eta(U)\eta(R(Y,V)W)\\
&-g(Y,V)\eta(R(U,\zeta)W)+\eta(V)\eta(R(U,Y)W)\\
&-g(Y,W)\eta(R(U,V)\zeta)+\eta(W)\eta(R(U,V)Y)\big].
\end{aligned}
\tag{4.65}
$$

Using (1.26) and (1.80), (4.65) reduces to

$$L_Zbk[\eta(W)\eta(U)g(V,Y)-\eta(W)\eta(V)g(U,Y)]=0,$$

which implies (since $b\neq 0$),

$$L_Zk=0,$$

i. e., $L_Z=0$ or $k=0$.

This leads to the theorem:

**Theorem 4.7** *A $Z$-generalized pseudosymmetric $N(k)$-quasi Einstein manifold is either semisymmetric or $k=0$.*

## 4.6 Pseudo projective curvature tensor in an $N(k)$-quasi Einstein manifold

Suppose $M$ satisfies the curvature condition $\bar{P}(\zeta,X)\cdot W_2=0$. Then,

$$
\begin{aligned}
\bar{P}(\zeta,X)W_2(U,V)Z \quad &- \quad W_2(\bar{P}(\zeta,X)U,V)Z-W_2(U,\bar{P}(\zeta,X)V)Z)\\
&- \quad W_2(U,V)\bar{P}(\zeta,X)Z=0,
\end{aligned}
\tag{4.66}
$$

for all vector fields $U, V, Z, X \in M$.

Using equation (4.14) in (4.66), we have

$$\begin{aligned} b\Bigg[&\frac{(\alpha-\beta)}{n}\Big\{W_2'(U,V,Z,X)\zeta - \eta(W_2(U,V)Z)X \\ &-g(X,U)W_2(\zeta,V)Z + \eta(U)W_2(X,V)Z \\ &-g(X,V)W_2(U,\zeta)Z + \eta(V)W_2(U,X)Z \\ &-g(X,Z)W_2(U,V)\zeta + \eta(Z)W_2(U,V)X\Big\} \\ &+\beta\Big\{\eta(X)\eta(W_2(U,V)Z)\zeta - \eta(W_2(U,V)Z)X \\ &-\eta(X)\eta(U)W_2(\zeta,V)Z + \eta(U)W_2(X,V)Z \\ &-\eta(X)\eta(V)W_2(U,\zeta)Z + \eta(V)W_2(U,X)Z \\ &-\eta(X)\eta(Z)W_2(U,V)\zeta + \eta(Z)W_2(U,V)X\Big\}\Bigg] = 0. \end{aligned} \tag{4.67}$$

Since $b \neq 0$, equation (4.67) can be written as

$$\begin{aligned} &\frac{(\alpha-\beta)}{n}\Big\{W_2'(U,V,Z,X)\zeta - \eta(W_2(U,V)Z)X \\ &- g(X,U)W_2(\zeta,V)Z + \eta(U)W_2(X,V)Z \\ &- g(X,V)W_2(U,\zeta)Z + \eta(V)W_2(U,X)Z \\ &- g(X,Z)W_2(U,V)\zeta + \eta(Z)W_2(U,V)X\Big\} \\ &+ \beta\Big\{\eta(X)\eta(W_2(U,V)Z)\zeta - \eta(W_2(U,V)Z)X \\ &- \eta(X)\eta(U)W_2(\zeta,V)Z + \eta(U)W_2(X,V)Z \\ &- \eta(X)\eta(V)W_2(U,\zeta)Z + \eta(V)W_2(U,X)Z \\ &- \eta(X)\eta(Z)W_2(U,V)\zeta + \eta(Z)W_2(U,V)X\Big\} = 0. \end{aligned} \tag{4.68}$$

Taking inner product of (4.68) with respect to $\zeta$, we have

$$\begin{aligned}
&\frac{(\alpha-\beta)}{n}\Big\{W_2'(U,V,Z,X)-\eta(W_2(U,V)Z)\eta(X)\\
&-g(X,U)\eta(W_2(\zeta,V)Z)+\eta(U)\eta(W_2(X,V)Z)\\
&-g(X,V)\eta(W_2(U,\zeta)Z)+\eta(V)\eta(W_2(U,X)Z)\\
&-g(X,Z)\eta(W_2(U,V)\zeta)+\eta(Z)\eta(W_2(U,V)X)\Big\}\\
&+\beta\Big\{\eta(X)\eta(W_2(U,V)Z)-\eta(W_2(U,V)Z)\eta(X)\\
&-\eta(X)\eta(U)\eta(W_2(\zeta,V)Z)+\eta(U)\eta(W_2(X,V)Z)\\
&-\eta(X)\eta(V)\eta(W_2(U,\zeta)Z)+\eta(V)\eta(W_2(U,X)Z)\\
&-\eta(X)\eta(Z)\eta(W_2(U,V)\zeta)+\eta(Z)\eta(W_2(U,V)X)\Big\}=0. \qquad (4.69)
\end{aligned}$$

From equations (4.9) and (4.69), it follows that

$$\frac{(\alpha-\beta)}{n}W_2'(U,V,Z,X)=0.$$

Since $n>2$ this implies that

$$\alpha=\beta \quad \text{or} \quad W_2=0.$$

Thus, we can state:

**Theorem 4.8** *An $n$-dimensional $N(k)$-quasi Einstein manifold $M$ satisfies the curvature condition $\bar{P}(\zeta,X)\cdot W_2=0$ provided $\alpha=\beta$ or the manifold is $W_2$-flat.*

Suppose $M$ satisfies the curvature condition $\bar{P}(\zeta,X)\cdot H=0$. Then, we can write

$$\begin{aligned}
\bar{P}(\zeta,X)H(U,V)Z \;&-\; H(\bar{P}(\zeta,X)U,V)Z-H(U,\bar{P}(\zeta,X)V)Z\\
&-\; H(U,V)\bar{P}(\zeta,X)Z=0. \qquad (4.70)
\end{aligned}$$

Using (4.14) in (4.70), we have

$$\begin{aligned}
& b\Bigg[\frac{(\alpha-\beta)}{n}\Big\{H'(U,V,Z,X)\zeta-\eta(H(U,V)Z)X \\
& -g(X,U)H(\zeta,V)Z+\eta(U)H(X,V)Z \\
& -g(X,V)H(U,\zeta)Z+\eta(V)H(U,X)Z \\
& -g(X,Z)H(U,V)\zeta+\eta(Z)H(U,V)X\Big\} \\
& +\beta\Big\{\eta(X)\eta(H(U,V)Z)\zeta-\eta(H(U,V)Z)X \\
& -\eta(X)\eta(U)H(\zeta,V)Z+\eta(U)H(X,V)Z \\
& -\eta(X)\eta(V)H(U,\zeta)Z+\eta(V)H(U,X)Z \\
& -\eta(X)\eta(Z)H(U,V)\zeta+\eta(Z)H(U,V)X\Big\}\Bigg]=0. \qquad (4.71)
\end{aligned}$$

Since $b \neq 0$, equation (4.71) can be written as

$$\begin{aligned}
& \frac{(\alpha-\beta)}{n}\Big\{H'(U,V,Z,X)\zeta-\eta(H(U,V)Z)X \\
& -g(X,U)H(\zeta,V)Z+\eta(U)H(X,V)Z \\
& -g(X,V)H(U,\zeta)Z+\eta(V)H(U,X)Z \\
& -g(X,Z)H(U,V)\zeta+\eta(Z)H(U,V)X\Big\} \\
& +\beta\Big\{\eta(X)\eta(H(U,V)Z)\zeta-\eta(H(U,V)Z)X \\
& -\eta(X)\eta(U)H(\zeta,V)Z+\eta(U)H(X,V)Z \\
& -\eta(X)\eta(V)H(U,\zeta)Z+\eta(V)H(U,X)Z \\
& -\eta(X)\eta(Z)H(U,V)\zeta+\eta(Z)H(U,V)X\Big\}=0. \qquad (4.72)
\end{aligned}$$

Taking inner product of (4.72) with respect to $\zeta$, we have

$$\begin{aligned}
&\frac{(\alpha-\beta)}{n}\Big\{H'(U,V,Z,X)-\eta(H(U,V)Z)\eta(X)\\
&-g(X,U)\eta(H(\zeta,V)Z)+\eta(U)\eta(H(X,V)Z)\\
&-g(X,V)\eta(H(U,\zeta)Z)+\eta(V)\eta(H(U,X)Z)\\
&-g(X,Z)\eta(H(U,V)\zeta)+\eta(Z)\eta(H(U,V)X)\Big\}\\
&+\beta\Big\{\eta(X)\eta(H(U,V)Z)-\eta(H(U,V)Z)\eta(X)\\
&-\eta(X)\eta(U)\eta(H(\zeta,V)Z)+\eta(U)\eta(H(X,V)Z)\\
&-\eta(X)\eta(V)\eta(H(U,\zeta)Z)+\eta(V)\eta(H(U,X)Z)\\
&-\eta(X)\eta(Z)\eta(H(U,V)\zeta)+\eta(Z)\eta(H(U,V)X)\Big\}=0. \qquad (4.73)
\end{aligned}$$

Using equation (4.17) in (4.73), we get

$$\begin{aligned}
&\frac{(\alpha-\beta)}{n}\Big[H'(U,V,Z,X)+\frac{(na+b)}{(n-1)(n-2)}\big\{g(X,U)g(V,Z)\\
&-g(X,V)g(U,Z)\big\}\Big]-\beta\frac{(na+b)}{(n-1)(n-2)}\Big[\eta(X)\eta(U)g(V,Z)\\
&-\eta(X)\eta(V)g(U,Z)\Big]=0. \qquad (4.74)
\end{aligned}$$

Making use of (1.22) in (4.74), we obtain

$$\begin{aligned}
&\frac{(\alpha-\beta)}{n}\Big[R'(U,V,Z,X)-\frac{1}{(n-2)}\big\{S(V,Z)g(X,U)\\
&-S(U,Z)g(X,V)+g(V,Z)S(X,U)-g(U,Z)S(X,V)\big\}\\
&+\frac{(na+b)}{(n-1)(n-2)}\big\{g(V,Z)g(X,U)-g(U,Z)g(X,V)\big\}\Big]\\
&+\beta\frac{(na+b)}{(n-1)(n-2)}\Big[\eta(V)\eta(Z)g(X,U)-\eta(U)\eta(Z)g(X,V)\Big]=0. \qquad (4.75)
\end{aligned}$$

Taking $U=X=e_i$ in (4.75) and summing over $i, 1\leq i\leq n$, we get

$$\beta\Big(\frac{na+b}{n-2}\Big)\eta(V)\eta(Z)=0.$$

Since $n > 2$, $\beta \neq 0$ and $\eta \neq 0$, we have

$$na + b = 0.$$

Using this in equation (4.74), it follows that

$$\frac{(\alpha - \beta)}{n} H'(U, V, Z, X) = 0,$$
$$\Rightarrow \alpha = \beta = 0 \quad \text{or} \quad H'(U, V, Z, X) = 0,$$

which leads to the theorem:

**Theorem 4.9** *Let $M$ be an $n$-dimensional $N(k)$-quasi Einstein manifold. Then $M$ satisfies the curvature condition $\bar{P}(\zeta, X) \cdot H = 0$ if $\alpha = \beta$ or the manifold is conharmonically flat.*

Consider an $N(k)$-quasi Einstein manifold which is pseudo projectively semisymmetric. Then,

$$(R(X, Y) \cdot \bar{P})(U, V)W = 0,$$

$$R(X, Y)\bar{P}(U, V)W - \bar{P}(R(X, Y)U, V)W - \bar{P}(U, R(X, Y)V)W - \bar{P}(U, V)R(X, Y)W = 0, \tag{4.76}$$

Taking inner product of (4.76) with respect to $\zeta$, we have

$$g(R(X, Y)\bar{P}(U, V)W, \zeta) - g(\bar{P}(R(X, Y)U, V)W, \zeta) - g(\bar{P}(U, R(X, Y)V)W, \zeta) - g(\bar{P}(U, V)R(X, Y)W, \zeta) = 0,$$

Substituting $X = \zeta$, the above equation becomes

$$g(R(\zeta, Y)\bar{P}(U, V)W, \zeta) - g(\bar{P}(R(\zeta, Y)U, V)W, \zeta) - g(\bar{P}(U, R(\zeta, Y)V)W, \zeta) - g(\bar{P}(U, V)R(\zeta, Y)W, \zeta) = 0.$$

Using equation (4.3), we have

$$k\Bigg[\bar{P}'(U,V,W,Y) - \frac{(\alpha-\beta)}{n}\Big\{g(V,W)g(U,Y) - g(U,W)g(V,Y)\Big\}\Bigg] = 0. \tag{4.77}$$

Assuming $k \neq 0$ and making use of equation (1.24), (4.77) becomes

$$\begin{aligned}\alpha R'(U,V,W,Y) &= -\beta[S(V,W)g(U,Y) - S(U,W)g(V,Y)] \\ &\quad + \left\{\frac{r}{n}\Big(\frac{\alpha}{n-1}+\beta\Big) + \frac{(\alpha-\beta)b}{n}\right\}\big[g(V,W)g(U,Y) \\ &\quad - g(U,W)g(V,Y)\big].\end{aligned} \tag{4.78}$$

Contracting equation (4.78) with respect to $U$ and $Y$ , we get

$$S(V,W) = \Big[\frac{r}{n(n-1)} + \frac{(\alpha-\beta)b}{n(\alpha+\beta(n-1))}\Big]g(V,W)$$

showing that the manifold is Einstein, which is not possible since $M$ is an $N(k)$-quasi Einstein manifold. Thus, we can state:

**Theorem 4.10** *There does not exist a pseudo projectively semisymmetric $N(k)$-quasi Einstein manifold.*

Consider a pseudo projectively symmetric $N(k)$-quasi Einstein manifold. Then,

$$(\nabla_X \bar{P}')(Z,U,V,W) = 0.$$

By virtue of equation (1.24), the above equation can be written as

$$\begin{aligned}&\alpha(\nabla_X R')(Z,U,V,W) + \beta[(\nabla_X S)(U,V)g(Z,W) \\ &- (\nabla_X S)(Z,V)g(U,W)] - \frac{dr(X)}{n}\Big(\frac{\alpha}{n-1}+\beta\Big)\big[g(U,V)g(Z,W) \\ &- g(Z,V)g(U,W)\big] = 0.\end{aligned} \tag{4.79}$$

Contracting (4.79) with respect to $Z$ and $W$, we obtain

$$(\nabla_X S)(U,V) = \frac{dr(X)}{n} g(U,V). \tag{4.80}$$

Using (1.78) in equation (4.80), we have

$$\begin{aligned} da(X)g(U,V) + db(X)\eta(U)\eta(V) + b[(\nabla_X\eta)(U)\eta(V) \\ + (\nabla_X\eta)(V)\eta(U)] = \frac{dr(X)}{n} g(U,V). \end{aligned} \tag{4.81}$$

Substituting $U = V = \zeta$, (4.81) becomes

$$dr(X) = n[da(X) + db(X)]. \tag{4.82}$$

Also, taking covariant derivative of equation (4.6) with respect to $X$, we obtain

$$dr(X) = nda(X) + db(X). \tag{4.83}$$

From equations (4.82) and (4.83), it follows that

$$(n-1)db(X) = 0,$$

or

$$db(X) = 0,$$

which implies that $b$ is constant. This leads to the following theorem:

**Theorem 4.11** *An $N(k)$-quasi Einstein manifold is pseudo projectively symmetric provided the associated scalar $b$ is non-zero constant.*

## 4.7 Conharmonically pseudosymmetric $N(k)$-quasi Einstein manifold

Consider an $N(k)$-quasi Einstein manifold which is conharmonically pseudosymmetric. Then,

$$(R(X,Y)\cdot H)(Z,W)U = L_H Q(g,H)(Z,W,U;X,Y) \tag{4.84}$$

for a smooth function $L_H \in A_H = \{x \in M : Q(g,H) \neq 0 \text{ at } x\}$, where $X, Y, Z, W, U$ are arbitrary.

From equation (4.84) we have

$$\begin{aligned}
&R(X,Y)H(Z,W)U - H(R(X,Y)Z,W)U \\
&- H(Z,R(X,Y)W)U - H(Z,W)R(X,Y)U \\
&= L_H[(X \wedge_H Y)H(Z,W)U - H((X \wedge_H Y)Z,W)U \\
&- H(Z,(X \wedge_H Y)W)U - H(Z,W)(X \wedge_H Y)U].
\end{aligned} \tag{4.85}$$

Putting $X = \zeta$, (4.85) becomes

$$\begin{aligned}
&R(\zeta,Y)H(Z,W)U - H(R(\zeta,Y)Z,W)U \\
&- H(Z,R(\zeta,Y)W)U - H(Z,W)R(\zeta,Y)U \\
&= L_H[(\zeta \wedge_H Y)H(Z,W)U - H((\zeta \wedge_H Y)Z,W)U \\
&- H(Z,(\zeta \wedge_H Y)W)U - H(Z,W)(\zeta \wedge_H Y)U].
\end{aligned} \tag{4.86}$$

Making use of equation (4.3) in (4.86) and

$$(X \wedge_H Y)Z = g(Y,Z)X - g(X,Z)Y,$$

we have

$$(k - L_H)[H'(Z,W,U,Y)\zeta - \eta(H(Z,W)U)Y - g(Y,Z)H(\zeta,W)U$$
$$+ \eta(Z)H(Y,W)U - g(Y,W)H(Z,\zeta)U + \eta(W)H(Z,Y)U$$
$$- g(Y,U)H(Z,W)\zeta + \eta(U)H(Z,W)Y] = 0.$$

Assuming $k \neq L_H$ and taking inner product of the above equation with respect to $\zeta$, we have

$$H'(Z,W,U,Y) + \frac{na+b}{(n-1)(n-2)}\Big[g(Z,Y)g(W,U) - g(Z,U)g(W,Y)\Big] = 0. \tag{4.87}$$

From equations (1.22) and (4.87), we obtain

$$R'(Z,W,U,Y) = a_1[g(Z,Y)g(W,U) - g(Z,U)g(W,Y)]$$
$$+ a_2[\eta(W)\eta(U)g(Y,Z) - \eta(Z)\eta(U)g(W,Y)$$
$$+ \eta(Z)\eta(Y)g(W,U) - \eta(W)\eta(Y)g(Z,U)],$$

where $a_1 = \frac{a}{n-1} - \frac{b}{(n-1)(n-2)}$ and $a_2 = \frac{b}{n-2}$ . This leads to the following theorem:

**Theorem 4.12** *An $n$-dimensional $N(k)$-quasi Einstein manifold which is conharmonically pseudosymmetric and $k \neq L_H$ is of quasi-constant curvature.*

## 4.8 Examples of $N(k)$-quasi Einstein manifolds

### Example 1:

Consider a Riemannian metric $g$ on $\mathbb{R}^3$ by

$$ds^2 = g_{ij}dx^i dx^j = e^{x^3}cos(x^3)[(dx^1)^2 + (dx^2)^2] - (dx^3)^2.$$

Then, we have

$$g_{11} = g_{22} = e^{x^3} cos(x^3), \quad g_{33} = -1,$$

$$g^{11} = g^{22} = e^{-x^3} sec(x^3), \quad g^{33} = -1.$$

Then, the non-vanishing components of the Christoffel's symbols and the curvature tensors are

$$\Gamma^3_{11} = \Gamma^3_{22} = e^{x^3} \frac{(cos(x^3) - sin(x^3))}{2},$$

$$\Gamma^1_{13} = \Gamma^2_{23} = \frac{cos(x^3) - sin(x^3)}{2cos(x^3)},$$

$$R_{1221} = -e^{2x^3} \frac{(1 - sin(2x^3))}{4}, \quad R_{1331} = R_{2332} = -e^{x^3} \frac{(1 + sin(2x^3))}{4cos(x^3)}.$$

Also, the non-vanishing components of the Ricci tensors are

$$S_{11} = S_{22} = -e^{x^3} sin(x^3), \quad S_{33} = \frac{(1 + sin(2x^3))}{2cos^2(x^3)}.$$

Using these results in

$$r = g^{ij} S_{ij}, \tag{4.88}$$

we get

$$r = -\frac{(sec^2(x^3) - 6tan^2(x^3))}{2},$$

which is non-zero.

To show that the manifold is $N(k)$-quasi Einstein, we choose the scalar functions $a$ and $b$ and the 1-form $\eta$ as

$$a = -tan(x^3), \quad b = \frac{1}{2} sec^2(x^3),$$

$$\eta_i(x) = \begin{cases} 1, & i = 3, \\ 0, & otherwise, \end{cases}$$

at any point $x \in \mathbb{R}^3$.

From (1.78), we have

$$S_{11} = ag_{11} + b\eta_1\eta_1, \tag{4.89}$$

$$S_{22} = ag_{22} + b\eta_2\eta_2, \tag{4.90}$$

$$S_{33} = ag_{33} + b\eta_3\eta_3 \tag{4.91}$$

and all others hold trivially.

$$\begin{aligned}
\text{R. H. S of (4.91)} &= ag_{33} + b\eta_3\eta_3 \\
&= -tan(x^3)(-1) + \frac{1}{2}sec^2(x^3)(1) \\
&= \frac{(1 + sin(2x^3))}{2cos^2(x^3)} = S_{33} \\
&= \text{L.H.S of (4.91).}
\end{aligned}$$

Similarly, it can be shown that equations (4.89) and (4.90) hold. Using equation (4.1), we get

$$k = \frac{a+b}{n-1} = \frac{sin(2x^3) - 1}{4}.$$

So, $(\mathbb{R}^3, g)$ is an $N\big(\frac{sin(2x^3)-1}{4}\big)$–quasi Einstein manifold.

## Example 2:

Consider a pseudo projectively flat quasi Einstein manifold. Then, from (1.24), we have

$$\begin{aligned}
\alpha R(X,Y)Z = &-\beta\big[S(Y,Z)X - S(X,Z)Y\big] \\
&+ \frac{r}{n}\Big[\frac{\alpha}{n-1} + \beta\Big]\Big[g(Y,Z)X - g(X,Z)Y\Big]
\end{aligned} \tag{4.92}$$

Using equations (1.77) in (4.92), we get

$$\alpha R(X,Y)Z = -\Big\{\beta a - \frac{r}{n}\Big(\frac{\alpha}{n-1} + \beta\Big)\Big\}\Big[g(Y,Z)X - g(X,Z)Y\Big] \qquad (4.93)$$
$$- \beta b\big[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\big].$$

Replacing $Z$ by $\zeta$ in (4.93) we have,

$$R(X,Y)\zeta = \Big[\frac{r}{n(n-1)} - \frac{\beta b}{n}\big(\frac{n-1}{n}\big)\Big]\big[\eta(Y)X - \eta(X)Y\big],$$

which shows that $\zeta$ belongs to the $\Big(\frac{r}{n(n-1)} - \frac{\beta b}{n}\big(\frac{n-1}{n}\big)\Big)$-nullity distribution. Therefore, we can state:

**Theorem 4.13** *A pseudo projectively flat quasi Einstein manifold is an* $N\Big(\frac{r}{n(n-1)} - \frac{\beta b}{n}\big(\frac{n-1}{n}\big)\Big)$*-quasi Einstein manifold.*

## Example 3:

Consider a quasi Einstein manifold which is conharmonically flat. Then by equation (1.22), we have

$$R(X,Y)Z = \frac{1}{(n-2)}\Big[S(X,Z)Y - S(Y,Z)X \qquad (4.94)$$
$$+ g(Y,Z)QX - g(X,Z)QY\Big].$$

Using (1.77) and (4.6) in (4.94), we have

$$R(X,Y)Z = \frac{1}{(n-2)}\Big[2a\big\{g(Y,Z)X - g(X,Z)Y\big\} + b\big\{\eta(Y)\eta(Z)X \qquad (4.95)$$
$$- \eta(X)\eta(Z)Y + g(Y,Z)\eta(X)\zeta) - g(X,Z)\eta(Y)\zeta\big\}\Big].$$

Substituting $Z = \zeta$, equation (4.95) reduces to

$$R(X,Y)\zeta = \Big(\frac{2a+b}{n-2}\Big)\big[\eta(Y)X - \eta(X)Y\big],$$

showing that the manifold is an $N\Big(\frac{2a+b}{n-2}\Big)$-quasi Einstein manifold. Thus, we have the following theorem:

**Theorem 4.14** *A conharmonically flat quasi Einstein manifold is an* $N\Big(\dfrac{2a+b}{n-2}\Big)$-*quasi Einstein manifold.*

## Example 4:

Consider $\mathbb{R}^4$ with the Riemannian metric $g$ defined by

$$ds^2 = g_{ij}dx^i dx^j = (x^3)^2[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2.$$

Then, we have

$$g_{11} = g_{22} = g_{33} = (x^3)^2, \quad g_{44} = 1,$$
$$g^{11} = g^{22} = g^{33} = \frac{1}{(x^3)^2}, \quad g^{44} = 1.$$

The non-vanishing components of the Christoffel's symbols, the curvature tensors and the Ricci tensors are

$$\Gamma^3_{11} = \Gamma^3_{22} = -\frac{1}{x^3}, \quad \Gamma^3_{33} = \Gamma^1_{13} = \Gamma^2_{23} = \frac{1}{x^3},$$
$$R_{1331} = R_{2332} = -1, \quad S_{11} = S_{22} = S_{44} = 0, \quad S_{33} = \frac{2}{(x^3)^2}.$$

Using (4.88) and the above results, we get

$$r = 2,$$

which is non-vanishing. To show that the manifold under consideration is an $N(k)$-quasi Einstein manifold, we choose the scalar functions $a, b$ and the 1-form $\eta$ as

$$a = 0, \quad b = 2,$$

$$\eta_i(x) = \begin{cases} \frac{1}{x^3}, & i = 3, \\ 0, & otherwise, \end{cases}$$

at any point $x \in \mathbb{R}^4$. From (1.78), we have

$$S_{11} = ag_{11} + b\eta_1\eta_1, \tag{4.96}$$

$$S_{22} = ag_{22} + b\eta_2\eta_2, \tag{4.97}$$

$$S_{33} = ag_{33} + b\eta_3\eta_3, \tag{4.98}$$

$$S_{44} = ag_{44} + b\eta_4\eta_4 \tag{4.99}$$

and all others hold trivially.

$$\begin{aligned} \text{R. H. S of (4.98)} &= ag_{33} + b\eta_3\eta_3 \\ &= -0 + 2\Big(\frac{1}{x^3}\Big)\Big(\frac{1}{x^3}\Big) \\ &= \frac{2}{(x^3)^2} = S_{33} \\ &= \text{L.H.S of (4.98)}. \end{aligned}$$

Similarly, it can be shown that equations (4.96), (4.97) and (4.99) hold. Using (4.1), we get

$$k = \frac{a+b}{n-1} = \frac{2}{3}.$$

So, $(\mathbb{R}^4, g)$ is an $N\big(\frac{2}{3}\big)$-quasi Einstein manifold.

# Chapter 5

# Weakly $Z$-symmetric Manifolds

In this chapter, we considered weakly $Z$-symmetric manifolds. Weakly $Z$-symmetric manifolds with Codazzi type and cyclic parallel $Z$ tensor are studied. We considered Einstein weakly $Z$-symmetric manifolds and conformally flat weakly $Z$-symmetric manifolds. Also, we showed that a totally umbilical hypersurface of a conformally flat weakly $Z$-symmetric manifolds is of quasi constant curvature. Decomposable weakly $Z$-symmetric manifolds are studied and some examples are constructed to support the existence of such manifolds.

## 5.1 Introduction

A non-flat Riemannian manifold $(M^n, g)$ is said to be weakly symmetric (Tamassy and Binh, 1989) if the curvature tensor $R'$ given by $R'(X,Y,U,V) = g(R(X,Y)U,V)$ satisfies (1.69). The $Z$ tensor in a Riemannian manifold is defined by

$$Z(X,Y) = S(X,Y) + \phi g(X,Y),$$

which on contraction reduces to

$$Z = r + n\phi. \tag{5.1}$$

Lalnunsiami, K. and Singh J. P. (2020). On some classes of weakly $Z$-symmetric manifolds, *Communications of the Korean Mathematical Society,* **35(2)**, 935-951.

(Mantica and Suh, 2012). A Riemannian manifold is called a weakly $Z$-symmetric manifold (Mantica and Molinari, 2012) if the $Z$ tensor satisfies

$$(\nabla_X Z)(U,V) = A(X)Z(U,V) + B(U)Z(X,V) + D(V)Z(U,X), \tag{5.2}$$

where $A, B, D$ are simultaneously non-zero 1-forms defined by

$$A(X) = g(X,\rho_1), \quad B(X) = g(X,\rho_2), \quad D(X) = g(X,\rho_3). \tag{5.3}$$

Here, $\rho_1, \rho_2, \rho_3$ are known as the basic vector fields of $M^n$ corresponding to $A, B, D$ respectively. We denote this manifold by $(WZS)_n$. An Einstein manifold (Besse, 1987) is a Riemannian manifold $(M^n, g)$ whose Ricci tensor satisfies (1.77).

An Einstein manifold can be generalized to a quasi Einstein manifold. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ whose Ricci tensor is not identically zero and satisfies

$$S(U,V) = ag(U,V) + b\eta(U)\eta(V),$$

for smooth functions $a, b(\neq 0)$, and $\eta$ is a 1-form which is non zero defined by

$$\eta(X) = g(X,\zeta), \quad \eta(\zeta) = 1,$$

for all vector fields $X$ is called a quasi Einstein manifold (Chaki and Maity, 2000).

A Riemannian manifold is said to have cyclic parallel Ricci tensor if $S$ is non-zero and satisfies (1.81) (Gray, 1978). Also, the Ricci tensor $S$ in a Riemannian manifold is said to be of Codazzi type (Gray, 1978) if $S$ is not zero and satisfies (1.82). We have the following important lemma:

**Lemma 5.1** *(Walker's Lemma) (Walker, 1950): If $a_{ij}, b_i$ are numbers satisfying*

$$a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0,$$

*for $i, j, k = 1, 2, ........n$, then either all $a_{ij} = 0$ or all $b_i = 0$.*

Let $S$ denote the Ricci tensor of the manifold defined by

$$S(X,Y) = g(QX,Y),$$

where $Q$ is the Ricci operator. We define

$$\bar{A}(X) = A(QX), \quad \bar{B}(X) = B(QX), \quad \bar{D}(X) = D(QX), \tag{5.4}$$

called the auxiliary 1-forms corresponding to $A, B$ and $D$. From (1.20), we have

$$Z(U,V) = Z(V,U),$$

$$Z(U,\rho_1) = S(U,\rho_1) + \phi g(U,\rho_1),$$

or,

$$Z(U,\rho_1) = \bar{A}(U) + \phi A(U).$$

Similarly,

$$Z(U,\rho_2) = \bar{B}(U) + \phi B(U),$$

$$Z(U,\rho_3) = \bar{D}(U) + \phi D(U).$$

Now, equation (5.2) yields

$$(\nabla_X Z)(U,V) - (\nabla_V Z)(U,X) = E(X)Z(U,V) - E(V)Z(U,X), \tag{5.5}$$

where

$$E(X) = A(X) - D(X) = g(X,\rho), \quad \rho = \rho_1 - \rho_3. \tag{5.6}$$

Making use of equation (1.20) in (5.5), we have

$$\begin{aligned}(\nabla_X S)(U,V) + (X\phi)g(U,V) \quad &- \quad (\nabla_V S)(U,X) - (V\phi)g(U,X) \\ &= \quad E(X)Z(U,V) - E(V)Z(U,X). \end{aligned} \tag{5.7}$$

Contracting equation (5.7) with respect to $U$ and $V$ and using (5.1) and (5.6), we get

$$dr(X) = 2\big[r + (n-1)\phi\big]E(X) - 2\bar{E}(X) - 2(n-1)(X\phi). \tag{5.8}$$

A conformally flat Riemannian manifold $(M^n, g)(n > 2)$ is said to be of quasi constant curvature (Chen and Yano, 1972) if its curvature tensor $R'$ of type (0, 4) satisfies

$$\begin{aligned} R'(X,Y,U,V) &= l\Big[g(Y,U)g(X,V) - g(X,U)g(Y,V)\Big] \\ &+ m\Big[g(X,V)H(Y)H(U) + g(Y,U)H(X)H(V) \\ &- g(X,U)H(Y)H(V) - g(Y,V)H(X)H(U)\Big], \end{aligned} \tag{5.9}$$

where $l, m$ are scalar functions, $m \neq 0$ called the associated scalars, $H \neq 0$ is called the associated 1-form given by $g(X,\mu) = H(X)$ and $\mu$ is a unit vector field known as the generator of the manifold.

A Riemannian manifold whose curvature tensor $R'$ satisfies

$$R'(X,Y,U,V) = F(Y,U)F(X,V) - F(X,U)F(Y,V), \tag{5.10}$$

where $F$ is a symmetric (0, 2) tensor is known as a special manifold (Chern, 1956) with the associated symmetric tensor $F$ and denoted by $\psi(F)_n$.

## 5.2 Weakly $Z$-symmetric manifolds

In this section, we consider Einstein $(WZS)_n$, $(WZS)_n$ with Codazzi and cyclic parallel $Z$ tensor and conformally flat $(WZS)_n$.

Suppose the $Z$ tensor in a $(WZS)_n$ is of Codazzi type. Then,

$$(\nabla_X Z)(U,V) = (\nabla_V Z)(U,X). \tag{5.11}$$

Using (5.2) in (5.11), we get

$$\big[A(X) - D(X)\big]Z(U,V) = \big[A(V) - D(V)\big]Z(U,X),$$

or,

$$E(X)Z(U,V) = E(V)Z(U,X). \tag{5.12}$$

Contraction of (5.12) over $U$ and $X$ yields

$$E(V)[r + (n-1)\phi] = E(QV). \tag{5.13}$$

Also, on substituting $V = \rho$ in (5.12) and using equations (1.20), (5.4) and (5.6), we obtain

$$E(X)\big[E(QU) + \phi E(U)\big] = E(\rho)\big[S(X,U) + \phi g(X,U)\big]. \tag{5.14}$$

Equations (5.13) and (5.14) gives

$$E(X)E(U)(r + (n-1)\phi) = E(\rho)S(X,U) + \phi E(\rho)g(X,U). \tag{5.15}$$

If $\rho$ is a unit vector field, then equation (5.15) becomes

$$S(X,U) = -\phi g(X,U) + (r + (n-1)\phi)E(X)E(U),$$

which implies that the manifold is quasi Einstein. Thus, we can state:

**Theorem 5.1** *In a $(WZS)_n$, if the $Z$ tensor is of Codazzi type, then the manifold is quasi Einstein provided that the vector field $\rho$ defined by*

$$E(X) = A(X) - D(X) = g(X,\rho), \quad \rho = \rho_1 - \rho_3,$$

*is a unit vector field.*

Consider an Einstein $(WZS)_n$. Then, from (1.77), we have

$$dr(X) = 0, \tag{5.16}$$

which implies that,

$$(\nabla_X S)(U,V) = 0. \tag{5.17}$$

Making use of equations (5.16) and (5.17) in (5.2), we obtain

$$\begin{aligned}(X\phi)g(U,V) &= \Big(\frac{r}{n}+\phi\Big)\big[A(X)g(U,V)\\ &+ B(U)g(X,V)+D(V)g(U,X)\big].\end{aligned} \tag{5.18}$$

Contracting (5.18) over $U$ and $V$, we get

$$n(X\phi) = \Big(\frac{r}{n}+\phi\Big)\big[nA(X)+B(X)+D(X)\big]. \tag{5.19}$$

Similarly,

$$n(U\phi) = \Big(\frac{r}{n}+\phi\Big)\big[nB(X)+A(X)+D(X)\big], \tag{5.20}$$

and

$$n(V\phi) = \Big(\frac{r}{n}+\phi\Big)\big[nD(X)+B(X)+A(X)\big]. \tag{5.21}$$

Replacing $U,V$ by $X$ in (5.20) and (5.21) and adding (5.19), (5.20) and (5.21), we have

$$3n(X\phi) = \Big(\frac{r}{n}+\phi\Big)(n+2)\big[A(X)+B(X)+D(X)\big]. \tag{5.22}$$

If $\phi$ is constant, we have from (5.22),

$$A(X)+B(X)+D(X)=0. \tag{5.23}$$

Conversely, if (5.23) holds, then $\phi$ is constant which proves the theorem:

**Theorem 5.2** *The sum of the associated 1-forms $A, B, D$ in an Einstein $(WZS)_n$ is zero if and only if $\phi$ is constant.*

Now, suppose (5.23) holds. Then, from (5.2),

$$(\nabla_X Z)(X,X) = [A(X)+B(X)+D(X)]Z(X,X),$$

which gives

$$(\nabla_X Z)(X, X) = 0, \tag{5.24}$$

i. e., $Z$ is covariantly constant in the direction of $X$.

Further, if (5.24) holds, then (5.23) follows if $Z(X, X) \neq 0$. This leads to the following corollary:

**Corollary 5.1** *The $Z$ tensor in an Einstein $(WZS)_n$ is covariantly constant in the direction of $X$ if and only if* (5.24) *holds.*

Interchanging $X, U, V$ in equation (5.2) and adding, we have

$$\begin{aligned}(\nabla_X Z)(U, V) + (\nabla_U Z)(V, X) \;&+\; (\nabla_V Z)(U, X) = F(X)Z(U, V)\\ &+\; F(U)Z(V, X) + F(V)Z(U, X),\end{aligned} \tag{5.25}$$

where $F(X) = A(X) + B(X) + D(X)$.

Suppose the $Z$ tensor is cyclic parallel, i. e.,

$$(\nabla_X Z)(U, V) + (\nabla_U Z)(V, X) + (\nabla_V Z)(U, X) = 0.$$

Then, (5.25) becomes

$$F(X)Z(U, V) + F(U)Z(V, X) + F(V)Z(U, X) = 0. \tag{5.26}$$

From Walker's Lemma, it follows that $F(X) = 0$ or $Z(X, Y) = 0$. But $Z(X, Y) \neq 0$ which implies that $F(X) = 0$. i. e.,

$$A(X) + B(X) + D(X) = 0.$$

Conversely, if $A(X) + B(X) + D(X) = 0$, then from (5.25), $Z$ is cyclic parallel. This leads to the theorem:

**Theorem 5.3** *In a $(WZS)_n$, the $Z$ tensor is cyclic parallel if and only if $A(X) + B(X) + D(X) = 0$.*

The Weyl conformal curvature tensor $\bar{C}$ in a Riemannian manifold $(M^n, g)(n > 3)$ is given by equation (1.27). Consider a conformally flat $(WZS)_n$. Then,

$$div\ \bar{C} = 0,$$

where "*div*" denotes the divergence. Hence,

$$\begin{aligned}(\nabla_X S)(U,V) - (\nabla_V S)(U,X) &= \frac{1}{2(n-1)}\Big[g(U,V)dr(X) \\ &- g(U,X)dr(V)\Big]. \end{aligned} \tag{5.27}$$

Using equations (5.7) and (5.8), (5.27) becomes

$$\begin{aligned} E(X)Z(U,V) &- E(V)Z(U,X) - (X\phi)g(U,V) + (V\phi)g(U,X) \\ &= \frac{1}{2(n-1)}\Big[2\{r + (n-1)\phi\}\{E(X)g(U,V) \\ &- E(V)g(U,X)\} - 2(n-1)\{(X\phi)g(U,V) \\ &- (V\phi)g(U,X)\} - 2\{\bar{E}(X)g(U,V) \\ &- \bar{E}(V)g(U,X)\}\Big], \end{aligned} \tag{5.28}$$

where $\bar{E}(X) = E(QX)$.

Substituting $X = \rho$ in (5.28), we get

$$E(X)\bar{E}(V) = E(V)\bar{E}(X). \tag{5.29}$$

Again, replacing $X$ by $\rho$ in (5.29), we get

$$\bar{E}(V) = \frac{\bar{E}(\rho)}{E(\rho)}E(V),$$

or

$$\bar{E}(V) = sE(V), \tag{5.30}$$

where

$$s = \frac{\bar{E}(\rho)}{E(\rho)}. \tag{5.31}$$

Using (5.31) in (5.8), we obtain

$$dr(X) = 2\{r - s + (n-1)\phi\}E(X) - 2(n-1)(X\phi). \tag{5.32}$$

Assume that $E \neq 0$. Substituting $X = \rho$ in (5.28) and using (5.30), we have

$$\begin{aligned} E(\rho)Z(U,V) - E(V)Z(U,\rho) &= \frac{1}{(n-1)}\Big\{r - s + (n-1)\phi\Big\}\Big[E(\rho)g(U,V) \\ &- E(V)E(\rho)\Big]. \end{aligned} \tag{5.33}$$

Making use of (1.20) and (5.30), (5.33) becomes

$$\begin{aligned} &E(\rho)S(U,V) + \phi E(\rho)g(U,V) - sE(U)E(V) - \phi E(U)E(V) \\ = &\frac{1}{(n-1)}\Big\{r - s + (n-1)\phi\Big\}\Big[E(\rho)g(U,V) - E(V)E(\rho)\Big], \end{aligned} \tag{5.34}$$

or

$$S(U,V) = \left(\frac{r-s}{n-1}\right)g(U,V) + \left(\frac{ns-r}{n-1}\right)T(U)T(V),$$

where $T(X) = \frac{E(X)}{E(\rho)}$. Thus, we can write

$$S(U,V) = ag(U,V) + bT(U)T(V), \tag{5.35}$$

where $a = \dfrac{r-s}{n-1}$ and $b = \dfrac{ns-r}{n-1}$. Thus, the manifold is quasi-Einstein which leads to the theorem:

**Theorem 5.4** *A conformally flat $(WZS)_n$ is quasi Einstein provided that the 1-form $E$ given by*

$$E(X) = A(X) - D(X) = g(X,\rho), \quad \rho = \rho_1 - \rho_3,$$

*is non-zero.*

Now, from (5.34), we have

$$S(U,V) = \Big(\frac{r-s}{n-1}\Big)g(U,V) + \Big(\frac{ns-r}{n-1}\Big)\frac{E(U)E(V)}{\rho}.$$

Putting $V = \rho$, we get

$$S(U, \rho) = sE(U) = sg(U, \rho),$$

i. e., $\rho$ is an eigenvector of $S$ with eigenvalue $s$. Thus, we can state the following corollary:

**Corollary 5.2** *In a conformally flat $(WZS)_n$, the vector field $\rho$ corresponding to the 1-form $E$ is an eigenvector of the Ricci tensor $S$ corresponding to the eigenvalue $s$.*

Suppose the $(WZS)_n$ is conformally flat. Then, we have,

$$\bar{C}(X, Y, U, V) = 0,$$

which implies that,

$$\begin{aligned} R'(X, Y, U, V) &= \frac{1}{(n-2)}\Big[S(Y, U)g(X, V) - S(X, U)g(Y, V) \\ &+ S(X, V)g(Y, U) - S(Y, V)g(X, U)\Big] \\ &- \frac{r}{(n-1)(n-2)}\Big[g(Y, U)g(X, V) - g(X, U)g(Y, V)\Big]. \end{aligned}$$

Using (5.35) and assuming that $\rho$ is a unit vector field, the above equation becomes

$$\begin{aligned} R'(X, Y, U, V) &= \Big[\frac{2a}{(n-2)} - \frac{r}{(n-1)(n-2)}\Big]\Big[g(Y, U)g(X, V) - g(X, U)g(Y, V)\Big] \\ &+ \frac{b}{(n-2)}\Big[g(X, V)E(Y)E(U) + g(Y, U)E(X)E(V) \\ &- g(X, U)E(Y)E(V) - g(Y, V)E(X)E(U)\Big], \end{aligned}$$

or,

$$\begin{aligned} R'(X, Y, U, V) &= l\Big[g(Y, U)g(X, V) - g(X, U)g(Y, V)\Big] \\ &+ m\Big[g(X, V)E(Y)E(U) + g(Y, U)E(X)E(V) \\ &- g(X, U)E(Y)E(V) - g(Y, V)E(X)E(U)\Big], \end{aligned} \tag{5.36}$$

where $l = \dfrac{2a}{(n-2)} - \dfrac{r}{(n-1)(n-2)}$ and $m = \dfrac{b}{(n-2)}$, i. e., the manifold is of quasi constant curvature. Thus, we have the theorem:

**Theorem 5.5** *If the vector field $\rho$ associated with $E$ in a conformally flat $(WZS)_n$ is a unit vector field, then the manifold is of quasi constant curvature.*

Suppose that

$$F(X,Y) = \sqrt{l}g(X,Y) + \frac{m}{\sqrt{l}}E(X)E(Y). \tag{5.37}$$

Then, $F$ is a symmetric tensor. Using equation (5.37) in (5.36), we get

$$R'(X,Y,U,V) = F(Y,U)F(X,V) - F(X,U)F(Y,V).$$

Thus, the manifold is a $\psi(F)_n$. i.e.,

**Proposition 5.1** *A conformally flat $(WZS)_n$ is a $\psi(F)_n$.*

**Definition 5.1** *Consider a hypersurface $(\bar{M}^{n-1}, g)$ of a conformally flat $(WZS)_n$ whose curvature tensor is denoted by $\bar{R}$. Then, for any vector field $X, Y, U, V \in \chi(\bar{M}^{n-1})$, we have (Yano and Kon, 1984)*

$$\begin{aligned} g(R(X,Y)U,V) &= g(\bar{R}(X,Y)U,V) - g(B(X,V),B(Y,U)) \\ &+ g(B(Y,V),B(X,U)), \end{aligned} \tag{5.38}$$

*where $B$ is the second fundamental form of $\bar{M}$. If*

$$B(X,Y) = \tau g(X,Y), \tag{5.39}$$

*where $\tau$ is the mean curvature of $M$, then $M$ is totally umbilical.*

In a conformally flat $(WZS)_n$,

$$\begin{aligned} g(R(X,Y)U,V) &= l\Big[g(Y,U)g(X,V) - g(X,U)g(Y,V)\Big] \\ &+ m\Big[g(X,V)T(Y)T(U) + g(Y,U)T(X)T(V) \\ &- g(X,U)T(Y)T(V) - g(Y,V)T(X)T(U)\Big], \end{aligned} \tag{5.40}$$

where $l = \dfrac{2a}{(n-2)} - \dfrac{r}{(n-1)(n-2)}$, $m = \dfrac{b}{(n-2)}$ and $T(X) = \dfrac{E(X)}{\sqrt{E(\rho)}}$.

From equations (5.38) and (5.40), we have

$$
\begin{aligned}
g(\bar{R}(X,Y)U,V) = l\Big[g(Y,U)g(X,V) - g(X,U)g(Y,V)\Big]\\
+m\Big[g(X,V)T(Y)T(U) + g(Y,U)T(X)T(V)\\
-g(X,U)T(Y)T(V) - g(Y,V)T(X)T(U)\Big]\\
+g(B(X,V),B(Y,U)) - g(B(Y,V),B(X,U)).
\end{aligned}
$$

By hypothesis $\bar{M}$ is totally umbilical, so the above equation becomes

$$
\begin{aligned}
g(\bar{R}(X,Y)U,V) &= (l+\mid \tau \mid^2)\Big[g(Y,U)g(X,V) - g(X,U)g(Y,V)\Big]\\
&+ m\Big[g(X,V)T(Y)T(U) + g(Y,U)T(X)T(V)\\
&- g(X,U)T(Y)T(V) - g(Y,V)T(X)T(U)\Big],
\end{aligned}
$$

or

$$
\begin{aligned}
g(\bar{R}(X,Y)U,V) &= p\Big[g(Y,U)g(X,V) - g(X,U)g(Y,V)\Big]\\
&+ q\Big[g(X,V)E(Y)E(U) + g(Y,U)E(X)E(V)\\
&- g(X,U)E(Y)E(V) - g(Y,V)E(X)E(U)\Big], \qquad (5.41)
\end{aligned}
$$

where $p = l+\mid \tau \mid^2$ and $q = m$ and assuming that $\rho$ is a unit vector field. Since $b \neq 0$, i. e., $q \neq 0$, we can state:

**Theorem 5.6** *A totally umbilical hypersurface of a conformally flat $(WZS)_n$ is of quasi constant curvature.*

## 5.3 Examples of $(WZS)_n$

### Example 1:

Consider $\mathbb{R}^3$ with the Riemannian metric $g$ given by

$$ds^2 = g_{ij}dx^i dx^j = e^{x^1}x^3(dx^1)^2 + (dx^2)^2 + (dx^3)^2,$$

which gives the non-vanishing components of the Lorentzian metric and its associated components as

$$g_{11} = e^{x^1}x^3, \quad g_{22} = g_{33} = 1,$$
$$g^{11} = \frac{1}{e^{x^1}x^3}, \quad g^{22} = g^{33} = 1.$$

Then, the non-zero components of the Christoffel's symbols, the curvature tensors and the Ricci tensors are

$$\Gamma^1_{11} = \frac{1}{2}, \quad \Gamma^3_{11} = -\frac{1}{2e^{x^1}}, \quad \Gamma^1_{13} = \frac{1}{2x^3},$$
$$R_{1331} = -\frac{1}{4x^3}e^{x^1},$$

$$S_{11} = \frac{1}{4x^3}e^{x^1}, \quad S_{33} = \frac{1}{4(x^3)^2}.$$

Using

$$r = g^{ij}S_{ij}, \tag{5.42}$$

we get $r = \frac{1}{2(x^3)^2}$ which is non-vanishing and non-constant.

Take the function $\phi = \frac{1}{4(x^3)^2}$ such that

$$Z_{11} = \frac{1}{2x^3}e^{x^1}, \quad Z_{33} = \frac{1}{2(x^3)^2},$$

$$Z_{11,1} = \frac{1}{2x^3}e^{x^1}, \quad Z_{11,3} = -\frac{1}{2(x^3)^2}e^{x^1}, \quad Z_{33,3} = -\frac{1}{(x^3)^3}.$$

Also, we choose the 1-forms as

$$A_i(x) = \begin{cases} -\frac{1}{x^3}, & i = 3 \\ 0, & otherwise, \end{cases}$$

$$B_i(x) = \begin{cases} -\frac{2}{x^3}, & i = 3 \\ 0, & otherwise, \end{cases}$$

$$D_i(x) = \begin{cases} \frac{1}{x^3}, & i = 3 \\ 0, & otherwise, \end{cases}$$

at any point $x$ in $\mathbb{R}^3$. From equation (5.2), we have

$$Z_{11,1} = A_1 Z_{11} + B_1 Z_{11} + D_1 Z_{11}, \tag{5.43}$$

$$Z_{11,3} = A_3 Z_{11} + B_1 Z_{13} + D_1 Z_{13}, \tag{5.44}$$

$$Z_{33,3} = A_3 Z_{33} + B_3 Z_{33} + D_3 Z_{33}, \tag{5.45}$$

and all others hold trivially.

Now,

$$\begin{aligned} \text{R. H. S of (5.44)} &= A_3 Z_{11} + B_1 Z_{13} + D_1 Z_{13} \\ &= \Big(-\frac{1}{x^3}\Big).\Big(-\frac{1}{2x^3}e^{x^1}\Big) \\ &= -\frac{1}{2(x^3)^2}e^{x^1} = Z_{11,3} \\ &= \text{L.H.S of (5.44)}, \end{aligned}$$

and,

$$\begin{aligned}
\text{R. H. S of (5.45)} &= A_3Z_{33} + B_3Z_{33} + D_3Z_{33} \\
&= \left(-\frac{1}{x^3} - \frac{2}{x^3} + \frac{1}{x^3}\right)\frac{1}{2(x^3)^2} \\
&= -\frac{1}{(x^3)^3} = Z_{33,3} \\
&= \text{L.H.S of (5.45)}
\end{aligned}$$

and (5.43) holds. So, $(\mathbb{R}^3, g)$ is a $(WZS)_n$.

## Example 2:

Consider the Riemannian metric $g$ in $\mathbb{R}^4$ defined by

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^{\frac{4}{3}}[(dx^2)^2 + (dx^3)^2 + (dx^4)^2],$$

which gives the non-zero components of the Lorentzian metric and its associated components as

$$g_{11} = 1, \quad g_{22} = g_{33} = g_{44} = (x^1)^{\frac{4}{3}},$$

$$g^{11} = 1, \quad g^{22} = g^{33} = g^{44} = \frac{1}{(x^1)^{\frac{4}{3}}}.$$

Then, the non-vanishing components of the Christoffel's symbols and the curvature tensors are

$$\Gamma^4_{14} = \Gamma^3_{13} = \Gamma^2_{12} = \frac{2}{3x^1}, \qquad \Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = -\frac{2}{3}(x^1)^{\frac{1}{3}},$$

$$R_{1221} = R_{1331} = R_{1441} = -\frac{2}{9(x^1)^{\frac{2}{3}}}, \qquad R_{2332} = R_{3443} = R_{2442} = \frac{4}{9}(x^1)^{\frac{2}{3}}.$$

From these, we get the non-zero components of the Ricci tensors and their covariant derivatives as

$$S_{11} = \frac{2}{3(x^1)^2}, \quad S_{22} = S_{33} = S_{44} = -\frac{2}{3(x^1)^{\frac{2}{3}}}.$$

Using (5.42), we get $r = -\frac{4}{3(x^1)^2}$ which is non-vanishing and non-constant. Take the function $\phi = -\frac{1}{(x^1)^2}$. Then, using (1.20), we obtain

$$Z_{11} = \frac{1}{3(x^1)^2}, \quad Z_{22} = Z_{33} = Z_{44} = -\frac{5}{3(x^1)^{\frac{2}{3}}},$$

$$Z_{11,1} = -\frac{2}{3(x^1)^3}, \quad Z_{22,1} = Z_{33,1} = Z_{44,1} = \frac{10}{9(x^1)^{\frac{5}{3}}}.$$

Also, we choose the 1-forms

$$A_i(x) = \begin{cases} -\frac{2}{3x^1}, & i = 1 \\ 0, & otherwise, \end{cases}$$

$$B_i(x) = \begin{cases} -\frac{1}{3x^1}, & i = 1 \\ 0, & otherwise, \end{cases}$$

$$D_i(x) = \begin{cases} -\frac{1}{x^1}, & i = 1 \\ 0, & otherwise, \end{cases}$$

at any point $x$ in $\mathbb{R}^3$. From (5.2), we have

$$Z_{11,1} = A_1 Z_{11} + B_1 Z_{11} + D_1 Z_{11}, \tag{5.46}$$

$$Z_{22,1} = A_1 Z_{22} + B_2 Z_{12} + D_2 Z_{12}, \tag{5.47}$$

$$Z_{33,1} = A_3 Z_{33} + B_3 Z_{13} + D_3 Z_{13}, \tag{5.48}$$

$$Z_{44,1} = A_1 Z_{44} + B_4 Z_{14} + D_4 Z_{14} \tag{5.49}$$

and all others hold trivially.

Therefore,

$$\begin{aligned}
\text{R. H. S of (5.46)} &= A_1Z_{11} + B_1Z_{11} + D_1Z_{11} \\
&= \Big( -\frac{2}{3} - \frac{1}{3} - 1\Big)\frac{1}{3(x^1)^2} \\
&= -\frac{2}{3(x^1)^2} = Z_{11,1} \\
&= \text{L.H.S of (5.46)},
\end{aligned}$$

and,

$$\begin{aligned}
\text{R. H. S of (5.47)} &= A_1Z_{22} + B_2Z_{12} + D_2Z_{12} \\
&= -\frac{2}{3x^1}.\Big( -\frac{5}{3(x^1)^{\frac{2}{3}}}\Big) \\
&= \frac{10}{9(x^1)^{\frac{5}{3}}} = Z_{22,1} \\
&= \text{L.H.S of (5.47)}.
\end{aligned}$$

Similarly, equations (5.48) and (5.49) can be proved. So, $\mathbb{R}^4$ with the given metric is a $(WZS)_n$.

## Example 3:

Define a Riemannian metric $g$ on $\mathbb{R}^4$ as

$$ds^2 = g_{ij}dx^i dx^j = e^{x^4}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2.$$

Then, the non-vanishing components of the Lorentzian metric and its associated components are

$$g_{11} = g_{22} = g_{33} = e^{x^4}, \quad g_{44} = 1,$$

$$g^{11} = g^{22} = g^{33} = \frac{1}{e^{x^4}}, \quad g^{44} = 1.$$

Also, the non-zero components of the Christoffel's symbols, the curvature tensors and the Ricci tensors are

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{1}{2}, \quad \Gamma^4_{11} = \Gamma 4_{22} = \Gamma^4_{33} = -\frac{1}{2}e^{x^4},$$

$$R_{1441} = R_{2442} = R_{3443} = \frac{1}{4}e^{x^4}, \quad R_{1221} = R_{2332} = R_{1331} = \frac{1}{4e^{2x^4}}.$$

$$S_{11} = S_{22} = S_{33} = -\frac{3}{4e^{x^4}}, \quad S_{44} = -\frac{3}{4}.$$

From (5.42), we get $r = -3$ which is non-vanishing.

Let us take $\phi = 1$. Thus, the components of the $Z$ tensor and their derivatives are

$$Z_{11} = Z_{22} = Z_{33} = \frac{1}{4e^{x^4}}, \quad Z_{44} = \frac{1}{4}$$
$$Z_{11,4}, = Z_{22,4} = Z_{33,4} = \frac{1}{4e^{x^4}}.$$

Also, we choose the 1-forms as

$$A_i(x) = \begin{cases} 1, & i = 4 \\ 0, & otherwise, \end{cases}$$

$$B_i(x) = \begin{cases} 2, & i = 4 \\ 0, & otherwise, \end{cases}$$

$$D_i(x) = \begin{cases} -3, & i = 4 \\ 0, & otherwise, \end{cases}$$

at any point $x$ in $\mathbb{R}^4$. From (5.2), we have

$$Z_{11,4} = A_4 Z_{11} + B_1 Z_{14} + D_1 Z_{14}, \tag{5.50}$$

$$Z_{22,4} = A_4 Z_{22} + B_2 Z_{42} + D_2 Z_{42}, \tag{5.51}$$

$$Z_{33,4} = A_4 Z_{33} + B_3 Z_{43} + D_3 Z_{43}, \tag{5.52}$$

$$Z_{44,1} = A_1 Z_{44} + B_4 Z_{14} + D_4 Z_{14} \tag{5.53}$$

and all others hold trivially. Now,

$$\begin{aligned}
\text{R. H. S of (5.50)} &= A_4Z_{11} + B_1Z_{14} + D_1Z_{14} \\
&= 1.\frac{1}{4e^{x^4}} \\
&= \frac{1}{4e^{x^4}} = Z_{11,4} \\
&= \text{L.H.S of (5.50)},
\end{aligned}$$

and,

$$\begin{aligned}
\text{R. H. S of (5.53)} &= A_1Z_{44} + B_4Z_{14} + D_4Z_{14} \\
&= (1 + 2 - 3)\frac{1}{4} \\
&= 0 = Z_{44,4} \\
&= \text{L.H.S of (5.53)}.
\end{aligned}$$

Similarly, we can show that (5.51) and (5.52) holds. So, $(\mathbb{R}^4, g)$ is a $(WZS)_n$.

## 5.4 Decomposable $(WZS)_n$

Consider a decomposable manifold $M = M_1^p \times M_2^{n-p}$ for $2 \leq p \leq n-p$, i. e., in some coordinate neighbourhood of the manifold,

$$ds^2 = g_{ij}dx^idx^j = \bar{g}_{ab}dx^adx^b + g^*_{\alpha\beta}dx^\alpha dx^\beta,$$

where $\bar{g}_{ab}$ are functions of $\bar{x} = x^1, x^2, ...., x^p$ and $g^*_{\alpha\beta}$ are functions of $x^* = x^{p+1}, x^{p+2}, ...., x^n$, $a, b$ runs from 1 to $p$ and $\alpha, \beta$ runs from $p+1$ to $n$. Here, $M_1^p$ and $M_2^{n-p}$ are called the components of $M^n$.

Let $\bar{X}, \bar{Y}, \bar{Z}, \bar{U}, \bar{V} \in \chi(M_1)$, $X^*, Y^*, Z^*, U^*, V^* \in \chi(M_2)$. Since the manifold is decomposable, we have (Yano and Kon, 1984)

$$S(\bar{X}, \bar{Y}) = \bar{S}(\bar{X}, \bar{Y}),$$

$$S(X^*, Y^*) = S^*(X^*, Y^*),$$

$$(\nabla_{\bar{X}} S)(\bar{Y}, \bar{Z}) = (\bar{\nabla}_{\bar{X}} S)(\bar{Y}, \bar{Z}),$$

$$(\nabla_{X^*} S)(Y^*, Z^*) = (\nabla^*_{X^*} S)(Y^*, Z^*)$$

and $r = \bar{r} + r^*$.

Now, using (1.20) and (5.2), we have

$$\begin{aligned}(\nabla_X S)(U,V) + (X\phi)g(U,V) &= A(X)\Big[S(U,V) + \phi g(U,V)\Big] \\ &+ B(U)\Big[S(X,V) + \phi g(X,V)\Big] \\ &+ D(V)\Big[S(U,X) + \phi g(U,X)\Big].\end{aligned}$$

Then, we have

$$\begin{aligned}(\nabla^*_X S)(\bar{U},\bar{V}) + (X^*\phi)g(\bar{U},\bar{V}) &= A(X^*)\Big[S(\bar{U},\bar{V}) + \phi g(\bar{U},\bar{V})\Big] \\ &+ B(\bar{U})\Big[S(X^*,\bar{V}) + \phi g(X^*,\bar{V})\Big] \\ &+ D(\bar{V})\Big[S(\bar{U},X^*) + \phi g(\bar{U},X^*)\Big].\end{aligned}$$

Suppose $\phi$ is constant in $M_2$, then the above equation becomes

$$A(X^*)\Big[S(\bar{U},\bar{V}) + \phi g(\bar{U},\bar{V})\Big] = 0,$$

which implies that $A = 0$ in $M_2$ or $M_1$ is Einstein. Similarly, we can show that if $\phi$ is constant in $M_1$, then $A = 0$ in $M_1$ or $M_2$ is Einstein. This leads to the theorem:

**Theorem 5.7** *In a decomposable $(WZS)_n$ $(M^n, g)$, where $M^n = M_1^p \times M_2^{n-p}, 2 \leq p \leq n-2$, if $\phi$ is constant in $M_2$, then $A = 0$ in $M_2$ or $M_1$ is Einstein. Similarly, if $\phi$ is constant in $M_1$, then either $A = 0$ in $M_1$ or $M_2$ is Einstein.*

# Chapter 6

# Summary and Conclusion

In the present thesis, we studied the structures of some almost contact manifolds. The following objectives are taken up in the study:

1. To study properties of metric/non metric connection.

2. To establish the geometrical properties of semi-generalized recurrent almost contact manifolds.

3. To study inter-relations and applications of certain curvature conditions on Quasi-Einstein manifolds.

4. To characterize weak symmetry of $Z$ tensor in almost contact manifolds.

In Chapter 1, we give the general introduction of the study which includes the basic definitions and formulae of differential geometry such as topological space, differentiable manifolds, tangent vector, tangent space, vector field, Lie bracket, Lie derivative, connection, covariant derivative, contraction, Riemannian manifold, Riemannian connection, torsion tensor, semi-symmetric and quarter symmetric connection, different curvature tensors, almost contact metric manifolds, almost contact para-contact metric manifolds, recurrent manifolds and symmetric manifolds. Some methods used for solving problems and the review of literature are also included in this chapter.

In Chapter 2, we study weak symmetries of Kenmotsu and Para-Sasakian manifolds admitting a semi-symmetric metric connection. Weakly symmetric and weakly Ricci symmetric Kenmotsu manifolds with respect to a semi-symmetric metric connection have been studied. We obtained the sum of the associated 1-forms in weakly concircular and weakly concircular Ricci symmetric Kenmotsu manifold admitting a semi-symmetric metric connection. A necessary and sufficient condition for the Ricci tensor $\tilde{S}$ in a weakly $m$-projectively symmetric Kenmotsu manifold with respect to the semi-symmetric metric connection to be of Codazzi type is obtained. Also, a necessary condition for a Para-Sasakian manifold to be weakly symmetric and weakly Ricci symmetric with respect to a semi-symmetric metric connection are obtained. Lastly, we construct an example of a 3-dimensional weakly symmetric and weakly Ricci symmetric Para-Sasakian manifold admitting a semi-symmetric metric connection.

In Chapter 3, we investigated semi-generalized $W_3$ recurrent manifolds. A necessary and sufficient condition for the scalar curvature to be constant in such a manifold is obtained. Also, we proved that a semi-generalized $W_3$ recurrent manifold with constant scalar curvature is semi-generalized Ricci recurrent. Later, we showed that a Ricci symmetric semi-generalized $W_3$ recurrent manifold is an Einstein manifold. A sufficient condition for a semi-generalized $W_3$ recurrent manifold to be quasi-Einstein is obtained. Decomposable semi-generalized $W_3$ recurrent manifolds are studied. Finally, we have given two examples of a semi-generalized $W_3$ recurrent manifold.

In Chapter 4, some curvature properties of $N(k)$-quasi einstein manifolds are studied. We proved that the associated 1-form $\eta$ in an $n(n > 3)$-dimensional $N(k)$-quasi Einstein manifold which is $W^*$-conservative and $b$ is non-zero constant is closed and the integral curves of the generator $\zeta$ are geodesics. Also, we proved that an $n$-dimensional $W^*$-Ricci pseudosymmetric $N(k)$-quasi Einstein manifold satisfies the relation $L_S = \frac{b}{2(n-1)}$. A sufficient condition for an $N(k)$-quasi Einstein manifold to be $W_2$-pseudosymmetric is obtained. $Z$-generalized pseudosymmetric $N(k)$-quasi Einstein manifolds are studied. We considered the properties of the pseudo projective, $W_2$ and conharmonic curvature tensors in an $N(k)$-quasi Einstein manifold. We studied pseudo projectively symmetric $N(k)$-quasi Einstein manifolds and showed that there does not exist a pseudo projectively semi-symmetric $N(k)$-quasi Einstein manifold. Lastly, we constructed examples to support our results.

In Chapter 5, we studied weakly $Z$-symmetric manifolds. We showed that in a weakly $Z$-symmetric manifolds with Codazzi type $Z$ tensor, the manifold is quasi Einstein provided that the vector field $\rho$ defined by

$$E(X) = A(X) - D(X) = g(X, \rho), \quad \rho = \rho_1 - \rho_3,$$

is a unit vector field. Einstein weakly $Z$-symmetric manifolds and conformally flat weakly $Z$-symmetric manifolds are studied. A necessary condition for the $Z$ tensor in a weakly $Z$-symmetric manifolds to be cyclic parallel is obtained. Also, we showed that a totally umbilical hypersurface of a conformally flat weakly $Z$-symmetric manifolds is of quasi constant curvature. Decomposable weakly $Z$-symmetric manifolds are studied and some examples are constructed to support the existence of such manifolds.

Finally, we concluded that the whole work of this thesis give some geometrical properties and structures of almost contact manifolds with respect to semi-symmetric metric connection, semi-generalized recurrent properties in almost contact manifolds, curvature conditions in $N(k)$-quasi Einstein manifolds and weak symmetries of the $Z$-tensor in almost contact manifolds.

## Appendices

## (A) LIST OF RESEARCH PUBLICATIONS

(1) J. P. Singh and **K. Lalnunsiami** (2019). Some results on weakly symmetric Kenmotsu manifolds, *Science and Technology Journal*, **7(1)**, 13-21

(2) J. P. Singh and **K. Lalnunsiami** (2020). Certain curvature properties of $N(k)$-quasi Einstein manifolds, *SUT Journal of Mathematics*, **56(1)**, 55-69.

(3) **K. Lalnunsiami** and J. P. Singh (2020). On a type of $N(k)$-quasi Einstein manifolds, *Bulletin of the Transilvania University of Braşov Series III: Mathematics, Informatics, Physics*, **13(62), No. 1**, 219-236.

(4) **K. Lalnunsiami** and J. P. Singh (2020). On some classes of weakly $Z$-symmetric manifolds, *Communications of the Korean Mathematical Society*, **35(2)**, 935-951.

(5) **K. Lalnunsiami** and J. P. Singh (2021). On semi-generalized $W_3$ recurrent manifolds, *Ganita*, **71(1)**, 275-289.

## (B) CONFERENCES/ SEMINARS/ WORKSHOPS

(1) Presented a paper "On Weak Symmetries of Para-Sasakian manifolds admitting a semi-symmetric metric connection" in Multidisciplinary International Seminar On "A perspective of Global Research Process: Presented Scenario and Future Challenges" organized by Manipur University, Manipur, on $19^{th}$ and $20^{th}$ January, 2019.

(2) Attended "National Workshop On Ethics in Research and Preventing Plagiarism (ERPP 2019)" organized by Department of Physics, School of Physical Sciences, Mizoram University, Aizawl, Mizoram, on $3^{rd}$ October, 2019.

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(5) Presented a paper "On Some Classes of weakly $Z$-symmetric Manifolds" in "2nd Annual Convention of North East (India) Academy of Science and Technology (NEAST) and International Seminar on Recent Advances in Science and Technology (IRSRAST)" organized by North East (India) Academy of Science and Technology (NEAST), Mizoram University, Aizawl, Mizoram, on $16^{th}$-$18^{th}$ November, 2020.

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# ABSTRACT

# A STUDY OF CERTAIN STRUCTURES ON ALMOST CONTACT MANIFOLDS

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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## ABSTRACT

The study of geometric structures on smooth manifolds has broad applications in classical mechanics, thermodynamics, geometric quantization, integrable systems and to control theory. Contact geometry is an important tool to study systems of differential equations. Focusing on the study of almost contact manifolds, the following objectives are taken up in the thesis:

1. To study properties of metric/non metric connection.
2. To establish the geometrical properties of semi-generalized recurrent almost contact manifolds.
3. To study inter-relations and applications of certain curvature conditions on quasi-Einstein manifolds.
4. To characterize weak symmetry of $Z$ tensor in almost contact manifolds.

Chapter 1 is the General Introduction of the problems which includes basic definitions, topological space, differentiable manifolds, tangent vector, tangent space, vector field, Lie bracket, Lie derivative, connection, covariant derivative, contraction, Riemannian manifold, Riemannian connection, torsion tensor, semi-symmetric and quarter symmetric connection, different curvature tensors, almost contact metric manifolds, almost contact para-contact metric manifolds, recurrent manifolds, symmetric manifolds and review of literatures.

In Chapter 2, we studied semi-symmetric metric connection in weakly symmetric almost contact manifolds. Weakly symmetric, weakly Ricci symmetric, weakly concircular symmetric, weakly concircular Ricci symmetric and weakly m-projectively symmetric Kenmotsu manifolds with respect to such a connection are considered. We investigated the properties of weakly symmetric and weakly Ricci symmetric Para-Sasakian manifolds admitting a semi-symmetric metric connection. Lastly, we give an example of a Para-Sasakian manifold which is weakly symmetric and weakly Ricci symmetric with respect to such a connection.

In Chapter 3, we investigated semi-generalized $W_3$ recurrent manifolds. A necessary and sufficient condition for the scalar curvature to be constant in such a manifold is obtained. Ricci symmetric and decomposable semi-generalized $W_3$ recurrent manifolds are studied. We obtained a sufficient condition for a semi-generalized $W_3$ recurrent manifold to be quasi Einstein. We studied decomposable semi-generalized $W_3$ recurrent manifolds and constructed examples to support our results.

In Chapter 4, some curvature properties of $N(k)$-quasi Einstein manifolds are studied. The nature of the associated 1-form in a $W^*$-conservative $N(k)$-quasi Einstein manifold with constant associated scalar $b$ is investigated. We studied $W^*$-Ricci pseudosymmetric, $W_2$-pseudosymmetric and $Z$-generalized pseudosymmetric $N(k)$-quasi Einstein manifolds. The curvature properties of the pseudo projective, $W_2$ and conharmonic curvature tensors in an $N(k)$-quasi Einstein manifold are considered. Finally, we give examples of $N(k)$-quasi Einstein manifolds.

In Chapter 5, we studied weakly $Z$-symmetric manifolds. We investigated weakly $Z$-symmetric manifolds with Codazzi type and cyclic parallel $Z$ tensor, Einstein weakly $Z$-symmetric manifolds and conformally flat weakly $Z$-symmetric manifolds. A totally umbilical hypersurface of a conformally flat weakly $Z$-symmetric manifold is considered. Decomposable weakly $Z$-symmetric manifolds are studied and examples are given to support our results.

Chapter 6 is on the summary and conclusion.