A STUDY OF ELASTIC WAVE IN DIFFERENT THERMOELASTIC MATERIALS

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A STUDY OF ELASTIC WAVE IN DIFFERENT THERMOELASTIC MATERIALS

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Submitted

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CERTIFICATE

This is to certify that the thesis entitled "A Study of Elastic Wave in Different Thermoelastic Materials" submitted by Mr. Lalawmpuia for the award of the degree of Doctor of Philosophy (Ph. D.) in Mathematics, is a bonafide record of the original research carried out by him under my supervision. He has been duly registered and the thesis is worthy of being considered for the award of the Ph. D. degree.

I, the undersigned has declared that this research work has been done under my supervision and has not been submitted for any degree of any other universities.

(Dr. S. SARAT SINGH)
Supervisor

DECLARATION

Mizoram University

November, 2021

I, **Lalawmpuia**, hereby declare that the subject matter of this thesis is the record of work done by me, that the contents of this thesis do not form basis of the award of any previous degree to me or to do the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in other University/Institute.

This is being submitted to the Mizoram University for the degree of Doctor of Philosophy in Mathematics.

(LALAWMPUIA)

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Dated: ....

Place: Aizawl (LALAWMPUIA)

PREFACE

The present thesis entitled "**A Study of Elastic Wave in Different Thermoelastic Materials**" is an outcome of the research carried out by me under the supervision of Dr. S. Sarat Singh, Department of Mathematics & Computer Science, Mizoram University, Aizawl - 796 004, Mizoram, INDIA.

This thesis consists of problem related with the propagation of elastic waves in different thermoelastic materials. The amplitude and energy ratios of body waves and the dispersion relations of surface waves are derived with the help of appropriate boundary conditions of the materials. It consists of six chapters. The first chapter is the introduction of the thesis which includes the basic definitions, elastic waves, thermoelasticity and theories, application of wave propagation and review of literature.

Second chapter deals with the propagation of surface waves (Stoneley and Rayleigh waves) in thermoelastic materials with voids. The frequency equations of the Stoneley waves at the bonded and unbonded interfaces between two dissimilar half-spaces of thermoelastic materials with voids were obtained. The numerical values of the determinant for bonded and unbonded interfaces are calculated for a particular model. We also derived the frequency equation of Rayleigh wave in thermoelastic materials with voids. The phase velocity and attenuation coefficients have shown that there are two modes of vibration. These two modes are computed and they are depicted graphically. The effect of thermal parameters on these surface waves are also discussed.

Third chapter studied the reflection/transmission of elastic waves in initially stressed transversely isotropic thermoelastic materials. Three quasi type coupled longitudinal(QL), transverse(QT) and thermal waves were found to propagate in initially stressed transversely isotropic thermoelastic materials. For incident QL and QT-waves at a plane interface, boundary conditions were implemented for obtaining the coefficients of reflection/transmission, the distribution of energy in the reflected and transmitted waves are also discussed. We have observed that the results vary with direction of incidence as well as the parameters due to elasticity, thermal and initial stresses. Numerical computations have been performed and analyzed the impact of initial stresses on the results. We have observed critical angles at $\theta_0 = 30^0$ and 58^0 for the reflected and transmitted QL-waves for incident QT-wave.

In Chapter 4, we have investigated 'how do Rayleigh waves propagate on the surface of heat conducting saturated porous materials?'. There exist three couple longitudinal and a shear wave in such materials. The phase velocities of these body waves are obtained and used for calculating the numerical results of phase velocities of the Rayleigh type waves and attenuation coefficients. The dispersion relation for Rayleigh type waves is obtained with the help of boundary conditions. We have observed that two modes of Rayleigh waves, i.e., Type - I and II exist in the thermoelastic saturated porous medium. The phase velocity and attenuation coefficients of these waves are computed to verify the model. We have presented the results through velocity curves. We have observed that the velocities depend upon the porosity, voids, elastic and thermal parameters of the materials.

Chapter 5 discuss the problem of reflection/transmission of elastic waves in incompresssible transversely isotropic thermoelastic materials. Due to the incompressibility condition, two coupled quasi-shear waves are found to propagate in such materials. At the plane interface, appropriate boundary conditions have been implemented to obtain the amplitude ratios of the reflected and transmitted quasi-shear waves. It has been observed that these ratios are functions of angle of incidence, elastic and thermal parameters of the material. To analyze the effect of thermal expansion and specific heat of the material, we have depicted the results graphically.

Chapter 6 is summary and conclusions of the thesis.

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Chapter 1

General Introduction

1.1 Basic definition

Let $\boldsymbol{R} = (X_1, X_2, X_3)$ and $\boldsymbol{r} = (x_1, x_2, x_3)$ be the position vector of a particle in the undeformed body V_0 at time t_0 and deformed body V at time $t_0 + t$ respectively. This vector $\boldsymbol{R}$ refers particles of the body, while vector $\boldsymbol{r}$ describes the motion of the particles. Thus

$$\mathbf{r} = \mathbf{r}(\mathbf{R}, t), \tag{1.1}$$

represents the deformation (motion) of all the particles in V_0 as shown in Figure 1.1. In component form, we can write

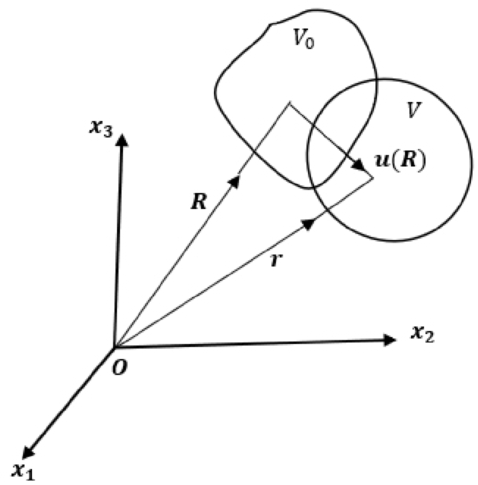

Figure 1.1: Deformation of a body.

$$r_i = x_i = x_i(X_1, X_2, X_3, t), \qquad i = 1, 2, 3. \tag{1.2}$$

It is possible to write **R** in terms of **r** (See Pujol, 2003)

$$\mathbf{R} = \mathbf{R}(\mathbf{r}, t) \tag{1.3}$$

if the Jacobian is different from zero, i.e.,

$$J = \frac{\partial\,(x_1, x_2, x_3)}{\partial\,(X_1, X_2, X_3)} \neq 0.$$

Equations (1.1) and (1.3) corresponds to the *Lagrangian* and *Eulerian* description of motion respectively. This deformation is caused due to an external force, which is either a *body force* that acts at a distance within a body or between bodies or a *surface force* which acts upon a surface element of the body, regardless of whether that element is a part of the boundary surface or an arbitrary element of surface within the body.

If the distance between two neighborhood particles of a body is negligibly small in comparison to its dimension, then the body is called *continuum body.* When an elastic continuum body which is in equilibrium is subjected to some external forces, then the body undergoes deformation. In this process, the particles act against internal resistive forces of the material, that exist even in the absence of external forces. The force per unit area set up inside the body to resist the deformation is called *stress.*

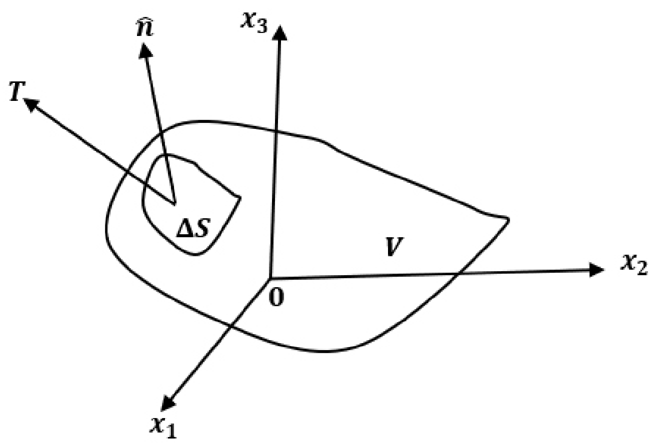

Figure 1.2: Stress vector.

Consider a surface element ΔS, inside or on the boundary of a material having volume V as shown in Figure 1.2. Let the force acting on the surface element ΔS be denoted by $\mathbf{F}$. The stress vector $\mathbf{T}$ which represents the surface force per unit area acting at the point (x_i) across the surface element with outward unit normal vector $\hat{\mathbf{n}}$ is given by (Sokolnikoff, 1946)

$$\mathbf{T}(x_i; \hat{\mathbf{n}}) = \lim_{\Delta S \to 0} \frac{\mathbf{F}}{\Delta S}. \tag{1.4}$$

If $\mathbf{T}^n$ is the stress vector acting at a point of surface to which $\hat{\mathbf{n}}$ is normal, then

$$\mathbf{T}^n = \tau_{ij} n_j,$$

where τ_{ij} is a stress tensor and it represents the j^{th} component of the stress vector acting across a plane to which x_i-axis is normal.
The deformation of the body accompanying stress is called *strain.* The deformation is called *dilatation* when the strain set up in the body is such that there is a change in volume of the body without change in its shape. But the deformation is called *shear* if the strain setup in the body is such that the shape of the body changes without any change in its volume. Thus, stress and strain occur simultaneously. The relationship between stress and strain for a deformable body is given by the generalized *Hooke's law* which states that stress and strain are linearly related. The tensor form of this law is given as

$$\tau_{ij} = c_{ijkl} e_{kl}, \qquad (i,\ j,\ k,\ l = 1,\ 2,\ 3) \tag{1.5}$$

where c_{ijkl} is the elastic constant and also known as stiffness tensor. These are 81 which reduce to 54 due to symmetric nature of stress tensor and then reduce to 36 due to the symmetry of strain tensor. The existence of strain energy function give further reduction of these constants to 21 independent elastic constants. Such an elastic body is called *anisotropic material.* For a monoclinic anisotropic materials, there are 13 independent elastic constants. The number of independent elastic constants in orthotropic materials and transversely isotropic materials are 9 and 5 respectively.

1.2 Elastic waves

Wave is a disturbance or variation that carries energy from one point of the medium to another without actual transfer of particles. It may take the form of elastic deformation, change in temperature, pressure, electric potential, electric or magnetic intensity. The *amplitude* of a wave is the maximum displacement of any particle of the medium from its equilibrium position. The time taken by any particle of the medium to complete one vibration is called *period* of a wave. *Wavelength* is equal to the distance between two consecutive particles of the medium which are in the same state of vibration. It is also equal to the distance traveled by a wave by its period. *Frequency* of a wave is the number of vibration made per second by any particles of the medium. *Phase* or *phase angle* and *phase difference* represent the state of vibration of the particle of a medium with respect to its mean position and the different state of vibration of a particle at two different instants respectively. *Path difference* indicates the distance between two points measured along the direction of propagation of the wave through the medium, *time difference* is the time taken by the wave to travel from one point of the medium to another.

The consequence of a rapid disturbance in an elastic material is transmitted from one part of the body to another parts of the body. The remote parts of the body may not be effected by the disturbance produced at a point as soon as it is produced. This disturbance propagates through the body in the form of waves which are called *elastic waves*. The elastic waves are based on the principle of restoring forces acting on the particles of the medium, when the material is deformed by some external force. Those waves found in the earth's crust due to earthquakes are known as *seismic waves*. When an elastic wave propagates in the material, the energy associated with the deformation of the material gets transferred in the absence of a flow of matter. The properties of elastic waves depend on the properties of the material in which they propagate. Elastic waves are mainly divided into two types: *Body waves* and *Surface waves.*

Elastic waves which propagate through the interior of an elastic body are known as *body waves.* These waves are classified into *Primary waves (P-waves)* and *Secondary waves (S-waves)* according to their modes of propagation. The primary waves are compressional waves which are longitudinal in nature, they are associated with pushing or pulling of the particles along the direction of the energy. These waves are able to travel through solid as well as liquid materials and can travel through the material with the greatest velocity. The secondary waves are shear/shaking waves that are transverse in nature shearing the particles of the material along the direction perpendicular to the direction of the wave. These waves have different effects on the surface of the material depending on their polarization and direction of propagation. Horizontally polarized S-waves known as *shear horizontal (SH)* waves move the material from side to side relative to the direction of propagation of the waves. Vertically polarized S-waves known as *shear vertical (SV)* waves move the particle of the material up and down relative to the direction of propagation. The velocity of the body wave depends not only on the elastic property of the medium but also on the density of the medium. Since it is not possible to shear or twist a liquid, the secondary waves can travel only through solids and their speeds of propagation are slower than those of P-waves.

Surface waves are produced at the surface of material discontinuity in an elastic body. These waves are propagating along the direction parallel to the surface of discontinuity and the amplitudes of these type of waves decrease with the increase in the distance from the free surface. Surface waves are produced by energy carried by body waves incident at the free surface of the material. There are mainly three types of surface waves: *Rayleigh waves, Stoneley waves* and *Love waves.*

It was Rayleigh (1885) who first found the existence of an elastic wave in the vicinity of the free surface of a semi infinite, homogeneous and isotropic elastic solid.

The effect of these waves decrease rapidly with depth from the free surface and these waves are known as *Rayleigh waves*. These waves are the result of fusion of longitudinal and transverse waves and the particle motion in these waves are confined to the neighborhood of the free surface. It is found that the surface particles describe an elliptical path in the retrograde fashion and the maximum displacement parallel to the direction of transmission is about two third of that in the vertical direction.

Stoneley (1924) observed that Rayleigh type waves could be transmitted along the interface between two semi-infinite elastic solids having almost similar elastic properties in welded contact. These waves are known as generalized Rayleigh waves or *Stoneley waves*. Later on it was confirmed that generalized Rayleigh waves do to exist at the interface of a solid medium and a liquid medium and their phase speeds are lesser than that of regular Rayleigh waves.

Love waves are horizontally polarized surface waves which travel along the free surface of an elastic material. These type of waves are the result of interference of many S- waves guided by an elastic layer between elastic material and a stress free surface. Love wave is the fastest surface wave and causes horizontal shifting of the particles of an elastic material at the right angle to the direction of propagation.

1.3 Thermoelasticity

The theory of *thermoelasticity* deals with the influence of temperature of an elastic solid on the distribution of stress and strain, and the inverse effect of deformation on the distribution of temperature. Let δ represents the first variation of a function in terms of its variables, the first law of thermodynamics also known as Law of Conservation of Energy, states that within an unit volume of the system, the first variation of the heat absorbed Q is equal to the difference between the differential of internal energy U and the first variation of the work done on the system W.

Mathematically, it can be represented by (See, Hetnarski and Eslami, 2009)

$$\delta Q = dU - \delta W. \tag{1.6}$$

Here, δ-operation is assumed as path dependent, with the energy interactions between two end states depending upon the end states as well as the path of variations. On the other hand in d-operation, the variation of the function is with respect to all the variables involved in the function including time. When a system completes a cyclic process, Eq. (1.6) reduces to

$$\delta Q + \delta W = 0. \tag{1.7}$$

The thermodynamic process in a system is classified into *reversible* and *irreversible* process. A process is said to be *reversible* when the system can return to its starting state from the final state by following the same path of intermediate states from the initial state to the final state. Otherwise, the process is called *irreversible.* According to second law of thermodynamics, when a thermodynamic process is complete, the equality sign in the *Clausius inequality*

$$\oint \frac{\delta Q}{T} \leq 0, \tag{1.8}$$

refers to the reversible process. Where T is the absolute temperature. The left hand side of the above inequality introduces a thermodynamic property called *entropy.* The third law of thermodynamics states that for each system in equilibrium, the entropy becomes zero when the temperature approaches the absolute zero.

In general, the variation of the temperature field within an elastic continuum produced thermal stresses. The effect of temperature field in the governing equations of thermoelasticity is through the constitutive law. The theory of linear thermoelasticity is based on linear addition of thermal strains to mechanical strains. The equilibrium and compatibility equations remain the same as in the theory of elasticity and the main difference is in the constitutive law.

The paper on thermoelasticity by Duhamel(1837) derived the equations of motion which involved the coupling between temperature field and the deformation of the body. This theory of thermoelasticity is also known as *theory of uncoupled thermoelasticity*. This theory claims that the temperature of an elastic body is not effect by change in mechanical state of the body, which is not in accordance with the physical experiments. The limitation of this theory is that the equation which governs the temperature is in the form of parabola which indicate that even at an infinite distance from the heat source, the thermal signals of infinite speed and the thermal disturbances have impact. Biot (1956a) formulated the *theory of coupled thermoelasticity* considering the coupling equations of elasticity and heat conduction. This theory includes the theory of heat conduction, thermal stresses, and strains set up due to the flow of temperature in elastic bodies and the inverse effect of elastic deformation on the distribution of temperature which give rise to thermoelastic dissipation. This theory introduces the concept of thermoelastic potential which represents the elastic and thermoelastic properties of the material and the concept of thermal force which considers the generalized force as the product of temperature and virtual entropy displacement. Just as in the uncoupled theory, this coupled theory also shows infinite speed for the thermal signal. These two theories of thermoelasticity are also known as *classical theories*.

1.3.1 Lord-Shulman Model ($L - S$ model)

The thermoelastic model of Lord and Shulman (1967) used modify Fourier's law of heat conduction with the concept of a relaxation time. This relaxation time represents the time needed to accelerate the heat flow. The energy equations for an isotropic elastic material having thermal conduction are given as

$$\tau_{ij}e_{ij} + \rho T\dot{S} = \rho\dot{E}, \tag{1.9}$$

$$\rho T\dot{S} = -q_{i,i}, \tag{1.10}$$

where τ_{ij} and e_{ij} represent stress and strain tensor respectively of an elastic material having mass density ρ, change in temperature T, entropy density S, internal energy E and heat flux vector q_i. The superposed dot represents the time derivative. The modified form of heat conduction law is given as

$$-kT_{,i} = q_i + \tau_0 \dot{q}_i, \tag{1.11}$$

where k and τ_0 represent the coefficient of heat conduction and relaxation time respectively.

The governing equations for a homogeneous isotropic heat conducting elastic material are given by

$$\begin{aligned} (\lambda + 2\mu)\, u_{i,ij} + \mu u_{i,ij} - (3\lambda + 2\mu)\, \alpha T_{,i} &= \rho \ddot{u}_i, & (1.12) \\ \rho C_e \left(\dot{T} + \tau_0 \ddot{T}\right) + (3\lambda + 2\mu)\, \alpha T_0 \left(\dot{e}_{kk} + \tau_0 \ddot{e}_{kk}\right) &= kT, ii, & (1.13) \end{aligned}$$

where λ and μ are Lamé parameters, u_i is the displacement vector, C_e is the specific heat, T_0 is absolute temperature of the reference state and α represents the coefficient of linear thermal expansion.

1.3.2 Green-Lindsay Model ($G - L$ model)

This theory of thermoelasticity was introduced by Green and Lindsay(1972) using the concept of entropy production inequality of Green and Laws(1972). The stress tensor, heat flux vector and energy equation are given as

$$\begin{aligned} \tau_{ik} &= K_{ikrj} e_{rj} + a_{ik} T + b_{ik} \dot{T} + a_{ikr} T_{,r}, & (1.14) \\ q_i &= \frac{-T_0}{\alpha} \left[a_i T + \alpha b_i \dot{T} + a_{rs} e_{rs} + \alpha K_{ij} T_{,j} \right], & (1.15) \\ \rho_0 \eta &= \frac{1}{\alpha} \left[b + \left(e - \frac{b\beta}{\alpha} \right) T + \left(f - \frac{b\gamma}{\alpha} \right) \dot{T} - \alpha b_i T_{ji} - b_{ij} e_{ij} \right], & (1.16) \end{aligned}$$

where ρ_0 is the density on an elastic material having specific entropy η and $h\alpha = f - \frac{b\gamma}{\alpha}$. The entropy inequality leads to the following restrictions in the coefficients of the above equations as

$$\begin{aligned} a_i &= 0; \qquad a_{ijk} = 0; \qquad b = \alpha a; \qquad b_{ij} = \alpha a_{ij}; \\ (d\alpha &- h)\dot{T}^2 + 2b_i \dot{T} T_{,i} + K_{ij} T_{,i} T_{,j} \geq 0. \end{aligned}$$

1.3.3 Green-Naghdi Model ($G - N$ model)

The thermoelastic theory proposed by Green and Naghdi(1993), is described by a system of partial differential equations in which the Fourier law of heat conduction is replaced by the relation

$$\dot{\mathbf{q}} = -\mathbf{k}^* \nabla T, \tag{1.17}$$

where $\mathbf{q}$ and T are the heat flux vector and temperature change fields respectively, and $\mathbf{k}^*$ is a symmetric positive definite second-order tensor field of dimension

$$[\mathbf{k}^*] = \left[\mathbf{k} T_0^{-1}\right], \tag{1.18}$$

where $\mathbf{k}$ is the second-order tensor field for heat conductivity and T_0 is a time unit. In $G - N$ model, thermoelastic wave corresponding to a displacement-temperature pair $(\mathbf{u}, T)$ satisfies the following equations

$$\begin{aligned} \mathrm{div}\mathbf{C}\left[\nabla \mathbf{u}\right] - \rho \ddot{\mathbf{u}} + \mathrm{div}\left(T\mathbf{M}\right) &= -b, \\ \mathrm{div}\left(\mathbf{k}^* \nabla T\right) - C_e \ddot{T} + T_0 \mathbf{M} \cdot \nabla \ddot{\mathbf{u}} &= -\dot{r}, \end{aligned} \tag{1.19}$$

on $\mathbb{R}^3 \times [0, \infty)$ with the initial conditions

$$\mathbf{u}\left(\mathbf{x}, 0\right) = u_0, \quad \dot{\mathbf{u}}\left(\mathbf{x}, 0\right) = \dot{u}_0, \quad T\left(\mathbf{x}, 0\right) = \nu_0, \quad \dot{T}\left(\mathbf{x}, 0\right) = \dot{\nu}_0, \tag{1.20}$$

on $\mathbb{R}^3$ and the boundary conditions

$$\mathbf{u} = \mathbf{u}' \quad \text{and} \quad T = T', \tag{1.21}$$

on $\partial\mathbb{R}^3 \times [0, \infty)$. Here, $\mathbf{C}$ is the elasticity tensor field, $\mathbf{M}$ is the stress-temperature field. The absence of $\dot{T}$ in the energy equation (1.19) implies that a pair $(\mathbf{u}, T)$ represents an undamped thermoelastic wave. Due to this reason, $G - N$ theory is also known as *thermoelasticity without energy dissipation.*

1.3.4 Dual Phase Lag Model (DPL Model)

The concept of dual phase lag was introduced by Tzou(1995). This model allows macroscopic formulation to represent the microscopic interactions in the heat transport mechanism. Such interactions yield macroscopic lagging (or delayed) with delaying times τ_T and τ_0 which represent the phase-lag of temperature gradient (T) and the phase-lag of heat flux ($\mathbf{q}$) respectively. This theory describes the lagging behavior with the constitutive equation

$$\mathbf{q}(\mathbf{x}, t + \tau_0) = -k\boldsymbol{\nabla} T(\mathbf{x}, t + \tau_T). \tag{1.22}$$

Applying Taylor series expansion with respect to t, Eq.(1.22) gives

$$\left(1 + \tau_0 \frac{\partial}{\partial t}\right) \mathbf{q} \equiv -k \left(1 + \tau_T \frac{\partial}{\partial t}\right) \boldsymbol{\nabla} T. \tag{1.23}$$

Combining Eq.(1.23) with the energy equation which is given by

$$-\boldsymbol{\nabla} \cdot \mathbf{q} = C_e \frac{\partial}{\partial t} T, \tag{1.24}$$

the following equations are obtained

$$\begin{gathered}\left(1 + \tau_T \frac{\partial}{\partial t}\right) \boldsymbol{\nabla}^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t} + \frac{\tau_0}{\alpha}\frac{\partial^2 T}{\partial t^2}, \\ \left(1 + \tau_T \frac{\partial}{\partial t}\right) \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \mathbf{q}\right) = \frac{1}{\alpha}\frac{\partial \mathbf{q}}{\partial t} + \frac{\tau_0}{\alpha}\frac{\partial \mathbf{q}}{\partial t}.\end{gathered} \tag{1.25}$$

These equations reduce to thermal wave in heat conduction when $\tau_T = 0$. If $\tau_0 = \tau_T$, Eq.(1.22) reduces to Fourier law in heat conduction and Eq.(1.25) reduces to the classical diffusion equation.

1.3.5 Hetnarski-Ignaczak Model ($H - I$ Model)

Hetnarski and Ignaczak (1996) proposed a non-linear thermoelastic model for soliton like thermoelastic waves at low temperature. This model obeys the following system of field equations and inequalities:

(i) The geometric relations

$$E = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathbf{T}}\right). \tag{1.26}$$

(ii) The laws of balance of forces and moments

$$\nabla \cdot \mathbf{S} + \mathbf{b} = \rho \ddot{\mathbf{u}}, \qquad \mathbf{S} = \mathbf{S}^{\mathbf{T}}. \tag{1.27}$$

(iii) Law of conservation of energy

$$\dot{\mathbf{e}} = \mathbf{S} \cdot \dot{\mathbf{E}} - \nabla \cdot \mathbf{q} + \mathbf{r}. \tag{1.28}$$

(iv) Dissipation inequality

$$\dot{\eta} \geq -\nabla \cdot \frac{\mathbf{q}}{\theta} + \frac{\mathbf{r}}{T} \qquad (T > 0). \tag{1.29}$$

(v) The constitutive laws

$$\eta = -\frac{\partial \psi}{\partial T}, \qquad \mathbf{S} = \frac{\partial \psi}{\partial \mathbf{E}}, \qquad \mathbf{q} = -k\nabla T + \boldsymbol{\beta}, \tag{1.30}$$

where $\mathbf{u}$, $\mathbf{E}$, $\mathbf{S}$, $\mathbf{q}$, T, η, e, $\mathbf{b}$, $\mathbf{r}$ and ρ represent displacement, strain, stress, heat flux, absolute temperature, entropy, internal energy, body force, heat supply and density fields respectively. It may be noted that all are function of reference position vector $\mathbf{x}$ and time t except ρ. The function ψ is defined as

$$\psi(\theta, \boldsymbol{\beta}, \mathbf{E}) = e - \eta\theta, \tag{1.31}$$

where $\boldsymbol{\beta}$ is a new constitutive variable and

$$\dot{\boldsymbol{\beta}} = -A\frac{\nabla \theta}{\theta}, \quad (|A| > 0). \tag{1.32}$$

The constant A represents an elastic heat flow vector field.

The material constants k, C_e, θ_0, μ, λ and α satisfy the following conditions

$$k > 0, \qquad C_e > 0, \qquad \theta_0 > 0, \qquad \mu > 0, \qquad 3\lambda + 2\mu > 0, \qquad |\alpha| > 0. \tag{1.33}$$

These material constants are analogous to those of a linear classical homogeneous isotropic thermoelasticity and if $\boldsymbol{\beta} = 0$ they reduce to the thermal conductivity, specific heat for zero deformation, reference temperature, Lamè moduli, and coefficient of linear thermal expansion respectively.

1.3.6 Chandrasekharaiah-Tzou Model ($C-T$ Model)

Chandrasekharaiah (1998) extended the work of Tzou (1995) on dual phase lag by retaining τ_0 upto second order and τ_T upto first order in the Taylor's expansion of Eq.(1.22). This expansion is given by

$$\left(1+\tau_0\frac{\partial}{\partial t}+\frac{1}{2}\tau_0^2\frac{\partial^2}{\partial t^2}\right)\mathbf{q} = -k\left(1+\tau_T\frac{\partial}{\partial t}\right)\boldsymbol{\nabla}T. \tag{1.34}$$

This equation produces a hyperbolic-type heat transport equation predicting wave-like thermal signals propagating with the finite speed.

1.3.7 Three Phase Lag Model (TPL Model)

A generalized mathematical model of a coupled thermoelasticity theory using three-phase lags in the heat flux vector, the temperature gradient and in the thermal displacement gradient was introduced by Choudhuri (2007). This theory uses three phase-lags τ_0, τ_T and τ_ν to the heat flux vector ($\mathbf{q}$), the temperature gradient ($\boldsymbol{\nabla}T$) and the thermal displacement gradient ($\boldsymbol{\nabla}\nu$) and is also known as *three phase lag* model. This model has generalized constitutive equation for heat conduction as

$$\mathbf{q}\left(\mathbf{x},t+\tau_0\right) = -\left[k\boldsymbol{\nabla}T(\mathbf{x},t+\tau_T)+k^*\boldsymbol{\nabla}\nu(\mathbf{x},t+\tau_\nu)\right], \tag{1.35}$$

where k^* is material constant.

The Taylor's series expansion of (1.35) upto first order gives the following generalized heat conduction law as

$$\left(1+\tau_0\frac{\partial}{\partial t}\right)\mathbf{q} = -\left[\left(k+\tau_\nu k^*\right)\boldsymbol{\nabla}T+\tau_T\frac{\partial\boldsymbol{\nabla}T}{\partial t}+k^*\boldsymbol{\nabla}\nu\right]. \tag{1.36}$$

It may be noted that the above equation reduces to the classical Fourier law when $k^* = 0, \tau_0 = \tau_T$, it reduces to the heat conduction law of L–S theory when $k^* = 0, \tau_T = 0$, and it becomes $G - N$ theory when $\tau_0 = 0$, $\tau_T = 0$ and $\tau_\nu = 0$.

1.4 Application of wave propagation

We observe the practical application of propagation of elastic waves in many activities from our daily life. The process of sharpening of knife is the result of stress wave produced at the 'cone of percussion' which breaks the particle of the knife in very specific pattern. The analysis of the effect of thermal on the deformation of the material plays important role while constructing railway tracks. In the study of structural materials, the main interest is in the response to impact or blast loads. When a material is under load of moderate strength, the theory of elastic waves may be enough to explain all the aspects of the response. Under more severe loads, the material may undergo permanent deformation, fracture or perforation. In this type of situations, the theory of elastic waves is still applicable in predicting the response away from the impact.

The general aspects of *ultrasonics* are based on the introduction of a very low energy level, high frequency stress pulse or wave packet into a material and observing the subsequent propagation and reflection of that energy. The study of propagation, reflection and attenuation of ultrasonic pulses determine many fundamental properties of materials such as elastic constants and damping characteristics.

The study of propagation of seismic waves considered the Earth as an inhomogeneous isotropic elastic half-space for short epicentral distances or a ball for large epicentral distances. The study of the generation and propagation of seismic waves, produced in the Earth due to earthquakes is one of the important part of seismology. Elastic waves carry lots of information about the characteristics of the medium through which they travel and thus, it becomes a very reliable tool in the exploration cites. They give valuable information about the interior of the material body. The body waves (P and S-waves) are useful in earthquake engineering for predicting earthquake in the dynamic response of soils and man-made structures.

In the areas like mining and quarrying, we can find numerous applications of elastic waves. Drilling process is perform by transmitting longitudinal waves produced by an air hammer down drilling rod into the rock. In these type of works, the purpose of blasting is to produce intense stress waves. The interactions of these waves with each other and with the boundaries create fracture or remove large quantities of rock. Many problems under the fields of Seismology, geophysics, Earthquake engineering, tele-communication, medicines (echography), metallurgy (non-destructive testing) and signal processing are associated with the study of elastic waves. This study plays an important role in the process of detection of notches and faults in different types of materials such as in railway tracks, buried land-mines, etc. The study of propagation of longitudinal waves, shear waves and surface waves have been utilized in various detecting applications.

1.5 Review of Literature

The subject of wave propagation and their phenomena of reflection and transmission from boundary surface and interface is an interesting area of research. The study of properties of different elastic materials and the propagation of elastic waves can be explored through books such as Love (1944), Ewing *et al.* (1957), Biot (1965), Payton (1983), Hanyga (1985), Graff (1991), Nayfeh (1995), Udias (1999), Chapman (2004), Sato and Fehler (2009), Barber (2010), Jeffrey (2010), Singh (2013a), Hetnarski (2014), Altenbach and Öchsner (2020) and many others.

Chadwick and Seet (1970) considered the effects of heat conduction on plane harmonic waves of small amplitude propagating in a transversely isotropic heat conducting elastic material. The linear and nonlinear theory of elastic material containing voids were explored by Cowin and his co-workers (1979, 1983). Their theory was based on the idea of Goodman and Cowin (1972) which presented a continuum theory for granular materials. They derived the constitutive relations, established the thermodynamic restrictions on material moduli and presented governing equations. They have showed that the internal dissipation in the material arises due to the changes in the void volume fraction and derived the propagation condition for the acceleration waves. Dhaliwal and Sherief (1980) used generalized theory of thermoelasticity in an anisotropic medium for obtaining variational principle for the equations of motion. Chandrasekharaiah (1986a) discussed the theory of heat conduction with second sound and derived the governing equations of the conventional thermoelastic theory. Sharma (1988) showed the existence of three types of plane waves quasi-longitudinal (QL), quasi-transverse (QT) and Thermal (T-mode) wave in a thermally conducting homogeneous transversely isotropic elastic solid. Singh (2003) considered the problem of plane wave propagation in a homogeneous transversely isotropic thermally conducting elastic solid with two thermal relaxation times and obtained the amplitude ratios of the reflected waves. Singh and Tomar (2007, 2011) explored the problem of plane wave propagation in an infinite thermo-elastic and rotating generalized thermo-elastic material with voids. Kumar and Kansal (2008) derived the constitutive relations and field equations for anisotropic generalized thermoelastic diffusion and also reduced the results for transversely isotropic materials. Ciarletta *et al.* (2009) investigated the linear theory of micropolar thermoelasticity for isotropic and homogeneous materials with voids with the consideration of thermal relaxation time. The equations of motion for thermoporoelastic solids with two temperatures were derived by Manjula and Reddy (2015) and they obtained the frequency equation.

Singh (2010a) studied the problem of the reflection of plane waves from a thermally insulated stress-free monoclinic half-space of thermoelastic solid using $L-S$ and $G-L$ theories. Abbas and Abd-Alla (2011) investigated the thermoelastic interaction in an infinite fibre-reinforced anisotropic plate containing a circular hole by showing effects of the presence and absence reinforcement on temperature, stress and displacement. Sharma and Grover (2011) discussed the effects of voids, relaxation times, thermomechanical coupling, surface conditions and beam dimensions on energy dissipation induced by thermoelastic damping in micro-electromechanical systems ($MEMS$) or nano-electromechanical systems ($NEMS$) resonators for beams under clamped and simply supported conditions. Othman *et al.* (2012) presented a mathematical study of thermoelasticity in solid materials using Eringen coupled (micropolar) theory, $G-L$ theory and $L-S$ theory. Bucur *et al.* (2014) studied the problem of the damped effects of the thermal field on the behavior of plane harmonic waves and Rayleigh waves in a linear thermoelastic material with voids. Pal *et al.* (2014) discussed the problem of plane wave propagation in an inhomogeneous anisotropic thermally conducting elastic solid with two thermal relaxation times. Pazera and Jedrysiak (2015) considered the problem of thermoelasticity in composite, made of two components non-periodically distributed as microlaminas along one direction x_1, which macroscopic properties change continuously along this direction perpendicular to the laminas. Chirita and Danescuca (2016) investigated the propagation of plane time harmonic waves in an infinite space filled by a thermoelastic material with microtemperatures. Abd-Alla *et al.* (2017) obtained the temperature, displacement components and stresses components in the physical domain using Lame's potential method in a homogeneous orthotropic, thermo-elastic medium under the effect of gravity field. Othman and Abd-Elaziz (2017) investigated the effect of hall current and rotation on a magneto-thermoelastic solid with microtemperatures and voids. Lianngenga and Singh (2018) studied the effect of linear thermal expansion and micro-inertia on the refraction of elastic waves at a plane interface between two dissimilar half-spaces of micropolar thermoelastic materials with voids. They obtained the amplitude and energy ratios of the reflected and refracted waves.

Chadwick and Currie (1974) explained the possibility of transformation of the general secular equation for Stoneley waves propagating at the interface between elastic crystals into a form which reduced to a single real condition on the wave speed. Currie (1974) analyzed the properties of Rayleigh waves traveling on a free surface of an anisotropic elastic half-space. Harinath (1974) described the conditions for propagation of interface waves between thermo-elastic solid and an elastic solid by deriving the frequency equations. Murty (1975a, 1975b) examined the existence of Stoneley waves at an unbonded and loosely bonded interface between two elastic half-spaces. Barnett *et al.* (1985) shown the existence of Stoneley waves using the notion of the impedance tensor at the bonded interface between the half-spaces of anisotropic elastic materials. Chandrasekharaiah (1986b, 1987) studied the problem of surface waves of general type propagating in a homogeneous isotropic linear elastic half-space containing a distribution of voids and also obtained the corresponding frequency equations. Abd-Alla (1999) investigated the effect of gravity and initial stress on Rayleigh wave propagation in an orthotropic elastic solid medium by solving the frequency equation. Mondal and Acharya (2006) determined the effect of voids on the propagation of surface waves in a homogeneous micropolar elastic solid material containing distribution of vacuous pores. Singh and Tomar (2007) studied the problem of Rayleigh–Lamb waves propagation in an infinite plate of microstretch elastic material with finite thickness. They derived frequency equations of the surface waves using suitable boundary conditions. Kumar and Kumar (2010) considered the effect of voids on the surface wave propagation in a layer of transversely isotropic thermoelastic material with voids lying over an isotropic elastic half-space. Singh (2010) discussed the problem of the propagation of a Love wave in a corrugated isotropic layer over a homogeneous isotropic half-space. In the context of dual-phase-lag model, Abouelregal (2011) discussed the problem of propagation of Rayleigh waves in a thermo-elastic homogeneous isotropic solid half-space. Kumar and Kumar (2011) analyzed the effect of voids on the surface wave propagation in a layer of orthotropic thermoelastic material with voids lying over an isotropic elastic half-space and derived the frequency equation on the basis of the developed mathematical model under the boundary conditions for welded and smooth contacts.

Kumar *et al.* (2013) examined the propagation of Stoneley waves at the interface between two couple stress thermo-elastic half-spaces. Sharma (2014a) discussed the problem of propagation of Rayleigh waves in a thermoelastic half-space and plotted the velocity curves. The frequency equation of Stoneley waves in magneto-thermoelastic materials with voids and two thermal relaxation times was examined by Abo-Dahab (2015). The properties of Stoneley and Scholte waves in a multilayer of liquid/solid and solid/solid interfaces were investigated by Onen and Uz (2015). Abbas *et al.* (2016) obtained the frequency equation for Rayleigh–Lamb wave propagating in a plate of generalized thermoelasticity with one relaxation time. Gupta and Ahmed (2017) developed a mathematical model on the propagation of Rayleigh wave in a self-reinforced layer over an incompressible inhomogeneous elastic half-space under the conditions of quadratically varying rigidity and linearly varying density. Khurana and Tomar (2017) derived the frequency equations of two modes of Rayleigh type waves in the half-space of non-local micropolar elastic solid and explained the condition of existence of these two modes. Biswas and Mukhopadhyay (2018) examined the problem of Rayleigh surface waves in a homogeneous transversely isotropic thermoelastic material in the context of three-phase-lag model. Farhan and Abd-Alla (2018) discussed the effects of magnetic field and rotation on the propagation of surface wave in a generalized magneto-thermoelastic materials with voids. Kaur *et al.* (2018) derived the frequency equation for Rayleigh-type surface wave in an isotropic homogeneous non-local elastic solid half-space with voids. They have illustrated the graphical presentation of the variations of phase speed and corresponding attenuation of Rayleigh-type wave against frequency, non-locality and void parameters. Kumar *et al.* (2018a) analyzed the propagation of Stoneley waves at the interface between two dissimilar half-spaces of isotropic modified couple stress thermo-elastic materials. Lianngenga and Singh (2019) investigated the problem of symmetric and anti-symmetric vibrations in micropolar thermoelastic plate with voids and they obtained the dispersive frequency equations for different surface waves propagating in the plate.

Biot (1962a) analyzed the mechanics of deformation and acoustic propagation in porous media using the specific relaxation models. Chang (1971) investigated the stress field around a finite closed crack in an elastic material by applying a plane dilatational wave to the crack. McCarthy (1972) discussed the propagation of waves in generalized thermoelasticity and found the existence of four principal waves. Bedford and Sutherland (1973) studied the problem of reflection and transmission of plane waves at the interface between elastic material and fiber-reinforced material. They compared the results with the experimental data obtained from ultrasonic measurements of waves transmitted through water into an aluminum plate containing tungsten fibers. Sharma and Sidhu (1986) derived the secular equation for the propagating plane harmonic waves in a homogeneous anisotropic generalized thermoelastic material. Green (1991) discussed the problem of reflection and transmission of transient stress waves in fiber composite laminates. Chattopadhyay *et al.* (2002) computed the numerical values of reflection coefficients of qP and qSV waves at the free rigid boundary of a fibre-reinforced material. Sharma and Pal (2004) investigated the problem of propagation of the magnetic-thermoelastic plane wave in an initially unstressed homogeneous isotropic conducting plate under uniform static magnetic field. Singh and Singh (2004) presented the expressions for the phase velocity of quasi-P and quasi-SV waves propagating in the half-space of fibre-reinforced anisotropic elastic materials. They considered the reinforcement direction as functions of the angle between the propagation and reinforcement directions. Singh and Tomar (2006) examined the problem of reflection and transmission of obliquely incident plane transverse wave at a plane interface between two porous elastic half-spaces in welded contact. Singh (2006) discussed the propagation of plane waves in a thermally conducting linear fibre-reinforced composite materials and obtained the phase velocity of coupled waves, namely qP, qSV and quasi-thermal waves. Singh and Tomar (2007a, 2007b) investigated the problem of propagation of qP, qSV and

qSH waves incident at a corrugated interface between two dissimilar fibre-reinforced elastic half-spaces. They obtained amplitude and energy ratios of the reflected and refracted waves. Chattopadhyay *et al.* (2007) derived the amplitude ratios of the reflected and refracted waves at the plane interface between two monoclinic half-spaces. Kumar and Singh (2008) obtained the reflection and transmission coefficients of quasi-longitudinal, quasi-thermal and quasi-transverse waves for different incident waves at an imperfectly bonded interface between two orthotropic generalized thermoelastic half-spaces having different elastic and thermal properties. Kumar and Gupta (2010) studied the problem of propagation of elastic waves in transversely isotropic micropolar generalized thermoelastic half-space and also derived the amplitude ratios.

Abbas (2011) studied the problem of propagation of plane waves in the fibre-reinforced anisotropic thermoelastic half-space. He used finite element method for numerical computation of displacement, temperature and components of stress to compare the results of the theories($GN-II$ and $GN-III$). Ponnusamy and Rajagopal (2011) explored the problem of wave propagation in a transversely isotropic thermoelastic solid cylinder of arbitrary cross-section using Fourier expansion collocation method. They obtained frequency equations for longitudinal and flexural (symmetric and antisymmetric) modes of vibrations. Singh (2011, 2013b) investigated the problem of reflection and transmission of elastic waves due to incident plane couple longitudinal waves and transverse wave at a plane interface between two dissimilar half-spaces of thermo-elastic materials with voids. Chattopadhyay and Singh (2012) explained the possibility of propagation of shear wave at the interface of two different types of fibre reinforced media. Placidi *et al.* (2013) examined the problem of reflection and transmission of compression and shear waves at structured interfaces between second gradient continua and obtained the general balance equations for the bulk system as well as jump duality conditions. Othman and Song (2014) described the properties of reflected plane harmonic waves in a thermo-microstretch elastic half-space under the effect of rotation. Choudhury *et al.* (2015) analyzed the propagation of elastic waves in an infinite granular thermoelastic medium rotating with a constant speed. Kumar and Gupta (2015) discussed the problem of reflection and refraction of obliquely incident plane wave at the interface of elastic and thermoelastic diffusion media with fractional order derivative. Sahu *et al.* (2015) studied the problem of the scattering of elastic wave in a composite bedded structure in which isotropic layer is sandwiched between two highly anisotropic media. They have found the reflection/transmission coefficients and energy ratios of different reflected and transmitted waves. Othman and Hilal (2016) determined the effects of the gravity and the magnetic fields on the plane waves in an isotropic thermoelastic materials under thermal loading due to laser pulse. Pal and Kanoria (2016) analyzed the gravitational response in the propagation of elastic waves in an infinite, homogeneous, transversely isotropic thick plate. Abd-Alla *et al.* (2017) analyzed the effects of relaxation times, rotation and magnetic field on incident and reflected plane waves in a transversely isotropic magneto-thermoelastic medium. Lianngenga (2017) computed the numerical values of the phase velocities, attenuations and reflection coefficients of plane body waves in the half-space of micropolar porous solid. Deswal *et al.* (2018) analyzed the reflection of plane waves from the free surface of a homogeneous anisotropic fiber-reinforced thermoelastic rotating medium with dual-phase-lag model. Khurana and Tomar (2018) investigated the problem of reflection and transmission of a longitudinal displacement wave and a set of coupled transverse waves at plane discontinuity separating the two distinct non-local micropolar solids. Li *et al.* (2018) discussed the problem of reflection and refraction of thermoelastic waves at an imperfect interface between two semi-infinite homogeneous isotropic thermoelastic couple stress solids. Lalvohbika and Singh (2019) examined the problem of reflection and transmission of elastic waves for incident qP and qSV-wave at a corrugated interface between two dissimilar nematic elastomer half-spaces.

Chadwick (1979) described the problem of propagation of plane harmonic waves of small amplitude in a heat-conducting elastic body of unrestricted symmetry. Benveniste (1981) analyzed the propagation of one-dimensional waves in an initially deformed incompressible medium with different moduli of tension and compression. Rogerson (1991) presented various dynamic properties and the condition of propagation of elastic waves in the incompressible transversely isotropic elastic materials. Dhaliwal and Wang (1993) established a generalized theory for a thermoelastic dipolar pre-stressed body. Chadwick (1993,1994) developed the constitutive theory governing small deformations of an incompressible transversely isotropic elastic material and discussed the nature of homogeneous and inhomogeneous plane waves in the material. Ogden and Sotiropoulos (1997) examined the effect of pre-stress and finite strain on the reflection of homogeneous plane waves in an incompressible isotropic elastic solid. Itskov and Aksel (2002) explained the difficulty of deriving the constitutive relations for anisotropic incompressible materials. Singh (2007) analyzed the problem of wave propagation in free surface of an incompressible transversely isotropic elastic half-space. The reflection coefficients are obtained for the case when outer slowness section is re-entrant.

Abd-Alla *et al.* (2011a, 2011b) discussed the effect of initial stress on the propagation of elastic waves in the half-spaces of different thermoelastic materials. Abo-Dahab (2014) obtained the frequency equation for surface waves in a generalized magneto-thermoelastic materials with voids and initial stress. Guo and Wei (2004) analyzed the effects of initial stress on the numerical value of the amplitude and energy ratios of the reflected and transmitted waves at the interface between two piezoelectric half-spaces. Prikazchikov and Rogerson (2004) attempted the problem of propagation of surface wave in a pre-stressed transversely isotropic incompressible half-space and obtained the secular equation. Othman and Atwa (2012) constructed the basic equations of generalized thermoelastic isotropic materials under hydrostatic initial stress in the context of the GN theory of types II and III. They used normal mode analysis to obtain the exact expressions of temperature, displacement and stress. Chatterjee *et al.* (2016) examined the reflection and refraction phenomena of plane waves incident at the interface between two distinct triclinic media under the initial stresses. Othman et al. (2017) investigated the effect of rotation and initial stress on the P, T and SV-waves in the generalized magneto-thermoelastic materials using the normal mode analysis. Biswas and Abo-Dahab (2018) analyzed the effect of initial stress and magnetic field on the propagation of Rayleigh waves in the context of three-phase-lag model in the homogeneous magneto-thermoelastic orthotropic materials. Kundu *et al.* (2019) studied the effect of thickness and initial stress on the propagation of Rayleigh waves in an anisotropic crustal layer lying over an elastic half-space containing void pores.

Biot (1956b) discussed the propagation of stress waves in a porous elastic solid containing a compressible viscous fluid. Biot (1962b) extended the theory of acoustic propagation in porous media to include anisotropy, viscoelasticity and solid dissipation. Hosten (1991) computed the numerical values of reflection and transmission coefficients of elastic waves through immersed composite layers at any incidence plane. Hirai (1992) presented a numerical analysis on the propagation of Rayleigh waves in a saturated porous elastic medium using finite element method. Boer *et al.* (1993) analyzed transient wave propagation in fluid-staturated porous media and obtained the exact solution by taking the Laplace transform of the governing equations with the initial and boundary conditions. Levy *et al.* (1995) developed a mathematical model for saturated flow of a Newtonian fluid in a homogeneous isotropic thermoelastic porous medium under non-isothermal conditions. Kumar *et al.* (2002) investigated a problem of surface wave propagation in a micropolar liquid-saturated porous layer over a micropolar liquid-saturated porous half-space. Sharma and Pathania (2004) studied the problem of generalized thermoelastic waves in anisotropic plates sandwiched between liquid layers using $L-S$, $G-L$ and $G-N$ theories of thermoelasticity. Kumar and Hundal (2005) presented the analysis of symmetric wave propagation in the half-space of fluid-saturated incompressible porous material and they obtained the secular equation. Sharma (2007) explored an equivalence relation between the mathematical models of wave propagation derived from Biot's theories as well as homogenisation theory. He derived phase velocity and attenuation of waves propagation in an anisotropic fluid-saturated porous medium.

Markov (2009) determined the velocity and attenuation of Stoneley wave at the interface between two dissimilar half-spaces of fluid-saturated porous media. Kumar *et al.* (2011) discussed the problem of reflection and transmission of plane waves between two different fluid saturated porous half-spaces for incident longitudinal and transverse waves. They depicted the amplitude ratios of various reflected and transmitted waves. Sharma (2012a, 2012b) obtained the frequency equations of the Rayleigh surface waves in the partially saturated porous materials and poro-viscoelastic media. Sharma and Bhargava(2014) investigated the problem of reflection and transmission of thermoelastic plane waves at an imperfect interface between the half-spaces of thermal conducting viscous-liquid and generalized thermoelastic solid. Sharma (2014b) considered the effects of wave-induced fluid flow on the numerical values of phase velocity and attenuation of Rayleigh waves in an elastic solid having double porosity. Bucur (2016) examined the dissipative nature of the porous thermoviscoelastic materials in the propagation of the Rayleigh waves and obtained the secular equation in the explicit and implicit form. Zorammuana and Singh (2016) presented graphically the variations of amplitude and energy ratios of reflected waves with angle of incidence. They analyzed both for the incident longitudinal and transverse waves at the free surface of thermoelastic saturated porous material. Barak and Kaliraman (2018) performed numerical computation for the reflection and transmission coefficients of elastic waves at an imperfect interface between the micropolar elastic and fluid saturated porous solid half-spaces. Kumar *et al.* (2018b) investigated the disturbances in a homogeneous transversely isotropic magneto-visco thermoelastic rotating medium with two temperature due to thermomechanical sources. Painuly and Arora (2019) analyzed the problem of propagation of Rayleigh wave along the free surface of a composite porous half-space saturated by two immiscible fluids.

The extensive studies of the wave propagation can be explored through Toupin (1962), Sinha and Sinha (1974), Yew and Jogi (1976), Tajuddin (1984), Sorek *et al.* (1992), Zhang and Shinozuka (1996), Sinha and Elsibai (1996, 1997), Xia *et al.* (1999), Verma (2001), Tomar and Singh (2005), Zhu and Tsvankin (2006), Tomar and Singh (2006), Tomar and Kaur (2007a, 2007b), Yu and Dravinski (2009), Reddy and Tajuddin (2010), Singh and Pal (2011), Zakharov (2011), Vinh and Giang (2012), Singh and Zorammuana (2013), Steeb *et al.* (2013), Tomar and Khurana (2013), Tomar *et al.* (2013), Vinh (2013), Zenkour et al. (2013), Singh et al. (2014), Tomar and Ogden (2014), Yang et al. (2014), Singh (2015, 2017), Zorammuana and Singh (2015), Srisailam *et al.* (2016), Vinh *et al.* (2016), Singh and Lianngenga (2017), Sudheer *et al.* (2017), Singh *et al.* (2018), Singh and Lalvohbika (2018), Tong *et al.* (2018) and Tomar and Kumar (2020).

Chapter 2

Propagation of surface waves in thermoelastic materials with voids[1]

2.1 Introduction

Rayleigh (1885) introduced surface waves which travel along the stress free boundary of an elastic half-space such that the disturbance is largely confined to the neighborhood of the free boundary surface. In the propagation of Rayleigh wave, the surface particles move in counterclockwise elliptical (retrograde) which change from retrograde at the surface to prograde (clockwise elliptical) at depth passing through a node at which there is no horizontal motion. Surface waves do to exist along the interface between solid and solid half-spaces and they are called *Stoneley waves*, named after Stoneley (1924). The amplitudes of these waves are maximum at the surface of the material and decay exponentially towards the depth of each of the elastic solids. Iesan (1986) developed the linear theory of thermoelastic material with voids by using Green and Rivlin (1964) and obtained the basic equations from the balance of energy under the rigid body motions. Iesan also studied the propagation of acceleration waves in homogeeous and isotropic bodies. Tomar and Singh(2006) derived the frequency equations for Stoneley waves at unbonded and bonded interfaces between two dissimilar microstretch elastic half-spaces.

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In this chapter, we study the propagation of surface waves in thermoelastic materials containing voids and obtain the secular equations for Stoneley waves for the bonded and unbonded interfaces. We also computed the numerical values of the determinants corresponding to secular equations of the Stoneley waves. The phase velocity and attenuation of Rayleigh waves in thermoelastic material with voids are obtained. We have depicted the velocity curves of these surface waves by computing the phase velocity and attenuation coefficients.

2.2 Basic Equations

The field equations in thermo-elastic materials with voids in the absence of body forces and external heat sources are given by (Iesan, 1986)

$$\begin{aligned} c_1^2\nabla(\nabla\cdot\mathbf{u}) - c_2^2\nabla\times(\nabla\times\mathbf{u}) + \frac{\beta^*}{\rho}\nabla\phi - \frac{\beta}{\rho}\nabla\nu &= \ddot{\mathbf{u}}, & (2.1)\\ \alpha^*\nabla^2\phi - \beta^*\nabla\cdot\mathbf{u} - \xi^*\phi + m\nu &= \rho\kappa^*\ddot{\phi}, & (2.2)\\ \kappa\nabla^2\nu - \beta T_0\nabla\cdot\dot{\mathbf{u}} - a_e T_0\dot{\nu} - mT_0\dot{\phi} &= 0, & (2.3)\end{aligned}$$

where $c_1^2 = (\lambda+2\mu)/\rho$ and $c_2^2 = \mu/\rho$, λ and μ are Lamé parameters, ρ is the density of the medium, κ, β, a_e and m are thermal parameters, T_0 is the absolute temperature of the reference state, α^*, β^*, ξ^* and κ^* are voids parameters, $\mathbf{u}$ is the displacement vector, ϕ is the change in void volume fraction and ν is the change of temperature from the reference state.

The constitutive relations in the thermo-elastic materials with voids are given by

$$\begin{aligned} &\tau_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + (\beta^*\phi - \beta\nu)\delta_{ij}, \quad h_i = \alpha^*\phi_{,i}, \quad q_i = \kappa\nu_{,i},\\ &e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \rho\eta = \beta e_{kk} + a_e\nu + m\phi, \quad i,j,k = 1,\ 2,\ 3 \qquad (2.4)\end{aligned}$$

where τ_{ij} are stress tensors, e_{ij} are strain tensors, δ_{ij} are Kronecker's delta, h_i are equilibrated stress vectors, q_i are heat flux vectors and η is the specific entropy. Commas in the subscript denote the spatial derivative.

The Helmholtz representation of the vector field $\mathbf{u}$ is

$$\mathbf{u} = \nabla p + \nabla \times \boldsymbol{\psi}, \quad \nabla \cdot \boldsymbol{\psi} = 0, \tag{2.5}$$

where p and $\boldsymbol{\psi}$ are scalar and vector potential respectively.

2.3 Wave Propagation

Consider the Cartesian co-ordinate with x and y-axis lying horizontally and z-axis vertically with positive direction pointing downward. Let us take two dissimilar semi-infinite half-spaces of thermoelastic solids with voids ($M : -\infty < z \leq 0$) and ($M' : 0 \leq z < \infty$) separated by $z = 0$. We will denote all the parameters in M without prime and M' with prime.

We consider the two dimensional wave propagation in xz-plane. The equations of motion in the half-space, M may be obtained from Eq.(2.1)-(2.3) by using harmonicity of the traveling waves and (2.5) as

$$(\nabla^2 + l_1^2)\ p(x,z) + l_2^2\ \phi(x,z) - l_3^2\ \nu(x,z) = 0, \tag{2.6}$$

$$(c_2^2 \nabla^2 + \omega^2)\ \boldsymbol{\psi}(x,z) = 0, \tag{2.7}$$

$$(\nabla^2 + l_4^2)\ \phi(x,z) - l_5^2 \nabla^2\ p(x,z) + l_6^2\ \nu(x,z) = 0, \tag{2.8}$$

$$(\nabla^2 + l_7)\ \nu(x,z) + l_8 \nabla^2\ p(x,z) + l_9\ \phi(x,z) = 0, \tag{2.9}$$

where ω is the angular frequency,

$$l_1^2 = \frac{\omega^2}{c_1^2},\ l_2^2 = \frac{c_3^2}{c_1^2},\ l_3^2 = \frac{c_4^2}{c_1^2},\ l_4^2 = \frac{\omega^2 - c_7^2}{c_5^2},\ l_5^2 = \frac{c_6^2}{c_5^2},\ l_6^2 = \frac{c_8^2}{c_5^2},$$

$$l_7 = c_9\omega,\ l_8 = c_{10}\omega,\ l_9 = c_{11}\omega,\ c_3^2 = \frac{\beta^*}{\rho},\ c_4^2 = \frac{\beta}{\rho},\ c_5^2 = \frac{\alpha^*}{\rho\kappa^*},$$

$$c_6^2 = \frac{\beta^*}{\rho\kappa^*},\ c_7^2 = \frac{\xi^*}{\rho\kappa^*},\ c_8^2 = \frac{m}{\rho\kappa^*},\ c_9 = \frac{\imath c^*}{\kappa},\ c_{10} = \frac{\imath\beta T_0}{\kappa},\ c_{11} = \frac{\imath m T_0}{\kappa},\ c^* = a_e T_0.$$

Similarly, the equations of motion for the other half-space, M' are

$$
\begin{aligned}
(\nabla^2 + l_1'^2)\ p'(x,z) + l_2'^2\ \phi'(x,z) - l_3'^2\ \nu'(x,z) &= 0, && (2.10)\\
(c_2'^2\nabla^2 + \omega^2)\ \boldsymbol{\psi}'(x,z) &= 0, && (2.11)\\
(\nabla^2 + l_4'^2)\ \phi'(x,z) - l_5'^2\nabla^2\ p'(x,z) + l_6'^2\ \nu'(x,z) &= 0, && (2.12)\\
(\nabla^2 + l_7')\ \nu'(x,z) + l_8'\nabla^2\ p'(x,z) + l_9'\ \phi'(x,z) &= 0, && (2.13)
\end{aligned}
$$

where

$$
l_1'^2 = \frac{\omega^2}{c_1'^2},\ l_2'^2 = \frac{c_3'^2}{c_1'^2},\ l_3'^2 = \frac{c_4'^2}{c_1'^2},\ l_4'^2 = \frac{(\omega^2 - c_7'^2)}{c_5'^2},\ l_5'^2 = \frac{c_6'^2}{c_5'^2},\ l_6'^2 = \frac{c_8'^2}{c_5'^2},\ l_7' = c_9'\omega,
$$

$$
l_8' = c_{10}'\omega,\ l_9' = c_{11}'\omega,\ c_1'^2 = \frac{(\lambda' + 2\mu')}{\rho'},\ c_3'^2 = \frac{\beta^{*\prime}}{\rho'},\ c_4'^2 = \frac{\beta'}{\rho'},\ c_5'^2 = \frac{\alpha^{*\prime}}{\rho'\kappa^{*\prime}},\ c_6'^2 = \frac{\beta^{*\prime}}{\rho'\kappa^{*\prime}},
$$

$$
c_7'^2 = \frac{\xi^{*\prime}}{\rho'\kappa^{*\prime}},\ c_8'^2 = \frac{m'}{\rho'\kappa^{*\prime}},\ c_9' = \frac{\imath c^{*'}}{\kappa'},\ c_{10}' = \frac{\imath\beta' T_0'}{\kappa'},\ c_{11}' = \frac{\imath m' T_0'}{\kappa'},\ c^{*'} = a_e' T_0'.
$$

The coupled dilatational waves (p, ϕ, ν) and (p', ϕ', ν') in Eqs.(2.6)-(2.9) and (2.10)-(2.13) respectively satisfy the following equations

$$
(\nabla^6 + A\ \nabla^4 + B\ \nabla^2 + C)\{p, \phi, \nu\}(x,z) = 0, \tag{2.14}
$$

$$
(\nabla^6 + A'\ \nabla^4 + B'\ \nabla^2 + C')\{p', \phi', \nu'\}(x,z) = 0, \tag{2.15}
$$

where

$$
\begin{aligned}
A &= l_1^2 + l_4^2 + l_7 + l_2^2\ l_5^2 + l_3^2\ l_8,\\
B &= l_1^2(l_4^2 + l_7) + l_4^2\ l_7 - l_6^2\ l_9 + l_2^2(l_5^2\ l_7 + l_6^2\ l_8) + l_3^2(l_4^2\ l_8 + l_5^2\ l_9),\\
C &= l_1^2(l_4^2\ l_7 - l_6^2\ l_9),\\
A' &= l_1'^2 + l_4'^2 + l_7' + l_2'^2\ l_5'^2 + l_3'^2\ l_8',\\
B' &= l_1'^2(l_4'^2 + l_7') + l_4'^2\ l_7' - l_6'^2\ l_9' + l_2'^2(l_5'^2\ l_7' + l_6'^2\ l_8') + l_3'^2(l_4'^2\ l_8' + l_5'^2\ l_9'),\\
C' &= l_1'^2(l_4'^2\ l_7' - l_6'^2\ l_9').
\end{aligned}
$$

The full structure of the surface waves may be written as

(In the half-space, M)

$$\begin{aligned} \{p,\ \phi,\ \nu\}(x,z,t) =& \sum_{n=1}^{3}\{A_n,\ V_nA_n,\ L_nA_n\}e^{(\imath kx-m_nz-\imath\omega t)}, \\ \psi(x,z,t) =& A_4\ e^{(\imath kx-m_4z-\imath\omega t)}, \end{aligned} \tag{2.16}$$

where ψ is the y component of $\boldsymbol{\psi}$, k is the wavenumber, A_i are amplitudes, $m_i^2 = k^2 - k_i^2,\ (i = 1,\ 2,\ 3)$ and $m_4^2 = k^2 - k_t^2$ are the penetration depth of surface waves which decay exponentially in the medium M, k_i^2 are obtained from Eq.(2.14) and $k_t = \omega/c_2$. The coupling parameters are given as

$$V_i = \frac{(l_1^2 - k_i^2)}{(l_3^2\ H_i^* - l_2^2)},\quad L_i = H_i^*\ V_i,\quad (i = 1,\ 2,\ 3)$$

and

$$H_i^* = \frac{(k_i^2 - l_1^2)\{(l_7 - k_i^2)(l_4^2 - k_i^2) - l_6^2\ l_9\} - l_2^2\{l_6^2\ l_8\ k_i^2 + l_5^2\ k_i^2(l_7 - k_i^2)\}}{l_3^2\{l_6^2\ l_8\ k_i^2 + l_5^2\ k_i^2(l_7 - k_i^2)\}}.$$

(In the half-space, M')

$$\begin{aligned} \{p',\ \phi',\ \nu'\}(x,z,t) =& \sum_{n=1}^{3}\{A'_n,\ V'_nA'_n,\ L'_nA'_n\}e^{(\imath kx+m'_nz-\imath\omega t)}, \\ \psi'(x,z,t) =& A'_4\ e^{(\imath kx+m'_4z-\imath\omega t)}, \end{aligned} \tag{2.17}$$

where ψ' is the y component of $\boldsymbol{\psi}'$, A'_i are amplitudes, $m_i'^2 = k^2 - k_i'^2$ and $m_4'^2 = k^2 - k_t'^2$ are the penetration depth of surface waves, which decay exponentially in the medium M', $k_i'^2$ are obtained from Eq.(2.15) and $k'_t = \omega/c'_2$, where $c'_2 = \mu'/\rho'$. The coupling parameters in M' are

$$V'_i = \frac{(l_1'^2 - k_i'^2)}{(l_3'^2 H_i^{*\prime} - l_2'^2)},\quad L'_i = H_i^{*\prime}V'_i,$$

and

$$H_i^{*\prime} = \frac{(k_i'^2 - l_1'^2)\{(l'_7 - k_i'^2)(l_4'^2 - k_i'^2) - l_6'^2\ l'_9\} - l_2'^2\{l_6'^2\ l'_8\ k_i'^2 + l_5'^2\ k_i'^2(l'_7 - k_i'^2)\}}{l_3'^2\{l_6'^2\ l'_8\ k_i'^2 + l_5'^2\ k_i'^2(l'_7 - k_i'^2)\}}.$$

2.4 Boundary Conditions

The stress traction, displacements, equilibrated stress vectors and heat flux vectors are continuous at $z = 0$. We consider the problem for both bonded interface and unbonded interface between the two dissimilar half-spaces. These conditions at $z = 0$ may be written as

(i) Continuity of normal stress

$$\lambda(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2}) + 2\mu(\frac{\partial^2 p}{\partial z^2} + \frac{\partial^2 \psi}{\partial x \partial z}) + \beta^* \phi - \beta \nu$$

$$= \lambda'(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial z^2}) + 2\mu'(\frac{\partial^2 p'}{\partial z^2} + \frac{\partial^2 \psi'}{\partial x \partial z}) + \beta^{*\prime} \phi' - \beta' \nu', \tag{2.18}$$

(ii) Continuity of displacement component

$$\frac{\partial p}{\partial x} - \frac{\partial \psi}{\partial z} = \frac{\partial p'}{\partial x} - \frac{\partial \psi'}{\partial z}, \tag{2.19}$$

(iii) Continuity of equilibrated stress vectors

$$\frac{\partial \phi}{\partial z} = \chi_2 \frac{\partial \phi'}{\partial z}, \qquad \frac{\partial \phi}{\partial x} = \chi_2 \frac{\partial \phi'}{\partial x}, \tag{2.20}$$

(iv) Continuity of heat flux vectors

$$\frac{\partial \nu}{\partial z} = \chi_1 \frac{\partial \nu'}{\partial z}, \qquad \frac{\partial \nu}{\partial x} = \chi_1 \frac{\partial \nu'}{\partial x}, \tag{2.21}$$

(For the bonded interface)

(v) Continuity of displacement component

$$\frac{\partial p}{\partial z} + \frac{\partial \psi}{\partial x} = \frac{\partial p'}{\partial z} + \frac{\partial \psi'}{\partial x}, \tag{2.22}$$

(vi) Continuity of shear stress

$$\mu(2\frac{\partial^2 p}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2}) = \mu'(2\frac{\partial^2 p'}{\partial x \partial z} + \frac{\partial^2 \psi'}{\partial x^2} - \frac{\partial^2 \psi'}{\partial z^2}), \tag{2.23}$$

(For the unbonded interface)

(v) Shearing stresses vanish

$$2\frac{\partial^2 p}{\partial x \partial z}+\frac{\partial^2 \psi}{\partial x^2}-\frac{\partial^2 \psi}{\partial z^2}=0, \qquad 2\frac{\partial^2 p'}{\partial x \partial z}+\frac{\partial^2 \psi'}{\partial x^2}-\frac{\partial^2 \psi'}{\partial z^2}=0, \tag{2.24}$$

where $\chi_1=\kappa'/\kappa$, and $\chi_2=\alpha^{*\prime}/\alpha^*$.

Using Eqs.(2.4),(2.5), (2.16) and (2.17) into boundary conditions, we get the following two sets of equations

$$\sum_{j=1}^{4} a_{ij}A_j+\sum_{j=5}^{8} a_{ij}A'_{j-4}=0, \quad i=1,2,3,4,5,6,7,8 \tag{2.25}$$

where non-zero values of a_{ij} are given by

$$a_{1i}=(\lambda+2\mu)m_i^2-\lambda k^2+\beta^* V_i-\beta L_i, \qquad (i=1,2,3), \qquad a_{14}=-2\imath\mu k m_4,$$

$$a_{1i}=-(\lambda'+2\mu')m'^2_{i-4}+\lambda' k'^2-\beta^{*\prime} V'_{i-4}+\beta' L'_{i-4}, \quad (i=5,6,7), \quad a_{18}=-2\imath\mu' k m'_4,$$

$$a_{2i}=k, \quad (i=1,2,3), \qquad a_{24}=-\imath m_4, \qquad a_{2i}=-k, \quad (i=5,6,7), \qquad a_{28}=-\imath m'_4,$$

$$a_{3i}=L_i m_i, \quad (i=1,2,3), \quad a_{3i}=\chi_1 L'_{i-4}m'_{i-4}, \quad (i=5,6,7), \quad a_{4i}=L_i, \ (i=1,2,3),$$

$$a_{4i}=-\chi_1 L'_{i-4}, (i=5,6,7), \ a_{5i}=V_i m_i, (i=1,2,3), \ a_{5i}=\chi_2 V'_{i-4}m'_{i-4}, (i=5,6,7),$$

$$a_{6i}=V_i, \quad (i=1,2,3), \qquad a_{6i}=-\chi_2 V'_{i-4}, \quad (i=5,6,7),$$

(for bonded interface)

$$a_{7i}=m_i, \quad (i=1,2,3), \quad a_{74}=-\imath k, \quad a_{7i}=m'_{i-4}, \quad (i=5,6,7),$$

$$a_{78}=\imath k, \qquad a_{84}=-\imath\mu(m_4^2+k^2), \qquad a_{8i}=2\mu k m_i, \qquad (i=1,2,3),$$

$$a_{8i}=2\mu' k m'_{i-4}, \qquad (i=5,6,7), \qquad a_{88}=\imath\mu'(m'^2_4+k^2).$$

(for unbonded interface)

$$a_{7i}=2k m_i, \qquad (i=1,2,3), \qquad a_{74}=-\imath(m_4^2+k^2),$$

$$a_{8i}=2k m'_{i-4}, \qquad (i=5,6,7), \qquad a_{88}=\imath\mu'(m'^2_4+k^2).$$

These equations give the boundary conditions for the surface wave propagation both for the bonded and unbonded interfaces between two dissimilar half-spaces of thermoelastic materials with voids.

2.5 Stoneley Waves

We know that Stoneley waves propagate along the interfaces of solid-solid materials. The secular equations of such waves for bonded and unbonded interface between two dissimilar half-spaces of thermoelastic materials with voids are derived from (2.25) as

$$|a_{ij}| = 0, \qquad (i,\ j = 1, 2, 3, 4, 5, 6, 7, 8) \tag{2.26}$$

The expression of 8×8 determinant $|a_{ij}|$ may be written as
(bonded interface)

$$\Delta = \alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) - \alpha_{12}(\alpha_{21}\alpha_{33} - \alpha_{23}\alpha_{31}) + \alpha_{13}(\alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31}) \tag{2.27}$$

and (unbonded interface)

$$\Delta = \beta_{11}\beta_{22} - \beta_{12}\beta_{21}, \tag{2.28}$$

where

$$\begin{aligned}
\alpha_{11} &= g_{41}m_2' - g_{42}m_1' + g_{43}m_3 + g_{44}m_2 + g_{45}m_1 - g_{46}m_3m_4m_2' + g_{47}m_2m_4m_2' + \\
&\quad g_{48}m_1m_4m_2' + g_{49}m_3m_4m_1' - g_{50}m_2m_4m_1' - g_{51}m_1m_4m_1', \\
\alpha_{12} &= g_{41}m_3' - g_{52}m_1' + g_{53}m_3 + g_{54}m_2 + g_{55}m_1 - g_{56}m_3m_4m_3' + g_{57}m_2m_4m_3' + \\
&\quad g_{58}m_1m_4m_3' + g_{59}m_3m_4m_1' - g_{60}m_2m_4m_1' - g_{61}m_1m_4m_1', \\
\alpha_{13} &= g_{62}m_4m_1' + g_{63}m_1'm_4' - g_{64}m_3m_4 - g_{65}m_3m_4' + g_{66}m_2m_4 + g_{67}m_2m_4' + \\
&\quad g_{68}m_1m_4 + g_{69}m_1m_4' - g_{70}, \\
\alpha_{21} &= g_{74}m_2' - g_{75}m_1' + g_{76}m_3 + g_{77}m_2 + g_{78}m_1 + g_{79}m_4m_1'm_2' - g_{80}m_3m_4m_2' + \\
&\quad g_{81}m_2m_4m_2' + g_{82}m_1m_4m_2' + g_{83}m_3m_4m_1' - g_{84}m_2m_4m_1' - g_{85}m_1m_4m_1',
\end{aligned}$$

$$\begin{aligned}
\alpha_{22} &= g_{86}m_3' - g_{87}m_1' + g_{88}m_3 + g_{89}m_2 + g_{90}m_1 + g_{91}m_4m_1'm_3' - g_{80}m_3m_4m_3' + \\
&\quad g_{81}m_2m_4m_3' + g_{82}m_1m_4m_3' + g_{95}m_3m_4m_1' - g_{96}m_2m_4m_1' - g_{97}m_1m_4m_1', \\
\alpha_{23} &= g_{98}m_4m_1' + g_{99}m_1'm_4' - h_{11}m_3m_4 - h_{12}m_3m_4' + h_{13}m_2m_4 + h_{14}m_2m_4' + \\
&\quad h_{15}m_1m_4 + h_{16}m_1m_4', \\
\alpha_{31} &= h_{17}m_2' - h_{18}m_1' + h_{19}m_3 + h_{20}m_2 + g_{78}m_1 + g_{79}m_4m_1'm_2' - g_{80}m_3m_4m_2' + \\
&\quad g_{81}m_2m_4m_2' + g_{82}m_1m_4m_2' + g_{83}m_3m_4m_1' - g_{84}m_2m_4m_1' - g_{85}m_1m_4m_1', \\
\alpha_{32} &= h_{29}m_3' - h_{30}m_1' + h_{31}m_3 + h_{32}m_2 + h_{33}m_1 + h_{34}m_4m_1'm_3' - h_{35}m_3m_4m_3' + \\
&\quad h_{36}m_2m_4m_3' + h_{37}m_1m_4m_3' + h_{38}m_3m_4m_1' - h_{39}m_2m_4m_1' - h_{40}m_1m_4m_1', \\
\alpha_{33} &= h_{41}m_4m_1' + h_{42}m_1'm_4' - h_{43}m_3m_4 - h_{44}m_3m_4' + h_{45}m_2m_4 + h_{46}m_2m_4' + \\
&\quad h_{47}m_1m_4 + h_{48}m_1m_4',
\end{aligned}$$

$$h_{11} = f_{32}g_{15}, \quad h_{12} = f_{32}g_{16}, \quad h_{13} = f_{33}g_{15}, \quad h_{14} = f_{33}g_{16}, \quad h_{15} = f_{34}g_{15},$$

$$h_{16} = f_{34}g_{16}, \quad h_{17} = f_{55}g_{11}, \quad h_{18} = f_{51}g_{13}, \quad h_{19} = (f_{52}g_{13} - f_{56}g_{11}),$$

$$h_{20} = (f_{57}g_{11} - f_{53}g_{13}), \quad h_{21} = (f_{58}g_{11} - f_{54}g_{13}), \quad h_{22} = (f_{51}g_{12} - f_{55}g_{12}),$$

$$h_{23} = f_{52}g_{12}, \quad h_{24} = f_{53}g_{12}, \quad h_{25} = f_{54}g_{12}, \quad h_{26} = f_{56}g_{12}, \quad h_{27} = f_{57}g_{12},$$

$$h_{28} = f_{58}g_{12}, \quad h_{29} = f_{61}g_{11}, \quad h_{30} = f_{51}g_{14}, \quad h_{31} = (f_{52}g_{14} - f_{62}g_{11}),$$

$$h_{32} = (f_{63}g_{11} - f_{53}g_{14}), \quad h_{33} = (f_{64}g_{11} - f_{54}g_{14}), \quad h_{34} = (f_{51}g_{12} - f_{61}g_{12}),$$

$$h_{35} = f_{52}g_{12}, \quad h_{36} = f_{53}g_{12}, \quad h_{37} = f_{54}g_{12}, \quad h_{38} = f_{62}g_{12}, \quad h_{39} = f_{63}g_{12},$$

$$h_{40} = f_{64}g_{12}, \quad h_{41} = f_{51}g_{15}, \quad h_{42} = f_{51}g_{16}, \quad h_{43} = f_{52}g_{15}, \quad h_{44} = f_{52}g_{16},$$

$$h_{45} = f_{53}g_{15}, \quad h_{46} = f_{53}g_{16}, \quad h_{47} = f_{54}g_{15}, \quad h_{48} = f_{54}g_{16}, \quad g_{11} = f_{12}k_t^2,$$

$$g_{12} = 2f_{11}g_{41}, \quad g_{13} = f_{13}k_t^2, \quad g_{14} = f_{14}k_t^2, \quad g_{15} = 2f_{11}g_{44}, \quad g_{16} = f_{11}k_t^2,$$

$$g_{21} = (f_{11}k_t^2 - 2f_{15}b_{41}), \quad g_{22} = f_{16}k_t^2, \quad g_{23} = f_{17}k_t^2, \quad g_{24} = f_{18}k_t^2, \quad g_{26} = f_{19}k_t^2,$$

$$g_{27} = f_{20}k_t^2, \quad g_{28} = f_{21}k_t^2, \quad g_{32} = f_{22}k_t^2, \quad g_{33} = f_{23}k_t^2, \quad g_{34} = f_{24}k_t^2, \quad g_{35} = 2f_{15}b_{44},$$

$$g_{41} = g_{11}g_{21}, \quad g_{42} = g_{21}g_{13}, \quad g_{43} = (g_{22}g_{13} - g_{11}g_{26}), \quad g_{44} = (g_{27}g_{11} - g_{13}g_{23}),$$

$g_{45} = (g_{28}g_{11} - g_{13}g_{24}),\quad g_{46} = g_{12}g_{22},\quad g_{47} = g_{23}g_{12},\ g_{48} = g_{12}g_{24},\ g_{49} = g_{26}g_{12},$

$g_{50} = g_{27}g_{12},\quad g_{51} = g_{28}g_{12},\quad g_{53} = (g_{22}g_{14} - g_{11}g_{32}),\quad g_{54} = (g_{33}g_{11} - g_{14}g_{23}),$

$g_{52} = g_{14}g_{21},\quad g_{55} = (g_{11}g_{34} - g_{12}g_{22}),\ g_{56} = g_{46},\ g_{57} = g_{47},\ g_{58} = g_{48},\ g_{59} = g_{49},$

$g_{60} = g_{50},\quad g_{61} = g_{51},\quad g_{62} = (g_{21}g_{15} + g_{12}g_{35}),\quad g_{63} = g_{21}g_{16},\quad g_{64} = g_{15}g_{22},$

$g_{65} = g_{22}g_{16},\ g_{66} = g_{15}g_{23},\ g_{67} = g_{23}g_{16},\ g_{68} = g_{24}g_{15},\ g_{69} = g_{16}g_{24},\ g_{70} = g_{11}g_{35},$

$g_{74} = g_{11}f_{35},\quad g_{75} = g_{13}f_{31},\quad g_{76} = (g_{13}f_{32} - g_{11}f_{36}),\quad g_{77} = (g_{11}f_{37} - g_{13}f_{33}),$

$g_{78} = (g_{11}f_{38} - g_{13}f_{34}),\quad g_{79} = (g_{12}f_{31} - g_{12}f_{35}),\quad g_{80} = g_{12}f_{32},\quad g_{81} = g_{12}f_{33},$

$g_{82} = g_{12}f_{34},\ g_{83} = g_{12}f_{36},\ g_{84} = g_{12}f_{37},\ g_{85} = g_{12}f_{38},\ g_{86} = g_{11}f_{41},\ g_{87} = g_{14}f_{31},$

$g_{88} = (g_{14}f_{32} - g_{11}f_{42}),\quad g_{89} = (g_{11}f_{43} - g_{14}f_{33}),\quad g_{95} = g_{12}f_{42},\quad g_{96} = g_{12}f_{43},$

$g_{90} = (g_{11}f_{44} - g_{14}f_{34}),\quad g_{91} = (g_{12}f_{31} - g_{12}f_{41}),\quad g_{97} = g_{13}f_{44},\quad g_{98} = g_{15}f_{31},$

$g_{99} = g_{16}f_{31},\quad f_{11} = e_{11}e_{22},\quad f_{12} = (e_{11}e_{23} - e_{21}e_{12}),\quad f_{13} = (e_{11}e_{24} - e_{21}e_{13}),$

$f_{14} = (e_{11}e_{25} - e_{21}e_{14}),\quad f_{15} = e_{11}e_{33},\quad f_{16} = e_{12}e_{22},\quad f_{17} = (e_{31}e_{12} - e_{11}e_{34}),$

$f_{18} = (e_{11}e_{35} - e_{12}e_{32}),\quad f_{19} = e_{13}e_{22},\quad f_{22} = e_{22}e_{14},\quad f_{20} = (e_{31}e_{13} - e_{11}e_{36}),$

$f_{21} = (e_{11}e_{37} - e_{32}e_{13}),\quad f_{23} = (e_{31}e_{14} - e_{11}e_{38}),\quad f_{31} = e_{11}e_{44},\quad f_{32} = e_{41}e_{12},$

$f_{24} = (e_{11}e_{39} - e_{32}e_{14}),\quad f_{33} = (e_{42}e_{12} - e_{11}e_{45}),\quad f_{35} = e_{11}e_{47},\quad f_{36} = e_{13}e_{41},$

$f_{34} = (e_{11}e_{46} - e_{22}e_{14}),\quad f_{37} = (e_{42}e_{13} - e_{11}e_{48}),\quad f_{41} = e_{11}e'_{41},\quad f_{42} = e_{41}e_{14},$

$f_{38} = (e_{11}e_{49} - e_{43}e_{13}),\quad f_{43} = (e_{42}e_{14} - e_{11}e'_{42}),\quad f_{51} = e_{11}e_{64},\quad f_{52} = e_{12}e_{61},$

$f_{44} = (e_{11}e'_{43} - e_{43}e_{14}),\quad f_{53} = (e_{12}e_{62} - e_{11}e_{65}),\quad f_{55} = e_{11}e_{67},\quad f_{56} = e_{13}e_{61},$

$f_{54} = (e_{11}e_{66} - e_{12}e_{63}),\quad f_{57} = (e_{13}e_{62} - e_{11}e_{68}),\quad f_{61} = e_{11}e'_{61},\quad f_{62} = e_{61}e_{14},$

$f_{58} = (e_{11}e_{69} - e_{14}e_{63}),\quad f_{63} = (e_{14}e_{62} - e_{11}e'_{62}),\quad f_{64} = (e_{11}e'_{63} - e_{63}e_{14}),$

$e_{11} = (d_{11}d_{22} - d_{21}d_{12}),\quad e_{12} = (d_{11}d_{23} - d_{21}d_{13}),\quad e_{13} = (d_{11}d_{24} - d_{21}d_{14}),$

$e_{14} = (d_{11}d_{25} - d_{21}d_{15}),\quad e_{21} = (d_{11}d_{32} - d_{31}d_{12}),\quad e_{23} = (d_{11}d_{33} - d_{31}d_{13}),$

$$e_{22}=d_{11}b_{11},\quad e_{24}=(d_{11}d_{34}-d_{31}d_{14}),\quad e_{25}=(d_{11}d_{35}-d_{31}d_{15}),\quad e_{31}=d_{12}b_{11},$$

$$e_{32}=(d_{12}b_{12}-d_{11}b_{13}),\quad e_{33}=d_{11}d_{41},\quad e_{34}=d_{13}b_{11},\quad e_{35}=(d_{13}b_{12}-d_{11}b_{15}),$$

$$e_{36}=d_{14}b_{11},\quad e_{37}=(d_{14}b_{12}-d_{11}b_{15}),\quad e_{38}=d_{15}b_{11},\quad e_{39}=(d_{15}b_{12}-d_{11}b_{17}),$$

$$e_{41}=d_{11}d_{53},\ e_{42}=d_{12}d_{51},\ e_{43}=(d_{52}d_{12}-d_{11}b_{54}),\ e_{44}=d_{11}d_{55},\ e_{45}=d_{13}d_{51},$$

$$e_{46}=(d_{13}d_{52}-d_{11}d_{56}),\quad e_{47}=d_{11}d_{57},\quad e_{48}=d_{14}d_{51},\quad e_{49}=(d_{14}d_{52}-d_{11}d_{58}),$$

$$e'_{41}=d_{11}d_{59},\ e'_{42}=d_{15}d_{51},\ e'_{43}=(d_{15}d_{52}-d_{11}b_{50}),\ e_{61}=d_{11}d_{63},\ e_{62}=d_{12}d_{61},$$

$$e_{63}=(d_{12}d_{62}-d_{11}d_{64}),\quad e_{64}=d_{11}d_{65},\quad e_{65}=d_{13}d_{61},\quad e_{66}=(d_{62}d_{13}-d_{11}b_{66}),$$

$$e_{67}=d_{11}d_{67},\ e_{68}=d_{14}d_{61},\ e_{69}=(d_{14}d_{62}-d_{11}d_{68}),\ e'_{61}=d_{11}d_{69},\ e'_{62}=d_{15}d_{61},$$

$$e'_{63}=(d_{15}d_{62}-d_{11}d_{60}),\quad d_{11}=(b_{11}L_2-b_{12}L_1),\quad d_{12}=(b_{11}L_3-b_{13}L_1),$$

$$d_{13}=(-\chi_1 b_{11}L'_1-b_{15}L_1),\ d_{14}=(-\chi_1 b_{11}L'_2-b_{16}L_1),\ d_{15}=(-\chi_1 b_{11}L'_3-b_{17}L_1),$$

$$d_{21}=(b_{11}V_2-b_{12}V_1),\quad d_{22}=(b_{11}V_3-b_{13}V_1),\quad d_{23}=(-\chi_2 b_{11}V'_1-b_{15}V_1),$$

$$d_{24}=(-\chi_2 b_{11}V'_2-b_{16}V_1),\quad d_{25}=(-\chi_2 b_{11}V'_3-b_{17}V_1),\quad d_{31}=k(b_{11}-b_{12}),$$

$$d_{32}=k(b_{11}-b_{13}),\ d_{33}=-k(b_{11}+b_{15}),\ d_{34}=-k(b_{11}+b_{16}),\ d_{35}=-k(b_{11}+b_{17}),$$

$$d_{41}=kb_{11},\ d_{51}=b_{11}V_2,\ d_{52}=b_{12}V_1,\ d_{53}=b_{11}V_3,\ d_{54}=b_{13}V_1,\ d_{55}=\chi_2 b_{11}V'_1,$$

$$d_{56}=b_{15}V_1,\ d_{57}=\chi_2 b_{11}V'_2,\ d_{58}=b_{16}V_1,\ d_{59}=\chi_2 b_{11}V'_3,\ d_{50}=b_{17}V_1,\ d_{61}=b_{11}L_2,$$

$$d_{62}=b_{12}L_1,\ d_{63}=b_{11}L_3,\ d_{64}=b_{13}L_1, d_{65}=\chi_1 b_{11}L'_1, d_{66}=b_{15}L_1,\ d_{67}=\chi_1 b_{11}L'_2,$$

$$d_{68}=b_{16}L_1,\ d_{69}=\chi_1 b_{11}L'_3,\ d_{60}=b_{17}L_1,\ b_{11}=(a_{11}-k^2),\ b_{12}=(a_{12}-k^2),$$

$$b_{13}=(a_{13}-k^2),\quad b_{15}=(k^2-a_{15}),\quad b_{16}=(k^2-a_{16}),\quad b_{17}=(k^2-a_{17}),$$

$$b_{41}=k(\mu-\mu')/\mu, b_{44}=(kb_{41}\mu+\mu' k_t'^2)/\mu,\ \beta_{11}=(q_{22}q_{26}-q_{25}q_{23}), q_{24}=-q_{11}a_{88},$$

$$\beta_{12}=(q_{22}q_{27}-q_{25}q_{24}),\quad \beta_{21}=(q_{22}q_{29}-q_{28}q_{23}),\quad \beta_{22}=(q_{22}q_{30}-q_{28}q_{24}),$$

$$q_{22}=(q_{11}a_{86}-q_{12}a_{85}),\quad q_{23}=(q_{11}a_{87}-q_{13}a_{85}),\quad q_{25}=(q_{11}q_{15}-q_{12}q_{14}),$$

$$q_{26}=(q_{11}q_{16}-q_{13}q_{14}),\quad q_{11}=(p_{74}p_{79}-p_{75}p_{78}),\quad q_{12}=(p_{74}p_{80}-p_{76}p_{78}),$$

$q_{13} = (p_{74}p_{81} - p_{77}p_{78}), \quad q_{14} = (p_{74}p_{83} - p_{75}p_{82}), \quad q_{15} = (p_{74}p_{84} - p_{76}p_{82}),$

$q_{16} = (p_{74}p_{85} - p_{77}p_{82}), \quad q_{27} = q_{11}q_{17}, \quad q_{17} = p_{74}p_{88}, \quad q_{18} = (p_{74}p_{88} - p_{75}p_{87}),$

$q_{19} = (p_{74}p_{89} - p_{76}p_{87}), \quad q_{20} = (p_{74}p_{80} - p_{77}p_{87}), \quad q_{21} = p_{74}p_{91}, \quad p_{13} = -a_{74}a_{31},$

$p_{11} = (a_{71}a_{32} - a_{72}a_{31}), \quad p_{12} = (a_{71}a_{33} - a_{73}a_{31}), \quad p_{14} = a_{71}a_{35}, \quad p_{15} = a_{71}a_{36},$

$p_{16} = a_{71}a_{37}, \quad p_{17} = (a_{71}a_{42} - a_{72}a_{41}), \quad p_{18} = (a_{71}a_{43} - a_{73}a_{41}), \quad p_{19} = -a_{74}a_{41},$

$p_{20} = a_{71}a_{45}, \quad p_{21} = a_{71}a_{46}, \quad p_{23} = (a_{71}a_{52} - a_{72}a_{51}), \quad p_{24} = (a_{71}a_{53} - a_{73}a_{51}),$

$p_{22} = a_{71}a_{47}, \quad p_{25} = -a_{74}a_{51}, \; p_{26} = a_{71}a_{55}, \; p_{27} = a_{71}a_{56}, \quad p_{29} = (a_{71}a_{62} - a_{72}a_{61}),$

$p_{28} = a_{71}a_{57}, \quad p_{30} = (a_{71}a_{63} - a_{73}a_{61}), \; p_{31} = -a_{74}a_{61}, \; p_{32} = a_{71}a_{65}, \quad p_{33} = a_{71}a_{66},$

$p_{34} = a_{71}a_{57}, \quad p_{35} = a_{21}(a_{71} - a_{72}), \quad p_{36} = a_{21}(a_{71} - a_{73}), \quad p_{37} = (a_{71}a_{24} - a_{74}a_{21}),$

$p_{38} = -a_{71}a_{21}, \quad p_{39} = a_{71}a_{28}, \quad p_{40} = (a_{71}a_{12} - a_{72}a_{11}), \quad p_{41} = (a_{71}a_{13} - a_{73}a_{11}),$

$p_{42} = (a_{71}a_{14} - a_{74}a_{11}), \quad p_{43} = -a_{71}a_{15}, \quad p_{44} = -a_{71}a_{16}, \quad p_{47} = (p_{11}p_{18} - p_{17}p_{12}),$

$p_{45} = -a_{71}a_{17}, \quad p_{46} = -a_{71}a_{18}, \quad p_{48} = (p_{11}p_{19} - p_{17}p_{13}), \quad p_{49} = (p_{11}p_{20} - p_{17}p_{14}),$

$p_{50} = (p_{11}p_{21} - p_{17}p_{15}), \quad p_{51} = (p_{11}p_{22} - p_{17}p_{16}), \quad p_{52} = (p_{11}p_{24} - p_{23}p_{12}),$

$p_{53} = (p_{11}p_{25} - p_{23}p_{13}), \quad p_{54} = (p_{11}p_{26} - p_{23}p_{14}), \quad p_{55} = (p_{11}p_{27} - p_{23}p_{15}),$

$p_{56} = (p_{11}p_{28} - p_{23}p_{16}), \quad p_{57} = (p_{11}p_{30} - p_{29}p_{12}), \quad p_{58} = (p_{11}p_{31} - p_{29}p_{13}),$

$p_{59} = (p_{11}p_{32} - p_{29}p_{14}), \quad p_{60} = (p_{11}p_{33} - p_{29}p_{15}), \quad p_{61} = (p_{11}p_{34} - p_{29}p_{16}),$

$p_{62} = (p_{11}p_{36} - p_{35}p_{12}), \quad p_{63} = (p_{11}p_{37} - p_{35}p_{13}), \quad p_{64} = (p_{11}p_{38} - p_{35}p_{14}),$

$p_{65} = (p_{11}p_{38} - p_{35}p_{15}), \quad p_{66} = (p_{11}p_{38} - p_{35}p_{16}), \quad p_{68} = (p_{11}p_{41} - p_{40}p_{12}),$

$p_{69} = (p_{11}p_{42} - p_{40}p_{13}), \quad p_{70} = (p_{11}p_{43} - p_{40}p_{14}), \quad p_{71} = (p_{11}p_{44} - p_{40}p_{15}),$

$p_{67} = p_{11}p_{39}, \quad p_{72} = (p_{11}p_{45} - p_{40}p_{16}), \quad p_{73} = p_{11}p_{46}, \quad p_{74} = (p_{47}p_{53} - p_{52}p_{48}),$

$p_{75} = (p_{47}p_{54} - p_{52}p_{49}), \quad p_{76} = (p_{47}p_{55} - p_{52}p_{50}), \quad p_{77} = (p_{47}p_{56} - p_{52}p_{51}),$

$p_{78} = (p_{47}p_{58} - p_{57}p_{48}), \quad p_{79} = (p_{47}p_{59} - p_{57}p_{49}), \quad p_{80} = (p_{47}p_{60} - p_{57}p_{50}),$

$$p_{81} = (p_{47}p_{61} - p_{57}p_{51}), \qquad p_{82} = (p_{47}p_{63} - p_{62}p_{48}), \qquad p_{83} = (p_{47}p_{64} - p_{62}p_{49}),$$

$$p_{84} = (p_{47}p_{65} - p_{62}p_{50}), \qquad p_{85} = (p_{47}p_{66} - p_{62}p_{51}), \qquad p_{87} = (p_{47}p_{69} - p_{68}p_{48}),$$

$$p_{88} = (p_{47}p_{70} - p_{68}p_{49}), \qquad p_{89} = (p_{47}p_{71} - p_{65}p_{50}), \qquad p_{90} = (p_{47}p_{72} - p_{68}p_{51}),$$

$$p_{86} = p_{47}p_{67}, \quad p_{91} = p_{47}p_{73}.$$

Thus, the frequency equations for bonded and unbonded interface are respectively given by

$$\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) - \alpha_{12}(\alpha_{21}\alpha_{33} - \alpha_{23}\alpha_{31}) + \alpha_{13}(\alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31}) = 0$$

and

$$\beta_{11}\beta_{22} - \beta_{12}\beta_{21} = 0.$$

2.6 Rayleigh Waves

Since Rayleigh waves propagate along the free surface of half-space, we neglect the half-space M' so that all the parameters corresponding to this materials are zero. In this case, the frequency equation of the Stoneley waves (2.26)-(2.28) are reduced to

$$f^2 - m_1^2 m_3^2 f_1^2 - f_2^2 m_2^2 m_4^2 - 2m_1 m_2 m_3 m_4 f_1 f_2 = 0, \tag{2.29}$$

where

$$f = m_1^2 m_3^2 (m_2^2 m_4^2 r_{51}^2 + r_{53}^2) - m_2^2 (m_1^2 r_{52}^2 + m_3^2 r_{54}^2), \;\; f_1 = 2r_{52}r_{54}m_2^2, \;\; f_2 = 2r_{53}r_{51}m_1^2 m_3^2,$$

$$r_{51} = 2k^2\{(V_1L_2 - V_2L_1)(V_1 - V_3) - (V_1L_3 - V_3L_1)(V_1 - V_2)\},$$

$$r_{52} = V_1 a_{13}(V_1L_2 - V_2L_1)(m_4^2 + k^2)/2, \quad r_{53} = V_1 a_{12}(V_1L_3 - V_3L_1)(m_4^2 + k^2)/2,$$

$$r_{54} = [a_{11}\{(V_1L_3 - V_3L_1)V_2 - (V_1L_2 - V_2L_1)V_3)\}](m_4^2 + k^2)/2.$$

This Equation (2.29) represents the frequency equation of the Rayleigh waves in the thermoelastic materials with voids.

2.7 Particular Cases

Case I: If the presence of voids are neglected, the problem reduces to the propagation of surface wave in thermoelastic materials. In this condition,

$$\alpha^* = \beta^* = \zeta^* = \kappa^* = \alpha'^* = \beta'^* = \zeta'^* = \kappa'^* = 0.$$

The frequency equations for the Stoneley waves for bonded and unbonded interface are obtained from (2.26) as

$$\begin{vmatrix} a_{11} & a_{12} & a_{14} & a_{15} & a_{16} & a_{18} \\ a_{21} & a_{22} & a_{24} & a_{25} & a_{26} & a_{28} \\ a_{31} & a_{32} & a_{34} & a_{35} & a_{36} & a_{38} \\ a_{41} & a_{42} & a_{44} & a_{45} & a_{46} & a_{48} \\ a_{71} & a_{72} & a_{74} & a_{75} & a_{76} & a_{78} \\ a_{81} & a_{82} & a_{84} & a_{85} & a_{86} & a_{88} \end{vmatrix} = 0$$

and the frequency equation for the Rayleigh wave is given by

$$a_{84}(a_{12}a_{31} - a_{11}a_{32}) + a_{14}(a_{32}a_{81} - a_{31}a_{82}) = 0 \tag{2.30}$$

with the void parameters in m_i, m_i', L_i, L_i', $(i = 1,\ 2)$ are vanished and the following modified values

$$a_{1i} = (\lambda+2\mu)m_i^2 - \lambda k^2 - \beta L_i, (i = 1, 2),\ a_{1i} = -(\lambda'+2\mu')m_{i-4}'^2 + \lambda' k'^2 + \beta' L_{i-4}', (i = 5, 6).$$

The frequency equation (2.30) for Rayleigh wave agrees with the result of Abouelregal (2011).

Case II : If the presence of thermal is neglected, the problem reduces to the propagation of surface wave in elastic materials containing voids. In this condition,

$$\kappa = \beta = a_e = m = T_0 = \kappa' = \beta' = a_e' = m' = T_0' = 0.$$

The frequency equations for the Stoneley waves for bonded and unbonded interface are obtained from (2.26) as

$$\begin{vmatrix} a_{11} & a_{13} & a_{14} & a_{15} & a_{17} & a_{18} \\ a_{21} & a_{23} & a_{24} & a_{25} & a_{27} & a_{28} \\ a_{51} & a_{53} & a_{54} & a_{55} & a_{57} & a_{58} \\ a_{61} & a_{63} & a_{64} & a_{65} & a_{67} & a_{68} \\ a_{71} & a_{73} & a_{74} & a_{75} & a_{77} & a_{78} \\ a_{81} & a_{83} & a_{84} & a_{85} & a_{87} & a_{88} \end{vmatrix} = 0$$

and the frequency equation for the Rayleigh wave is given by

$$a_{84}(a_{12}a_{51} - a_{11}a_{53}) + a_{14}(a_{53}a_{81} - a_{51}a_{83}) = 0 \qquad (2.31)$$

with the thermal parameters in m_i, m'_i, V_i, V'_i, $(i = 1, 3)$ are vanished and the following modified values

$$a_{1i} = (\lambda+2\mu)m_i^2 - \lambda k^2 + \beta^* V_i, (i = 1, 3),\ a_{1i} = -(\lambda'+2\mu')m'^2_{i-4} + \lambda' k'^2 - \beta^{*\prime} V'_{i-4}, (i = 5, 7).$$

The frequency equation (2.31) of the Rayleigh wave is similar with the result of Chandrasekharaiah (1987) for the relevant problem.

2.8 Numerical Results and Discussion

In order to discuss the frequency equations of Stoneley and Rayleigh waves in thermoelastic materials containing voids numerically, we consider the values of the parameters in the half-spaces, M (Dhaliwal and Singh, 1980) and M' (Singh, 2011) as given in Table 2.1.

We come to know that the frequency equation corresponding to the Stoneley waves are quite complicated and the exact value of the phase velocity can not be found easily. In this condition, we interpret the frequency curves with the help of determinant values given by Equations (2.27) for bonded and (2.28) for unbonded interface. Figures 2.1-2.4 and 2.5-2.8 discuss the effect of thermal parameter on the propagation of Stoneley and Rayleigh waves respectively.

Symbols(M)	Value	Symbols(M')	Value	Units
λ	2.17×10^{10}	λ'	2.12×10^{10}	Nm^{-2}
μ	3.278×10^{10}	μ'	3.17×10^{10}	Nm^{-2}
ρ	1.74×10^{3}	ρ'	3.8×10^{3}	kgm^{-3}
ξ^*	1.475×10^{10}	$\xi^{*\prime}$	2.25×10^{10}	Nm^{-2}
β^*	1.13849×10^{10}	$\beta^{*\prime}$	1.29×10^{10}	Nm^{-2}
α^*	3.688×10^{-5}	$\alpha^{*\prime}$	1.7×10^{-5}	N
κ^*	1.753×10^{-15}	$\kappa^{*\prime}$	1.57×10^{-15}	m^2
κ	1.7×10^{2}	κ'	1.14×10^{2}	$Wm^{-1}degree^{-1}$
β	2.68×10^{6}	β'	1.07×10^{6}	$Nm^{-2}degree^{-1}$
m	2×10^{6}	m'	2.28×10^{6}	$Nm^{-2}degree^{-1}$
c^*	1.8096×10^{6}	$c^{*\prime}$	1.04×10^{6}	$Jm^{-3}degree^{-1}$
T_0	298	T_0'	300	K

Table 2.1 Numerical values of the parameters

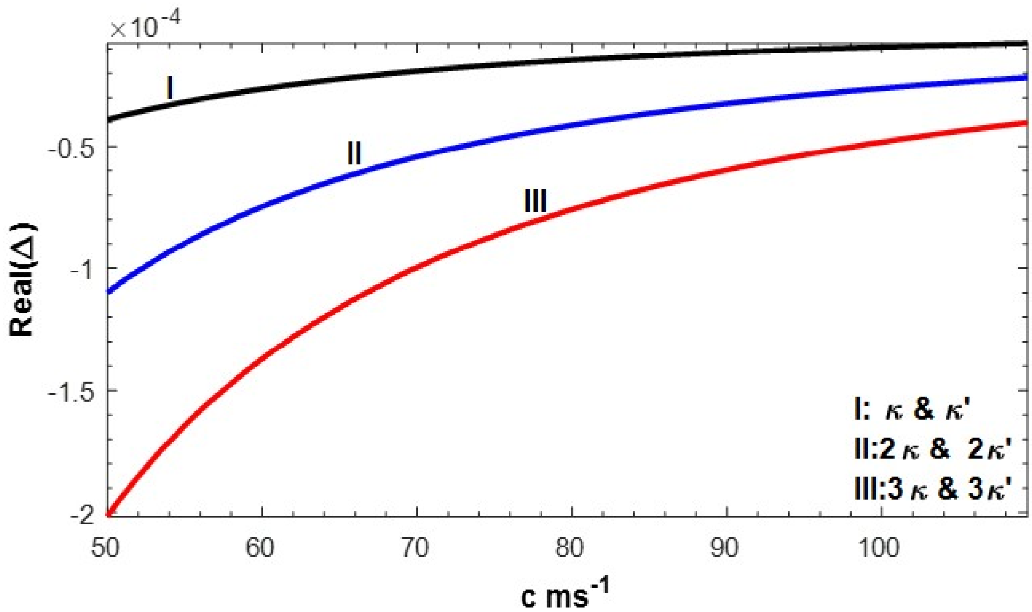


Figure 2.1: Variation of Real(Δ) with phase speed of Stoneley waves (bonded) at different values of κ & κ'.

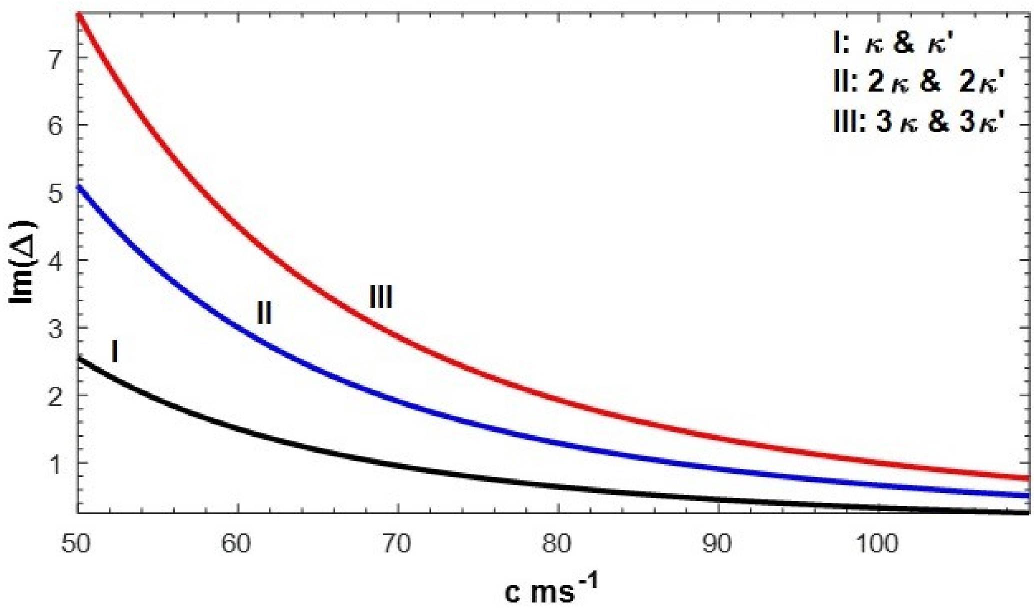


Figure 2.2: Variation of Image(Δ) with phase speed of Stoneley waves (bonded) at different values of κ & κ'.

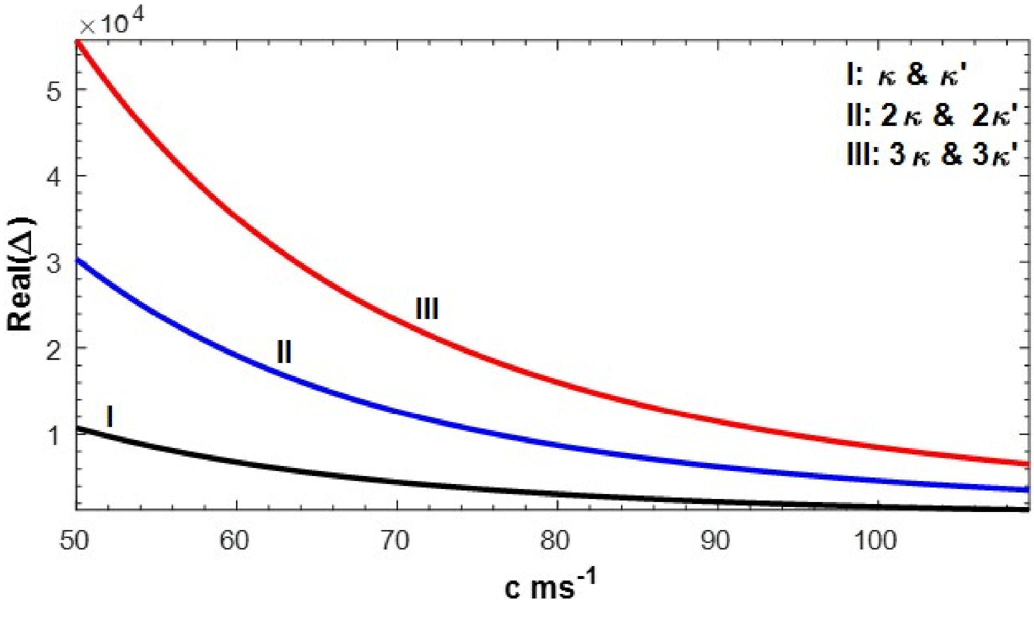


Figure 2.3: Variation of Real(Δ) with phase speed of Stoneley waves (unbonded) at different values of κ & κ'.

In Figures 2.1 and 2.2, the variation of the values of the real and imaginary part of the frequency equations of Stoneley waves for the bonded interface with the phase velocity at different values of ($\kappa,\ \kappa'$) are depicted. It may be noted that $\omega = 0.001$. It is observed that the values of real and imaginary values of Δ increases and decreases with the increase of phase velocity (c). With the increase of thermal parameters, Real(Δ) decreases, while Imag(Δ) increases. In the case of unbonded interface in Figures 2.3 and 2.4, the values of real and imaginary part of Δ decrease with the increase of phase velocity. With the increase of thermal parameters, these values increase. Thus, we have seen that the frequency equations of Stoneley waves depend on the thermal parameters.

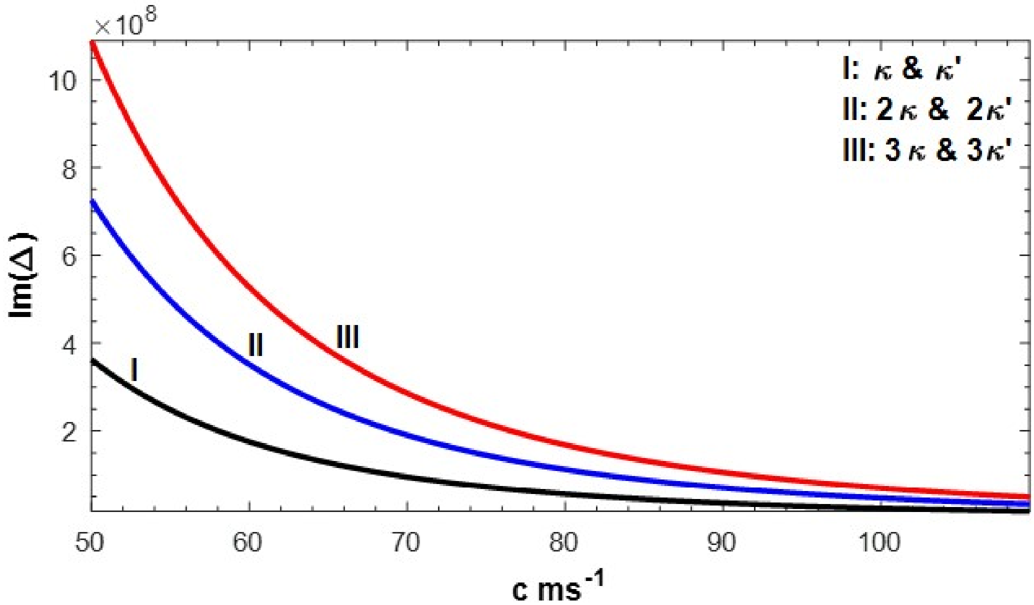


Figure 2.4: Variation of Image(Δ) with phase speed of Stoneley waves (unbonded) at different values of κ & κ'.

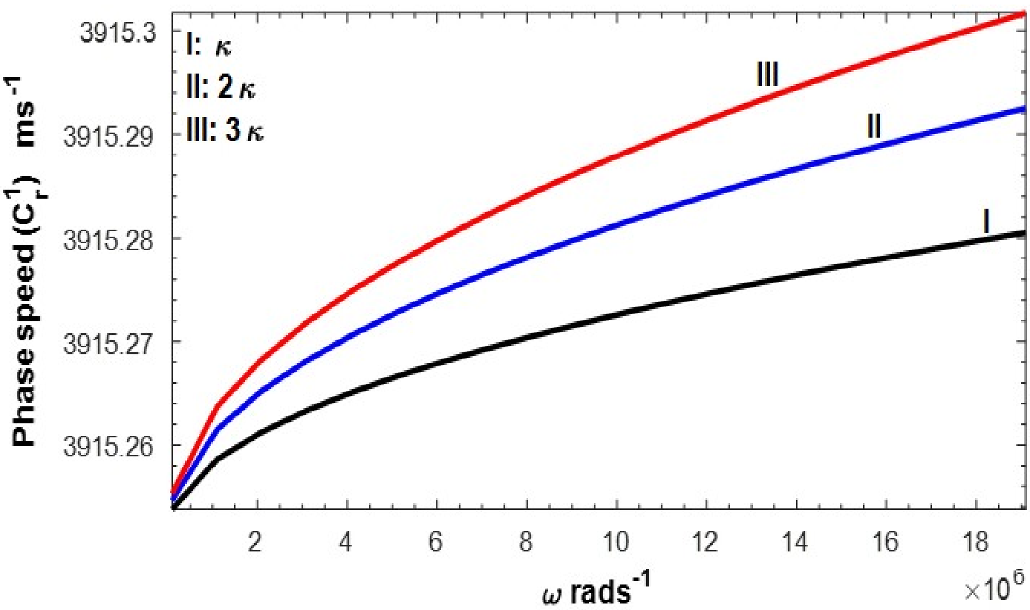


Figure 2.5: Variation of phase speed (C_r^1) of Rayleigh wave with angular frequency at different values of κ.

The velocity curves corresponding to the Rayleigh waves are given by Figures 2.5-2.8. We have observed two modes of phase velocity of this surface wave. In Figures 2.5 and 2.6, values of the two modes of phase velocitsy of the Rayleigh wave increase with the increase of angular frequency (ω) at different values of κ. These values increase with the increase of κ.

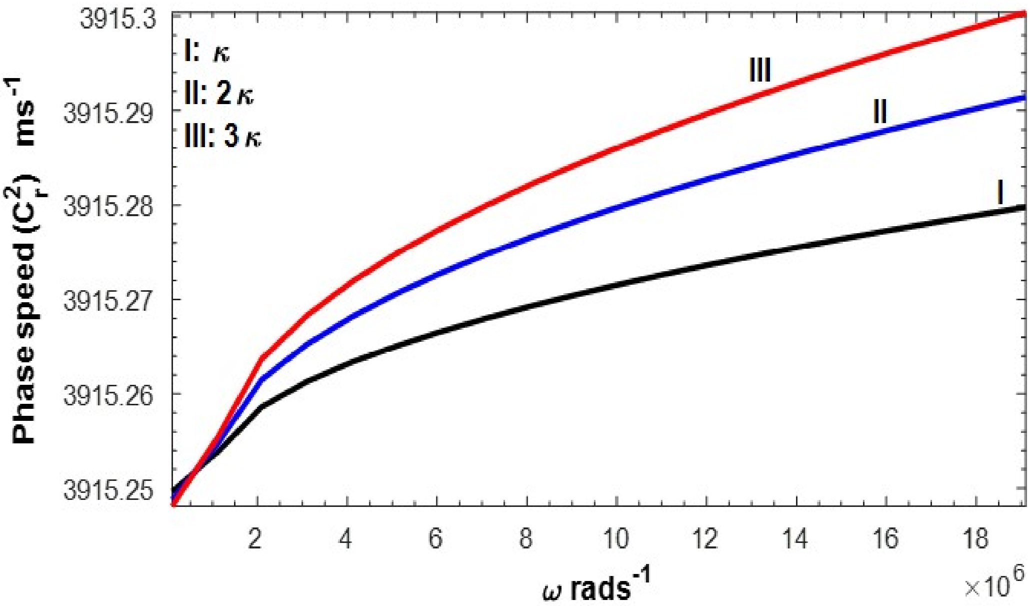


Figure 2.6: Variation of phase speed (C_r^2) of Rayleigh wave with angular frequency at different values of κ.

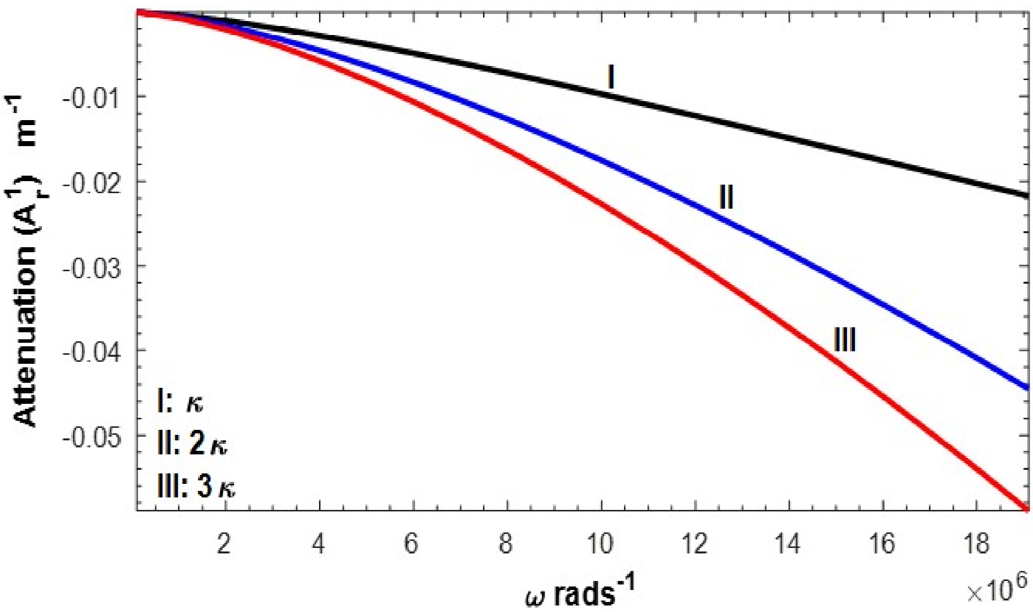


Figure 2.7: Variation of attenuation (A_r^1) of Rayleigh wave with angular frequency at different values of κ.

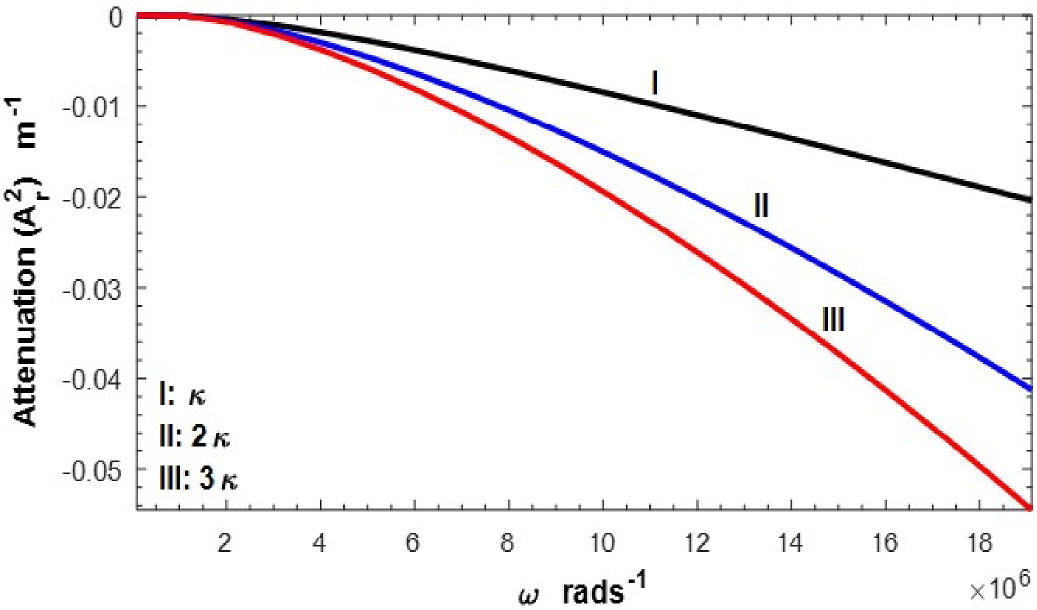


Figure 2.8: Variation of attenuation (A_r^2) of Rayleigh wave with angular frequency at different values of κ.

The attenuation coefficients of the Rayleigh waves corresponding to the two mode are depicted in Figures 2.7 and 2.8. These attenuation coefficients decrease with the increase of ω for different values of κ. With the increase of κ, the values of these coefficients decrease. Thus, the frequency equations depend on κ.

2.9 Conclusions

We have investigated the propagation of surface waves in the thermo-elastic materials with voids. The frequency equations of the Stoneley waves in the bonded and unbonded interface between two dissimilar half-spaces of thermo-elastic materials with voids are obtained. The frequency equation for the Rayleigh wave in such medium are also derived. The frequency curves for the Stoneley waves are examined numerically with the the help of determinant values. We also plot the velocity curves for the Rayleigh wave. We conclude with the following points

(i) The Stoneley waves between two dissimilar half-spaces of thermo-elastic materials with voids are dispersive in nature.

(ii) The real and imaginary values of determinant, Δ for the bonded interface of

Stoneley wave increases and decreases respectively with the increase of phase velocity.

(iii) The real part of Δ decreases with the increase of thermal parameters κ & κ', while imaginary value of Δ increases.

(iv) The real and imaginary values of Δ for the Stoneley wave in the unbonded interface decrease with the increase of phase velocity. These values increase with the increase of thermal parameters, κ & κ'.

(v) We have observed two modes of phase velocity for the Rayleigh wave in the thermoelastic materials with voids.

(vi) The two modes of phase velocity of the Rayleigh wave increase with the increase of angular frequency (ω) at different values of κ. These values also increase with the increase of κ.

(vii) The two modes of attenuation coefficients of the Rayleigh wave decrease with the increase of ω at different values of κ. It is observed that these coefficients decrease with the increase of κ.

Chapter 3

Transmission of elastic waves in initially stressed transversely isotropic thermoelastic solids[2]

3.1 Introduction

Thermoelasticity discusses heat conduction, strains, and thermal stresses in the materials with the inverse effect of temperature distribution. The study of thermoelastic material has been implemented in many important fields such as seismology, soil dynamics, physical sciences, aeronautics, atomic smasher, and nuclear reactors. Othman and Song (2007) formulated the governing equations for isotropic and homogeneous generalized thermoelastic half-space under hydrostatic initial stress using the Green and Naghdi theory of types II and III. They obtained the phase velocities of thermal, P and SV-waves. Singh (2010b) shown the existence of three plane quasi waves, namely, Quasi-Longitudinal (QL), Thermal (T-mode) and Quasi-Transverse (QT) waves in transversely isotropic thermoelastic solid with initial stresses and derived the amplitude ratios of the reflected waves from a plane free boundary of such medium.

In this chapter, we have studied the reflection/transmission of elastic waves in initially stressed transversely isotropic thermoelastic materials. Three quasi type

[2] ***Engineering Reports***, **e12104**, 1-14 (2020)

coupled longitudinal (QL), transverse(QT) and thermal waves are found to propagate in initially stressed transversely isotropic thermoelastic materials. For incident QL and QT-waves at a plane interface, boundary conditions were implemented for obtaining the coefficients of reflection/transmission, the distribution of energy in the reflected and transmitted waves are also discussed. Numerical computations have been performed and analyzed the impact of initial stresses on the amplitude and energy ratios of the reflected and transmitted waves. We have observed critical angles at $\theta_0 = 30^0$ and 58^0 for the reflected and transmitted QL-waves for incident QT-wave.

3.2 Basic Equations

Following Wang et al. (1997), the constitutive relations for prestressed bodies with generalized thermoelasticity are

$$\sigma_{ij} = c_{ijmn}e_{mn} + e_{jk}P_{ki} - \beta_{ij}T, \tag{3.1}$$

$$\rho\eta = \rho C_e T + \beta_{ij}e_{ij}, \tag{3.2}$$

$$q_i + \tau\dot{q}_i = -a_i T - K_{ij}T_{,j} - h_{ijk}e_{jk}, \tag{3.3}$$

$$e_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j}), \qquad (i,j,k,m,n = 1,2,3) \tag{3.4}$$

where σ_{ij}, q_i, η, P_{ki} and e_{ij} are stress, thermal flux, entropy, prestress and strain tensors respectively, c_{ijmn} and β_{ij}, K_{ij}, a_i, h_{ijk} are elastic and thermal coefficients respectively, the temperature is change from T_0 to T, u_i is the component of displacement vector of the material with density ρ, thermal relaxation time τ and specific heat C_e.

For the generalized thermoelastic materials under initial stresses with body force F_i and internal heat source S, the equations of motions are given as

$$\rho\ddot{u}_i = \sigma_{ji,j} + \rho F_i, \tag{3.5}$$

$$\rho T_0\dot{\eta} = -q_{i,i} + \rho S, \qquad (i,j = 1,2,3). \tag{3.6}$$

Without body forces as well as heat sources and using Eqs.(3.1)-(3.3) in (3.5) and (3.6), we get

$$\begin{aligned} (d_{ijmn}e_{mn} - \beta_{ij}T)_{,j} &= \rho\ddot{u}_i, && (3.7)\\ T_0(1+\tau\frac{\partial}{\partial t})(\beta_{ij}\dot{e}_{ij} + \rho C_e\dot{T}) &= (K_{ij}T_{,j} + h_{ijk}e_{jk} + a_iT)_{,i}, && (3.8) \end{aligned}$$

where $d_{ijmn} = c_{ijmn} + \delta_{jn}e_{mi}$ and δ_{jn} is the Kronecker's delta. Note that $a_i = 0$ and $h_{ijk} = 0$ for the uniform temperature prestressed bodies.

3.3 Wave Propagation

We consider Cartesian coordinates with x and y-axes lying horizontally and z-axis as vertically. Two half-spaces $M : 0 \le z < \infty$ and $M' : -\infty < z \le 0$ of transversely isotropic thermoelastic medium under initial stresses are assumed to analyze wave propagation in xz-plane. The diagrammatic structure of the problem is presented in Figure 3.1.

For half-space M, equations of motions are (Singh, 2010b)

$$\begin{aligned} d_{11}u_{1,11} + (d_{13}+d_{44})u_{3,13} + d_{44}u_{1,33} - \beta_1T_{,1} &= \rho\ddot{u}_1, && (3.9)\\ d_{44}u_{3,11} + (d_{13}+d_{44})u_{1,13} + d_{33}u_{3,33} - \beta_3T_{,3} &= \rho\ddot{u}_3, && (3.10)\\ T_0(1+\tau\frac{\partial}{\partial t})(\beta_1\dot{u}_{1,1} + \beta_3\dot{u}_{3,3} + d\dot{T}) = K_1T_{,11} &+ K_3T_{,33}, && (3.11) \end{aligned}$$

where $\mathbf{u} = (u_1, 0, u_3)$, $d_{11} = c_{11} + P_{11}$, $d_{13} = c_{13}$, $d_{33} = c_{33} + P_{33}$, $d_{44} = c_{44} + P_{11}$,$K_1 = K_{11}$, $K_3 = K_{33}$, $\beta_1 = \beta_{11} = (d_{11} + d_{12})\alpha_1 + d_{13}\alpha_3$, $\beta_3 = \beta_{33} = 2d_{13}\alpha_1 + d_{33}\alpha_3$, $d = \rho C_e$, α_1 and α_3 are linear thermal expansion coefficients.
Similarly, for M',

$$\begin{aligned} d'_{11}u'_{1,11} + (d'_{13}+d'_{44})u'_{3,13} + d'_{44}u'_{1,33} - \beta'_1T'_{,1} &= \rho'\ddot{u}'_1, && (3.12)\\ d'_{44}u'_{3,11} + (d'_{13}+d'_{44})u'_{1,13} + d'_{33}u'_{3,33} - \beta'_3T'_{,3} &= \rho'\ddot{u}'_3, && (3.13)\\ T'_0(1+\tau'\frac{\partial}{\partial t})(\beta'_1\dot{u}'_{1,1} + \beta'_3\dot{u}'_{3,3} + d'\dot{T}') = K'_1T'_{,11} &+ K'_3T'_{,33}, && (3.14) \end{aligned}$$

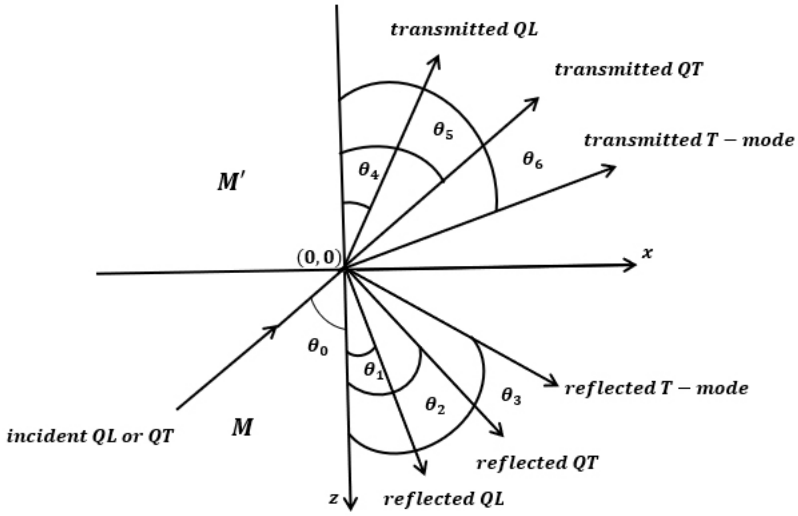


Figure 3.1: Geometry of the problem.

where $\beta_1' = \beta_{11}' = (d_{11}' + d_{12}')\alpha_1' + d_{13}'\alpha_3'$, $\beta_3' = \beta_{33}' = 2d_{13}'\alpha_1' + d_{33}'\alpha_3'$, α_1' and α_3' are due to thermal expansion, $\mathbf{u} = (u_1', 0, u_3')$, $d' = \rho' C_e'$, $d_{11}' = c_{11}' + P_{11}'$, $d_{13}' = c_{13}'$, $d_{33}' = c_{33}' + P_{33}'$, $d_{44}' = c_{44}' + P_{11}'$, $K_1' = K_{11}'$ and $K_3' = K_{33}'$.

For the incident, reflected and transmitted waves, we have

$$\langle u_1^{(n)}, u_3^{(n)}, T^{(n)} \rangle = \langle A_n d_1^{(n)}, A_n d_3^{(n)}, \imath k_n F^n A_n \rangle e^{\imath k_n \{x p_1^{(n)} + z p_3^{(n)} - v_n t\}}, \ n = 0 \text{ to } 6, \quad (3.15)$$

where A_n is the amplitude constant, $\langle d_1^{(n)}, 0, d_3^{(n)} \rangle$ and $\langle p_1^{(n)}, 0, p_3^{(n)} \rangle$ are unit displacement and propagation vectors respectively, k_n is wavenumber, v_n is phase velocity (Singh, 2010b). Note that $n = 0$ represents incident QL or QT wave, $n = 1, n = 2$ and $n = 3$ represent for reflected QL, QT and thermal waves (T-mode) respectively and $n = 4$, $n = 5$ and $n = 6$ represent for the transmitted QL, QT and thermal waves respectively. The coupling constant $F^{(n)}$ is given by

$$F^{(n)} = \begin{cases} \frac{(d_{11}p_1^{(n)^2} + d_{44}p_3^{(n)^2} - \rho v_n^2) d_1^{(n)} p_1^{(n)} + (d_{13} + d_{44}) p_1^{(n)^2} p_3^{(n)} d_3^{(n)}}{\beta_1} + \\ \frac{(d_{44}p_1^{(n)^2} + d_{33}p_3^{(n)^2} - \rho v_n^2) d_3^{(n)} p_3^{(n)} + (d_{13} + d_{44}) p_1^{(n)} p_3^{(n)^2} d_1^{(n)}}{\beta_3}, & n = 0, 1, 2, 3 \\ \frac{(d_{11}'p_1^{(n)^2} + d_{44}'p_3^{(n)^2} - \rho' v_n^2) d_1^{(n)} p_1^{(n)} + (d_{13}' + d_{44}') p_1^{(n)^2} p_3^{(n)} d_3^{(n)}}{\beta_1'} + \\ \frac{(d_{44}'p_1^{(n)^2} + d_{33}'p_3^{(n)^2} - \rho' v_n^2) d_3^{(n)} p_3^{(n)} + (d_{13}' + d_{44}') p_1^{(n)} p_3^{(n)^2} d_1^{(n)}}{\beta_3'}, & n = 4, 5, 6. \end{cases}$$

Using Snell's law, we can have (Singh, 2011)

$$k_0 \sin\theta_0 = k_r \sin\theta_r \text{ for } r = 1,2,3,4,5,6. \tag{3.16}$$

3.4 Boundary Conditions

The stress tractions, heat flow and displacement components are continuous at $z = 0$. We have

(i) Continuity of normal traction:

$$\sum_{r=0}^{3} d_{33}u_{3,3}^{(r)} + d_{13}u_{1,1}^{(r)} - \beta_3 T^{(r)} = \sum_{r=4}^{6} d'_{33}u_{3,3}^{(r)} + d'_{13}u_{1,1}^{(r)} - \beta'_3 T^{(r)}.$$

(ii) Continuity of shear traction:

$$\sum_{n=0}^{3} d_{44}(u_{1,3}^{(r)} + u_{3,1}^{(r)}) = \sum_{n=4}^{6} d'_{44}(u_{1,3}^{(r)} + u_{3,1}^{(r)}).$$

(iii) Continuity of heat flow:

$$\sum_{r=0}^{3} \dot{T}^{(r)} = \sum_{n=4}^{6} \dot{T}^{(r)}, \qquad \sum_{r=0}^{3} \frac{\partial T^{(r)}}{\partial z} = \sum_{r=4}^{6} \frac{\partial T^{(r)}}{\partial z}.$$

(iv) Continuity of displacement components:

$$\sum_{r=0}^{3} u_1^{(r)} = \sum_{r=4}^{6} u_1^{(r)}, \qquad \sum_{r=0}^{3} u_3^{(r)} = \sum_{r=4}^{6} u_3^{(r)}.$$

These boundary conditions may be reduced to

$$\sum_{r=0}^{3}\{k_r(d_{33}p_3^{(r)}d_3^{(r)} - \beta_3 F^{(r)}) + d_{13}k_0 p_1^{(0)} d_1^{(r)}\}A_r - \sum_{r=4}^{6}\{k_r(d'_{33}p_3^{(r)}d_3^{(r)} - \beta'_3 F^{(r)}) + d'_{13}k_0 p_1^{(0)} d_1^{(r)}\}A_r = 0, \tag{3.17}$$

$$\sum_{r=0}^{3} d_{44}(k_r p_3^{(r)} d_1^{(r)} + k_0 p_1^{(0)} d_3^{(r)})A_n - \sum_{r=4}^{6} d'_{44}(k_r p_3^{(r)} d_1^{(r)} + k_0 p_1^{(0)} d_3^{(r)})A_r = 0, \tag{3.18}$$

$$\sum_{r=0}^{3} k_r F^{(r)} A_r - \sum_{r=4}^{6} k_r F^{(r)} A_n = 0, \qquad \sum_{r=0}^{3} k_n^2 p_3^{(r)} F^{(r)} A_n - \sum_{r=4}^{6} k_r^2 p_3^{(r)} F^{(r)} A_n = 0, \tag{3.19}$$

$$\sum_{r=0}^{3} d_1^{(r)} A_r - \sum_{r=4}^{6} d_1^{(r)} A_r = 0, \qquad \sum_{r=0}^{3} d_3^{(r)} A_r - \sum_{r=4}^{6} d_3^{(r)} A_r = 0. \tag{3.20}$$

Equations (3.17)-(3.20) will help to find the reflection and transmission coefficients of the reflected and transmitted waves.

3.5 Amplitude Ratio

The matrix representation of Eqs (3.17)-(3.20) is given as

$$AZ = B, \tag{3.21}$$

where A is a 6×6 matrix, B and Z are 6×1 matrices with the following elements

$$a_{1r} = \begin{cases} k_r(d_{33}p_3^{(r)}d_3^{(r)} - \beta_3 F^{(r)}) + d_{13}k_0 p_1^{(0)} d_1^{(r)}, \; r = 1,2,3 \\ -k_r(d'_{33}p_3^{(r)}d_3^{(r)} - \beta'_3 F^{(r)}) - d'_{13}k_0 p_1^{(0)} d_1^{(r)}, \; r = 4,5,6 \end{cases},$$

$$a_{2r} = \begin{cases} d_{44}(k_r p_3^{(r)} d_1^{(r)} + k_0 p_1^{(0)} d_3^{(r)}), \; r = 1,2,3 \\ -d'_{44}(k_r p_3^{(r)} d_1^{(r)} + k_0 p_1^{(0)} d_3^{(r)}), \; r = 4,5,6 \end{cases}, \quad a_{3j} = \begin{cases} k_r F^{(r)}, \; r = 1,2,3 \\ -k_r F^{(r)}, \; r = 4,5,6 \end{cases},$$

$$a_{4r} = \begin{cases} d_1^{(r)}, \; r = 1,2,3 \\ -d_1^{(r)}, \; r = 4,5,6 \end{cases}, \quad a_{5r} = \begin{cases} d_3^{(r)}, \; r = 1,2,3 \\ -d_3^{(r)}, \; r = 4,5,6 \end{cases},$$

$$a_{6j} = \begin{cases} k_r^2 p_3^{(r)} F^{(r)}, \; r = 1,2,3 \\ -k_r^2 p_3^{(r)} F^{(r)}, \; r = 4,5,6 \end{cases}, \quad b_1 = -k_0(d_{33}p_3^{(0)}d_3^{(0)} + d_{13}p_1^{(0)}d_1^{(0)} - \beta_3 F^{(0)}),$$

$$b_2 = -d_{44}k_0(p_3^{(0)}d_1^{(0)} + p_1^{(0)}d_3^{(0)}), \quad b_3 = -k_0 F^{(0)}, \quad b_4 = -d_1^{(0)},$$

$$b_5 = -d_3^{(0)}, \quad b_6 = -k_0^2 p_3^{(0)} F^{(0)}, \quad Z_r = \frac{A_r}{A_0}.$$

Eq.(3.21) is solved for Z_r due to incident QL and QT-waves.

3.6 Energy Ratio

We have considered partition of energy at $z = 0$ and the rate of transmission is given by (Achenbach, 1976)

$$E^* = \langle \tau_{zz} \cdot \dot{u}_3 \rangle + \langle \tau_{zx} \cdot \dot{u}_1 \rangle. \tag{3.22}$$

Using Eq. (3.22), the energy ratios waves are

$$E_i = \frac{\eta_i}{\eta_0} Z_i^2, \qquad (i = 1, 2, 3, 4, 5, 6) \tag{3.23}$$

where

$$\eta_i = \begin{cases} d_3^{(i)}(d_{33}k_i d_3^{(i)} p_3^{(i)} + d_{13}k_0 d_1^{(i)} p_1^{(0)} - \beta_3 k_i F^{(i)}) + \\ d_{44} d_1^{(i)}(k_i d_1^{(i)} p_3^{(i)} + k_0 d_3^{(i)} p_1^{(0)}), & i = 0, 1, 2, 3 \\ d_3^{(i)}(d'_{33}k_i d_3^{(i)} p_3^{(i)} + d'_{13}k_0 d_1^{(i)} p_1^{(0)} - \beta'_3 k_i F^{(i)}) + \\ d'_{44} d_1^{(i)}(k_i d_1^{(i)} p_3^{(i)} + k_0 d_3^{(i)} p_1^{(0)}), & i = 4, 5, 6. \end{cases}$$

Note that E_r for $r = 1, 2, 3$ represent energy ratios of the reflected QL, QT and T-mode waves respectively and $r = 4, 5, 6$ represent for the transmitted QL, QT and T-mode waves respectively.

3.7 Particular Cases

CASE I: If $P_{11} = P_{33} = P'_{11} = P'_{33} = 0$, then $d_{ij} = c_{ij}$ and $d'_{ij} = c'_{ij}$. Eqs. (3.21) and (3.23) have the following modified values

$$a_{1j} = \begin{cases} k_j(c_{33}p_3^{(j)} d_3^{(j)} - \beta_3 F^{(j)}) + c_{13}k_0 p_1^{(0)} d_1^{(j)}, & j = 1, 2, 3 \\ -k_j(c'_{33}p_3^{(j)} d_3^{(j)} - \beta'_3 F^{(j)}) - c'_{13}k_0 p_1^{(0)} d_1^{(j)}, & j = 4, 5, 6 \end{cases},$$

$$a_{2j} = \begin{cases} c_{44}(k_j p_3^{(j)} d_1^{(j)} + k_0 p_1^{(0)} d_3^{(j)}), & j = 1, 2, 3 \\ -c'_{44}(k_j p_3^{(j)} d_1^{(j)} + k_0 p_1^{(0)} d_3^{(j)}), & j = 4, 5, 6 \end{cases},$$

$$b_1 = -k_0(c_{33}p_3^{(0)}d_3^{(0)} + c_{13}p_1^{(0)}d_1^{(0)} - \beta_3 F^{(0)}), \quad b_2 = -c_{44}k_0(p_3^{(0)}d_1^{(0)} + p_1^{(0)}d_3^{(0)}).$$

$$\eta_i = \begin{cases} d_3^{(i)}(c_{33}k_i d_3^{(i)}p_3^{(i)} + c_{13}k_0 d_1^{(i)}p_1^{(0)} - \beta_3 k_i F^{(i)}) + \\ c_{44}d_1^{(i)}(k_i d_1^{(i)}p_3^{(i)} + k_0 d_3^{(i)}p_1^{(0)}), & i = 0,1,2,3 \\ d_3^{(i)}(c'_{33}k_i d_3^{(i)}p_3^{(i)} + c'_{13}k_0 d_1^{(i)}p_1^{(0)} - \beta'_3 k_i F^{(i)}) + \\ c'_{44}d_1^{(i)}(k_i d_1^{(i)}p_3^{(i)} + k_0 d_3^{(i)}p_1^{(0)}), & i = 4,5,6. \end{cases}$$

CASE II: If M' is stress free, then

$$\begin{aligned} Z_1 &= \frac{b_1(a_{22}a_{63} - a_{23}a_{62}) - a_{12}(b_2a_{63} - a_{23}b_6) + a_{13}(b_2a_{62} - a_{22}b_6)}{a_{11}(a_{22}a_{63} - a_{23}a_{62}) - a_{12}(a_{21}a_{63} - a_{23}a_{61}) + a_{13}(a_{21}a_{62} - a_{22}a_{61})}, \\ Z_2 &= \frac{a_{11}(b_2a_{63} - a_{23}b_6) - b_1(a_{21}a_{63} - a_{23}a_{61}) + a_{13}(a_{21}b_6 - b_2a_{61})}{a_{11}(a_{22}a_{63} - a_{23}a_{62}) - a_{12}(a_{21}a_{63} - a_{23}a_{61}) + a_{13}(a_{21}a_{62} - a_{22}a_{61})}, \\ Z_3 &= \frac{a_{11}(a_{22}b_6 - b_2a_{62}) - a_{12}(a_{21}b_6 - b_2a_{61}) + b_1(a_{21}a_{62} - a_{22}a_{61})}{a_{11}(a_{22}a_{63} - a_{23}a_{62}) - a_{12}(a_{21}a_{63} - a_{23}a_{61}) + a_{13}(a_{21}a_{62} - a_{22}a_{61})}. \end{aligned} \tag{3.24}$$

These ratios exactly match Singh (2010b).

The distribution of energy $E_1,\ E_2$ and E_3 of the reflected waves are given by (3.23).

CASE III: If M' is stress free and $P_{11} = P_{33} = 0$. Equation(3.24) will be modified with the following changes

$$\begin{aligned} a_{1j} &= k_j(c_{33}p_3^{(j)}d_3^{(j)} - \beta_3 F^{(j)}) + c_{13}k_0 p_1^{(0)}d_1^{(j)}, \\ a_{2j} &= c_{44}(k_j p_3^{(j)}d_1^{(j)} + k_0 p_1^{(0)}d_3^{(j)}), \quad (j = 1,2,3) \\ b_1 &= -k_0(c_{33}p_3^{(0)}d_3^{(0)} + c_{13}p_1^{(0)}d_1^{(0)} - \beta_3 F^{(0)}), \\ b_2 &= -c_{44}k_0(p_3^{(0)}d_1^{(0)} + p_1^{(0)}d_3^{(0)}). \end{aligned}$$

The results are exactly same as Sharma (1988).

The energy ratios $E_1,\ E_2$ and E_3 are also given by (3.23) with the modified value of

$$\begin{aligned} \eta_i &= d_3^{(i)}(c_{33}k_i d_3^{(i)}p_3^{(i)} + c_{13}k_0 d_1^{(i)}p_1^{(0)} - \beta_3 k_i F^{(i)}) \\ &+ c_{44}d_1^{(i)}(k_i d_1^{(i)}p_3^{(i)} + k_0 d_3^{(i)}p_1^{(0)}), \quad (i = 0,1,2,3). \end{aligned}$$

3.8 Numerical Results

For evaluating the coefficients and energy distributions due to reflected and transmitted waves for incident QL and QT waves, we have used the relevant parametric values given in Table 3.1 (Chadwick and Seet, 1970).

Cobalt (M)	Value	Zinc (M')	Value	Units
ρ	8.836×10^3	ρ'	7.14×10^3	kgm^{-3}
c_{11}	3.071×10^{11}	c'_{11}	1.628×10^{11}	Nm^{-2}
c_{12}	1.650×10^{11}	c'_{12}	0.362×10^{11}	Nm^{-2}
c_{13}	1.027×10^{11}	c'_{13}	0.508×10^{11}	Nm^{-2}
c_{33}	3.581×10^{11}	c'_{33}	0.627×10^{11}	Nm^{-2}
c_{44}	0.755×10^{11}	c'_{44}	0.385×10^{11}	Nm^{-2}
β_1	7.04×10^6	β'_1	5.75×10^6	$Nm^{-2}degree^{-1}$
β_3	6.90×10^6	β'_3	5.17×10^6	$Nm^{-2}degree^{-1}$
C_e	4.27×10^2	C'_e	3.9×10^2	$Jkg^{-1}degree^{-1}$
K_1	0.690×10^2	K'_1	1.24×10^2	$Wm^{-1}degree^{-1}$
K_3	0.690×10^2	K'_3	1.24×10^2	$Wm^{-1}degree^{-1}$
T_0	298	T'_0	296	K
τ_0	0.05	τ'_0	0.06	

Table 3.1 Parametric Values

The unit propagation and displacement vectors are
(for incident quasi-longitudinal wave)

$$(p_1^{(0)}, 0, p_3^{(0)}) = (\sin\theta_0, 0, \cos\theta_0), \quad (d_1^{(0)}, 0, d_3^{(0)}) = (\sin\theta_0, 0, \cos\theta_0),$$

(for incident quasi-transverse wave)

$$(p_1^{(0)}, 0, p_3^{(0)}) = (\sin\theta_0, 0, \cos\theta_0), \quad (d_1^{(0)}, 0, d_3^{(0)}) = (\cos\theta_0, 0, -\sin\theta_0),$$

(for reflected waves)

$$(p_1^{(1)}, 0, p_3^{(1)}) = (\sin\theta_1, 0, -\cos\theta_1), \quad (d_1^{(1)}, 0, d_3^{(1)}) = (\sin\theta_1, 0, -\cos\theta_1),$$

$$(p_1^{(2)}, 0, p_3^{(2)}) = (\sin\theta_2, 0, -\cos\theta_2), \quad (d_1^{(2)}, 0, d_3^{(2)}) = (-\cos\theta_2, 0, -\sin\theta_2),$$

$$(p_1^{(3)}, 0, p_3^{(3)}) = (\sin\theta_3, 0, -\cos\theta_3), \quad (d_1^{(3)}, 0, d_3^{(3)}) = (\sin\theta_3, 0, -\cos\theta_3),$$

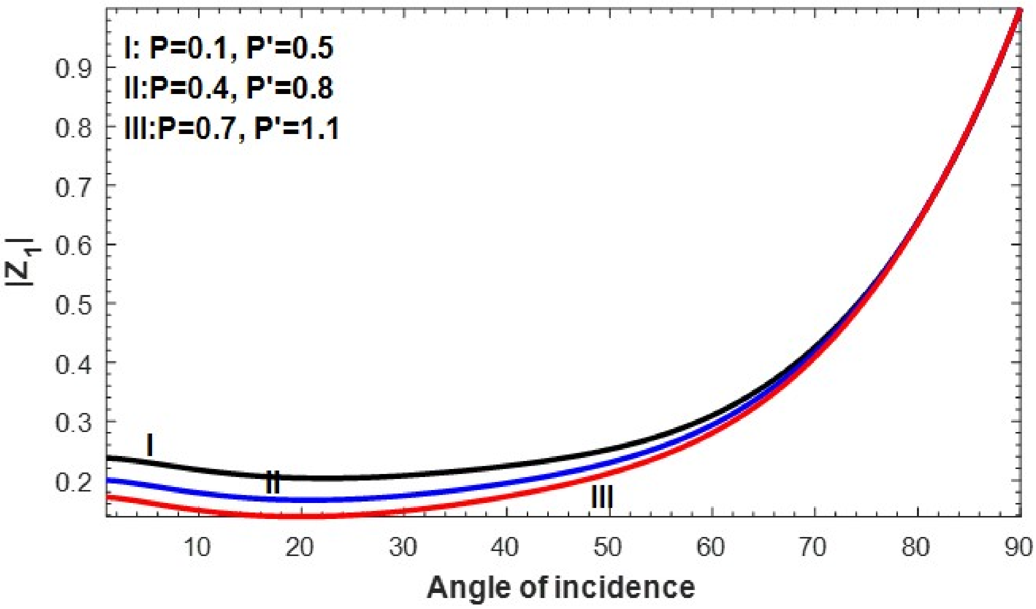


Figure 3.2: Variation of $|Z_1|$ with angle of incidence for different values of P and P'.

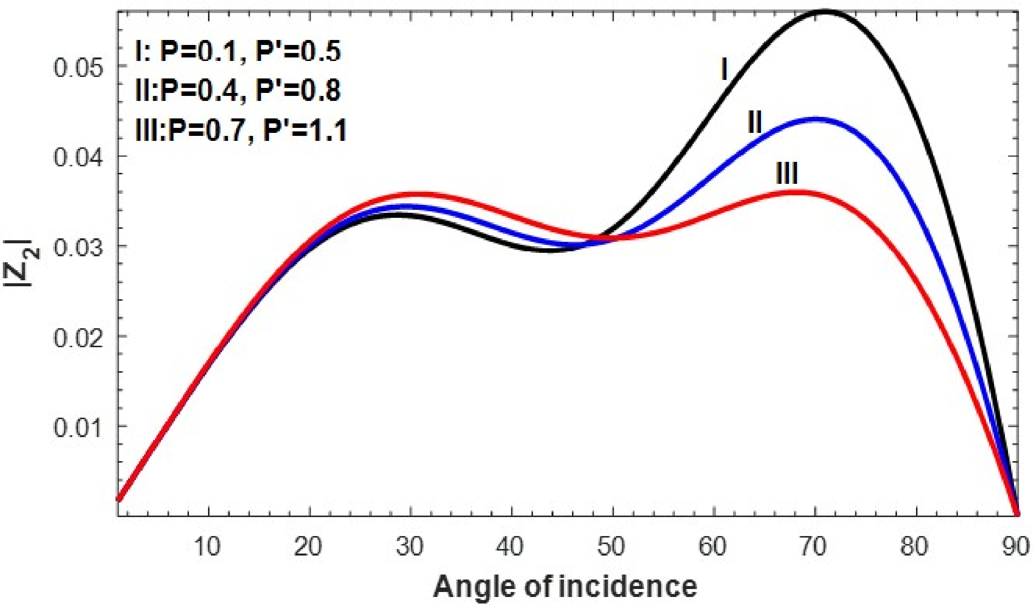


Figure 3.3: Variation of $|Z_2|$ with angle of incidence for different values of P and P'.

(for transmitted waves)

$$(p_1^{(4)}, 0, p_3^{(4)}) = (\sin\theta_4, 0, \cos\theta_4), \quad (d_1^{(4)}, 0, d_3^{(4)}) = (\sin\theta_4, 0, \cos\theta_4),$$

$$(p_1^{(5)}, 0, p_3^{(5)}) = (\sin\theta_5, 0, \cos\theta_5), \quad (d_1^{(5)}, 0, d_3^{(5)}) = (\cos\theta_5, 0, -\sin\theta_5),$$

$$(p_1^{(6)}, 0, p_3^{(6)}) = (\sin\theta_6, 0, \cos\theta_6), \quad (d_1^{(6)}, 0, d_3^{(6)}) = (\sin\theta_6, 0, \cos\theta_6).$$

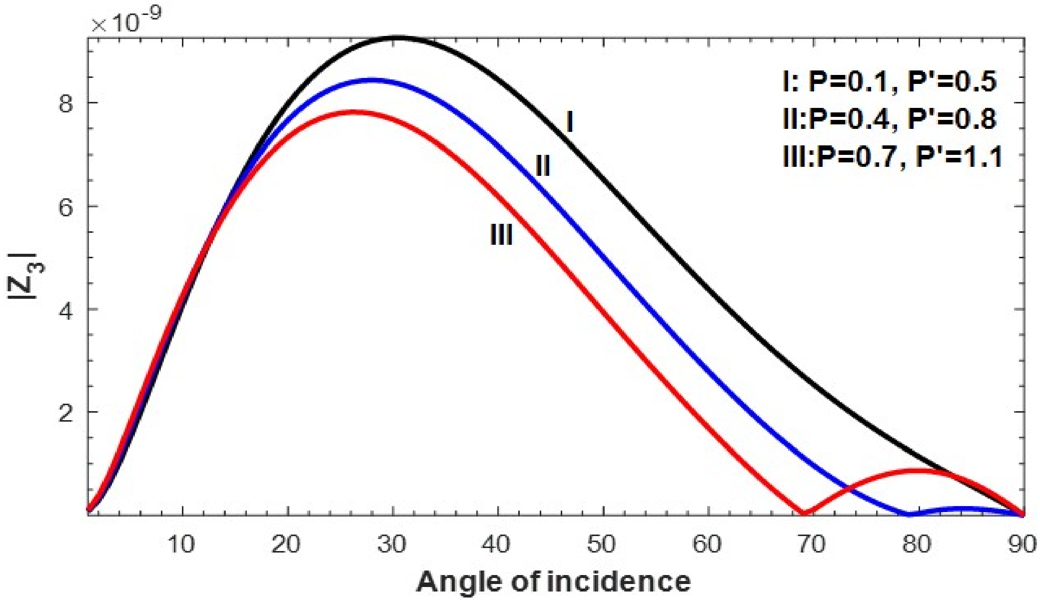


Figure 3.4: Variation of $|Z_3|$ with angle of incidence for different values of P and P'.

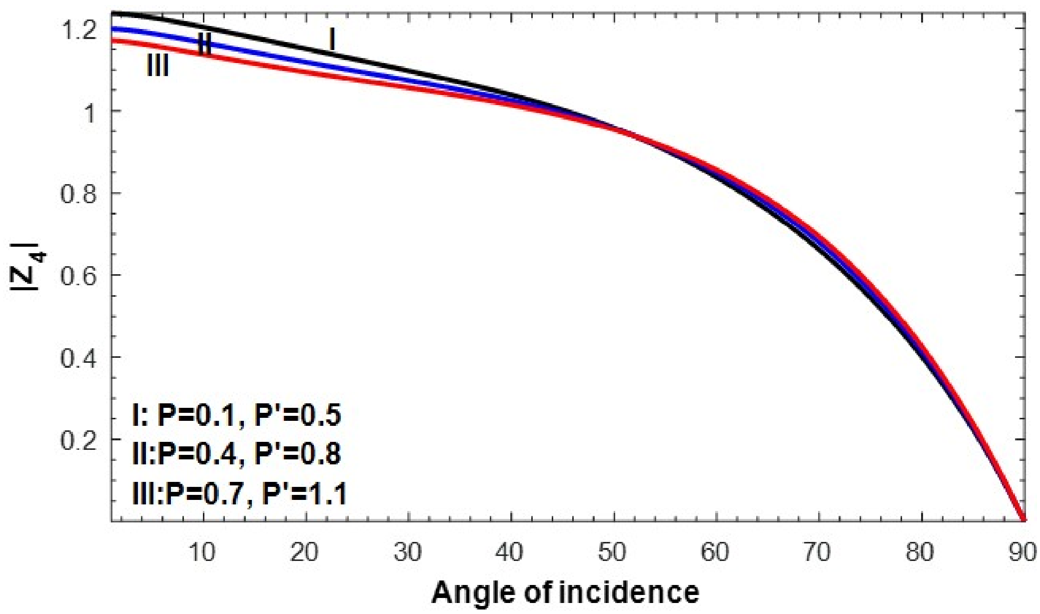


Figure 3.5: Variation of $|Z_4|$ with angle of incidence for different values of P and P'.

Figures 3.2-3.13 are due to incident QL wave, while Figures 3.14-3.25 represent for the incident QT wave. It may be noted that $P = P_{11} = P_{33}$, $P' = P'_{11} = P'_{33}$ and $\omega = 5$.

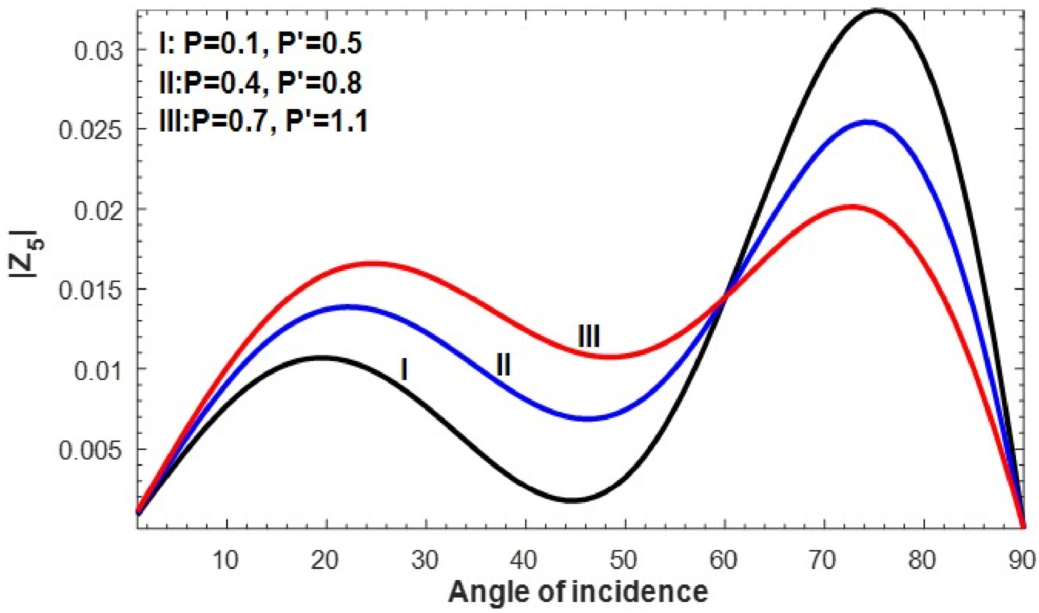


Figure 3.6: Variation of $|Z_5|$ with angle of incidence for different values of P and P'.

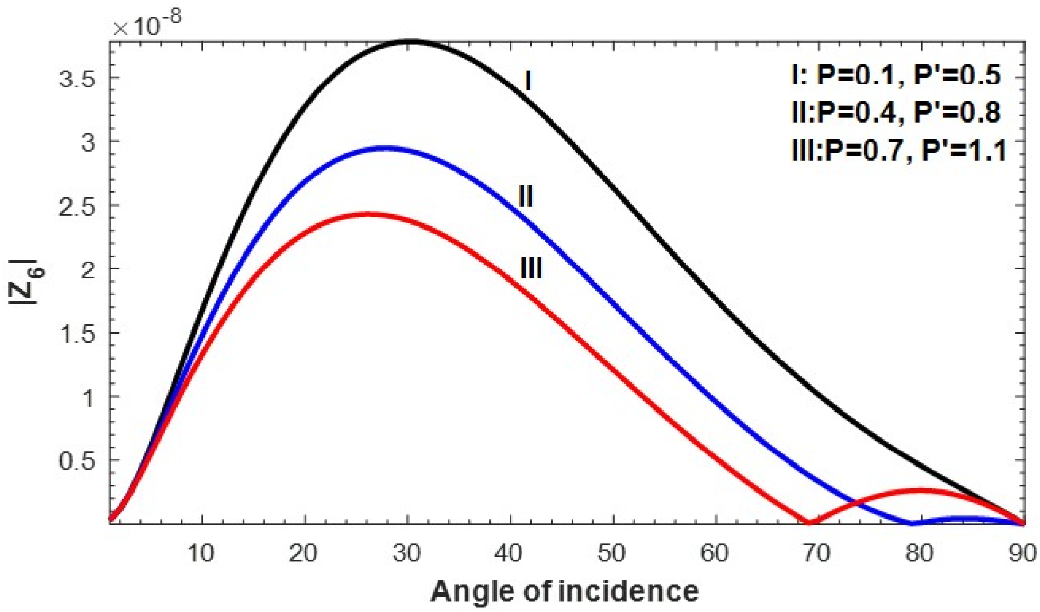


Figure 3.7: Variation of $|Z_6|$ with angle of incidence for different values of P and P'.

3.8.1 Incident QL-wave

Figures 3.2–3.4 explain the change in reflection coefficients with the change in θ_0 at different values of P and P'. We have seen that $|Z_1|$ in Figure 3.2 increases when θ_0 is increased.

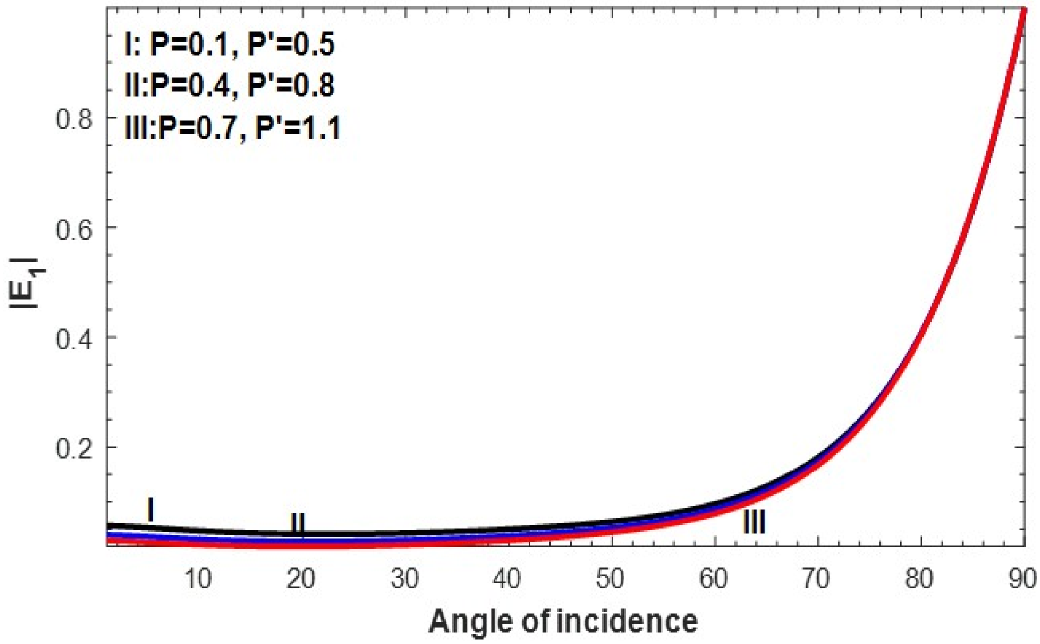


Figure 3.8: Variation of $|E_1|$ with angle of incidence for different values of P and P'.

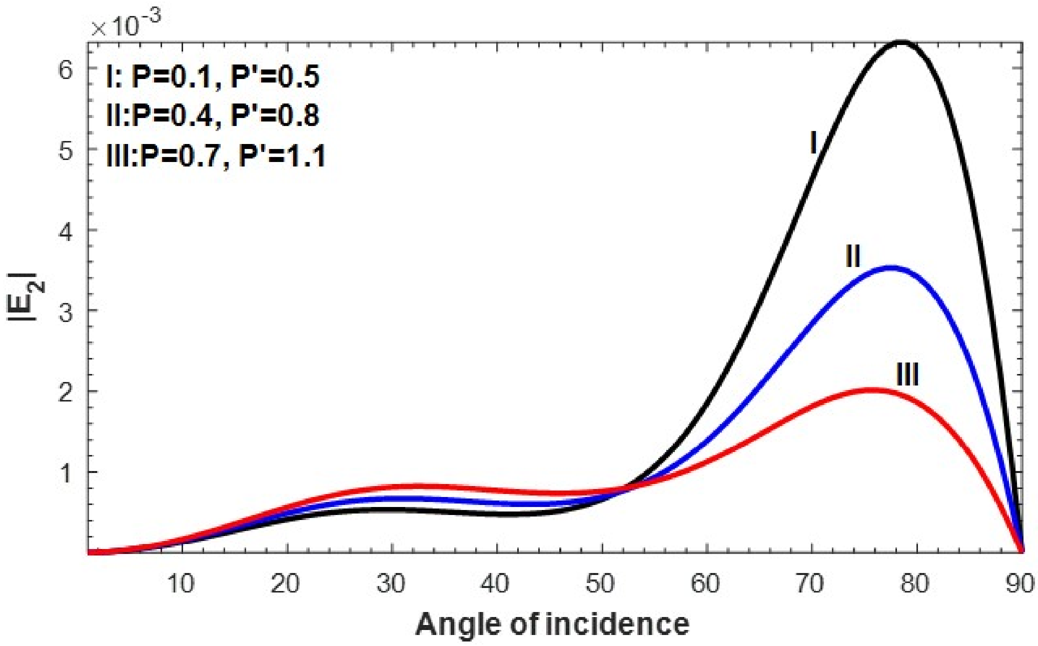


Figure 3.9: Variation of $|E_2|$ with angle of incidence for different values of P and P'.

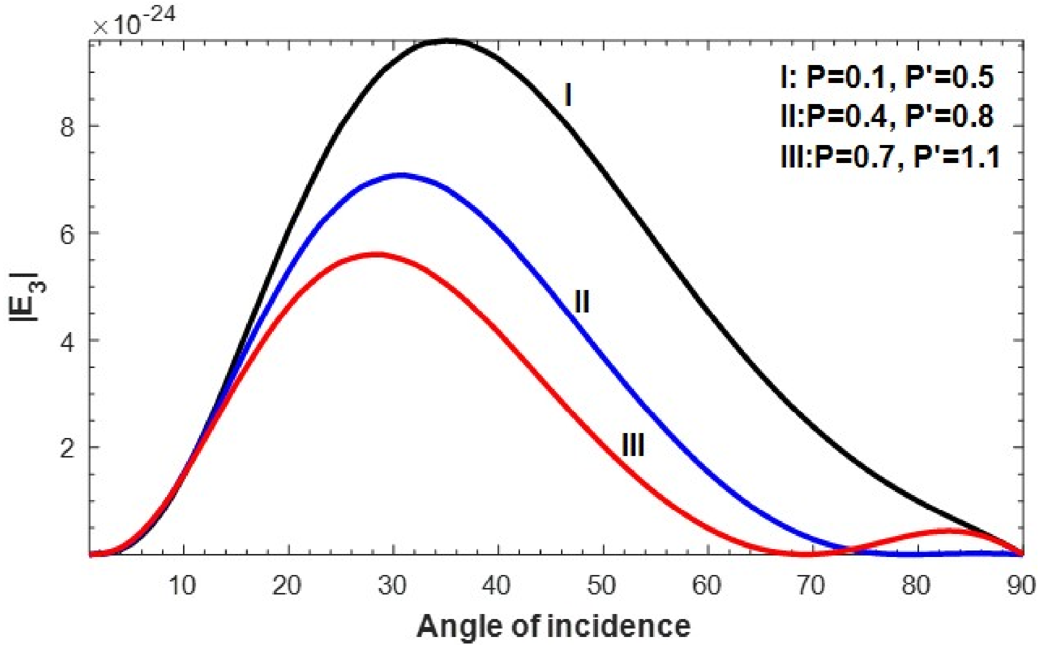


Figure 3.10: Variation of $|E_3|$ with angle of incidence for different values of P and P'.

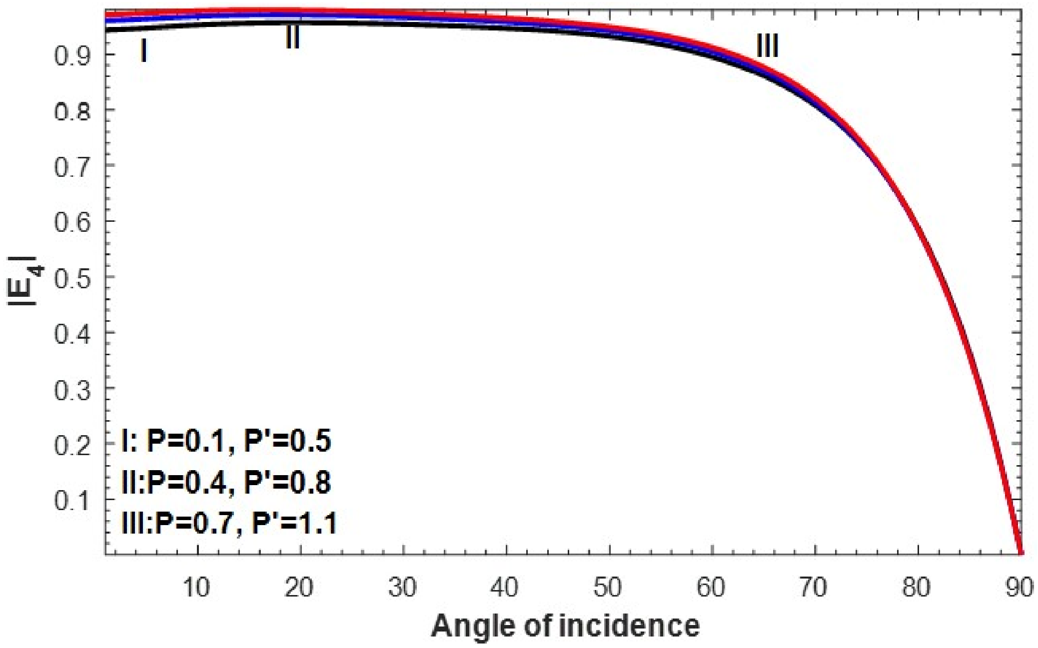


Figure 3.11: Variation of $|E_4|$ with angle of incidence for different values of P and P'.

In Figure 3.3, all the curves corresponding to $|Z_2|$ increase initially and decrease when the value of θ_0 get larger. Thereafter, Curve I, Curve II and Curve III increase to the maximum values at $\theta_0 = 71°$, $\theta_0 = 70°$ and $\theta_0 = 68°$ respectively and then decrease again. All the curves in Figure 3.4 for the amplitude ratio $|Z_3|$ increase to the maximum values at $\theta_0 = 30°$(Curve I), $\theta_0 = 28°$(Curve II) and $\theta_0 = 26°$(Curve III) which decrease with the increase of θ_0. Note that the minimum and maximum effects of initial stresses on $|Z_1|$ are near grazing and normal incidence respectively, while the minimum effect on $|Z_3|$ is near normal incidence.

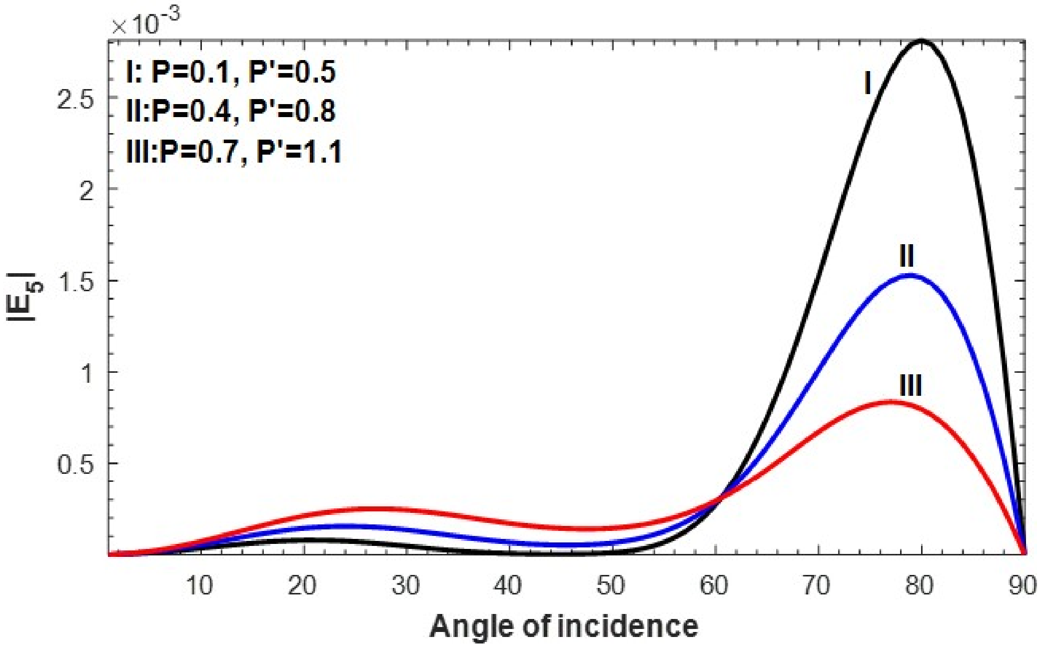


Figure 3.12: Variation of $|E_5|$ with angle of incidence for different values of P and P'.

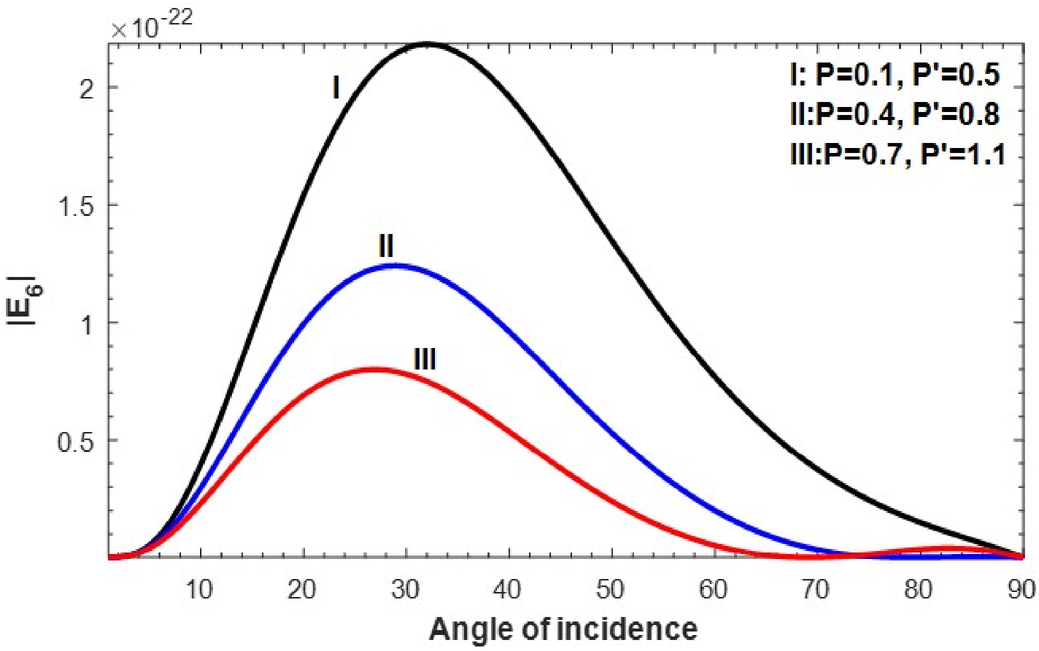


Figure 3.13: Variation of $|E_6|$ with angle of incidence for different values of P and P'.

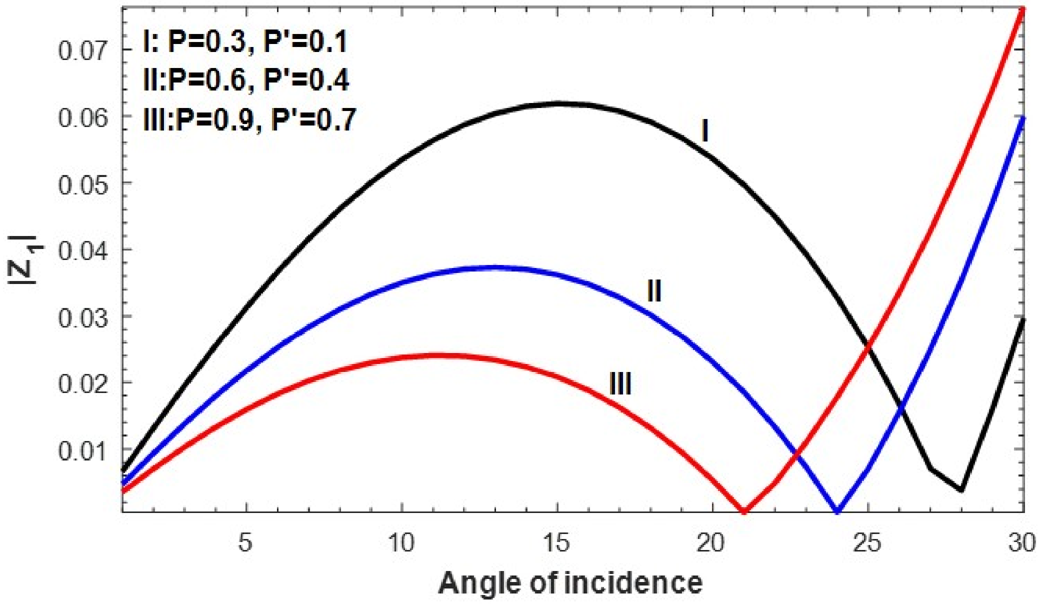


Figure 3.14: Variation of $|Z_1|$ with angle of incidence for different values of P and P'.

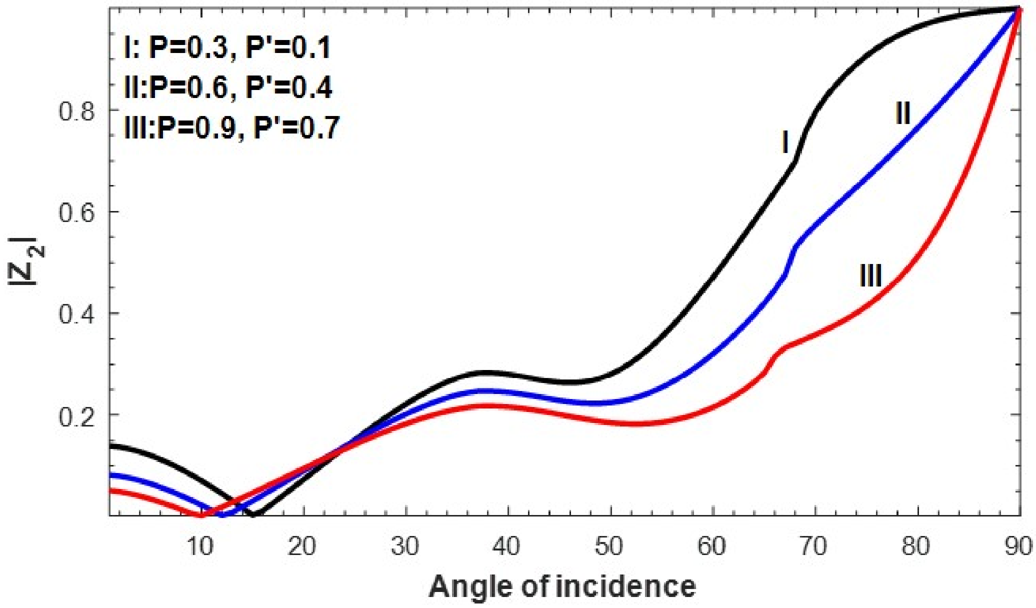


Figure 3.15: Variation of $|Z_2|$ with angle of incidence for different values of P and P'.

The variation of the transmission coefficients are depicted in Figures 3.5–3.7. We have seen that $|Z_4|$ decrease with the increase of θ_0, while $|Z_5|$ and $|Z_6|$ are similar to those of $|Z_2|$ and $|Z_3|$ respectively. All the curves in Figure 3.6 meet at a point $\theta_0 = 60°$. Here, also the effect of initial stresses on $|Z_4|$ is maximum when θ_0 is close to normal angle of incidence. The energy distribution on the reflected and transmitted waves are presented in Figures 3.8–3.10 and 3.11–3.13 respectively. In Figure 3.8, $|E_1|$ increases when the value of θ_0 is getting more. The effects of P and P' on $|E_1|$ are minimum and maximum at the grazing and normal angle of incidence respectively.

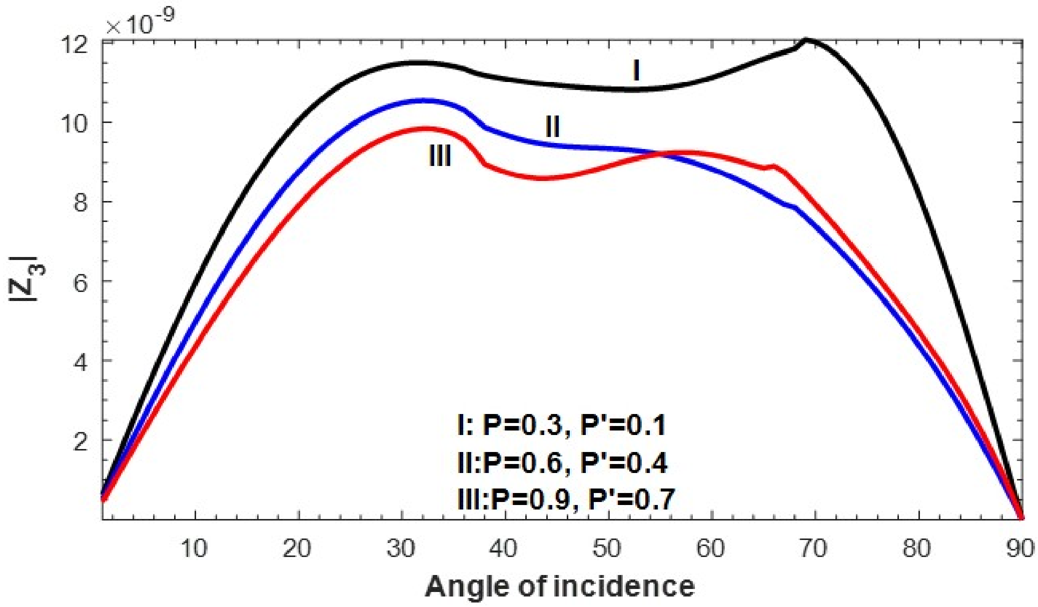


Figure 3.16: Variation of $|Z_3|$ with angle of incidence for different values of P and P'.

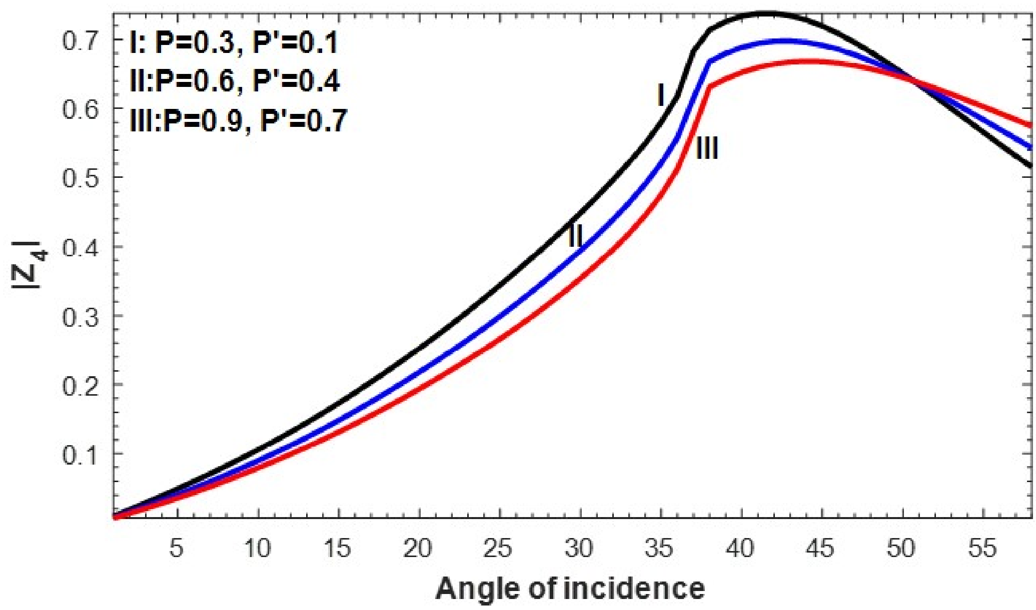


Figure 3.17: Variation of $|Z_4|$ with angle of incidence for different values of P and P'.

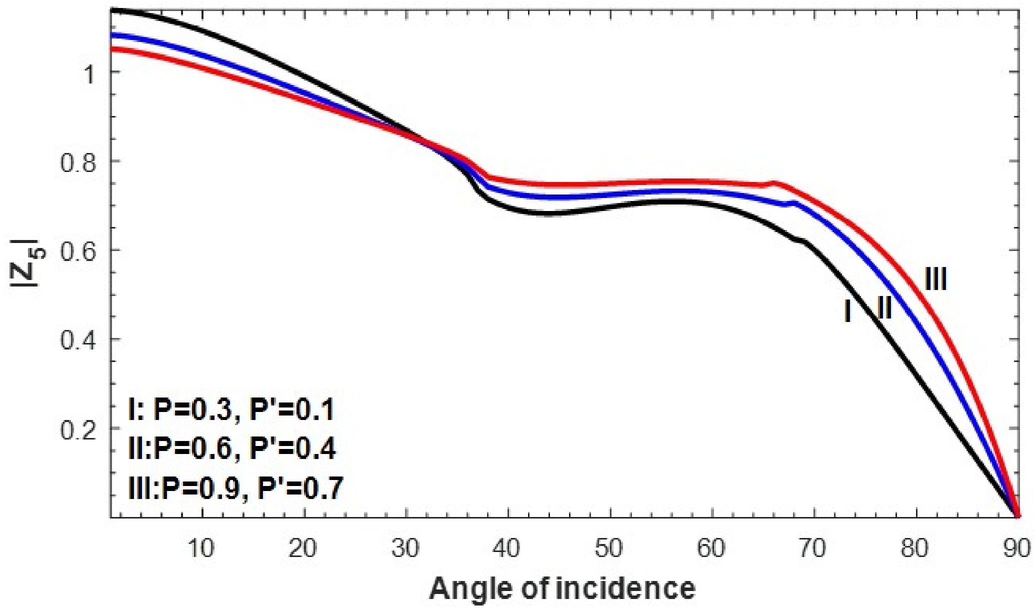


Figure 3.18: Variation of $|Z_5|$ with angle of incidence for different values of P and P'.

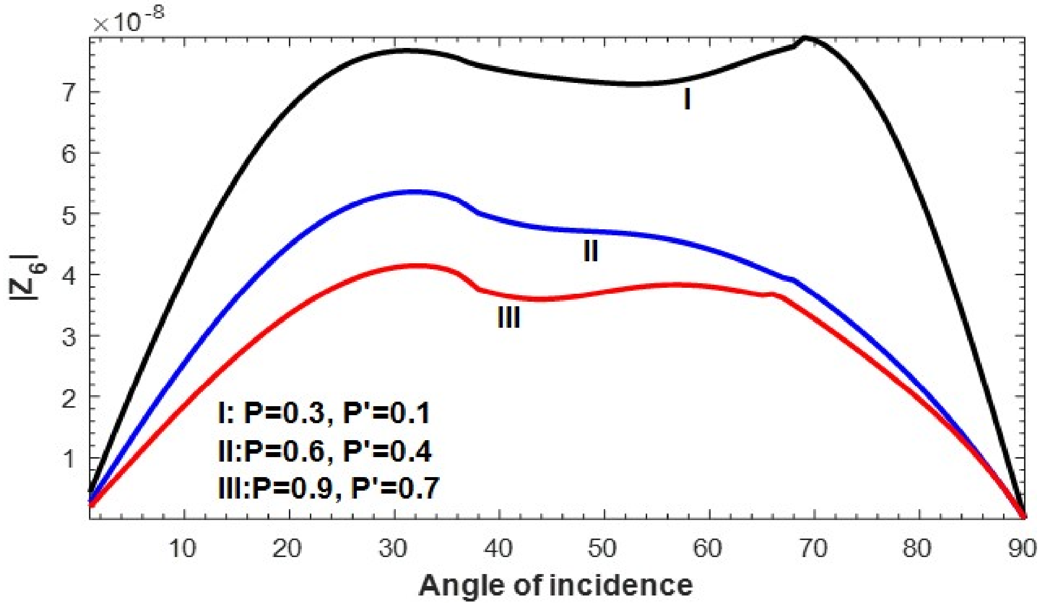


Figure 3.19: Variation of $|Z_6|$ with angle of incidence for different values of P and P'.

All the curves in Figure 3.9 for $|E_2|$ increase initially and meet at $\theta_0 = 52°$ which then increase to the maximum values at $\theta_0 = 77°$(Curve I), $\theta_0 = 78°$(Curve II) and $\theta_0 = 79°$(curve III). After these points, all the curves decrease with the rise of θ_0. In Figure 3.11, the value of $|E_4|$ falls when the value of θ_0 is increased. The values of $|E_3|$ in 3.10 increase to the maximum values for Curve I, Curve II and Curve III are observed at $\theta_0 = 35°$, $\theta_0 = 31°$ and $\theta_0 = 28°$ respectively and all decrease with the higher value of θ_0. We have observed that the minimum effect of P and P' on $|E_3|$ is near normal angle of incidence. $|E_5|$ and $|E_6|$ show similar pattern with $|E_2|$ and $|E_3|$ respectively. The sum of the energy ratios is close to one.

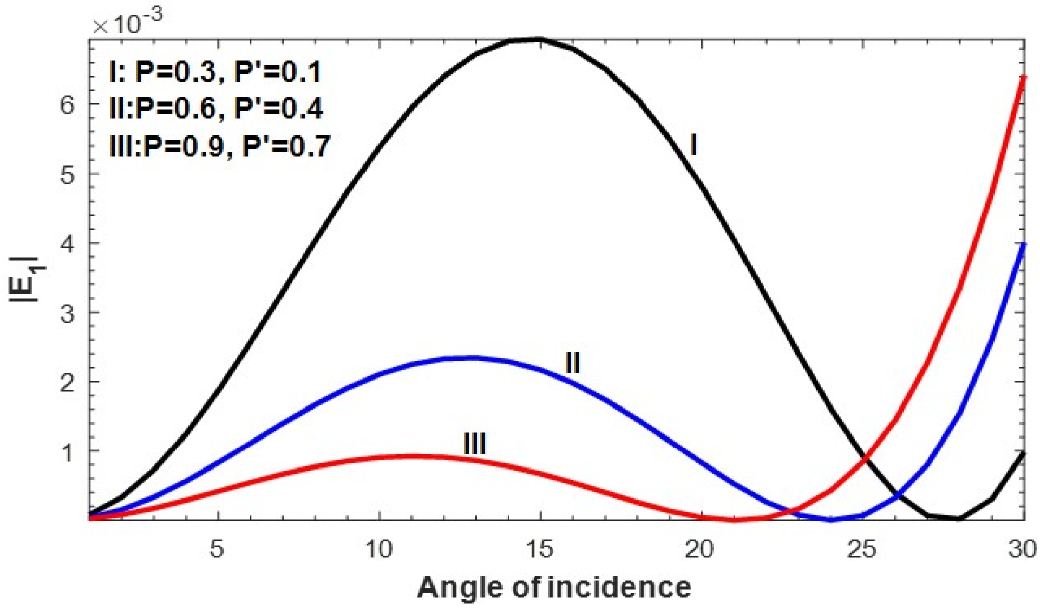


Figure 3.20: Variation of $|E_1|$ with angle of incidence for different values of P and P'.

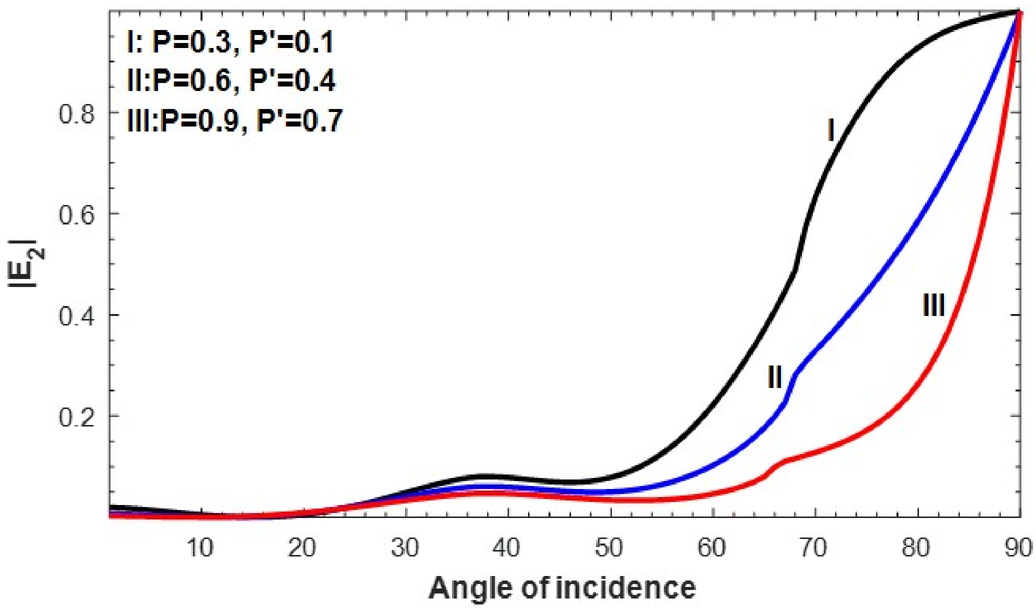


Figure 3.21: Variation of $|E_2|$ with angle of incidence for different values of P and P'.

3.8.2 Incident QT-wave

Figures 3.14–3.16 and 3.17–3.19 are corresponding to the coefficients of reflection and transmission respectively. In Figure 3.14, the values of $|Z_1|$ have parabolic paths in the regions, Curve I: $0° \leq \theta_0 \leq 28°$, Curve II: $0° \leq \theta_0 \leq 24°$, Curve III: $0° \leq \theta_0 \leq 21°$ and then increase with the increase of θ_0. In Figure 3.15, $|Z_2|$ starts decreasing to the minimum value which then increase with the higher value of θ_0.
The values of $|Z_3|$ in Figure 3.16 increase initially and decrease slightly which increase and decrease again when the value of θ_0 is getting larger. In Figure 3.17, $|Z_4|$ increases to the maximum value and then decreases with the rise in the value of θ_0.

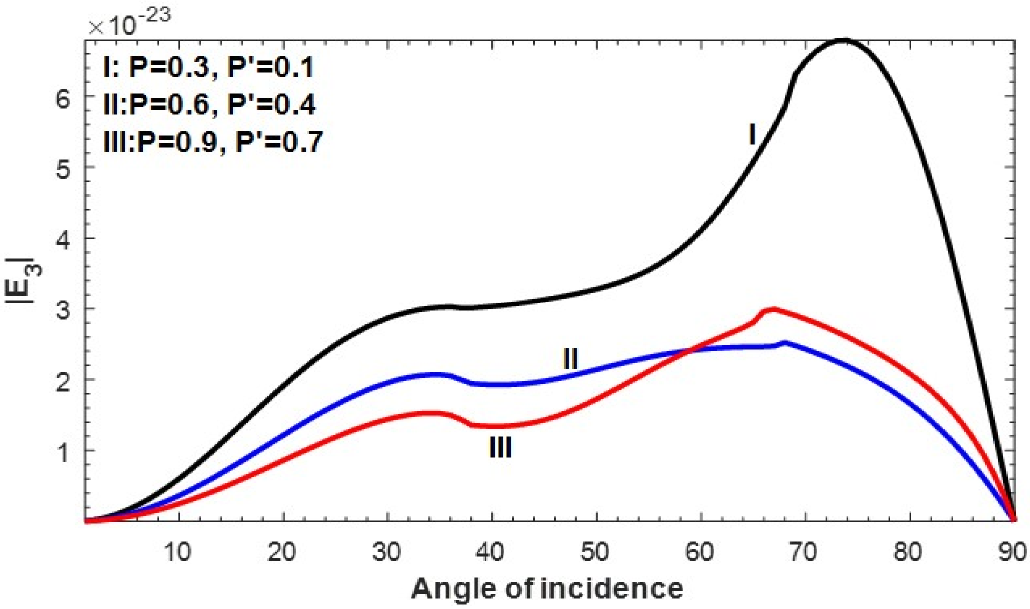


Figure 3.22: Variation of $|E_3|$ with angle of incidence for different values of P and P'.

The value of $|Z_5|$ in Figure 3.18 decreases when the value of θ_0 is increased. It is observed that $|Z_6|$ has similar pattern with $|Z_3|$. In this case, we have observed critical angles $\theta_0 = 30°$ for $|Z_1|$ and $\theta_0 = 58°$ for $|Z_4|$. We have seen that the effects of P and P' on $|Z_3|$ and $|Z_6|$ are minimum near normal as well as grazing angle of incidence. In Figure 3.20, all the curves show that $|E_1|$ increases to some point and then drops to the minimum value which then rises with the increase of θ_0. The values of $|E_2|$ in Figure 3.21 increase with the rise in the value of θ_0. We notice that the effect of initial stresses is very small near the normal incidence. It is observed that $|E_3|$ in Figure 3.22 increases up to certain value and then decreases with the increase of θ_0. In Figures 3.23–3.25, we have seen that the variation of $|E_4|$ and $|E_6|$ have similar pattern with $|Z_4|$ and $|Z_6|$ respectively.

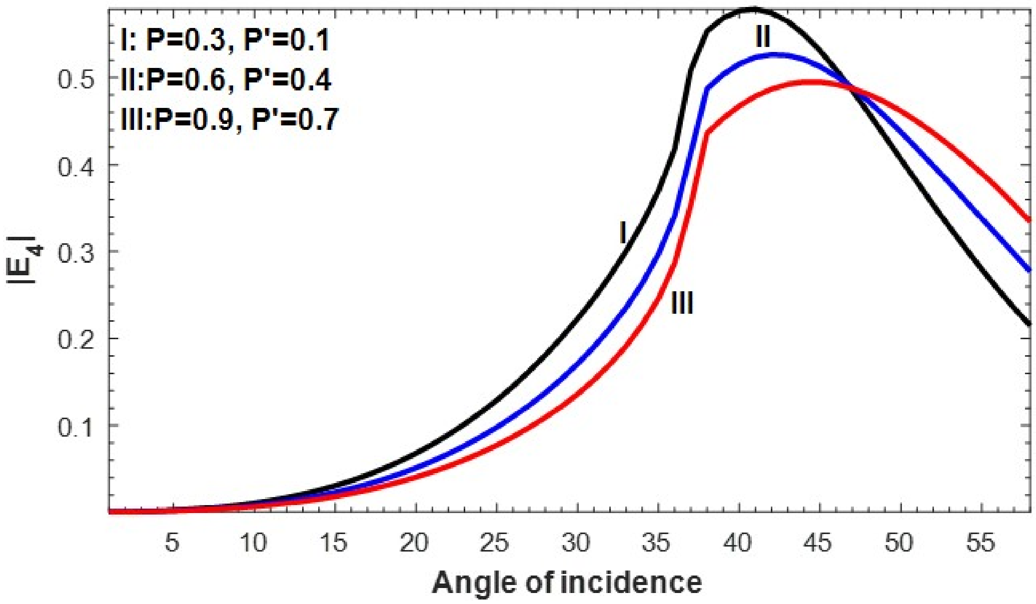


Figure 3.23: Variation of $|E_4|$ with angle of incidence for different values of P and P'.

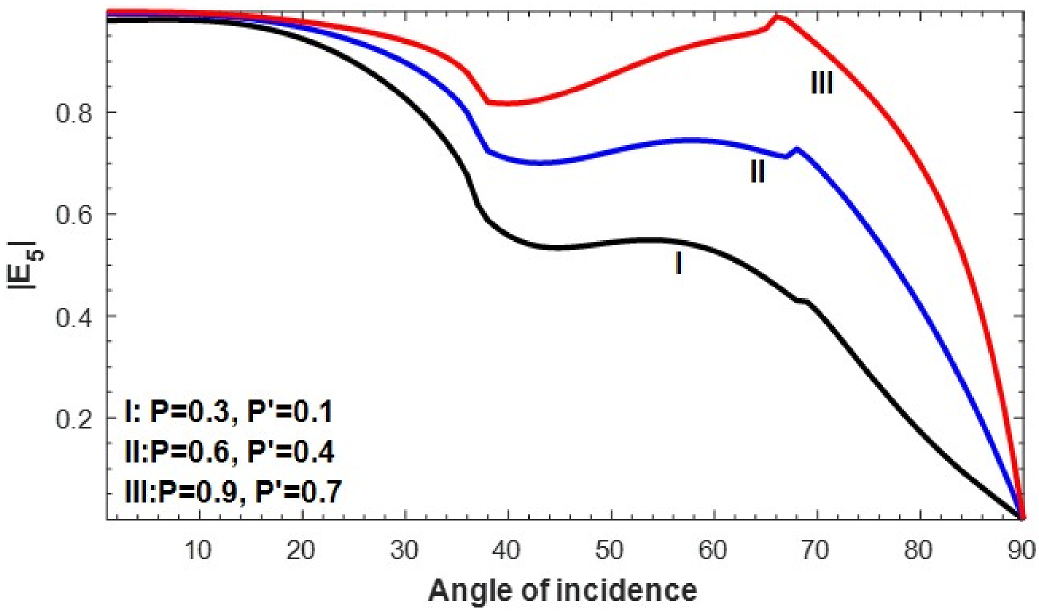


Figure 3.24: Variation of $|E_5|$ with angle of incidence for different values of P and P'.

The values of $|E_5|$ in Figure 3.24 decrease initially and then increase up to certain value which decreases again when the value of θ_0 is increased. Here, we have noticed critical angles, $\theta_0 = 30°$ for $|E_1|$ and $\theta_0 = 58°$ for $|E_4|$. The law of conservation of energy is also hold for this case. It is observed that the effect of initial stresses are very small near the grazing and normal incidence in most of the amplitude and energy ratios for incident QL and QT waves.

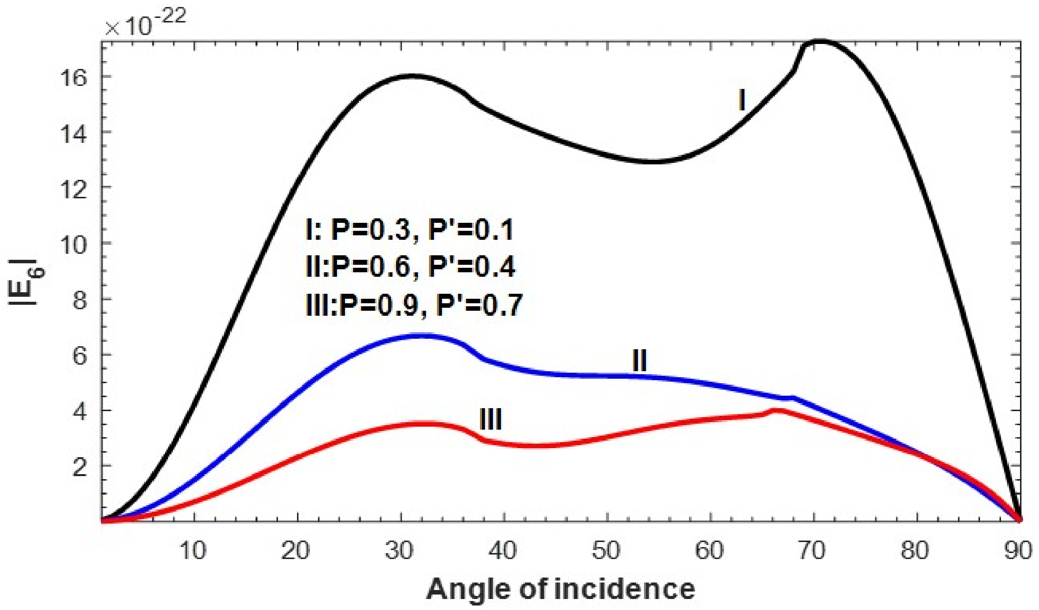


Figure 3.25: Variation of $|E_6|$ with angle of incidence for different values of P and P'.

3.9 Conclusion

For incident QL and QT-waves at the interface between two different half-spaces of initially stressed transversely isotropic thermo-elastic materials, the reflected and transmitted waves are analyzed. The formula corresponding to the coefficient of reflection/transmission and energy ratios are obtained with the help of appropriate boundary conditions. These formulas are computed numerically for a particular model. We have the following concluding remarks:

(i) The reflection/transmission coefficients and the energy distributions are found to depend on angle of incidence, elastic, thermal and initial stress parameters.

(ii) The ratios $|Z_1|$ and $|E_1|$ increase, while $|Z_4|$ and $|E_4|$ decrease with the rise in the value of θ_0 for the incident QL-wave.

(iii) For incident QL-wave, the maximum and minimum effects of initial stresses on $|Z_1|$, $|Z_4|$, $|E_1|$ and $|E_4|$ are found near normal and grazing angle of incidence.

(iv) The effect of initial stresses on $|Z_3|$, $|Z_6|$, $|E_3|$ and $|E_6|$ is minimum near normal angle of incidence for incident QL-wave and they have similar pattern.

(v) The reflected and transmitted QL-waves have critical angles at 30° and 58° respectively for the incident QT wave.

(vi) The addition of all the energy distributions for the incident QL and QT-waves is close to unity.

(vii) We have recovered the results of Singh (2010) and Sharma (1988) in the special cases of the present problem.

Chapter 4

Rayleigh waves in thermoelastic saturated porous medium[3]

4.1 Introduction

The solid and voids in porous materials are connected in a continuous form within the volume of the materials forming looseness between the particles. Nunziato and Cowin (1972) developed a non-linear theory for elastic material with voids and showed that an internal dissipation has been caused due to the changes in volume fraction of the materials. This theory was linearized by Cowin and Nunziato (1983) taking void volume fraction as independent kinematical variable. Currie et al.(1977) discussed the possibility of propagation of more than one surface waves in viscoelastic materials. Goyal *et al.* (2016) confirmed the existence of more than one type of Rayleigh waves in a swelling porous half-space. They obtained two modes of Rayleigh type surface waves. One of them is the counterpart of the classical Rayleigh wave and the second mode of Rayleigh-type surface waves arises due to the presence of either liquid or gas phases of the swelling porous medium.

This chapter investigates the propagation of surface waves in the heat conducting porous material saturated by non-viscous fluid. The frequency equation for Rayleigh type waves are obtained separately for thermally insulated and isothermal boundary

[3] ***International Journal of Advances in Applied Mathematics and Mechanics***, 8(4), 15-27 (2021)

conditions. We have observed two modes of dispersive Rayleigh type waves - I and II. The propagation speed, attenuation and specific loss due to these waves are computed numerically to see the effect of porosity and Biot's parameter. The velocity curves have been presented and it has been observed that they depend on the porosity, elastic, thermal and Biot's parameter of the material. The phase speed of first Rayleigh type wave is just lower than that of transverse waves and the second Rayleigh type wave is faster than those of body waves.

σ_{mn}	: Stress tensor in solid	τ_{mn}	: Stress tensor in porous
P_f	: Fluid pressure	α	: Biot's parameter
δ_{mn}	: Kronecker's delta	λ, μ	: Lamé parameters
M	: Bulk coupling parameter	w_m	: Average fluid motion
U_m	: Displacement in fluid	u_m	: Displacement in solid
T_0	: Reference temperature	f	: Porosity
β_f	: Thermal stress in fluid	β_s	: Thermal stress in solid
ρ	: Density of porous aggregate	ρ_f	: Density of pore fluid
τ_0, K	: Thermal parameters	q	: Inertial
ω	: Angular frequency	τ_0	: Thermal relaxation time
k	: Wavenumber		

Table 4.1: Symbol and Parameters

4.2 Basic equations

The stress tensors for a thermal conducting porous solid in which the voids are saturated by non-viscous fluids are given by Biot(1956a)

$$\tau_{mn} = \sigma_{mn} + \alpha(-P_f)\delta_{mn}. \tag{4.1}$$

The constitutive relations for an isotropic fluid saturated porous heat conducting materials are given as Bear *et al.* (1992)

$$\sigma_{mn} = \lambda u_{l,l}\delta_{mn} + \mu(u_{m,n} + u_{n,m}) - \beta_s(T - T_0)\delta_{mn}, \tag{4.2}$$

$$-P_f = \alpha M u_{l,l} + M w_{l,l} - \beta_f(T - T_0)\delta_{mn}. \tag{4.3}$$

The equation of motion in the absence of body and internal forces for such materials are

$$\tau_{mn,m} = \rho\ddot{u}_m + \rho_f\ddot{w}_m, \tag{4.4}$$

$$(-P_f)_{,m} = \rho_f\ddot{u}_m + q\ddot{w}_m, \tag{4.5}$$

$$KT_{,nn} - \rho C_e(\dot{T} + \tau_0\ddot{T}) = T_0\beta\{\tau_0(\ddot{u}_{n,n} + \ddot{w}_{n,n}) + \dot{u}_{n,n} + \dot{w}_{n,n}\}, \tag{4.6}$$

where $\beta = \beta_s + \alpha\beta_f$.

Using Eqs. (4.1)-(4.3) into Eqs. (4.4)-(4.6), we get

$$(\lambda + \mu + \alpha^2 M)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u} - \rho\ddot{\mathbf{u}} + \alpha M\nabla(\nabla\cdot\mathbf{w}) - \rho_f\ddot{\mathbf{w}} - \beta\nabla T = 0, \tag{4.7}$$

$$\alpha M\nabla(\nabla\cdot\mathbf{u}) - \rho_f\ddot{\mathbf{u}} + M\nabla(\nabla\cdot\mathbf{w}) - q\ddot{\mathbf{w}} - \beta_f\nabla T = 0, \tag{4.8}$$

$$\beta T_0\{(\nabla\cdot\dot{\mathbf{u}} + \tau_0\nabla\cdot\ddot{\mathbf{u}}) + (\nabla\cdot\dot{\mathbf{w}} + \tau_0\nabla\cdot\ddot{\mathbf{w}})\} - \{K\nabla^2 T - \rho C_e(\dot{T} + \tau_0\ddot{T}) = 0. \tag{4.9}$$

Using Helmholtz's theorem, $\mathbf{u}$ and $\mathbf{w}$ can be decomposed as

$$\mathbf{u} = \nabla\phi_s + \nabla\times\boldsymbol{\psi}_s, \quad \nabla\cdot\boldsymbol{\psi}_s = 0, \tag{4.10}$$

$$\mathbf{w} = \nabla\phi_f + \nabla\times\boldsymbol{\psi}_f, \quad \nabla\cdot\boldsymbol{\psi}_f = 0, \tag{4.11}$$

where ϕ_s and $\boldsymbol{\psi}_s$ are potentials representing solid phase and ϕ_f and $\boldsymbol{\psi}_f$ are representing fluid phase.

4.3 Surface wave

We consider Cartesian coordinates as x and z-axis lying horizontal with vertical y-axis. The two dimensional problem of surface wave propagation is considered in the

xy-plane of a half-space of thermoelastic saturated porous materials. Using harmonic nature of traveling waves that potentials vary with $\exp(-\imath\omega t)$ and inserting Eqs. (4.10) & (4.11) into Eqs. (4.7)-(4.9), we get two sets of equations

$$\begin{aligned} \{\Omega_3\nabla^6 - \omega^2\Omega_2\nabla^4 + \omega^4\Omega_1\nabla^2 - \omega^6\Omega_0\}\{\phi_s, \phi_f, T\}(x,y) &= 0, \quad (4.12)\\ \{\nabla^2 - \frac{\omega^2(\rho_f^2 - \rho q)}{\mu q}\}\{\boldsymbol{\psi}_s, \boldsymbol{\psi}_f\}(x,y) &= 0, \quad (4.13) \end{aligned}$$

where $\Omega_0 = \rho C_e \tau(\rho_f^2 - \rho q)$, $\Omega_3 = MK(\lambda + 2\mu)$, $\tau = 1 + \frac{\imath}{\omega}$, $D_1 = \lambda + 2\mu + \alpha^2 M$, $\Omega_1 = \rho C_e\tau(D_1 q + M\rho - 2\alpha M\rho_f) + K(\rho q - \rho_f^2) + T_0\tau\beta\beta_f(\rho - \rho_f) + \beta^2 T_0\tau(q - \rho_f)$ and $\Omega_2 = -K(D_1 q + M\rho - 2\alpha M\rho_f) - \rho C_e\tau(D_1 M - \alpha^2 M^2) - T_0\tau\beta\beta_f(D_1 - \alpha M) + \beta^2 T_0\tau(\alpha M - M)$.

The full potential structures for surface waves in the saturated heat conducting porous materials may be given as

$$\begin{aligned} \langle\phi_s, \phi_f, T - T_0\rangle(x,y,t) &= \sum_{n=1}^{3}\langle A_n, a_n A_n,\ b_n A_n\rangle e^{(\imath kx - m_n y - \imath\omega t)},\\ \langle\psi_s, \psi_f\rangle(x,y,t) &= \langle A_4, dA_4\rangle e^{(\imath kx - m_4 y - \imath\omega t)}, \end{aligned} \quad (4.14)$$

where d, a_n and b_n are the coupling constants given by

$$\begin{aligned} a_n &= \frac{\frac{\beta_f}{\beta}(D_1 - \rho c_n^2) - (\alpha M - \rho_f c_n^2)}{(M - q c_n^2) - \frac{\beta_f}{\beta}(\alpha M - \rho_f c_n^2)}, \quad d = \frac{\mu - \rho c_4^2}{\rho_f c_4^2},\\ b_n &= \frac{k_n^2\{(D_1 - \rho c_n^2)(M - q c_n^2) - (\alpha M - \rho_f c_n^2)^2\}}{\beta\{\frac{\beta_f}{\beta}(\alpha M - \rho_f c_n^2) - (M - q c_n^2)\}}, \quad (n = 1,2,3). \end{aligned}$$

Note that ψ_s and ψ_f are z-components of $\boldsymbol{\psi}_s$ and $\boldsymbol{\psi}_f$ respectively, A_n are amplitude constants, $m_n^2 = k^2 - k_n^2$ and k_n are wavenumbers of the body waves.

4.4 Frequency Equations

At the free surface of the thermo-elastic saturated porous medium, the stress tensors, gradient of temperature and fluid flux vanished. These conditions are (at $y = 0$)

$$(\lambda + \alpha^2 M)\nabla^2\phi_s + 2\mu\frac{\partial^2\phi_s}{\partial y^2} + \alpha M\nabla^2\phi_f - \beta(T - T_0) - 2\mu\frac{\partial^2\psi_s}{\partial x\partial y} = 0, \quad (4.15)$$

$$2\frac{\partial^2\phi_s}{\partial x\partial y} + \frac{\partial^2\psi_s}{\partial y^2} - \frac{\partial^2\psi_s}{\partial x^2} = 0, \quad (4.16)$$

$$\frac{\partial T}{\partial y} + hT = 0, \tag{4.17}$$

$$\frac{\partial \dot{\phi}_f}{\partial y} - \frac{\partial \dot{\psi}_f}{\partial x} = 0, \tag{4.18}$$

where $h \to 0$ for thermally insulated surface and $h \to \infty$ for isothermal surface.

Inserting Eq. (4.14) into (4.15)-(4.18), we have

$$a_{11}A_1 + a_{12}A_2 + a_{13}A_3 + \imath a_{14}m_4A_4 = 0, \tag{4.19}$$

$$m_1a_{21}A_1 + m_2a_{22}A_2 + m_3a_{23}A_3 + \imath a_{24}A_4 = 0, \tag{4.20}$$

$$m_1a_{31}A_1 + m_2a_{32}A_2 + m_3a_{33}A_3 = 0, \tag{4.21}$$

$$a_{31}A_1 + a_{32}A_2 + a_{33}A_3 = 0, \tag{4.22}$$

$$m_1a_{41}A_1 + m_2a_{42}A_2 + m_3a_{43}A_3 + \imath a_{44}A_4 = 0, \tag{4.23}$$

where

$$a_{1n} = (\lambda + \alpha^2 M + \alpha M a_n)(m_n^2 - k^2) + 2\mu m_n^2 - \beta b_n, \quad a_{14} = 2\mu k, \quad a_{2n} = 2k,$$

$$a_{24} = (m_4^2 + k^2), a_{3n} = b_n, \quad a_{4n} = a_n, \quad a_{44} = kd, \qquad (n = 1, 2, 3).$$

These equations help to derive the frequency equations of the Rayleigh waves corresponding to thermally insulated and isothermal surfaces respectively as

$$a_{11}m_2m_3D_{11} - a_{12}m_1m_3D_{12} + a_{13}m_1m_2D_{13} - a_{14}m_1m_2m_3D_{14} = 0, \tag{4.24}$$

and

$$D_{21}m_1m_2 + D_{22}m_1m_3 + D_{23}m_1m_4 - D_{24}m_2m_3 - D_{25}m_2m_4 - D_{26}m_3m_4 = 0, \tag{4.25}$$

where

$$D_{11} = (a_{22}a_{33} - a_{23}a_{32})a_{44} + (a_{32}a_{43} - a_{33}a_{42})a_{24}, \quad D_{12} = (a_{21}a_{33} - a_{23}a_{31})a_{44} +$$

$$(a_{31}a_{43} - a_{33}a_{41})a_{24}, \quad D_{13} = (a_{21}a_{32} - a_{22}a_{31})a_{44} + (a_{31}a_{42} - a_{32}a_{41})a_{24},$$

$$D_{14} = (a_{32}a_{43} - a_{33}a_{42})a_{21} + (a_{33}a_{41} - a_{31}a_{43})a_{22} + (a_{31}a_{42} - a_{32}a_{41})a_{23},$$

$$D_{21} = a_{33}a_{14}m_4^2(a_{41}a_{22} - a_{42}a_{21}), \quad D_{22} = a_{32}a_{14}m_4^2(a_{43}a_{21} - a_{41}a_{23}),$$

$$D_{23} = (a_{32}a_{13} - a_{33}a_{12})(a_{41}a_{24} - a_{44}a_{21}), \; D_{24} = a_{31}a_{14}m_4^2(a_{43}a_{22} - a_{42}a_{23}),$$

$$D_{25} = (a_{31}a_{13} - a_{33}a_{11})(a_{42}a_{24} - a_{44}a_{22}), \; D_{26} = (a_{31}a_{12} - a_{32}a_{11})(a_{44}a_{23} - a_{43}a_{24}).$$

Equations (4.24) & (4.25) contain radical powers in the expressions of m_n and difficult to solve directly. The radical powers are removed by squaring and obtained the frequency equations as

$$(\alpha_{11}^2 - \alpha_{12}^2 m_1^2 m_3^2 - \alpha_{13}^2 m_2^2 m_4^2)^2 - 4\alpha_{12}^2 \alpha_{13}^2 m_1^2 m_2^2 m_3^2 m_4^2 = 0, \tag{4.26}$$

$$\alpha_{21}^2 - \alpha_{22}^2 m_2^2 m_3^2 = 0, \tag{4.27}$$

where

$$\alpha_{11} = D_{11}^2 a_{11}^2 m_2^2 m_3^2 + D_{13}^2 a_{13}^2 m_1^2 m_2^2 - D_{12}^2 a_{12}^2 m_1^2 m_3^2 - D_{14}^2 a_{14}^2 m_1^2 m_2^2 m_3^2 m_4^2,$$

$$\alpha_{12} = 2D_{11}D_{13}a_{11}a_{13}m_2^2, \alpha_{13} = 2D_{12}D_{14}a_{12}a_{14}m_1^2 m_3^2, \alpha_{22} = 2(D_{33}D_{34}m_4^2 - D_{31}D_{32}),$$

$$\alpha_{21} = D_{31}^2 + D_{32}^2 m_2^2 m_3^2 - D_{33}^2 m_2^2 m_4^2 - D_{34}^2 m_3^2 m_4^2,\ \ D_{32} = 2(D_{21}D_{22}m_1^2 - D_{25}D_{26}m_4^2),$$

$$D_{31} = D_{21}^2 m_1^2 m_2^2 + D_{22}^2 m_1^2 m_3^2 + D_{23}^2 m_1^2 m_4^2 - D_{24}^2 m_2^2 m_3^2 - D_{25}^2 m_2^2 m_4^2 - D_{26}^2 m_3^2 m_4^2,$$

$$D_{33} = 2(D_{24}D_{26}m_3^2 - D_{21}D_{23}m_1^2), \quad D_{34} = 2(D_{24}D_{25}m_2^2 - D_{22}D_{23}m_1^2).$$

Eqs. (4.26) and (4.27) are equations of 48 powers and all roots do not satisfy the boundary conditions. The solutions of these equations are complex and assume that the wavenumber, $k = R + \imath Q$ be a solution, then the phase speed $c_r = \frac{\omega}{R}$ and attenuation $A_r = Q$ satisfy

$$c^{-1} = c_r^{-1} + \imath\omega^{-1}Q. \tag{4.28}$$

It may be noted that the exponent in Eq. (4.14) becomes $\imath R(x - c_r t) - Qx - m_n y$.

4.4.1 Specific Loss

The direct method to find internal friction for a material is finding the specific loss. It may be defined as the ratio of energy dissipated (ΔW) in a specimen through a stress cycle to the elastic energy (W) stored in the specimen at the maximum strain. Numerical values of this factor is calculated as

$$\text{Specific Loss} = 4\pi \left| \frac{c_r Q}{\omega} \right|. \tag{4.29}$$

4.5 Path of surface particles

The amplitude of displacement and temperature functions due to the propagation of Rayleigh type waves at the surface, $y = 0$ are obtained as

$$\{u_k, w_k, T\} = \left\{|U_k| e^{\imath\theta_k}, |W_k| e^{\imath\theta_{(k+2)}}, T_\theta e^{\imath\theta_5}\right\} A_1 e^{(\imath q - Qx)}, \quad k = 1, 2 \tag{4.30}$$

where

$$U_1 = \imath k\{1 + \frac{L_4}{L_5} - \frac{L_6}{L_3L_5}\} - \imath m_4\frac{L_7}{L_3L_5},\ U_2 = -m_1 - m_2\frac{L_4}{L_5} + m_3\frac{L_6}{L_3L_5} + \imath k\frac{L_7}{L_3L_5},$$
$$W_1 = \imath k\{a_1 + a_2\frac{L_4}{L_5} - a_3\frac{L_6}{L_3L_5}\} - \imath d m_4\frac{L_7}{L_3L_5},\ T_\theta = b_1m_1 + b_2m_2\frac{L_4}{L_5} - b_3m_3\frac{L_6}{L_3L_5},$$
$$W_2 = -a_1m_1 - a_2m_2\frac{L_4}{L_5} + a_3m_3\frac{L_6}{L_3L_5} + \imath dk\frac{L_7}{L_3L_5},\quad L_1 = a_{14}a_{21}m_1m_4 - a_{11}a_{24},$$
$$L_2 = a_{14}a_{22}m_2m_4 - a_{12}a_{24},\quad L_3 = a_{14}a_{23}m_3m_4 - a_{13}a_{24},\quad q = R(x - c_r t),$$
$$L_6 = L_1L_5 + L_2L_4,\quad L_7 = a_{11}L_3L_5 + a_{12}L_3L_4 - a_{13}\left(L_1L_5 + L_2L_4\right),$$
$$(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = \left(\arg(U_1), \arg(U_2), \arg(W_1), \arg(W_2), \arg(T_\theta)\right),$$
$$L_4 = L_1a_{33}m_3 - L_3a_{31}m_1, \text{(thermally insulated)},\quad L_1a_{33} - L_3a_{31}, \text{(isothermal)},$$
$$L_5 = L_2a_{33}m_3 - L_3a_{32}m_1, \text{(thermally insulated)},\quad L_2a_{33} - L_3a_{32}, \text{(isothermal)}.$$

We know that the surface of thermoelastic saturated porous material damped out the vibration of Rayleigh type wave. As a result of this, the phase differences are developed between u_1 and u_2 in the solid and w_1 and w_2 in the fluid phase. Eq. (4.30) retains the real part on the surface $y = 0$ as

$$\{u_k, w_k\} = \left\{|U_k| \cos(q + \theta_k), |W_k| \cos(q + \theta_{(k+2)})\right\} N, \quad k = 1, 2 \tag{4.31}$$

where $N = A_1 e^{-Qx}$.

Eliminating q from Eq. (4.31), we get

$$\left(\frac{u_1}{|U_1|}\right)^2 + \left(\frac{u_2}{|U_2|}\right)^2 - 2\left(\frac{u_1}{|U_1|}\right)\left(\frac{u_2}{|U_2|}\right)\cos(\theta_1 - \theta_2) = N^2 \sin^2(\theta_1 - \theta_2), \tag{4.32}$$
$$\left(\frac{w_1}{|W_1|}\right)^2 + \left(\frac{w_2}{|W_2|}\right)^2 - 2\left(\frac{w_1}{|W_1|}\right)\left(\frac{w_2}{|W_2|}\right)\cos(\theta_3 - \theta_4) = N^2 \sin^2(\theta_3 - \theta_4). \tag{4.33}$$

These Eqs. (4.32) and (4.33) represent ellipse in $u_1 - u_2$ plane and $w_1 - w_2$ plane respectively due to the fact that

$$\begin{aligned}
&\frac{4}{|U_1|^2\,|U_2|^2}\cos^2(\theta_1-\theta_2) - \frac{4}{|U_1|^2\,|U_2|^2} = -\frac{4}{|U_1|^2\,|U_2|^2}\sin^2(\theta_1-\theta_2) < 0 \quad \text{and} \\
&\frac{4}{|W_1|^2\,|W_2|^2}\cos^2(\theta_3-\theta_4) - \frac{4}{|W_1|^2\,|W_2|^2} = -\frac{4}{|W_1|^2\,|W_2|^2}\sin^2(\theta_3-\theta_4) < 0.
\end{aligned}$$

The semi major (X_s, X_f), minor axes (Y_s, Y_f) and eccentricities (e_s, e_f) of the elliptical path in Eqs. (4.32) and (4.33) are given by

$$\begin{aligned}
X_s^2 &= \frac{N^2}{2}\left[|U_1|^2 + |U_2|^2 + \sqrt{\left(|U_1|^2 - |U_2|^2\right)^2 + 4\,|U_1|^2\,|U_2|^2\cos^2(\theta_1-\theta_2)}\right], \\
Y_s^2 &= \frac{N^2}{2}\left[|U_1|^2 + |U_2|^2 - \sqrt{\left(|U_1|^2 - |U_2|^2\right)^2 + 4\,|U_1|^2\,|U_2|^2\cos^2(\theta_1-\theta_2)}\right], \\
e_s^2 &= \frac{2\sqrt{\left(|U_1|^2 - |U_2|^2\right)^2 + 4\,|U_1|^2\,|U_2|^2\cos^2(\theta_1-\theta_2)}}{|U_1|^2 + |U_2|^2 + \sqrt{\left(|U_1|^2 - |U_2|^2\right)^2 + 4\,|U_1|^2\,|U_2|^2\cos^2(\theta_1-\theta_2)}},
\end{aligned} \tag{4.34}$$

and

$$\begin{aligned}
X_f^2 &= \frac{N^2}{2}\left[|W_1|^2 + |W_2|^2 + \sqrt{\left(|W_1|^2 - |W_2|^2\right)^2 + 4\,|W_1|^2\,|W_2|^2\cos^2(\theta_3-\theta_4)}\right], \\
Y_f^2 &= \frac{N^2}{2}\left[|W_1|^2 + |W_2|^2 - \sqrt{\left(|W_1|^2 - |W_2|^2\right)^2 + 4\,|W_1|^2\,|W_2|^2\cos^2(\theta_3-\theta_4)}\right], \\
e_f^2 &= \frac{2\sqrt{\left(|W_1|^2 - |W_2|^2\right)^2 + 4\,|W_1|^2\,|W_2|^2\cos^2(\theta_3-\theta_4)}}{|W_1|^2 + |W_2|^2 + \sqrt{\left(|W_1|^2 - |W_2|^2\right)^2 + 4\,|W_1|^2\,|W_2|^2\cos^2(\theta_3-\theta_4)}}.
\end{aligned} \tag{4.35}$$

If α_s and α_f are the inclination of major-axes to the wave normal, then

$$\begin{aligned}
\tan(2\alpha_s) &= \frac{2\left\{(\tan^2\delta - 1)\,|U_1|\,|U_2|\cos(\theta_1-\theta_2) - \left(|U_1|^2 - |U_2|^2\right)\tan\delta\right\}}{(\tan^2\delta - 1)\left(|U_1|^2 - |U_2|^2\right) + 4\,|U_1|\,|U_2|\cos(\theta_1-\theta_2)\tan\delta}, \\
\tan(2\alpha_f) &= \frac{2\left\{(\tan^2\delta - 1)\,|W_1|\,|W_2|\cos(\theta_3-\theta_4) - \left(|W_1|^2 - |W_2|^2\right)\tan\delta\right\}}{(\tan^2\delta - 1)\left(|W_1|^2 - |W_2|^2\right) + 4\,|W_1|\,|W_2|\cos(\theta_3-\theta_4)\tan\delta},
\end{aligned} \tag{4.36}$$

where δ is angle of propagation. For Rayleigh waves propagating along x-axis, $\delta = \frac{\pi}{2}$, then Eq. (4.37) becomes

$$\begin{aligned}
\tan(2\alpha_s) &= \frac{2\,|U_1|\,|U_2|\cos(\theta_1-\theta_2)}{|U_1|^2 - |U_2|^2}, \\
\tan(2\alpha_f) &= \frac{2\,|W_1|\,|W_2|\cos(\theta_3-\theta_4)}{|W_1|^2 - |W_2|^2}.
\end{aligned} \tag{4.37}$$

The horizontal and vertical components of displacement in solid and fluid phase of the thermoelastic saturated porous material have equal magnitude, i.e., $|U_1| = |U_2|$ and $|W_1| = |W_2|$ when $\alpha_s = \alpha_f = \frac{\pi}{4}$. This means that the surface particles moving in the elliptical path of Eqs. (4.34) and (4.35) are parallel to the direction of propagation of Rayleigh waves. Since the semi-axes depend upon $N = A_1 e^{-Qx}$, they are increasing or decreasing exponentially. Thus, the decay of surface particles in thermoelastic saturated porous material are retrograde and prograde when X_s and Y_s in solid as well as X_f and Y_f in fluid have same sign and opposite sign respectively.

4.6 Particular Cases

Case I: If the porosity of the material is neglected, then the fluid flow in it will no more and the problem becomes Rayleigh waves propagation in the thermoelastic solid. Thus, $\alpha = f = q = \rho_f = M = \beta_f = a_n = 0$ and Eqs.(4.12) & (4.13) reduce to

$$\begin{aligned} \{\Omega_2\nabla^4 + \omega^2\Omega_1\nabla^2 + \omega^4\Omega_0\}\{\phi_s, T\}(x, y) =&\ 0, \\ \{\mu\nabla^2 + \rho\omega^2\}\{\boldsymbol{\psi}_s\}(x, y) =&\ 0, \end{aligned} \tag{4.38}$$

where

$$\Omega_0 = \rho^2 C_e \tau, \quad \Omega_1 = \rho C_e \tau(\lambda + 2\mu) + K\rho + \beta_s^2 T_0 \tau, \quad \Omega_2 = K(\lambda + 2\mu).$$

The coupling parameter b_n is given by

$$b_n = \frac{1}{\beta_s}\{(\lambda + 2\mu)(m_n^2 - k^2) + \rho\omega^2\}, \quad (n = 1, 2).$$

The frequency equations are also reduced as (for thermally insulated)

$$(\alpha_{11}^2 m_1^2 + \alpha_{13}^2 m_1^2 m_2^2 m_4^2 - \alpha_{12}^2 m_2^2)^2 - 4\alpha_{11}^2 \alpha_{13}^2 m_1^4 m_2^2 m_4^2 = 0, \tag{4.39}$$

(for isothermal)

$$\alpha_{21}^2 - \alpha_{22}^2 m_1^2 m_2^2 = 0, \tag{4.40}$$

with the following modified values

$\alpha_{11} = a_{12}a_{24}a_{31}, \quad \alpha_{12} = a_{11}a_{24}a_{32}, \quad \alpha_{13} = a_{14}(a_{21}a_{32} - a_{22}a_{31}), \quad \alpha_{22} = a_{14}^2 m_4^2 a_{21}a_{22}a_{31}a_{32},$

$\alpha_{21} = a_{24}^2(a_{31}a_{12} - a_{32}a_{11})^2 - a_{14}^2 m_4^2(a_{31}^2 a_{22}^2 m_2^2 + a_{32}^2 a_{21}^2 m_1^2),$

$a_{11} = (\lambda + \alpha^2 M + \alpha M)(m_1^2 - k^2) + 2\mu m_1^2, \quad a_{12} = (\lambda + \alpha^2 M + \alpha M)(m_2^2 - k^2) + 2\mu m_2^2.$

Eq. (4.39) is exactly match with the result of Abouelregal (2011) for the Lord and Shulman theory. If $\tau_0 = 0$, then Eqs. (4.39) and (4.40) reduced to the result of Chadwick(1960) for thermally insulated and isothermal condition respectively.

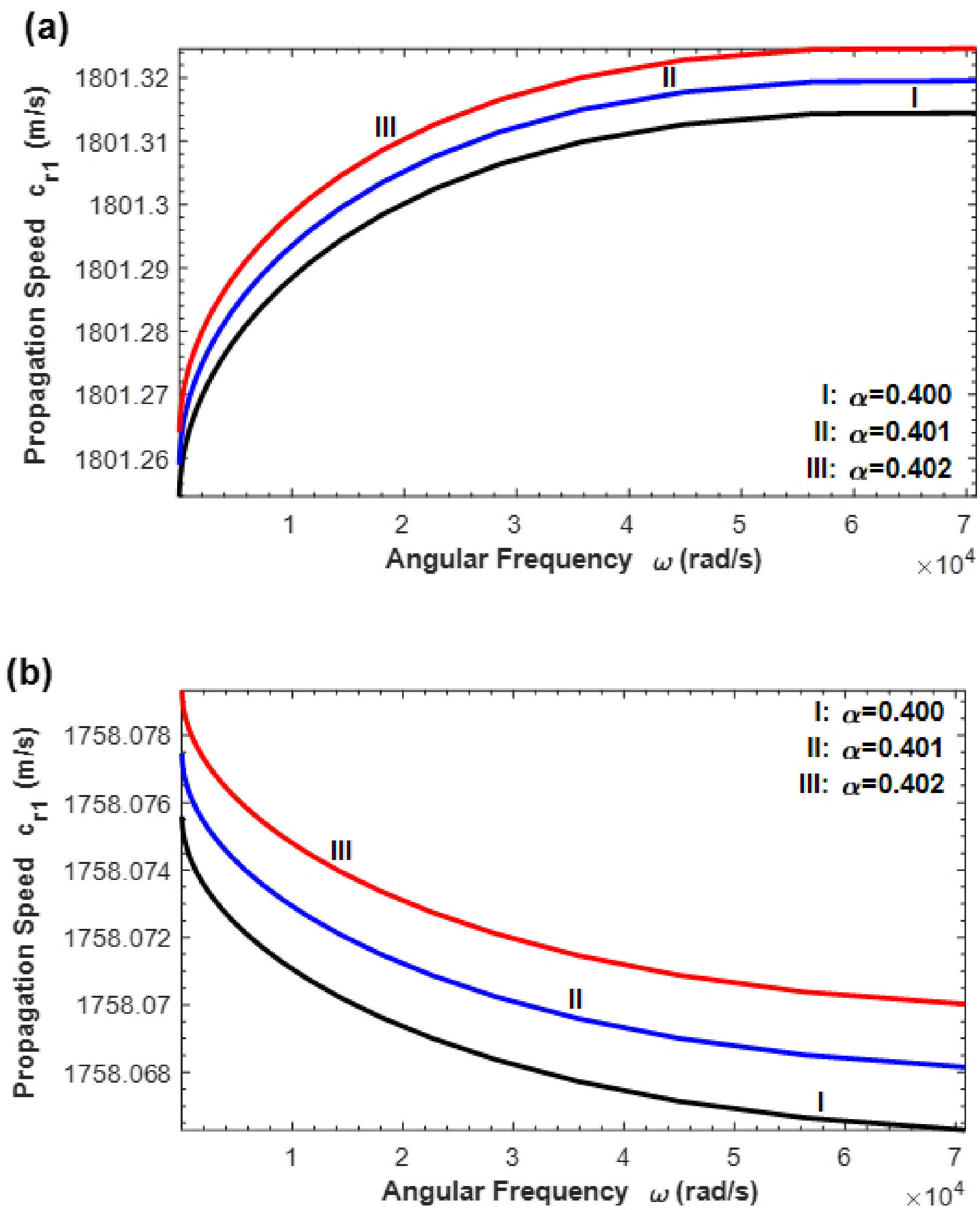


Figure 4.1: Effect of Biot's parameter on c_{r1} (a) Thermally Insulated, (b) Isothermal.

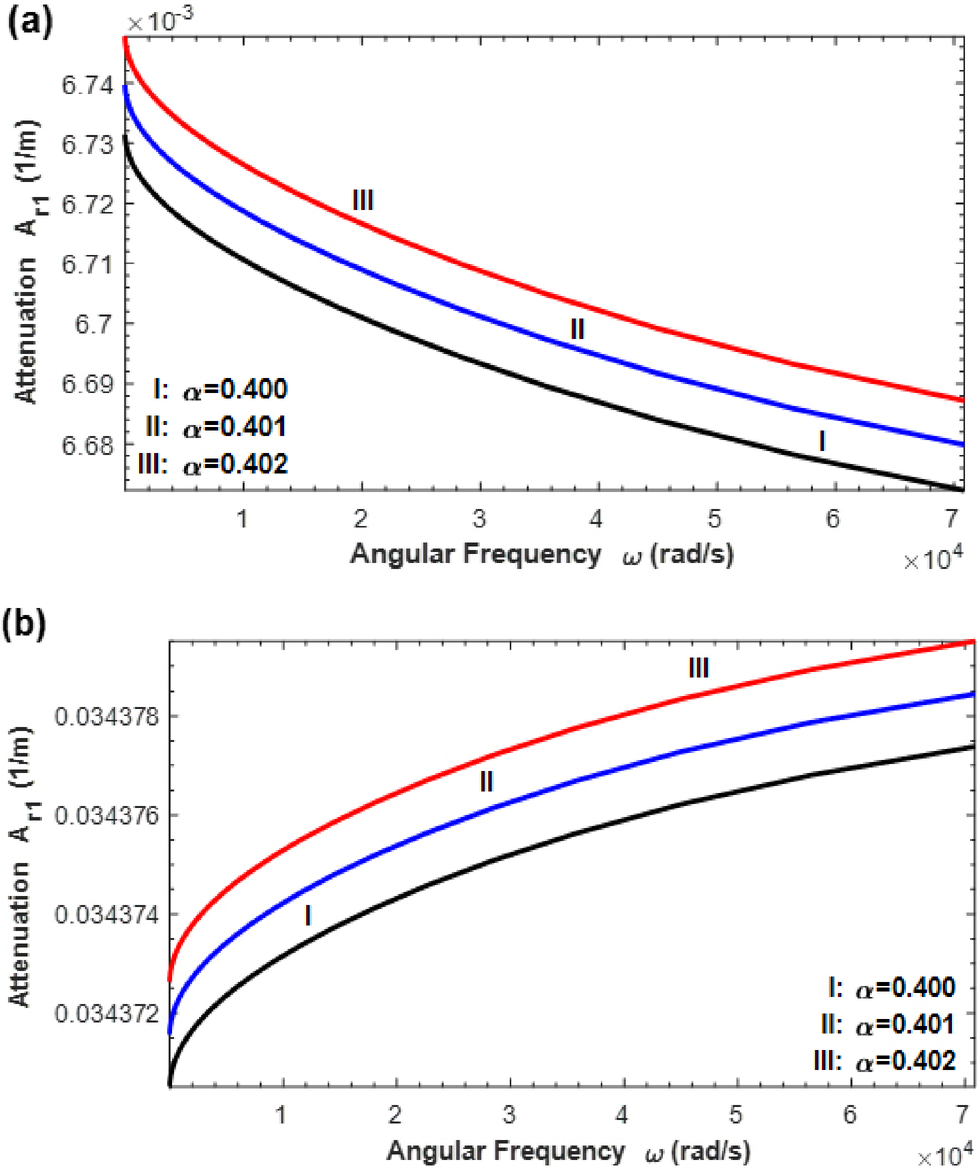


Figure 4.2: Effect of Biot's parameter on A_{r1} (a) Thermally Insulated, (b) Isothermal.

Case II: When we neglect thermal effect, $\beta_s = \beta_f = \tau_0 = K = C_e = b_n = 0$ and Eq.(4.12) becomes

$$\{\Omega_2 \nabla^4 + \omega^2 \Omega_1 \nabla^2 + \omega^4 \Omega_0\}\{\phi_s, \phi_f\}(x, y) = 0, \tag{4.41}$$

where $\Omega_0 = \rho q - \rho_f^2$, $\Omega_1 = D_1 q + M\rho - 2\alpha M \rho_f$, $\Omega_2 = D_1 M - \alpha M^2$ and the coupling parameter a_n reduces to

$$a_n = \frac{D_1(k^2 - m_n^2) - \rho\omega^2}{\alpha M(m_n^2 - k^2) + \rho_f \omega^2}, \quad (n = 1, 2).$$

The frequency equation, in this case, becomes

$$(\alpha_{11}^2 m_2^2 + \alpha_{13}^2 m_1^2 m_2^2 m_4^2 - \alpha_{12}^2 m_1^2)^2 - 4\alpha_{11}^2 \alpha_{13}^2 m_1^2 m_2^4 m_4^2 = 0, \tag{4.42}$$

with the modified values of

$$\alpha_{11} = a_{11}(a_{22}a_{44} - a_{24}a_{42}),\ \ \alpha_{12} = a_{12}(a_{21}a_{44} - a_{24}a_{41}),\ \ \alpha_{13} = a_{14}(a_{21}a_{42} - a_{22}a_{41}),$$

$$a_{11} = \lambda(m_1^2 - k^2) + 2\mu m_1^2 - b_1\beta, \quad a_{12} = \lambda(m_2^2 - k^2) + 2\mu m_2^2 - b_2\beta.$$

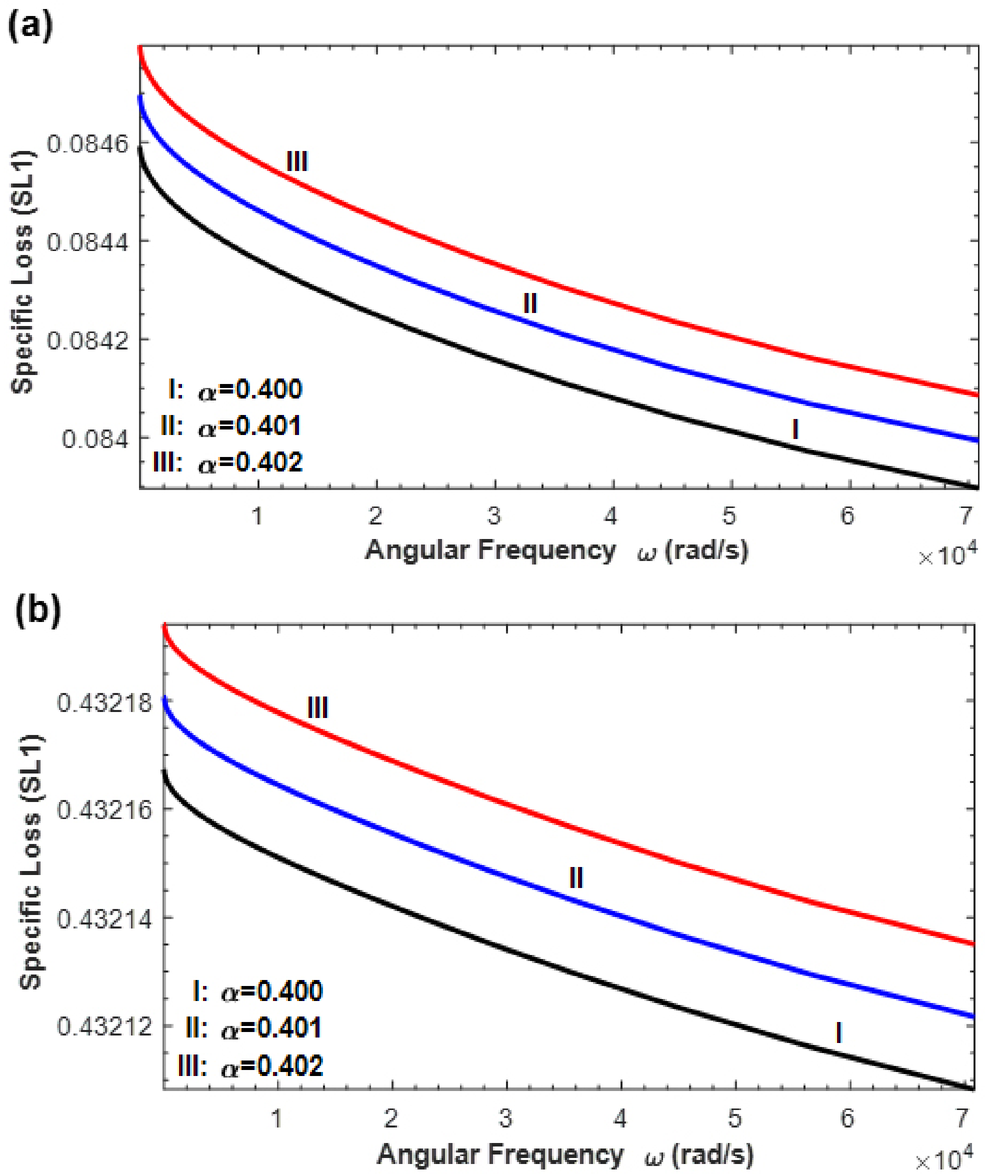


Figure 4.3: Effect of Biot's parameter on $SL1$ (a) Thermally Insulated, (b) Isothermal.

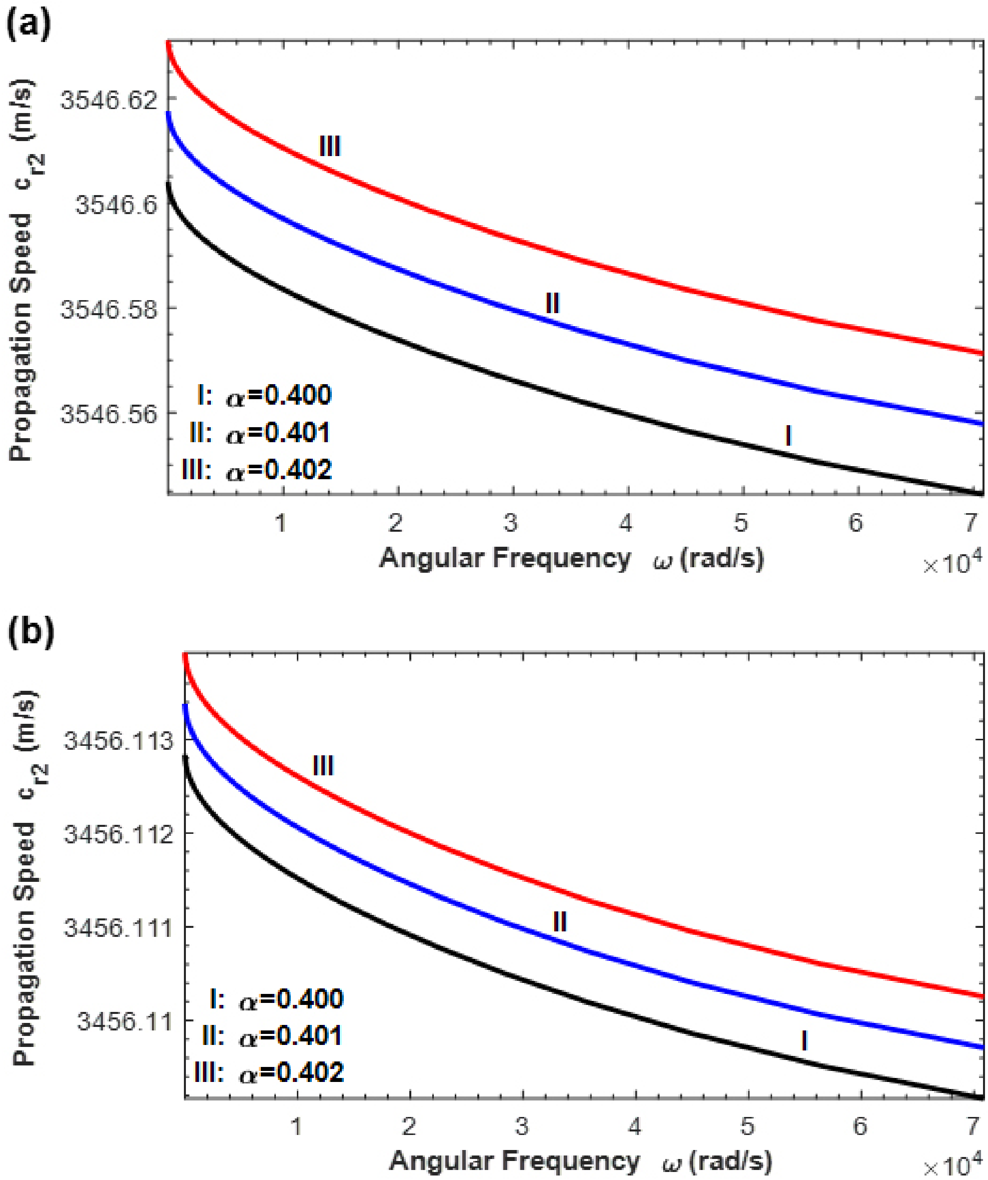


Figure 4.4: Effect of Biot's parameter on c_{r2} (a) Thermally Insulated, (b) Isothermal.

Case III: If we neglect the effect of thermal and porosity of the material, all the parameters except ρ, λ and μ are zero and Eqs.(4.12) & (4.13) reduce to

$$\begin{aligned} \{(\lambda+2\mu)\nabla^4+\rho\omega^2\}\{\phi_s\}(x,y) &= 0, \\ \{\mu\nabla^2+\rho\omega^2\}\{\boldsymbol{\psi}_s\}(x,y) &= 0. \end{aligned} \tag{4.43}$$

Eq. (4.25) also reduces to

$$\{\lambda(m_1^2-k^2)+2\mu m_1^2\}(m_4^2+k^2)-4\mu k^2 m_1 m_4=0,$$

which may be expressed as

$$\left(2 - \frac{c^2}{c_4^2}\right)^2 - 4\sqrt{1 - \frac{c^2}{c_1^2}}\sqrt{1 - \frac{c^2}{c_4^2}} = 0, \tag{4.44}$$

where $c_1^2 = \dfrac{\lambda + 2\mu}{\rho}$ and $c_4^2 = \dfrac{\mu}{\rho}$.

Eq. (4.44) is the frequency equation of Rayleigh wave in classical elasticity (Rayleigh, 1885).

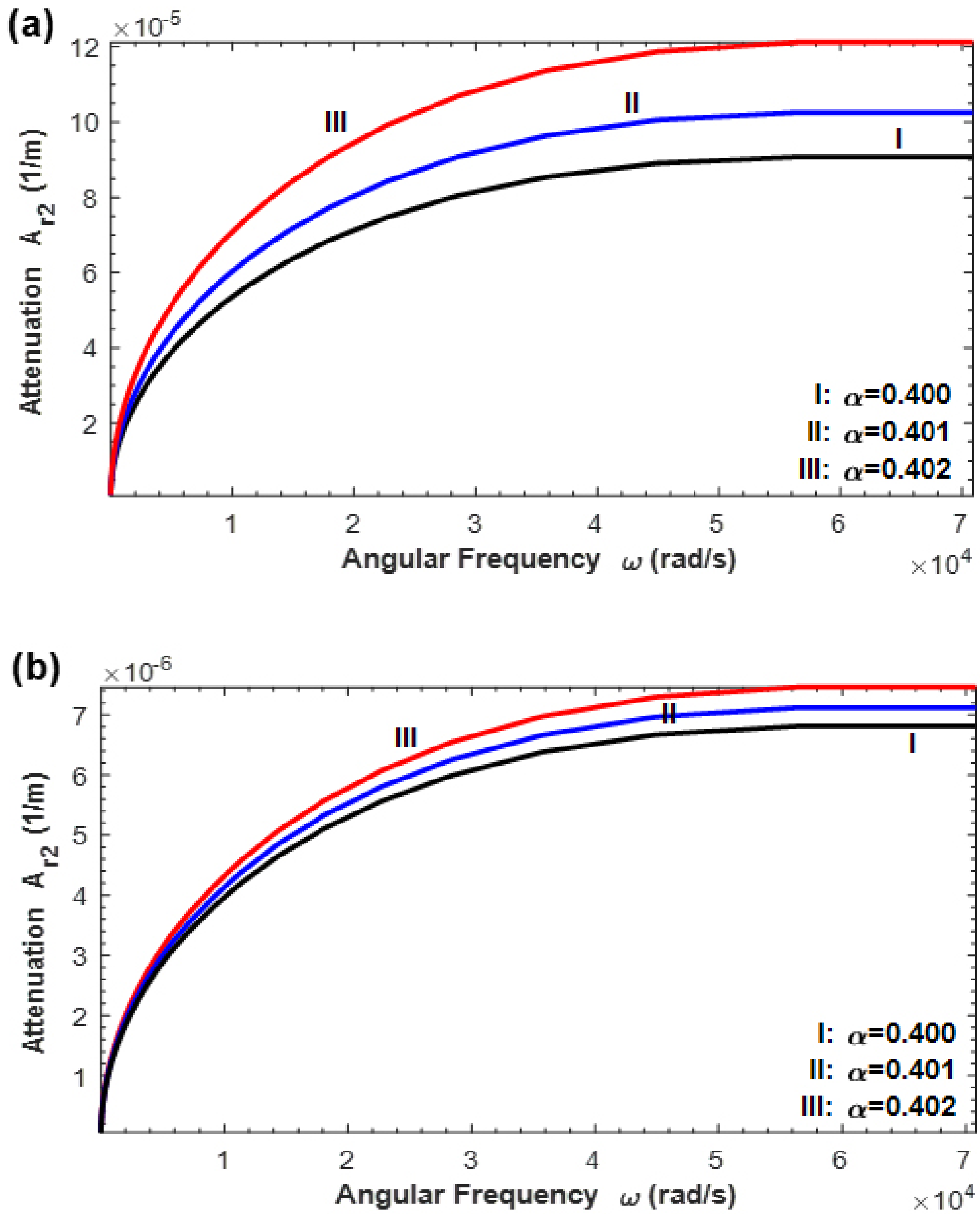
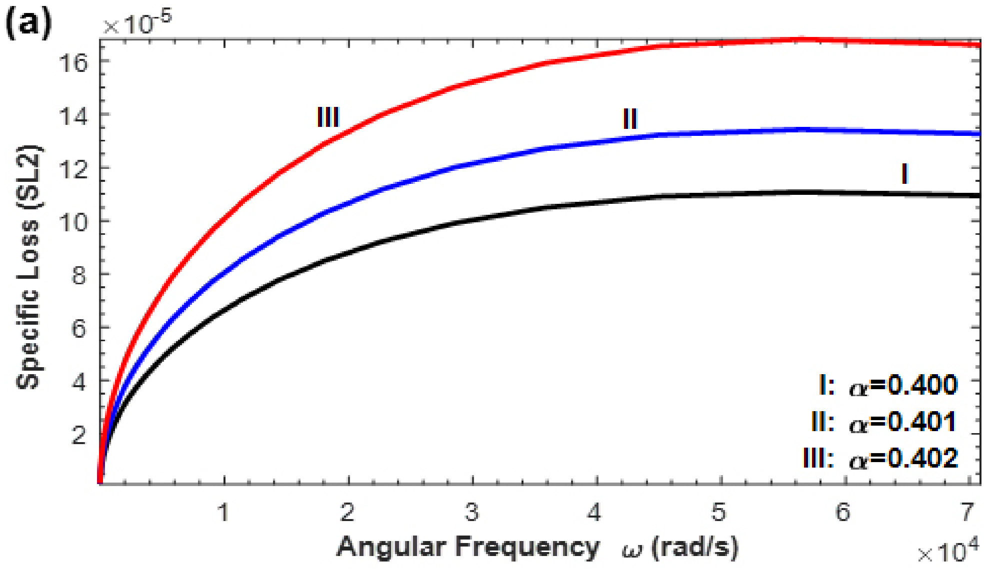


Figure 4.5: Effect of Biot's parameter on A_{r2} (a) Thermally Insulated, (b) Isothermal.



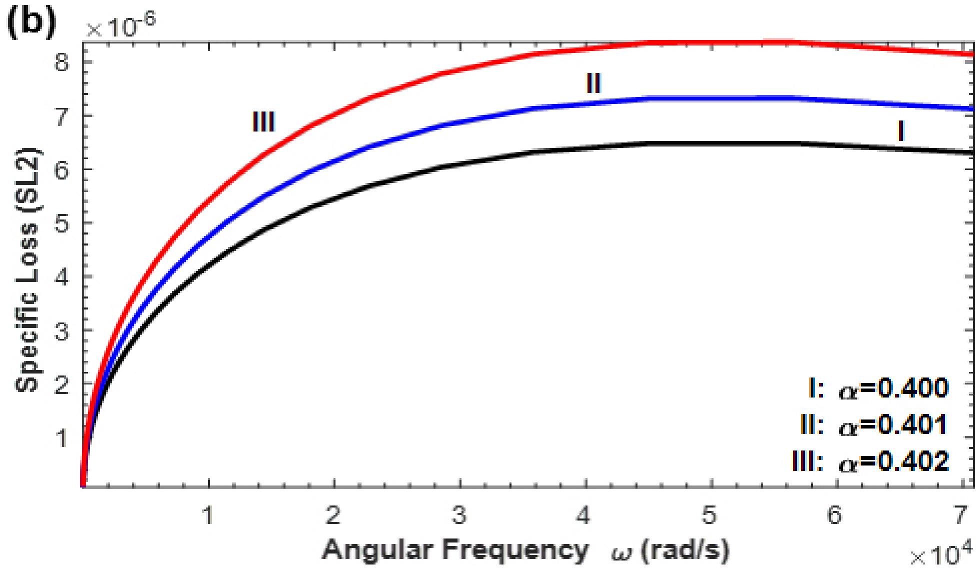


Figure 4.6: Effect of Biot's parameter on $SL2$ (a) Thermally Insulated, (b) Isothermal.

4.7 Numerical Results

For investigating the effect of porosity and Biot's parameter on Rayleigh waves, we consider the numerical values of liquid saturated reservoir rock particularly North-sea Sandstone as given in Table 4.2 (Sharma, 2008).

Parameters	Value
(λ, μ)	$(3.7\ Gpa,\ 7.9\ Gpa)$
$(\rho,\ \rho_f)$	$(2216\ kgm^{-3},\ 950\ kgm^{-3})$
$(\beta_f,\ \beta_s)$	$(2.37 \times 10^{-3}\ Gpa/K,\ 2\beta_f)$
$(C_e,\ K)$	$(1040\ JKg^{-1}/K,\ 170\ Wm^{-1}/K)$
$(\alpha,\ f)$	$(0.4,\ 0.16)$
$(M,\ q)$	$(6\ Gpa,\ 1.05\rho_f/f)$
$(T_0,\ \tau_0)$	$(300\ K, 10^{-10})$

Table 4.2: Parametric values

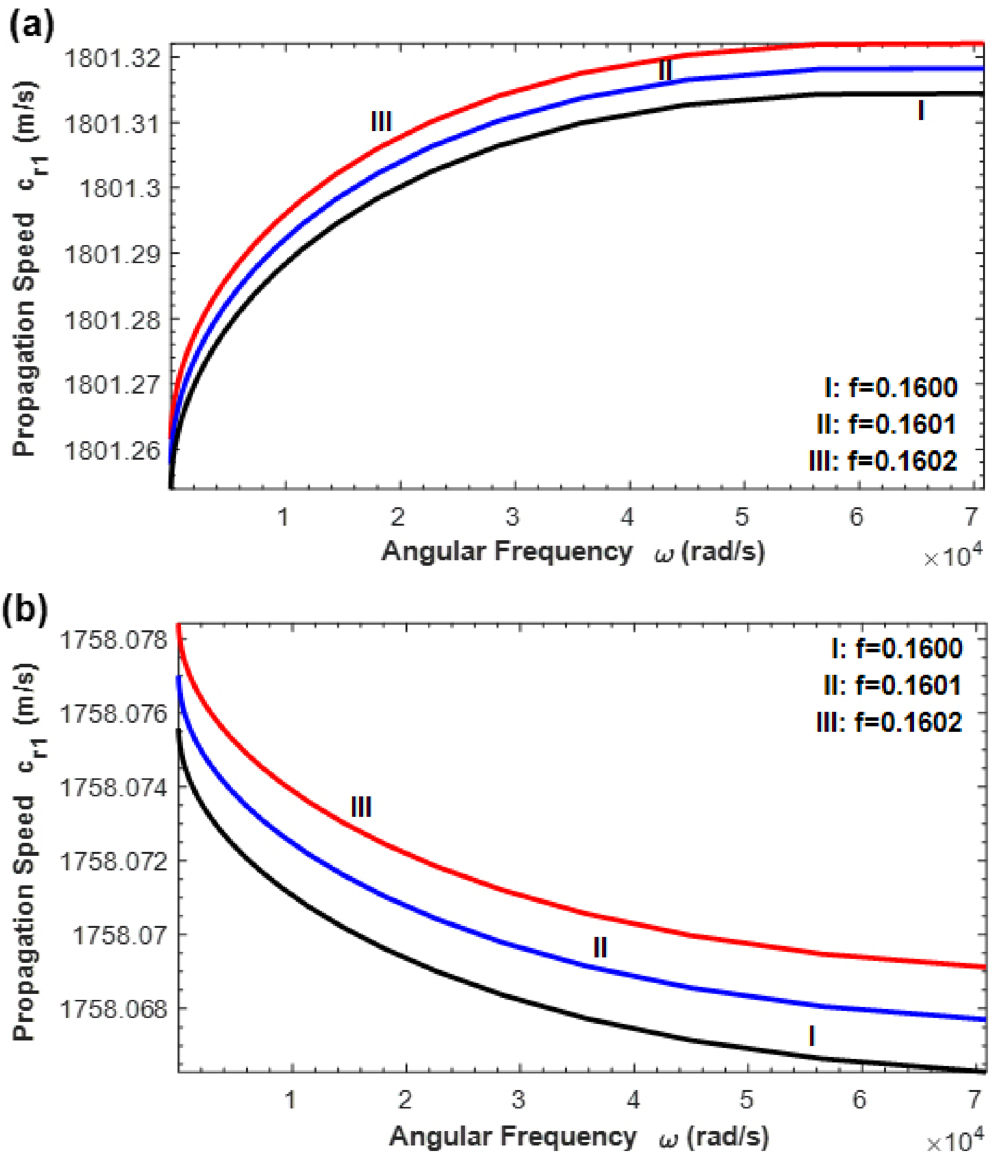


Figure 4.7: Effect of porosity on c_{r1} (a) Thermally Insulated, (b) Isothermal.

We have solved the polynomial Eqs.(4.26) and (4.27) numerically for the wavenumber of Rayleigh waves at the thermally insulated and isothermal surface respectively. There are 48 roots for each cases and we drop those roots that arise due to squaring. We get only two roots each for both the cases satisfying Eqs. (4.24) and (4.26) as well as Eqs. (4.25) and (4.27). These two roots correspond to Rayleigh type I and II which are of dispersive nature. Note that (c_{r1}, A_{r1}, $SL1$) and (c_{r2}, A_{r2}, $SL2$) correspond to Rayleigh type I and II respectively.

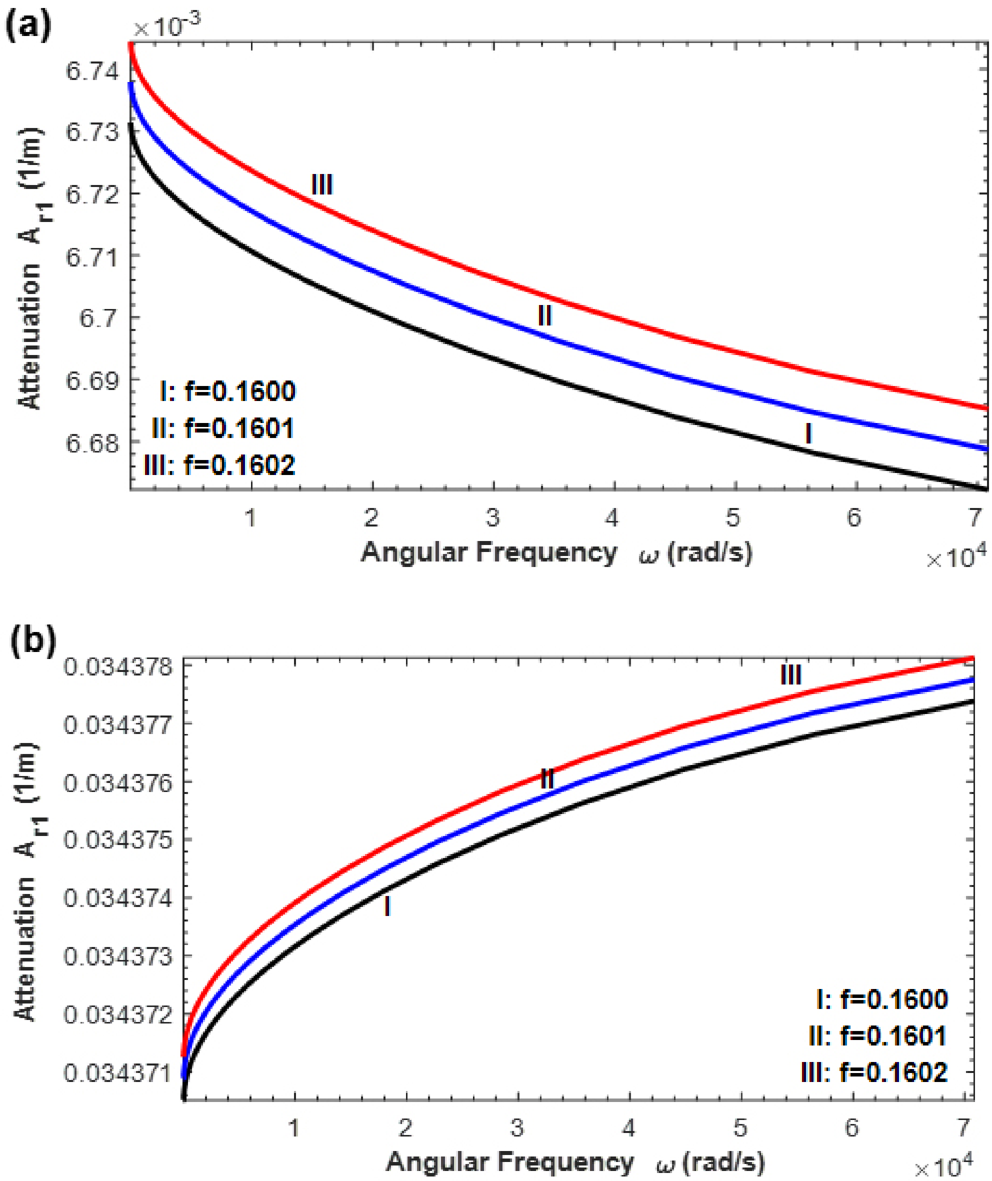
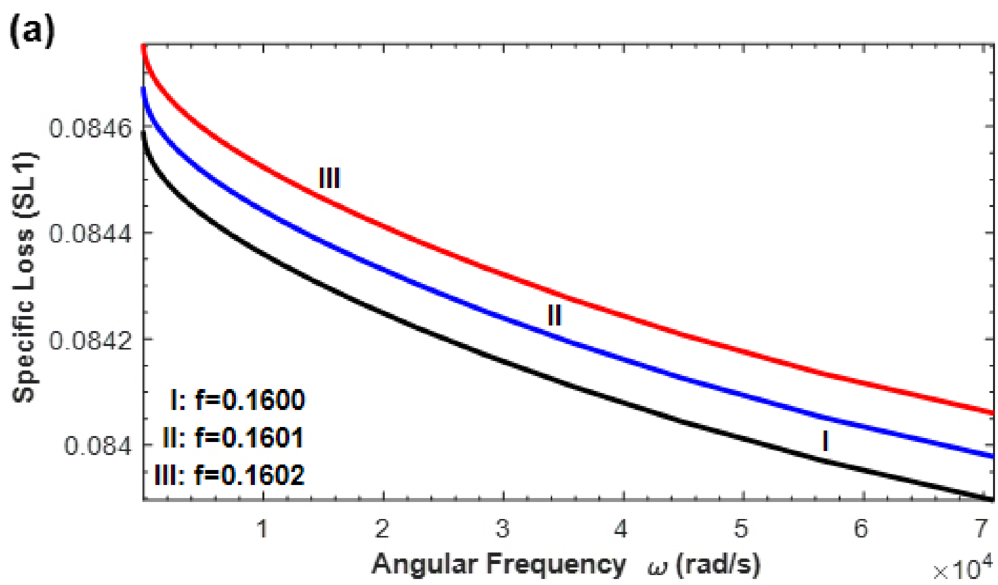


Figure 4.8: Effect of porosity on A_{r1} (a) Thermally Insulated, (b) Isothermal.

We have observed that Rayleigh type I is found to propagate with the speed just less than that of the transverse wave and the second type is faster than all the body waves in the thermoelastic saturated porous material. The velocity curves are depicted with angular frequency(ω) with different α and f in Figures 4.1-4.6 and 4.7-4.12 respectively. In each figures, (a) corresponds for thermally insulated and (b) represents for isothermal surface boundary.



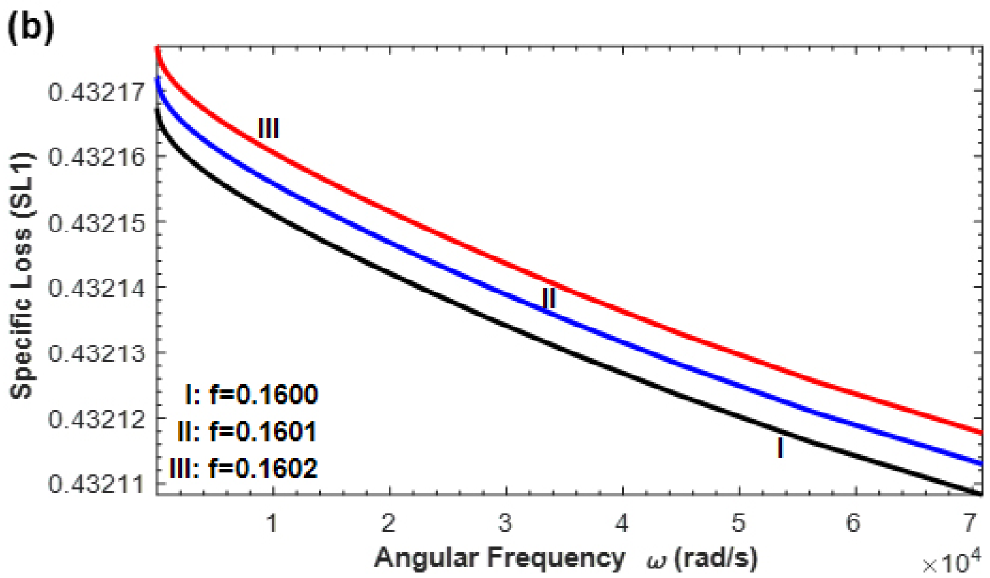


Figure 4.9: Effect of porosity on $SL1$ (a) Thermally Insulated, (b) Isothermal.

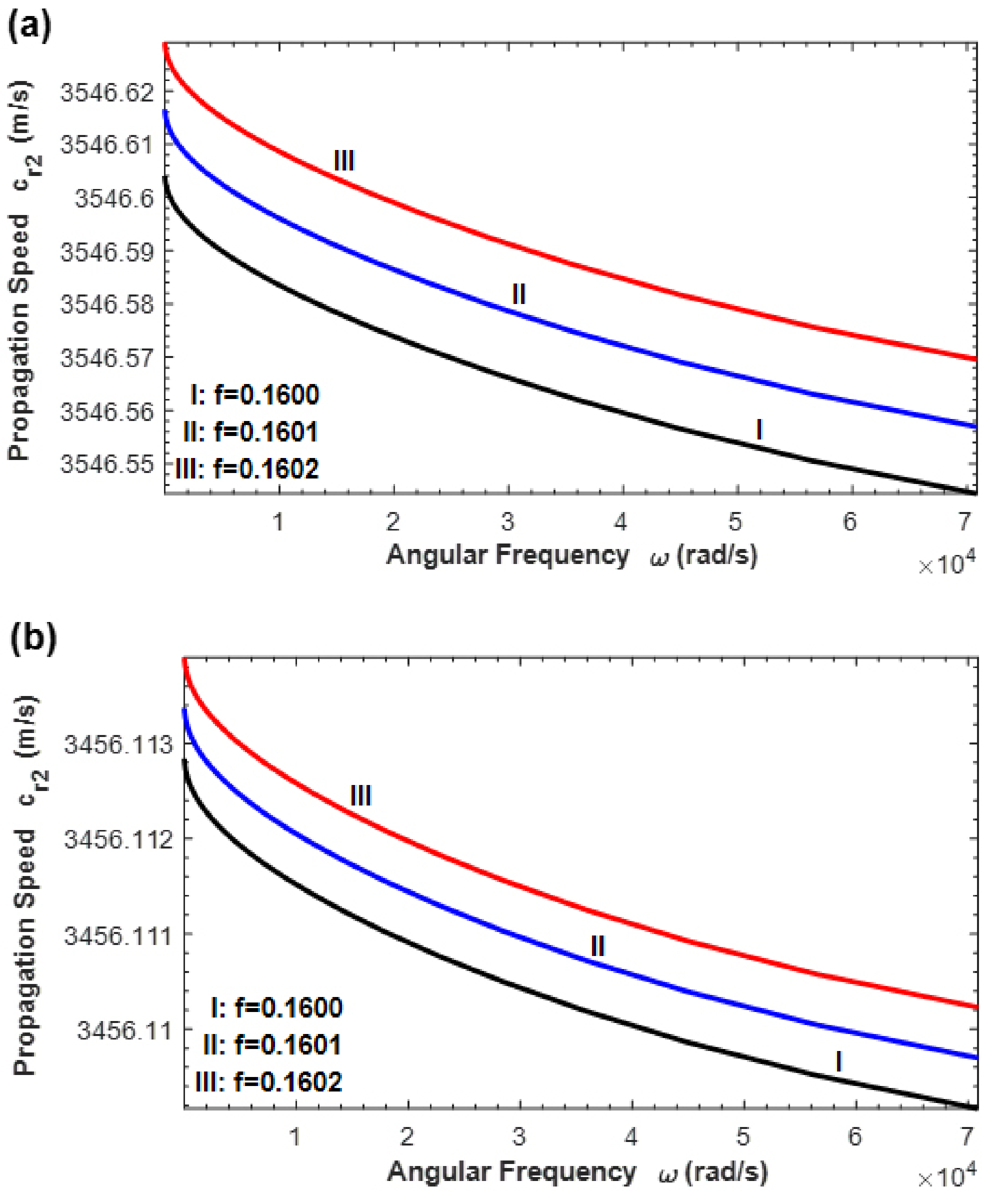


Figure 4.10: Effect of porosity on c_{r2} (a) Thermally Insulated, (b) Isothermal.

4.7.1 Effect of Biot's parameter

In Figure 4.1, the phase speed (c_{r1}) corresponding to Rayleigh type I increases with the increase of angular frequency (ω) for both the thermally insulated and isothermal surface boundary. We have seen that the Rayleigh type I is faster in isothermal than thermally insulated surface. Their values also increase with the increase of α. Similar nature of variations in attenuation A_{r1} are observed in Figure 4.2. The specific loss

($SL1$) in Figure 4.3 for thermally insulated surface increases with the increase of ω, while it is decreased for isothermal surface. The values of $SL1$ increase with the increase of α. For the thermally insulated and isothermal surface, the values of c_{r2}, A_{r2} and $SL2$ corresponding to Rayleigh type II in Figures 4.4, 4.5 and 4.6 are all increase with the increase of ω. Rayleigh type II is found to be faster in thermally insulated than isothermal surface.

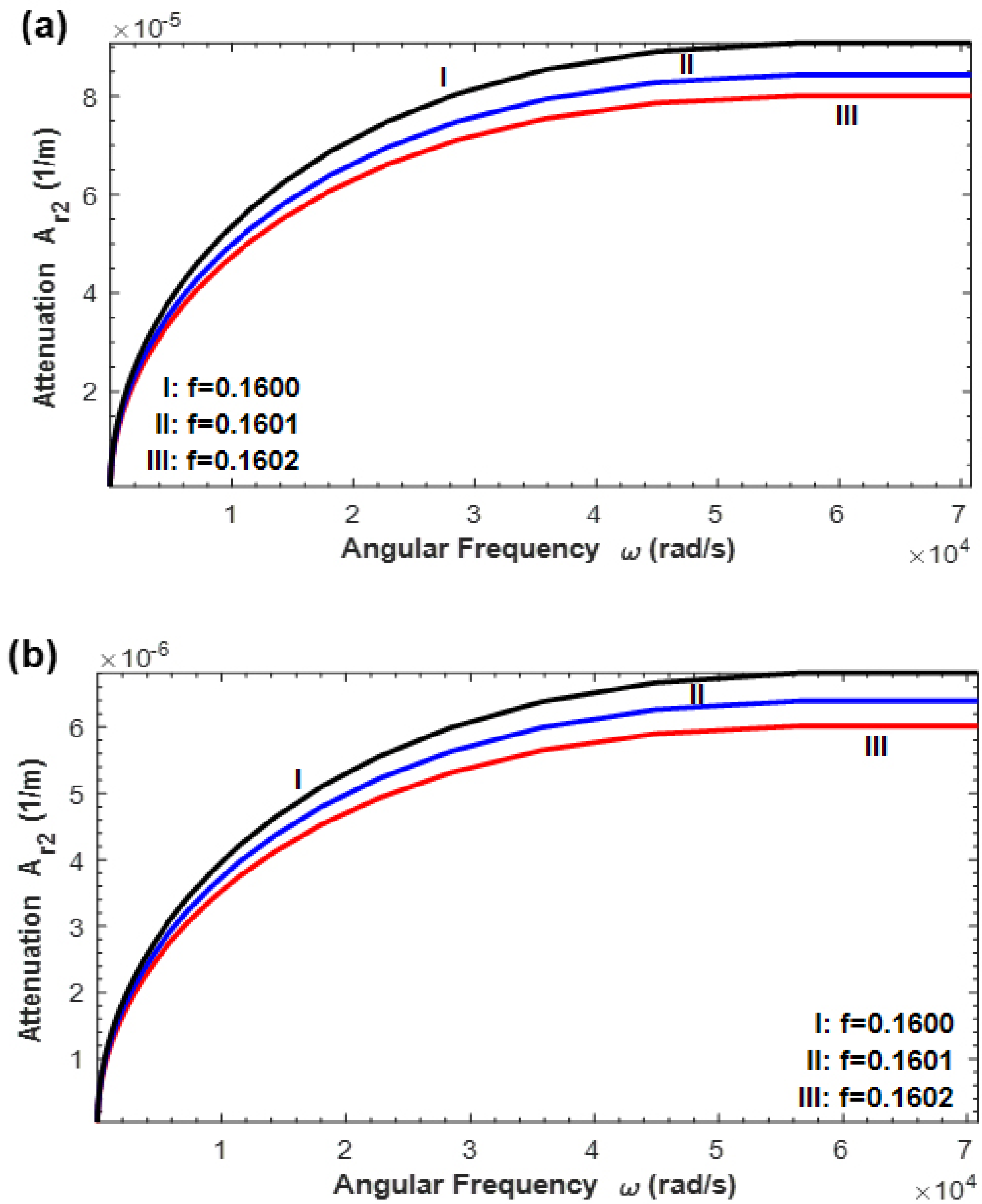


Figure 4.11: Effect of porosity on A_{r2} (a) Thermally Insulated, (b) Isothermal.

In Figure 4.4, we have observed that with the increase in the value of α, the values of c_{r2} for both the thermally insulated and isothermal surface decrease. The values of A_{r2} and $SL2$ for the thermally insulated surface in Figures.4.5(a) and 4.6(a) decrease, while those in Figures.4.5(b) and 4.6(b) for isothermal surface increase with the increase of α.

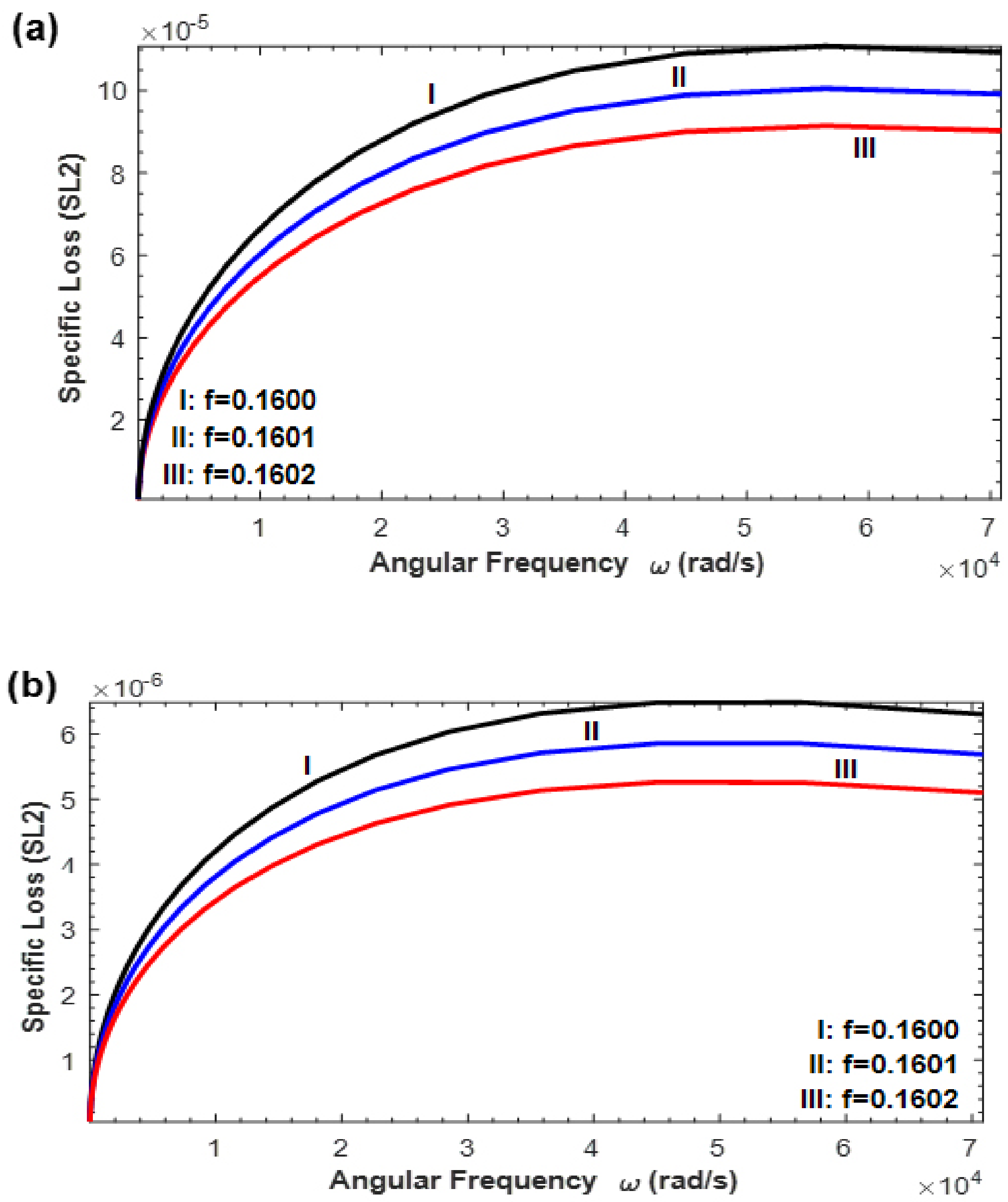


Figure 4.12: Effect of porosity on $SL2$ (a) Thermally Insulated, (b) Isothermal.

4.7.2 Effect of porosity

In Figures 4.7 and 4.10, the values of c_{r1} and c_{r2} for both thermally insulated and isothermal surface increase with the increase of porosity (f). We have observed that the values of A_{r1} and $SL1$ in Figures 4.8 and 4.9 decrease with the increase of f. In Figures 4.11 and 4.12, the values of A_{r2} and $SL2$ for thermally insulated surface decrease with the increase of f, while those of isothermal surface increase with the increase of f. The values of A_{r2} and A_{r1} are found to be smaller in the case of thermally insulated surface. In thermally insulated boundary, there is a linear relationship of $SL2$ with ω. Thus, the propagation speed, attenuation and specific loss of Rayleigh type waves in the thermally insulated and isothermal surface of thermoelastic saturated porous material depend on Biot's parameter and porosity.

4.8 Conclusions

We have analyzed the propagation of Rayleigh type waves at the thermally insulated and isothermal surface of thermo-elastic saturated porous medium. The frequency equations for the Rayleigh type waves have been derived separately using boundary conditions. The velocity curves have been depicted and the results are presented graphically. We conclude with the following points

(i) Two Rayleigh type waves - I, II exist in both the thermally insulated and isothermal boundary of thermo-elastic saturated porous medium. Notice that the propagation speed of first type is just lower than that of transverse waves and the second type is faster than those of body waves.

(ii) We have observed that Rayleigh type-I is faster in the case of isothermal surface, while Rayleigh type-II is faster in the case of thermally insulated boundary.

(iii) The propagation speeds, attenuation and specific loss except $SL1$ for isothermal surface increase with the increase of ω.

(iv) It is observed that the values of c_{r1}, A_{r1} and $SL1$ increase with the increase of

α, while c_{r2} decreases.

(v) The attenuation and specific loss of Rayleigh type II decrease with the increase of α and f for thermally insulated boundary which increase in the case of isothermal boundary.

(vi) The values of c_{r1} and c_{r2} increase, while A_{r1} and $SL1$ decrease with the increase of f for both the boundary conditions.

(vii) With a small change in f and α, there are drastic changed in velocity curves. Thus, the effect of these parameters on the velocity curves of the Rayleigh type waves are very high.

Chapter 5

Reflection and transmission of elastic waves at a plane interface between two dissimilar incompresssible transversely isotropic thermoelastic half spaces[4]

5.1 Introduction

The solutions of governing equations for an incompressible transversely isotropic thermoelastic solid have shown the existence of two plane shear waves. Leslie and Scott (1998, 2000) explored the wave stability for incompressibility at uniform temperature or entropy in isotropic generalized thermoelasticity. They have showed that the stability and unstability properties of waves propagating in a thermomechanically constrained material are found to be unchanged by the existence of the thermal relaxation time. Singh and Singla(2020) discussed the problem of surface wave in an incompressible, homogeneous, transversely isotropic and rotating thermoelastic medium in the context of the Green–Naghdi theory. They obtained the secular equation for Rayleigh waves for thermally insulated and isothermal boundaries.

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This chapter deals with the problem of reflection and transmission of shear waves at a plane interface between two dissimilar incompresssible transversely isotropic thermoelastic half-spaces. Two coupled quasi-shear waves are found to propagate due to the incompressibility of such materials. Applying appropriate boundary conditions at the plane interface, amplitude ratios of the reflected and transmitted quasi-shear waves are obtained. It has been observed that these ratios are functions of the angle of incidence, elastic and thermal parameters of the materials. These ratios are computed numerically for a particular model to see the effects of specific heat and thermal expansion.

5.2 Fundamental Equations

The non-deformed state of homogeneous thermal conducting incompressible elastic materials with transverse isotropy at uniform temperature, T_0 has the following set of equations (see Singh, 2015)

$$c_{11}u_{x,xx} + (c_{13} + c_{44})u_{z,xz} + c_{44}u_{x,zz} - \beta_1 T_{,x} - P_{,x} = \rho\ddot{u}_x, \quad (5.1)$$

$$c_{44}u_{z,xx} + (c_{13} + c_{44})u_{x,xz} + c_{33}u_{z,zz} - \beta_3 T_{,z} - P_{,z} = \rho\ddot{u}_z, \quad (5.2)$$

$$K_1 T_{,xx} + K_3 T_{,zz} - \rho C_e(\dot{T} + \tau_0\ddot{T}) = T_0\{\beta_1(\dot{u}_{x,x} + \tau_0\ddot{u}_{x,x}) + \beta_3(\dot{u}_{z,z} + \tau_0\ddot{u}_{z,z})\}, \quad (5.3)$$

where c_{ij} are elastic constants, P is the hydrostatic pressure, T is the increment in temperature, τ_0 and C_e are thermal relaxation times and specific heat respectively, ρ is the density, K_1 and K_3 are the coefficients of thermal conductivity. It may be noted that comma in the subscript denotes spatial derivatives, $\beta_1 = (c_{11} + c_{12})\alpha_1 + c_{13}\alpha_3$ and $\beta_3 = 2c_{13}\alpha_1 + c_{33}\alpha_3$ with α_1 and α_3 are coefficients of linear expansion.

The incompressibility condition may be given as

$$u_{x,x} + u_{z,z} = 0, \quad (5.4)$$

Eliminating the hydrostatic pressure from Eqs.(5.1) and (5.2), we have

$$\begin{aligned} c_{11}u_{x,xxz} + (c_{13}+c_{44})u_{z,xzz} + c_{44}u_{x,zzz} - \beta_1 T_{,xz} - \rho\ddot{u}_{x,z} = \\ c_{44}u_{z,xxx} + (c_{13}+c_{44})u_{x,xxz} + c_{33}u_{3,133} - \beta_3 T_{,xz} - \rho\ddot{u}_{z,x}. \end{aligned} \tag{5.5}$$

Due to the incompressibility condition (5.4), we can find a scalar function $\phi(x,z,t)$ such that

$$u_x = \phi_{,z} \quad \text{and} \quad u_z = -\phi_{,x}. \tag{5.6}$$

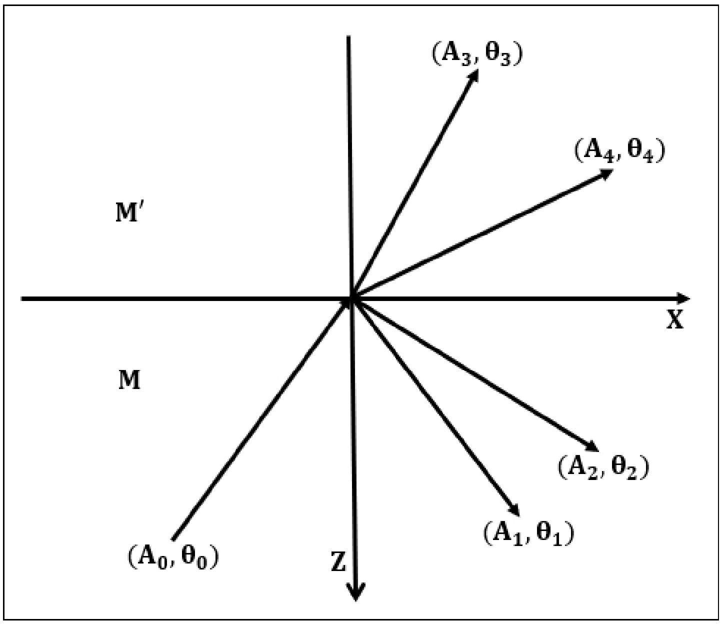

Figure 5.1: Incident, Reflected and Transmitted shear waves.

5.3 Wave Propagation

Consider the Cartesian co-ordinates system with x and y-axes lying horizontally and z-axis along the vertical direction. We aim to study the two dimensional problem of wave propagation in xz-plane in the half-spaces of two incompressible transversely isotropic materials $M : 0 \leq z < \infty$ and $M' : -\infty < z \leq 0$. The geometry of the problem is given in Figure 5.1.

The equations of motion for the half-spaces M and M' are

$$\begin{aligned} c_{44}\phi_{,xxxx} + 2\beta\phi_{,xxzz} + \beta_{13}T_{,xz} &= \rho(\ddot{\phi}_{,xx} + \ddot{\phi}_{,zz}), \\ K_1T_{,xx} + K_3T_{,zz} - \rho C_e(\dot{T} + \tau_0\ddot{T}) &= T_0\{(\beta_1 - \beta_3)(\dot{\phi}_{,xz} + \tau_0\ddot{\phi}_{,xz})\}, \end{aligned} \tag{5.7}$$

$$\begin{aligned} c'_{44}\phi'_{,xxxx} + 2\beta'\phi'_{,xxzz} + \beta'_{13}T'_{,xz} &= \rho'(\ddot{\phi}'_{,xx} + \ddot{\phi}'_{,zz}), \\ K'_1T'_{,xx} + K'_3T'_{,zz} - \rho'C'_e(\dot{T}' + \tau'_0\ddot{T}') &= T'_0\{(\beta'_1 - \beta'_3)(\dot{\phi}'_{,xz} + \tau'_0\ddot{\phi}'_{,xz})\}, \end{aligned} \tag{5.8}$$

where $\beta = (c_{11} + c_{33})/2 - c_{13} - c_{44}$, $\beta' = (c'_{11} + c'_{33})/2 - c'_{13} - c'_{44}$, $\beta_{13} = \beta_3 - \beta_1$, $\beta'_{13} = \beta'_3 - \beta'_1$.

When a quasi shear wave propagating in the half-space M be incident at the plane interface, $z = 0$ making an angle θ_0 with the normal, two quasi shear waves are reflected and transmitted in M and M' respectively. The structures of the wave field for the incident, reflected and transmitted waves may be written as

$$\langle \phi^{(n)}, T^{(n)} \rangle = \langle A_n, a_nA_n \rangle e^{\imath k_n\{xp_1^{(n)} + zp_3^{(n)} - c_nt\}}, \ n = 0, 1, 2, 3, 4 \tag{5.9}$$

where A_n is the amplitude constant, $\langle p_1^{(n)}, 0, p_3^{(n)} \rangle$ is the unit propagation vector, k_n is the wavenumber and c_n is the phase velocity. Note that $n = 0$ represents incident quasi shear wave, $n = 1, 2$ and $n = 3, 4$ represent for the reflected and transmitted quasi shear waves respectively. The coupling constant a_n is given by

$$a_n = \begin{cases} \dfrac{k_n^2\{c_{44}({p_1^{(n)}}^4 + {p_3^{(n)}}^4) + 2\beta {p_1^{(n)}}^2 {p_3^{(n)}}^2 - \rho c_n^2\}}{\beta_{13}p_1^{(n)}p_3^{(n)}}, & n = 0, 1, 2 \\ \\ \dfrac{k_n^2\{c'_{44}({p_1^{(n)}}^4 + {p_3^{(n)}}^4) + 2\beta' {p_1^{(n)}}^2 {p_3^{(n)}}^2 - \rho' c_n^2\}}{\beta'_{13}p_1^{(n)}p_3^{(n)}}, & n = 3, 4. \end{cases}$$

The Snell's law, in this case, is given as (Singh, 2011)

$$\frac{k_0}{k_n} = \frac{\sin\theta_n}{\sin\theta_0} \quad \text{for} \quad n = 1, 2, 3, 4. \tag{5.10}$$

5.4 Boundary Conditions

The tractions and displacement components are continuous at $z = 0$. These conditions may be written as

(i) Continuity of normal traction:

$$\begin{aligned}\sum_{n=0}^{2}\{c_{44}\phi^{(n)}_{,zzz} + c_{22}\phi^{(n)}_{,xxz} - \rho\ddot{\phi}^{(n)}_{,z} + \beta_{13}T^{(n)}_{,x}\} = \\ \sum_{n=3}^{4}\{c'_{44}\phi^{(n)}_{,zzz} + c'_{22}\phi^{(n)}_{,xxz} - \rho'\ddot{\phi}^{(n)}_{,z} + \beta'_{13}T^{(n)}_{,x}\},\end{aligned} \tag{5.11}$$

where $c_{22} = c_{11} + c_{33} - c_{44} - 2c_{13}$, $\; c'_{22} = c'_{11} + c'_{33} - c'_{44} - 2c'_{13}$.

(ii) Continuity of shear traction:

$$\sum_{n=0}^{2} c_{44}(\phi^{(n)}_{,zz} - \phi^{(n)}_{,xx}) = \sum_{n=3}^{4} c'_{44}(\phi^{(n)}_{,zz} - \phi^{(n)}_{,xx}). \tag{5.12}$$

(iii) Continuity of displacement components:

$$\sum_{n=0}^{2}\phi^{(n)}_{,z} = \sum_{n=3}^{4}\phi^{(n)}_{,z}, \qquad \sum_{n=0}^{2}\phi^{(n)}_{,x} = \sum_{n=3}^{4}\phi^{(n)}_{,x}. \tag{5.13}$$

Using Eqs.(5.9) and (5.10) into (5.11)-(5.13), these boundary conditions may be reduced to

$$\begin{aligned}\sum_{n=0}^{2}\{c_{44}k_n^3 {p_3^{(n)}}^3 + c_{22}k_n^3 {p_1^{(n)}}^2 p_3^{(n)} - \rho c_n^2 k_n^3 p_3^{(n)} - \beta_{13}a_n p_1^{(n)} k_n\}A_n - \\ \sum_{n=3}^{4}\{c'_{44}k_n^3 {p_3^{(n)}}^3 + c'_{22}k_n^3 {p_1^{(n)}}^2 p_3^{(n)} - \rho' c_n^2 k_n^3 p_3^{(n)} - \beta'_{13}a_n p_1^{(n)} k_n\}A_n = 0,\end{aligned} \tag{5.14}$$

$$\sum_{n=0}^{2} c_{44}k_n^2({p_3^{(n)}}^2 - {p_1^{(n)}}^2)A_n - \sum_{n=3}^{4} c'_{44}k_n^2({p_3^{(n)}}^2 - {p_1^{(n)}}^2)A_n = 0, \tag{5.15}$$

$$\sum_{n=0}^{2} k_n p_3^{(n)} A_n - \sum_{n=3}^{4} k_n p_3^{(n)} A_n = 0, \qquad \sum_{n=0}^{2} k_n p_1^{(n)} A_n - \sum_{n=3}^{4} k_n p_1^{(n)} A_n = 0. \tag{5.16}$$

Equations (5.14)-(5.16) will be used for evaluation of the amplitude ratios corresponding to the reflected and transmitted waves.

5.5 Amplitude Ratios

Equations (5.14)-(5.16) may be rewritten in matrix notation as

$$AZ = B, \tag{5.17}$$

where A is a matrix of order 4×4 and $B,\ Z$ are matrices of orders 4×1 with the following entries

$$a_{1j} = \begin{cases} c_{44}k_j^3 p_3^{(j)^3} + c_{22}k_j^3 p_1^{(j)^2} p_3^{(j)} - \rho c_j^2 k_j^3 p_3^{(j)} - \beta_{13} a_j p_1^{(j)} k_j,\ j = 1,2, \\ -\{c'_{44}k_j^3 p_3^{(j)^3} + c'_{22}k_j^3 p_1^{(j)^2} p_3^{(j)} - \rho' c_j^2 k_j^3 p_3^{(j)} - \beta'_{13} a_j p_1^{(j)} k_j\},\ j = 3,4, \end{cases}$$

$$a_{2j} = \begin{cases} c_{44}k_j^2 (p_3^{(j)^2} - p_1^{(j)^2}),\ j = 1,2, \\ -c'_{44}k_j^2 (p_3^{(j)^2} - p_1^{(j)^2}),\ j = 3,4, \end{cases}$$

$$a_{3j} = \begin{cases} k_j p_3^{(j)},\ j = 1,2, \\ -k_j p_3^{(j)},\ j = 3,4, \end{cases} \qquad a_{4j} = \begin{cases} k_j p_1^{(j)},\ j = 1,2, \\ -k_j p_1^{(j)},\ j = 3,4, \end{cases}$$

$$b_{11} = -\{c_{44}k_0^3 p_3^{(0)^3} + (c_{11} + c_{33} - c_{44} - 2c_{13})k_0^3 p_1^{(0)^2} p_3^{(0)} - \rho c_0^2 k_0^3 p_3^{(0)} - \beta_{13} a_0 p_1^{(0)} k_0\},$$

$$b_{21} = -c_{44}k_0^2 (p_3^{(0)^2} - p_1^{(0)^2}), \qquad b_{31} = -k_0 p_3^{(0)}, \qquad b_{41} = -k_0 p_1^{(0)}.$$

Eq.(5.17) is solved for $Z_j = \frac{A_j}{A_0}$ due to incident quasi shear wave. The amplitude ratio Z_j for $j = 1,2$ represent for the reflected quasi shear waves and for $j = 3,4$ represent for the transmitted quasi shear waves.

5.6 Particular Cases

CASE I: If we neglect the effect of thermal, the problem becomes reflection/transmission of plane waves at the interface of two dissimilar half-spaces of incompressible transversely isotropic materials. The amplitude ratios of the reflected and transmitted shear waves, in this case, are given by Eq.(5.17) with the following modified values

$$a_{1j} = \begin{cases} c_{44}k_j^3 p_3^{(j)^3} + c_{22}k_j^3 p_1^{(j)^2} p_3^{(j)} - \rho c_j^2 k_j^3 p_3^{(j)}, \ j = 1, 2, \\ -\{c'_{44}k_j^3 p_3^{(j)^3} + c'_{22}k_j^3 p_1^{(j)^2} p_3^{(j)} - \rho' c_j^2 k_j^3 p_3^{(j)}\}, \ j = 3, 4, \end{cases}$$

$$b_{11} = -\{c_{44}k_0^3 p_3^{(0)^3} + (c_{11} + c_{33} - c_{44} - 2c_{13})k_0^3 p_1^{(0)^2} p_3^{(0)} - \rho c_0^2 k_0^3 p_3^{(0)}\}.$$

CASE II: If the half-space M' is neglected, then the problem reduces to reflection of plane waves in an incompressible transversely isotropic thermoelastic materials. The amplitude ratios are given by Eq.(5.17) with the modification that A is a matrix of order 2×2, $B,\ Z$ are column matrices with the following entries

$$a_{1j} = c_{44}k_j^3 p_3^{(j)^3} + c_{22}k_j^3 p_1^{(j)^2} p_3^{(j)} - \rho c_j^2 k_j^3 p_3^{(j)} - \beta_{13} a_j p_1^{(j)} k_j, \quad a_{2j} = c_{44}k_j^2 (p_3^{(j)^2} - p_1^{(j)^2}),$$

$$b_{21} = -c_{44}k_0^2 (p_3^{(0)^2} - p_1^{(0)^2}), b_{11} = -c_{44}k_0^3 p_3^{(0)^3} - c_{22}k_0^3 p_1^{(0)^2} p_3^{(0)} + \rho c_0^2 k_0^3 p_3^{(0)} + \beta_{13} a_0 p_1^{(0)} k_0.$$

The amplitude ratios of the reflected waves depend on the angle of propagation, elastic and thermal parameters of the material.

CASE III: If we neglect the effect of thermal and the half-space M', the problem reduces to reflection of plane waves at the half-space of incompressible transversely isotropic material. In this case, the amplitude ratios of the reflected waves are given as in Case II with the following modified values

$$a_{1j} = c_{44}k_j^3 p_3^{(j)^3} + c_{22}k_j^3 p_1^{(j)^2} p_3^{(j)} - \rho c_j^2 k_j^3 p_3^{(j)}, \ b_{11} = \rho c_0^2 k_0^3 p_3^{(0)} - c_{44}k_0^3 p_3^{(0)^3} - c_{22}k_0^3 p_1^{(0)^2} p_3^{(0)}.$$

5.7 Numerical Results

We have computed the amplitude ratios of reflected and transmitted shear waves due to incident quasi shear waves. The relevant value of the parameters are given in Table 5.1(Chadwick and Seet, 1970).

It may be noted that $(p_1^{(0)}, 0, p_3^{(0)}) = (\sin\theta_0, 0, \cos\theta_0)$ for incident quasi shear wave, $(p_1^{(1)}, 0, p_3^{(1)}) = (\sin\theta_1, 0, -\cos\theta_1)$, $(p_1^{(2)}, 0, p_3^{(2)}) = (\sin\theta_2, 0, -\cos\theta_2)$ for reflected quasi shear wave and $(p_1^{(3)}, 0, p_3^{(3)}) = (\sin\theta_3, 0, \cos\theta_3)$, $(p_1^{(4)}, 0, p_3^{(4)}) = (\sin\theta_4, 0, \cos\theta_4)$ for transmitted quasi shear waves.

The variation of amplitude ratios with the angle of incidence, θ_0 at different values of (β, β') are depicted through Figures 5.2–5.5, while Figures 5.6 and 5.7 show the variation of amplitude ratios at different values of (C_e, C_e'). The values of the amplitude ratios $|Z_1|$ and $|Z_3|$ in Figures 5.2 and 5.4 of the reflected and transmitted shear waves increases and decreases respectively with the increase of θ_0. We have observed that the effects of (β, β') on $|Z_1|$ and $|Z_3|$ have minimum near the normal and grazing angle of incidence.

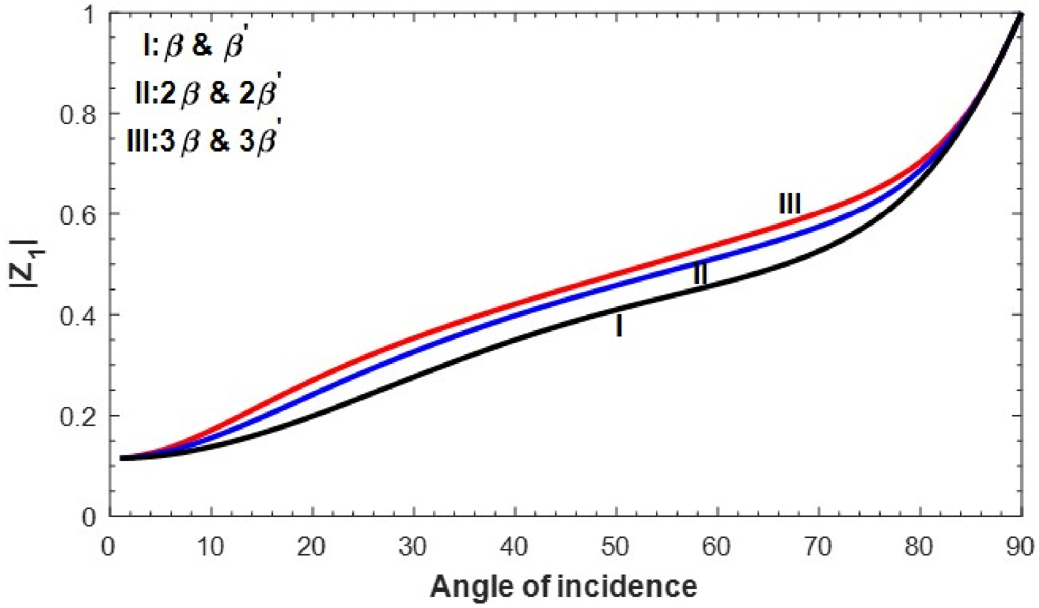

Figure 5.2: Variation of $|Z_1|$ with angle of incidence for different values of β and β'.

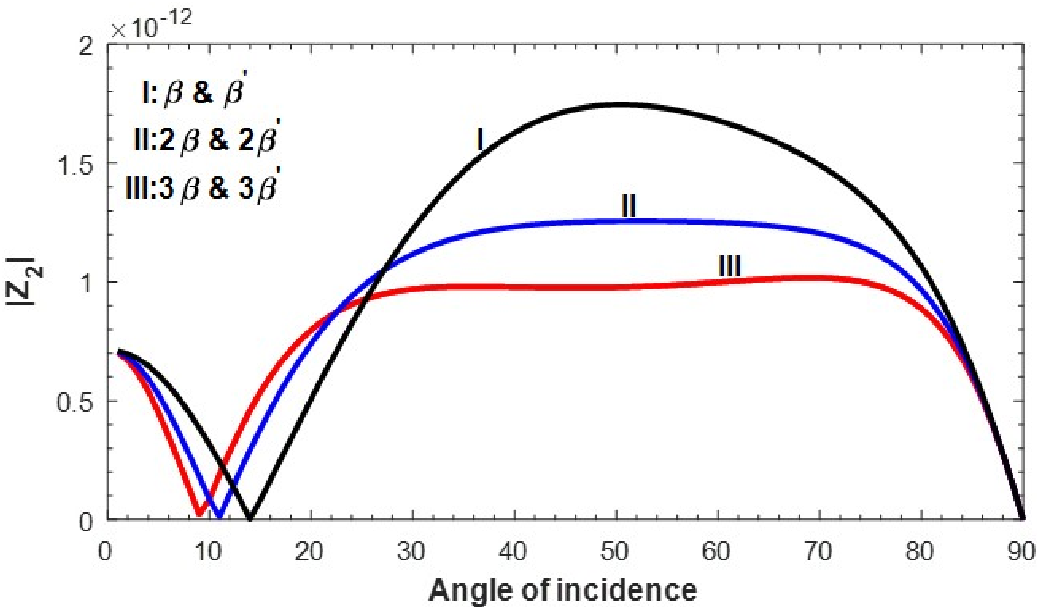

Figure 5.3: Variation of $|Z_2|$ with angle of incidence for different values of β and β'.

Cobalt (M)	Value	Zinc (M')	Value	Units
ρ	8.836×10^3	ρ'	7.14×10^3	kgm^{-3}
c_{11}	3.071×10^{11}	c'_{11}	1.628×10^{11}	Nm^{-2}
c_{12}	1.650×10^{11}	c'_{12}	0.362×10^{11}	Nm^{-2}
c_{13}	1.027×10^{11}	c'_{13}	0.508×10^{11}	Nm^{-2}
c_{33}	3.581×10^{11}	c'_{33}	0.627×10^{11}	Nm^{-2}
c_{44}	0.755×10^{11}	c'_{44}	0.385×10^{11}	Nm^{-2}
β_1	7.04×10^6	β'_1	5.75×10^6	$Nm^{-2}degree^{-1}$
β_3	6.90×10^6	β'_3	5.17×10^6	$Nm^{-2}degree^{-1}$
C_e	4.27×10^2	C'_e	3.9×10^2	$Jkg^{-1}degree^{-1}$
K_1	0.690×10^2	K'_1	1.24×10^2	$Wm^{-1}degree^{-1}$
K_3	0.690×10^2	K'_3	1.24×10^2	$Wm^{-1}degree^{-1}$
T_0	298	T'_0	296	K
τ_0	0.05	τ'_0	0.06	

Table 5.1: Value of the elastic and thermal parameters

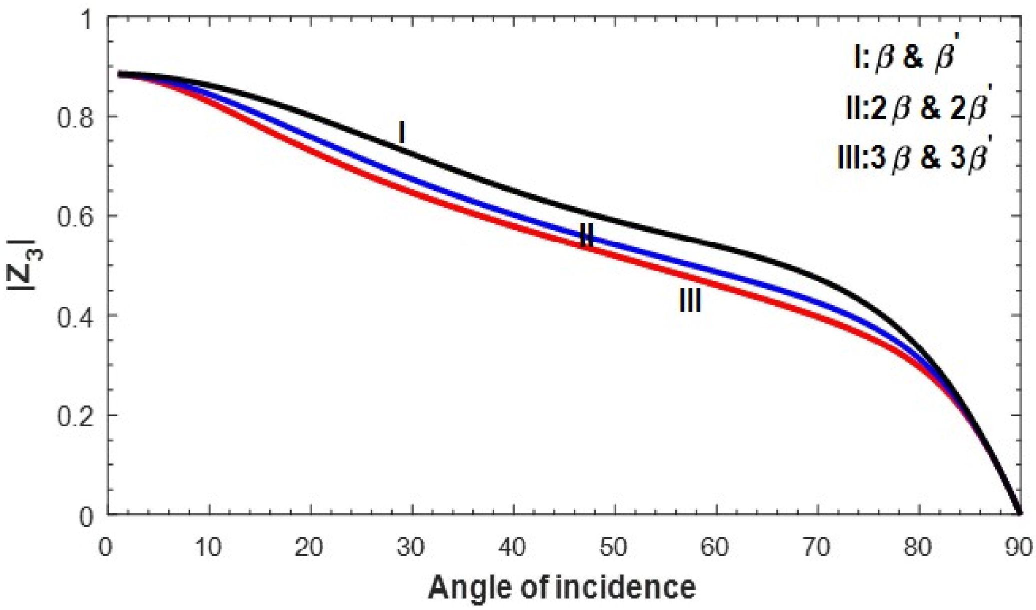

Figure 5.4: Variation of $|Z_3|$ with angle of incidence for different values of β and β'.

Figures 5.3 and 5.5 shown that the amplitude ratios $|Z_2|$ and $|Z_4|$ have similar fashion. They started from certain values which decrease with the increase of θ_0 and increase thereafter to the maximum value followed by decreasing with the increase of θ_0. Here also the minimum effect of (β, β') on $|Z_2|$ and $|Z_4|$ is observed near normal and grazing angle of incidence.

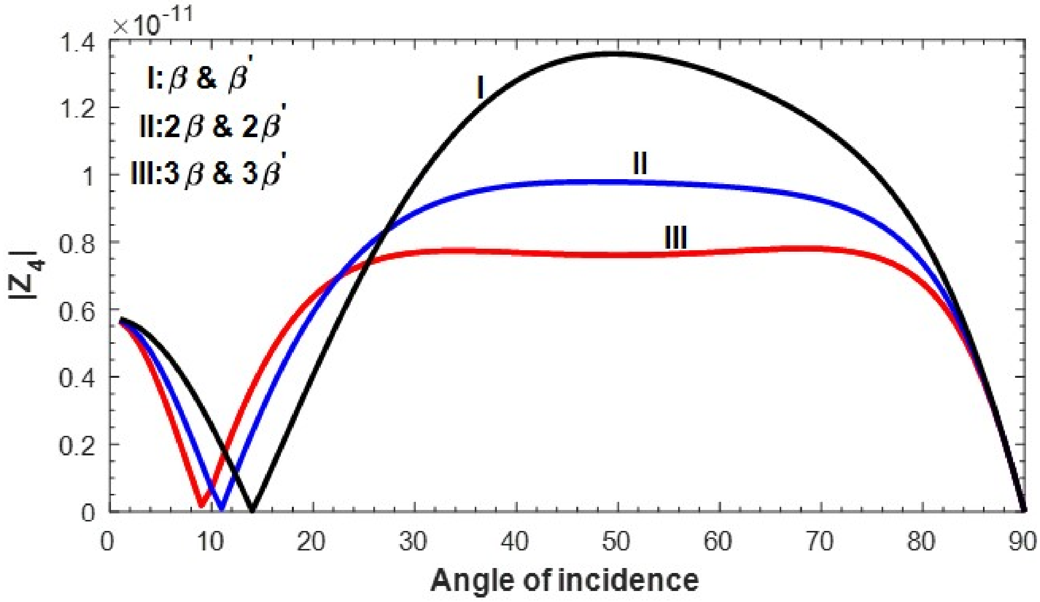

Figure 5.5: Variation of $|Z_4|$ with angle of incidence for different values of β and β'.

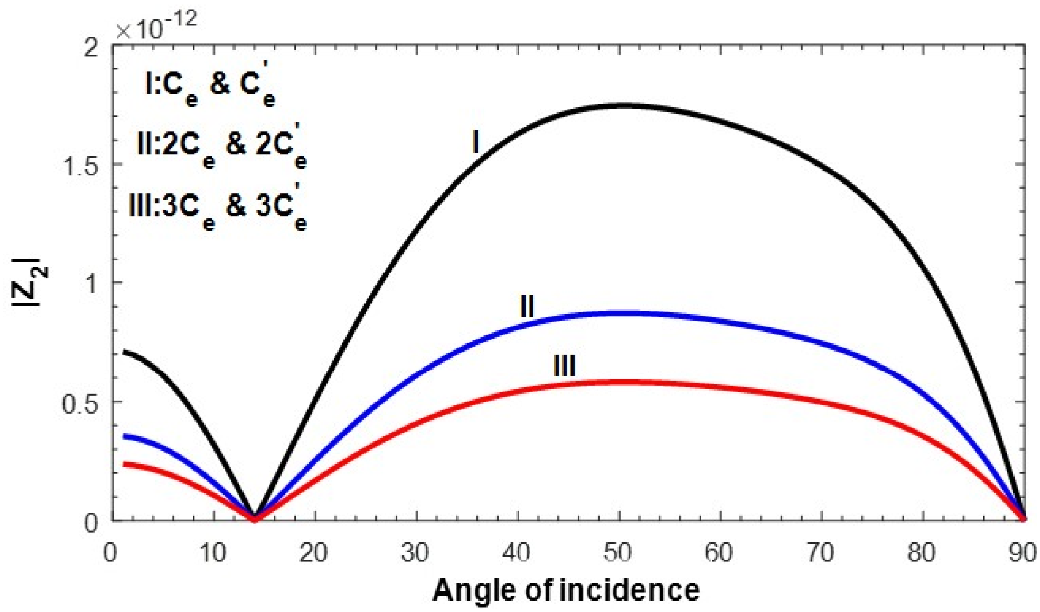

Figure 5.6: Variation of $|Z_2|$ with angle of incidence for different values of C_e and C_e'.

The effect of specific heats on the amplitude ratios $|Z_2|$ and $|Z_4|$ in Figures 5.6 and 5.7 have similar pattern. They have minimum effect of (C_e, C_e') near $\theta_0 = 14^0$ and grazing angle of incidence. It is also observed that the values of $|Z_2|$ and $|Z_4|$ decrease with the increase of specific heats. We also noticed very few effect of specific heats on $|Z_1|$ and $|Z_3|$. Thus, the amplitude ratios of the reflected and transmitted shear waves are found to be functions of angle of incidence, elastic and thermal parameters.

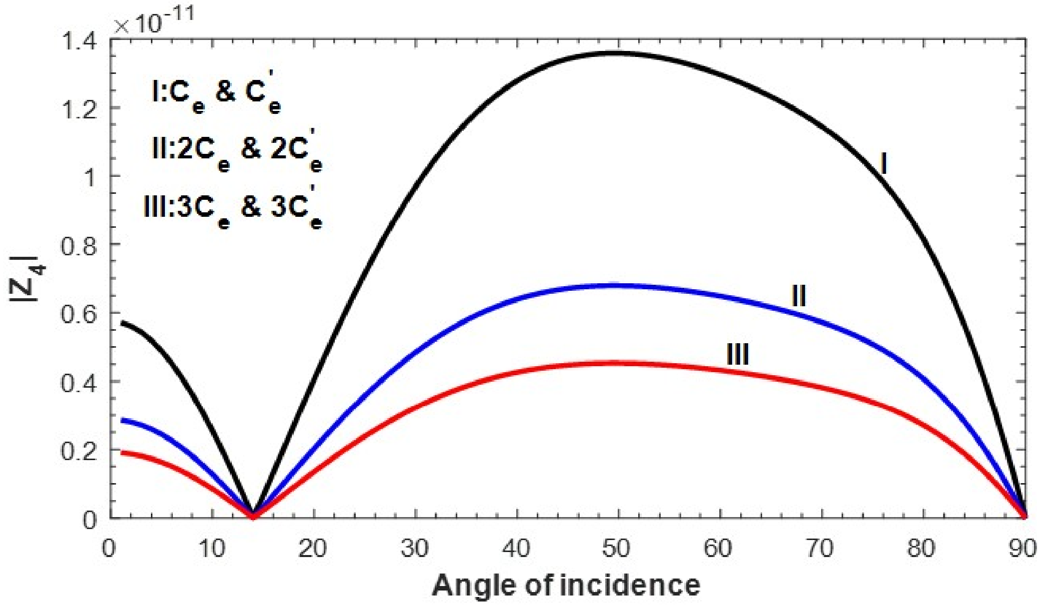

Figure 5.7: Variation of $|Z_4|$ with angle of incidence for different values of C_e and C_e'.

5.8 Conclusion

The problem of incident quasi shear wave at a plane interface between two dissimilar half-spaces of incompressible transversely isotropic thermoelastic materials has been investigated. The amplitude ratios of the reflected and transmitted shear waves are analytically and numerically obtained to analyze the effect of specific heats and coefficient of linear thermal expansion. We summarize the concluding remarks as

(i) The amplitude ratios of the reflected and transmitted waves are found to be functions of angle of incidence, elastic and thermal parameters of the materials.

(ii) The value of $|Z_1|$ and $|Z_3|$ increases and decreases respectively with the increase of θ_0.

(iii) The effect of (β, β') on the amplitude ratios has minimum near the normal and grazing angle of incidence.

(iv) The effect of (C_e, C_e') on $|Z_2|$ and $|Z_4|$ has minimum near $\theta_0 = 14^0$ and grazing angle of incidence.

(v) The values of $|Z_2|$ and $|Z_4|$ decrease with the increase of (C_e, C_e').

Chapter 6

Summary and Conclusions

In the present thesis, we study the propagation of elastic waves in different thermoelastic materials. We discuss the reflection and transmission of QL, QT and thermal waves (T-mode) from a plane interface between two dissimilar thermoelastic half-spaces. The amplitude and energy ratios of of the reflected and transmitted waves are obtained analytically and numerically. We also discuss the propagation of surface waves, particularly Rayleigh and Stoneley waves in such thermoelastic materials. The dispersion relations of Rayleigh and Stoneley waves are derived using suitable boundary conditions.

The first chapter is the general introduction of the thesis which includes the basic definitions, elastic waves, thermoelasticity and theories, application of wave propagation and review of literature.

In Chapter 2, the problem of the propagation of surface waves (Stoneley and Rayleigh waves) in thermoelastic materials with voids has been investigated. The dispersion relations of the Stoneley waves at the bonded and unbonded interfaces between two dissimilar half-spaces of thermoelastic materials with voids are derived. The numerical values of the determinant corresponding to the frequency equation of the Stoneley wave are calculated numerically for a particular model and they are represented graphically. We also derived the frequency equation of Rayleigh wave at the surface free boundary of thermoelastic materials with voids. The velocity curve and attenuation of Rayleigh wave in this material have shown that there are two modes of vibration. These two modes are computed and they are depicted graphically. The effect of thermal parameters in these surface waves are also discussed.

Third chapter deals with the reflection/transmission of elastic waves at a plane interface between two dissimilar half-space of initially stressed transversely isotropic thermoelastic materials. Three quasi coupled longitudinal(QL), transverse(QT) and thermal waves are found to propagate in initially stressed transversely isotropic thermoelastic materials. We use boundary conditions to obtain the reflection/transmission coefficients of the reflected/transmitted waves for incident QL and QT-waves. The distribution of energy for the reflected and transmitted waves are also discussed. Numerical computations have been performed for these coefficients and energy ratios to analyze the impact of initial stresses. In the case of incident QT-wave, critical angles are observed for reflected and transmitted QL-waves at $\theta_0 = 30^0$ and 58^0 respectively.

In Chapter 4, the problem of propagation of Rayleigh wave on the surface of heat conducting saturated porous materials has been discussed. The dispersion relations of the Rayleigh type waves are derived at the thermally insulated and isothermal boundary surface. The velocity curves, attenuation and specific loss of the two modes of Rayleigh waves are obtained for the thermoelastic saturated porous medium. The effect of porosity and Biot's parameters on these values are examined numerically for a particular model. The velocity curves of the Rayleigh type waves depend on the porosity, elastic, thermal and Biot's parameter of the material. The phase speed of Rayleigh type I is just lower than that of transverse waves and the Rayleigh type II wave is faster than those of body waves.

Chapter 5 discuss the problem of reflection and transmission of elastic waves between two dissimilar incompresssible transversely isotropic thermoelastic half-spaces. We have observed that two coupled quasi-shear waves can propagate through such materials due to the incompressibility condition. The amplitude ratios of the reflected and transmitted quasi-shear waves are obtained with the help boundary conditions.

These ratios are computed numerically for a particular model . These ratios are computed numerically and examined the effects of specific heat and thermal expansion. It has been observed that these ratios are functions of angle of incidence, elastic and thermal parameters of the material.

Chapter 6 is summary and conclusion.

Finally, list of references is given at the end.

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DATE OF ADMISSION	: 05.09.2014

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LIST OF PUBLICATIONS

1. SS Singh and Lalawmpuia Tochhawng (2019). Stoneley and Rayleigh waves in thermoelastic materials with voids, *Journal of Vibration and Control*, 25(14), 2053–2062.

2. Lalawmpuia Tochhawng and SS Singh (2020). Effect of initial stresses on the elastic waves in transversely isotropic thermoelastic materials, *Engineering Reports*, e12104, 1-14.

3. C Zorammuana, Lalawmpuia Tochhawng and SS Singh (2020). Elastic Waves in the Half-space of an Incompressible Thermoelastic Material with Transverse Isotropy, *Science and Technology Journal*, 8(1), 58-62.

4. Lalawmpuia Tochhawng and SS Singh (2021). Effect of porosity and Biot's parameters on Rayleigh waves in thermoelastic saturated porous materials, *International Journal of Advances in Applied Mathematics and Mechanics*, 8(4), 15-27.

5. Lalawmpuia Tochhawng, SS Singh and C. Zorammuana (2021). A Mathematical model of shear wave propagation in the incompressible transversely isotropic thermoelastic half-spaces, Accepted for publication to *International Journal of Engineering, Science and Technology.*

CONFERENCES/ SEMINARS/ WORKSHOPS

1. Attended "State Level Workshop on C^{++} Language and Numerical Methods" organized by Pachhunga University College, Aizawl-796001, Mizoram on 16^{th} and 20^{th} February, 2015.

2. Attended "Second Mizoram Mathematics Congress" organized by Mizoram Mathematics Society(MMS) in collaboration with Pachhunga University College and Mizoram University, Aizawl - 796004, Mizoram on 13^{th} and 14^{th} August, 2015.

3. Presented a paper "Elastic waves in initially stressed transversely isotropic thermoelastic solids" in Multidisciplinary International Seminar On "A perspective of Global Research Process: Presented Scenario and Future Challenges" organized by Manipur University, Manipur on 19^{th} and 20^{th} January, 2019.

4. Attended Instructional School for Teachers "Mathematical Modelling in Continuum Mechanics and Ecology" organized by National Centre for Mathematics and Mizoram University, Aizawl-796004, Mizoram on 3^{rd}-15^{th} June, 2019.

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ABSTRACT

A STUDY OF ELASTIC WAVE IN DIFFERENT THERMOELASTIC MATERIALS

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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ABSTRACT

A STUDY OF ELASTIC WAVE IN DIFFERENT THERMOELASTIC MATERIALS

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In partial fulfillment of the requirement of the Degree of Doctor of Philosophy in Mathematics of Mizoram University, Aizawl.

ABSTRACT

The study of wave propagation in thermoelastic materials is interesting due to its applications. The theories of thermoelasticity consist of the combined analysis of the effects of heat conduction as well as elasticity of the materials. The effect of heat on the deformation of an elastic solid and the inverse effect of deformation on the thermal state of the solid are considered. The study of wave propagation has wide applications in the fields of Seismology, geophysics, Earthquake engineering, tele-communication, medicines (echography), metallurgy and signal processing. It is useful to detect the notches and faults in railway tracks, buried land-mines, etc. The technique of wave propagation is also used in the exploration of valuable materials such as minerals, crystals, hydrocarbons, fluids (oils, water) etc. beneath the earth surface. The following objectives are taken up in the thesis:

1. Propagation of surface waves in thermoelastic materials with voids.

2. Transmission of elastic waves in initially stressed transversely isotropic thermoelastic solids.

3. Rayleigh waves in thermoelastic saturated porous medium.

4. Reflection and transmission of elastic waves at a plane interface between two dissimilar incompresssible transversely isotropic thermoelastic half spaces.

The first chapter is the general introduction of the thesis which includes the basic definitions, elastic waves, thermoelasticity and theories, application of wave propagation and review of literature.

In Chapter 2, the problem of the propagation of surface waves (Stoneley and Rayleigh waves) in thermoelastic materials with voids has been investigated. The dispersion relations of the Stoneley waves at the bonded and unbonded interfaces between two dissimilar half-spaces of thermoelastic materials with voids are derived. The numerical values of the determinant corresponding to the frequency equation of

the Stoneley wave are calculated numerically for a particular model and they are represented graphically. We also derive the frequency equation of Rayleigh wave at the surface free boundary of thermoelastic materials with voids. We have observed that there are two modes of vibration for the Rayleigh waves and obtained the velocity curves and attenuation. These two modes are computed and they are depicted graphically. The effect of thermal parameters on these surface waves are also discussed.

Third chapter deals with the reflection/transmission of elastic waves at a plane interface between two dissimilar half-spaces of initially stressed transversely isotropic thermoelastic materials. Three quasi coupled longitudinal(QL), transverse(QT) and thermal (T-mode) waves are found to propagate in initially stressed transversely isotropic thermoelastic materials. We use suitable boundary conditions at the interface to obtain the reflection/transmission coefficients of the reflected/transmitted waves for incident QL and QT-waves. The distribution of energy for the reflected and transmitted waves are also discussed. Numerical computations have been performed for these coefficients and energy ratios to analyze the impact of initial stresses. In the case of incident QT-wave, critical angles are observed for reflected and transmitted QL-waves at $\theta_0 = 30^0$ and 58^0 respectively.

In Chapter 4, the problem of propagation of Rayleigh wave on the heat conducting saturated porous materials has been discussed. The dispersion relations of the Rayleigh type waves are derived at the thermally insulated and isothermal boundary surface. The velocity curves, attenuation and specific loss of the two modes of Rayleigh waves are obtained for the thermoelastic saturated porous medium. The effect of porosity and Biot's parameters on these values are examined numerically for a particular model. The velocity curves of the Rayleigh type waves depend on the porosity, elastic, thermal and Biot's parameter of the material. The phase speed of Rayleigh type I is just lower than that of transverse waves and that of the Rayleigh type II wave is faster than those of body waves.

Chapter 5 discuss the problem of reflection and transmission of elastic waves between two dissimilar incompresssible transversely isotropic thermoelastic half-spaces. We have observed that two coupled quasi-shear waves can propagate through such materials. The amplitude ratios of the reflected and transmitted quasi-shear waves are obtained with the help of boundary conditions. These ratios are computed numerically and examined the effects of specific heat and thermal expansion. It has been observed that these ratios are functions of angle of incidence, elastic and thermal parameters of the material.

Chapter 6 is summary and conclusion.

Finally, list of references is given at the end.