A STUDY ON CERTAIN ALMOST CONTACT MANIFOLDS AND INVARIANT SUBMANIFOLDS

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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By

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Department of Mathematics and Computer Science

Supervisor: Prof. Jay Prakash Singh

Submitted

In partial fulfilment of the requirement of the Degree of Doctor of Philosophy in Mathematics of Mizoram University, Aizawl

DEPARTMENT OF MATHEMATICS & COMPUTER SCIENCE

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CERTIFICATE

This is to certify that the thesis entitled "A Study on Certain Almost Contact Manifolds and Invariant Submanifolds" submitted by Mr. Mohan Khatri (Registration No: MZU/Ph.D./1402 of 26.02.2019) for the degree of Doctor of Philosophy (Ph.D.) of the Mizoram University, embodies the record of original investigation carried out by him under my supervision. He has been duly registered and the thesis presented is worthy of being considered for the award of the Ph.D. degree. This work has not been submitted for any degree from any other university.

(Prof. JAY PRAKASH SINGH)

Supervisor

MIZORAM UNIVERSITY

TANHRIL

Month: September **Year: 2022**

DECLARATION

I Mohan Khatri, hereby declare that the subject matter of this thesis entitled "A Study on Certain Almost Contact Manifolds and Invariant Submanifolds" is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to do the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in other University/Institute.

This is being submitted to the Mizoram University for the degree of Doctor of Philosophy (Ph.D.) in Mathematics.

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PREFACE

The present thesis entitled **"A Study on Certain Almost Contact Manifolds and Invariant Submanifolds"** is an outcome of the research carried out by the author under the supervision of Prof. Jay Prakash Singh, Professor and Head, Department of Mathematics and Computer Science, Mizoram University, Aizawl, Mizoram.

This thesis has been divided into seven chapters and each chapter is subdivided into smaller sections. The first chapter is the general introduction which includes the literature reviews and the basic definitions such as topological manifold, smooth manifold, symmetric manifolds, almost contact metric manifold, (κ, μ)-contact metric manifold, Sasakian space forms, Kenmotsu manifold, almost Kenmotsu manifold, submanifolds, generalized m-quasi-Einstein structure, Ricci-Yamabe soliton and Lorentzian manifold.

The second chapter is dedicated to the study of (κ, μ)-contact metric manifold and the semiconformal curvature tensor. This chapter is divided into three sections. In the first section, we introduce two types of generalized ϕ-recurrent (κ, μ)-contact metric manifolds known as hyper generalized ϕ-recurrent (k, μ)-contact metric manifolds and quasi generalized ϕ-recurrent (k, μ)-contact metric manifolds, and investigate their properties. We prove their existence by constructing the non-trivial examples. Then in the second section, the geometric structures of generalized (k, μ)-space forms under certain curvature restrictions and their quasi-umbilical hypersurface are analyzed. Also, the results obtained are verified by constructing an example of 3-dimensional generalized (k, μ)-space form. Finally in the third section, we introduce a type of Riemannian manifold, namely, an almost pseudo semiconformally symmetric manifold which is denoted by $A(PSCS)_n$. Several geometric properties of such a manifold are studied under certain curvature conditions. Some results on Ricci symmetric $A(PSCS)_n$ and

Ricci-recurrent $A(PSCS)_n$ are obtained. Next, we consider the decomposability of $A(PSCS)_n$. Finally, two non-trivial examples of $A(PSCS)_n$ are constructed.

In the third chapter, we study the generalized m-quasi-Einstein structure. In the first section, we analyze the properties of H-contact manifolds and K-contact manifold admitting the generalized m-quasi-Einstein structure whose potential vector field satisfies certain conditions. Also, 3-dimensional normal almost contact manifold admitting generalized m-quasi-Einstein metric is considered. In the second section, we analyze the generalized m-quasi-Einstein structure in the context of almost Kenmotsu manifolds and gave its classification. Moreover, generalized m-quasi-Einstein metric (g, f, m, λ) in almost Kenmotsu 3-H-manifold is considered. Finally, some examples of generalized m-quasi-Einstein structures are constructed.

In the fourth chapter, we study the properties of almost Ricci-Yamabe solitons (shortly, ARYS). In the first section, ARYS in the context of a complete contact metric manifold with the Reeb vector field ξ as an eigenvector of the Ricci operator, K-contact and (κ, μ)-contact manifolds are analyzed. An illustrative example is given to support the obtained result. In the second section, we obtain some isometric results while examining ARYS in the Kenmotsu manifold, $(\kappa, \mu)'$-almost Kenmotsu manifold and 3-dimensional non-Kenmotsu almost Kenmotsu manifolds. Finally, a few non-trivial examples of Kenmotsu manifolds and almost Kenmotsu manifolds admitting ARYS are constructed.

The fifth chapter is divided into two sections. The first section is devoted to the study of invariant submanifolds of f-Kenmotsu manifolds under certain conditions on the second and third fundamental forms. We also consider the f-Kenmotsu space form and give two examples supporting the obtained results. In the second section, we obtain Chen's inequalities for the submanifolds of generalized Sasakian-space-forms endowed with a quarter-symmetric connection. As an application of the obtained inequality, we derive first Chen inequality for bi-slant submanifold of generalized Sasakian-space-forms.

In Chapter 6, we obtain some results on spacetime. This chapter includes two sections. In the first section, we study the geometrical aspects of a perfect fluid spacetime with torse-forming vector field ξ under certain curvature restrictions, and Ricci-Yamabe soliton and η-Ricci-Yamabe soliton in a perfect fluid spacetime. We also give a non-trivial example of perfect fluid spacetime admitting η-Ricci-Yamabe soliton. Then in the second section, we classify the Einstein-type metric on Kenmotsu, non-Kenmotsu $(\kappa, \mu)'$-almost Kenmotsu and almost Kenmotsu 3-H-manifolds. Finally, we construct some non-trivial examples to verify our main results.

In Chapter 7, we give the summary and the conclusion. The references of the mention papers have been given with the surname of the authors and the years of the publication, which are decoded in chronological order in the Bibliography.

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ABSTRACT

In 1828, Gauss formulated an important property of surfaces known as Gaussian curvature in his famous work,"*Theorema Egregium*".Then, Riemann extended Gauss's theorem to spaces known as manifolds. Later on Einstein used Riemannian geometry and its generalization, Finsler geometry, to formulate general relativity theory. Riemannian geometry are used in information geometry, probability and statistics, group theory, representation theory analysis and statistics. Contact geometry has been matured from the mathematical formalism of classical mechanics. It has broad applications in geometrical optics, integrable system, thermodynamics and control theory. In this thesis, we attempt to further understand the properties of Riemannian manifold and almost contact manifolds along with their submanifolds by considering the following objectives.

1. To study semiconformal curvature tensor.

2. To study geometrical properties of (κ, μ)-contact metric manifolds.

3. To study the properties of certain Ricci solitons.

4. To characterize the invariant submanifolds of certain almost contact manifolds.

In Chapter 1, we give the definitions of topological manifold, smooth manifold, Riemannian manifold, almost contact metric manifolds, Kenmotsu manifolds, f-Kenmotsu manifold, almost Kenmotsu manifolds, space forms, Lorentzian manifolds, generalized m-quasi-Einstein structure, almost Ricci-Yamabe soliton and Submanifolds, along with the review of literature.

In the first section of Chapter 2, we introduced and studied hyper generalized ϕ-recurrent (k, μ)-contact metric manifolds and quasi generalized ϕ-recurrent (k, μ)-contact metric manifolds. Then in the second section, the geometric structures of generalized (k, μ)-space forms and their quasi-umbilical hypersurface are analyzed.

Finally, the third section is devoted to the introduction of almost pseudo semiconformally symmetric manifold. Moreover, in each sections, we constructed non-trivial examples.

In Chapter 3, we considered generalized m-quasi-Einstein metric. In the first section, H-contact manifold, K-contact and 3-dimensional normal almost contact manifold admitting generalized m-quasi-Einstein metric are studied. In the second section, we showed that a complete Kenmotsu manifold admitting generalized m-quasi-Einstein metric is isometric to a hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or a warped product $\widetilde{M} \times_\gamma \mathbb{R}$. Then, $(\kappa, \mu)'$-almost Kenmotsu manifold with $h' \neq 0$ admitting generalized m-quasi-Einstein metric is locally isometric to some warped product spaces. Also, some examples of warped product manifolds admitting generalized m-quasi-Einstein metric are given. Finally, almost Kenmotsu 3-H-manifold are also considered.

In Chapter 4, we gave classification of almost Ricci-Yamabe solitons in the context of almost Kenmotsu manifolds as well as K-contact and (κ, μ)-contact metric manifold. In the first section, we focus on complete contact metric manifold with the Reeb vector field as an eigenvector of the Ricci operator whose metric admits an almost Ricci-Yamabe soliton and potential vector field collinear with the Reeb vector field. Then, complete K-contact manifold and non-Sasakian (k, μ)-contact metric manifold admitting gradient ARYS are studied. In the second section, ARYS in the context of almost Kenmotsu manifolds are considered. Non-trivial examples of manifolds whose metric admits ARYS are also constructed.

In Chapter 5, we derived Chen's inequalities for submanifolds of generalized Sasakian-space-forms endowed with a quarter-symmetric connection. Moreover, a detail study on invariant submanifold of f-Kenmotsu manifold is done.

In Chapter 6, the first section is focused on analyzing the geometrical properties of perfect fluid spacetime with torse-forming vector field admitting Ricci-Yamabe soliton and η-Ricci-Yamabe soliton. Then in second section, we classified Einstein-type metric in Kenmotsu manifold as well as non-Kenmotsu $(\kappa, \mu)'$-almost Kenmotsu manifolds.Also, almost Kenmotsu 3-H-manifold with Einstein-type metric are studied.

Chapter 7 is devoted for summary and conclusion.

Chapter 1

Introduction

Chapter 1

Introduction

1.1 Topological Manifold

Definition 1.1. *Suppose M is a topological space. We say that M is a topological manifold of dimension n or a topological n-manifold if it has the following properties:*

1. *M is a Hausdorff space: for every pair of distinct points $p, q \in M$, there are disjoint open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$.*

2. *M is second-countable: there exists a countable basis for the topology of M.*

3. *M is locally Euclidean of dimension n.*

1.2 Smooth Manifold

Let M be a topological n-manifold. A coordinate chart on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to \bar{U}$ is a homeomorphism from U to an open subset $\bar{U} = \varphi(U) \subseteq \mathbb{R}^n$. If (U, φ) and (V, ψ) are two charts such that $U \cap V \neq \varnothing$, the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called the transition map from φ to ψ. Two charts (U, φ) and (V, ψ) are said to be smoothly compatible if either $U \cap V = \varnothing$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism. An atlas $\mathcal{A}$ is a collection of charts whose domain cover M. If any two charts in $\mathcal{A}$ are smoothly compatible then $\mathcal{A}$ is said to be a smooth atlas. A smooth atlas $\mathcal{A}$ on M is maximal if it is not properly contained in any larger smooth atlas.

Definition 1.2. *If M is a topological manifold, a smooth structure or differentiable structure ($\mathbb{C}^\infty$-structure) on M is a maximal smooth atlas $\mathcal{A}$. A smooth manifold is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.*

1.3 Riemannian Manifold

The Riemannian metric allows us to define geometric concepts such as lengths, angles and distances on smooth manifolds. Similar to the inner product on vector space, for manifold, the appropriate structure is a Riemannian metric, which is essentially a choice of the inner product on each tangent space, varying smoothly from point to point.

Definition 1.3. *Let M be a smooth manifold with or without a boundary. A Riemannian metric on M is a smooth symmetric covariant 2-tensor field on M that is positive definite at each point. A Riemannian manifold is a pair (M, g), where M is a smooth manifold and g is a Riemannian metric on M.*

1.4 Connections on Riemannian Manifold

An affine or linear connection on a smooth manifold M is a R-bilinear mapping

$$\nabla : \chi(M) \times \chi(M) \to \chi(M),$$

which satisfies the following properties:

1. $\nabla_{fX}Y = f\nabla_X Y$,

2. $\nabla_X fY = f\nabla_X Y + (Xf)Y$,

for any $X, Y \in \chi(M)$ and smooth function f. On a Riemannian manifold M of dimension n, the affine connection ∇ is said to be Levi-Civita connection or Riemannian connection if it satisfies the following:

1. ∇ is symmetric or torsion-free i.e., $\nabla_X Y - \nabla_Y X = [X, Y]$ and
2. ∇ is a metric compatible i.e., $(\nabla_X g)(Y, Z) = 0$ for all $X, Y, Z \in \chi(M)$.

Let M^n be an n-dimensional Riemannian manifold with Riemannian metric g. A linear connection $\overline{\nabla}$ is known as a quarter-symmetric connection if its torsion tensor T is given by

$$T(X, Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y]$$

satisfies

$$T(X, Y) = \Lambda(Y)\varphi X - \Lambda(X)\varphi Y,$$

where Λ is a 1-form and P is a vector field given by $\Lambda(X) = g(X, P)$ and φ is $(1, 1)$-tensor. Qu and Wang (2015) introduced a special type of quarter-symmetric connection defined as:

$$\overline{\nabla}_X Y = \hat{\overline{\nabla}}_X Y + \psi_1 \Lambda(Y) X - \psi_2 g(X, Y) P, \tag{1.1}$$

where $\hat{\overline{\nabla}}$ denote the Levi-Civita connection. It is easy to see that the quarter-symmetric connection $\overline{\nabla}$ include the semi-symmetric metric connection ($\psi_1 = \psi_2 = 1$) and the semi-symmetric non-metric connection ($\psi_1 = 1, \psi_2 = 0$).

1.5 Symmetric Manifolds

Riemannian symmetric spaces have been a primary topic of research in differential geometry theories. It was Cartan, who first initiated the study of Riemannian symmetric spaces and gave its classification (Cartan, 1926). According to him, an n-dimensional Riemannian manifold M is said to be locally symmetric if its curvature tensor R satisfies $R_{hijk,l} = 0$, where "," represent the covariant differentiation with respect to the metric tensor and R_{hijk} are the components of the curvature tensor of the manifold M. The notion of locally symmetric spaces has been extended by many geometer throughout the year, some of which are Recurrent (Walker, 1950), locally φ-symmetric (Takahashi, 1977), pseudo symmetric (Chaki, 1987) and weakly symmetric (Tam$\acute{a}$ssy and Binh, 1989). Sen and Chaki (1967) obtained an expression for the covariant derivative of the curvature tensor while studying conformally flat space of class one with certain curvature restrictions on the curvature tensor, which is as follows:

$$R^h_{ijk,l} = 2\lambda_l R^h_{ijk} + \lambda_i R^h_{ljk} + \lambda_j R^h_{ilk} + \lambda_k R^h_{ijl} + \lambda^h R^l_{ijk}\,, \tag{1.2}$$

where λ_i is a non-zero covariant vector. Later, Chaki (1987) introduced a manifold whose curvature tensor satisfies (1.2) and called it a pseudo symmetric manifold $(PS)_n$. Extending $(PS)_n$, De and Gazi (2008) introduced almost pseudo symmetric manifold $(APS)_n$. A Riemannian manifold $(M^n, g), (n > 2)$ is said to be an almost pseudo symmetric (De and Gazi, 2008) if its curvature tensor R of type $(0, 4)$ satisfies the following relation:

$$\begin{aligned}(\nabla_E R)(X, Y, W, V) &= [A(E) + B(E)]R(X, Y, W, V) + A(X)R(E, Y, W, V) \\ &+ A(Y)R(X, E, W, V) + A(W)R(X, Y, E, V) \\ &+ A(V)R(X, Y, W, X),\end{aligned} \tag{1.3}$$

for all $X, Y, W, V \in \chi(M)$, where A, B are non-zero 1-forms given by

$$g(E, \rho) = A(E), g(E, \sigma) = B(E), \tag{1.4}$$

for all vector field $E \in \chi(M)$, where $\chi(M)$ being the Lie algebra of vector fields on M. Further extending the notion of $(APS)_n$, Tam$\acute{a}$ssy and Binh (1989) introduced weakly symmetric manifolds.

1.6 Recurrent Manifolds

Walker (1950) introduced the notion of a locally recurrent Riemannian manifold as an extension to locally symmetric spaces. A non flat Riemannian manifold is said to be locally recurrent (Walker, 1950) if there exists a non-zero 1-form A such that

$$(\nabla_X R)(Y, Z)U = A(X)R(Y, Z)U, \tag{1.5}$$

for all $X, Y, Z, U \in \chi(M)$. Then, De et al. (2003) studied φ-recurrent Sasakian manifolds as an extension to locally φ-symmetric manifolds (Takahashi, 1977). As a weaker version of the locally recurrent Riemannian manifold, Dubey (1979) introduced the notion of the generalized recurrent manifold. A non-flat Riemannian manifold is said to be a generalized recurrent manifold if its curvature tensor R satisfies

$$\nabla R = A \otimes R + B \otimes G, \tag{1.6}$$

where A and B are non-vanishing 1-forms defined as (1.4) and the tensor G is defined by

$$G(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \tag{1.7}$$

for any $X, Y, Z \in \chi(M)$. Shaikh et al. (2011) extended this concept to generalized φ-recurrent Sasakian manifold. Hui (2017) studied generalized φ-recurrent generalized (κ, μ)-contact metric manifold and obtained interesting results. A non-flat Riemannian manifold is said to be a generalized φ-recurrent manifold if the curvature tensor R satisfies the condition

$$\varphi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z + B(W)G(X,Y)Z, \tag{1.8}$$

for any $X, Y, Z, W \in \chi(M)$.

1.7 Almost Contact Metric Manifolds

A $(2n+1)$-dimensional smooth manifold M is called an almost contact metric manifold if it admits a $(1,1)$-tensor field φ, a unit vector field ξ (called the Reeb vector field) and a 1-form η such that

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \quad \eta \cdot \varphi = 0, \tag{1.9}$$

this is equivalent to a reduction of the structural group of the tangent bundle to $U(n) \times 1$ (see (Sasaki, 1960; Sasaki and Hatakeyama, 1961)). A Riemannian metric g is said to be an associated (or compatible) metric if it satisfies

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{1.10}$$

for all $X, Y \in \chi(M)$. An almost contact manifold $M^{2n+1}(\varphi, \xi, \eta)$ together with a compatible metric g is known as an almost contact metric manifold (Blair et al., 1995; Blair, 1976, 2010). Chinea and Gonzalez (1990) obtained a complete classification for almost contact metric manifolds through the study of the covariant derivative of the fundamental 2-form. The fundamental 2-form Φ of an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is defined by

$$\Phi(X,Y) = g(X, \varphi Y),$$

for all $X, Y \in \chi(M)$, and this form satisfies $\eta \wedge \Phi^n \neq 0$. This means that every almost contact metric manifold is orientable. Moreover, an almost contact metric manifold is said to contact metric manifold if $d\eta = \Phi$. The following formulas hold on a contact metric manifold (Blair, 2010)

$$\nabla_X \xi = -\varphi X - \varphi h X, \tag{1.11}$$

Further we define two self-adjoint operators h and ℓ by $h = \frac{1}{2}(\mathcal{L}_\xi \varphi)$, where $\mathcal{L}_\xi \varphi$ denotes the Lie-derivative of φ along ξ and $\ell = R(\cdot, \xi)\xi$ respectively, where R is the Riemannian curvature of M. These operators satisfy

$$h\xi = \ell\xi = 0, \quad h\varphi + \varphi h = 0, \quad Tr.h = Tr.h\varphi = 0. \tag{1.12}$$

$$Tr.\ell = S(\xi, \xi) = 2n - ||h||^2. \tag{1.13}$$

Here, "Tr." denotes trace. When a unit vector ξ is Killing (i.e. $h = 0$ or $Tr.\ell = 2n$) then the contact metric manifold is called K-contact. On the K-contact manifold, we have

$$R(X, \xi)\xi = X - \eta(X)\xi, \tag{1.14}$$

An almost contact structure (φ, η, ξ) and almost contact manifold M is said to be normal if the almost complex structure on $M \times \mathbb{R}$ defined by $J(X, fd/dt) = (\varphi X - f\xi, \eta(X)d/dt)$, where f is a real function on $M \times \mathbb{R}$ and t a coordinate on $\mathbb{R}$, is integrable (Blair, 1976, 2010). The necessary and sufficient condition for the almost contact structure (φ, η, ξ) to be normal is

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where the pair $[\varphi, \varphi]$ is the Nijenhuis tensor of φ defined by

$$[\varphi, \varphi](X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y],$$

for all $X, Y \in \chi(M)$. A normal almost contact metric manifold is a Sasakian manifold. It was shown that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X, \tag{1.15}$$

for any $X, Y \in \chi(M)$. A Sasakian manifold is K-contact but the converse is true only in dimension 3. Olszak (1986) showed that a 3-dimensional almost contact metric manifold M is normal if and only if $\nabla\xi \cdot \varphi = \varphi \cdot \nabla\xi$, or, equivalently,

$$\nabla_X \xi = -\alpha\varphi X + \beta(X - \eta(X)\xi), \tag{1.16}$$

where $2\alpha = div\xi$ and $2\beta = Tr.(\varphi\nabla\xi)$, $div\xi$ is the divergence of ξ defined by $div\xi = Tr.\{X \to \nabla_X \xi\}$ and $Tr.(\varphi\nabla\xi) = Tr.\{X \to \varphi\nabla_X\xi\}$. On a 3-dimensional normal almost contact metric manifold the following relations hold (Olszak, 1986)

$$S(X,\xi) = -X\alpha - (\varphi X)\beta - \{\xi\alpha + 2(\alpha^2 - \beta^2\}\eta(X), \tag{1.17}$$

$$\xi\alpha + 2\alpha\beta = 0. \tag{1.18}$$

for any $X \in \chi(M)$.

1.8 Generalized Sasakian space forms

A plane section π in the tangent bundle T_pM at a point p of a Riemannian manifold is called a φ-section if it is spanned by X and φX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a φ-section is called a φ-sectional curvature. A Sasakian manifold with constant φ-sectonal curvature c is said to be a Sasakian space form and is denoted by $M(c)$ (Blair, 1976). Some examples of Sasakian space forms are $\mathbb{R}^{2n+1}$ and S^{2n+1}, with the standard Sasakian structures (Blair, 1976). Alegre et al. (2004) introduced generalized Sasakian space forms as almost contact metric manifolds M whose curvature tensor R can be written as

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \tag{1.19}$$

for any vector fields X, Y, Z on M, where f_1, f_2 and f_3 are functions on M. Sasakian-space-forms appear as natural examples of generalized Sasakian-space-forms, with constant functions $f_1 = \frac{c+1}{4}, f_2 = f_3 = \frac{c-1}{4}$. Many authors have studied generalized Sasakian-space-forms in different context such as Alegre and Carriazo (2008), Carriazo et al. (2020), Rehman (2015) and Sarkar et al. (2015).

1.9 (κ, μ)-contact metric Manifold

The (κ, μ)-nullity distribution of almost contact metric manifold $M(\varphi, \xi, \eta, g)$ is a distribution (Blair et al., 1995):

$$N(\kappa,\mu) : p \to N_p(\kappa,\mu) = \{Z \in \chi(M) : R(X,Y)Z = \kappa\{g(Y,Z)X - g(X,Z)Y\} + \mu\{g(Y,Z)hX - g(X,Z)hY\}\},$$

for any $X, Y, Z \in \chi(M)$ and real numbers κ and μ. If $\mu = 0$, the (κ, μ)-nullity distribution $N(\kappa, \mu)$ is called the κ-nullity distribution $N(\kappa)$ (Koufogiorgos, 1993). An almost contact metric manifold M with $\xi \in N(\kappa, \mu)$ is called a (κ, μ)-contact metric manifold. A (κ, μ)-contact metric manifold becomes Sasakian manifold for $\kappa = 1$ and $\mu = 0$. In a (k, μ)-contact metric manifold the following relations hold (Blair et al., 1995; Papantoniou, 1993)

$$h^2 = (k-1)\varphi^2, \qquad k \leq 1, \tag{1.20}$$

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{1.21}$$

$$\begin{aligned} S(X,Y) = {} & [2(n-1) - n\mu]g(X,Y) + [2(n-1) + \mu]g(hX,Y) \\ & +[2(1-n) + n(2k+\mu)]\eta(X)\eta(Y), \end{aligned} \tag{1.22}$$

for any vector fields $X, Y \in \chi(M)$.

1.10 Curvatures on Riemannian Manifold

A conformal transformation is an angle preserving map. If g and $\bar{g}$ are two metrics of an n-dimensional Riemannian manifold M such that

$$\bar{g}(X,Y) = e^{2\sigma} g(X,Y),$$

for all vector fields X, Y on M and σ is a scalar function, then the angle between two tangent vectors at a point $p \in M$ does not change with respect to the change of metrics. Under such case M and $\bar{M}$ are conformally related and the corresponding between them is known as conformal transformation (Obata, 1970). One of the most important curvature tensors for analyzing the intrinsic properties of the Riemannian manifold is the Weyl conformal curvature tensor introduced by Yano and Kon (1984). This curvature is invariant under conformal transformation. The conformal curvature C of type (1,3) on a $(2n+1)$-dimensional Riemannian manifold $(M, g), n > 1$, is defined by

$$\begin{aligned} C(X,Y)Z = {} & R(X,Y)Z - \frac{1}{2n-1}\big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ & -g(X,Z)QY\big] + \frac{r}{2n(2n-1)}\big[g(Y,Z)X - g(X,Z)Y\big], \end{aligned} \tag{1.23}$$

where r is the scalar curvature of M, Q is the Ricci operator and S the Ricci tensor.

It is known that a harmonic function is defined as a function whose Laplacian vanishes. In conformal transformation, harmonic functions are not invariant, in general. To tackle this, Ishi (1957) obtained the condition under which a harmonic function becomes invariant by introducing conharmonic transformation as a subgroup of conformal transformation (1.23) satisfying

$$\sigma^{i}_{,i} + \sigma_{,i}\sigma^{i}_{,} = 0.$$

The tensor H which remains invariant under conharmonic transformation is known as conharmonic curvature tensor. For a Riemannian manifold M of dimension-$(2n+1)$, the conharmonic curvature tensor is given by

$$\begin{aligned} H(X,Y)Z &= R(X,Y)Z - \frac{1}{(2n-1)}\Big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY\Big], \end{aligned} \tag{1.24}$$

for all vector fields X, Y, Z on M.

According to Kim (2016), the semiconformal curvature tensor $\tilde{P}$ of type $(1,3)$ on a Riemannian manifold (M^n, g) is defined as follows:

$$\tilde{P}(X,Y)W = -(n-2)b\,C(X,Y)W + [a + (n-2)b]\,H(X,Y)W, \tag{1.25}$$

where a, b are constants not simultaneously zero, $C(X,Y)W$ denotes the conformal curvature tensor of type $(1,3)$, and $H(X,Y)W$ denotes the conharmonic curvature tensor of type $(1,3)$.

Mantica and Suh (2013) introduced and studied $\tilde{Q}$ curvature tensor. In a $(2n+1)$-dimensional Riemannian manifold (M, g), the $\tilde{Q}$ curvature tensor is given by

$$\tilde{Q}(X,Y)Z = R(X,Y)Z - \frac{v}{2n}\big[g(Y,Z)X - g(X,Z)Y\big], \tag{1.26}$$

for any $X, Y, Z \in \chi(M)$ and v is an arbitrary scalar function on M. If $v = \frac{r}{2n+1}$, then $\tilde{Q}$ curvature tensor reduces to concircular curvature tensor (Yano, 1910).

1.11 Kenmotsu Manifold

To study the manifolds of negative curvature Bishop and O'Neill (1969) introduced the warped product as a generalization of the Riemannian product. Tanno (1969) gave a classification of connected $(2n+1)$-dimensional almost contact metric manifold M based on its automorphism groups possessing the maximum dimension $(n+1)^2$. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, k(say). Then there are three classes.

i) $k > 0$, M is a homogeneous Sasakian manifold of constant holomorphic sectional curvature.

ii) $k = 0$, M is the global Riemannian product of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature.

iii) $k < 0$, M is warped product space $\mathbb{R} \times_f \mathbb{C}^n$.

Kenmotsu (1972) studied the third case and obtained its geometric properties. The structure so obtained is now known as the Kenmotsu structure and the manifold with a Kenmotsu structure is called the Kenmotsu manifold (Janssens and Vanhecke, 1981). In general, a Kenmotsu manifold is not Sasakian. A Kenmotsu manifold can be defined as a normal almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. Kenmotsu proved that such a manifold is locally a warped product $I \times_f N^{2n}$, where I is an open interval with coordinate t, $f = ce^t$ is the warping function for some positive constant c and N^{2n} is a Kählerian manifold (Kenmotsu, 1972). It is well known that a Kenmotsu manifolds can be characterized, through their Levi-Civita connection ∇ satisfying the following:

$$\nabla_X \xi = X - \eta(X)\xi, \tag{1.27}$$

$$(\nabla_X \varphi)Y = -\eta(Y)\varphi X - g(X, \varphi Y)\xi, \tag{1.28}$$

for any $X, Y \in \chi(M)$. On a Kenmotsu manifold M the following holds (Kenmotsu, 1972):

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{1.29}$$

$$Q\xi = -2n\xi, \tag{1.30}$$

for any vector fields X, Y on M.

1.12 Almost Kenmotsu Manifolds

Kim and Pak (2005) and Olszak (1989) studied almost contact metric manifolds such that η is closed and $d\Phi = 2\eta \wedge \Phi$ and called it as almost Kenmotsu manifold. A normal almost Kenmotsu manifold is a Kenmotsu manifold. In almost Kenmotsu manifold M, we have

$$\nabla_X \xi = -\varphi^2 X - \varphi h X, \tag{1.31}$$

for any vector field X on M. Dileo and Pastore (2009) studied almost Kenmotsu manifolds satisfying (κ, μ)-nullity distribution and $(\kappa, \mu)'$-nullity distribution. Later, Pastore and Saltarelli (2011) extended it to generalized nullity distribution. An almost Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is said to be a generalized (κ, μ)-almost Kenmotsu manifold if ξ belongs to the generalized (κ, μ)-nullity distribution, i.e.,

$$R(X,Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{1.32}$$

for all vector fields X, Y on M, where κ, μ are smooth functions on M. An almost Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is said to be a generalized $(\kappa, \mu)'$-almost Kenmotsu manifold if ξ belongs to the generalized $(\kappa, \mu)'$-nullity distribution, i.e.,

$$R(X,Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{1.33}$$

for all vector fields X, Y on M, where κ, μ are smooth functions on M. Moreover if both κ and μ are constants in (1.33), then M is called a $(\kappa, \mu)'$-almost Kenmotsu manifold (Pastore and Saltarelli, 2011; Wang and Liu, 2016a; Dileo and Pastore, 2009). On generalized (κ, μ) or $(\kappa, \mu)'$-almost Kenmotsu manifold with $h \neq 0$ (equivalently, $h' \neq 0$), the following relations hold (Dileo and Pastore, 2009):

$$h'^2 = (\kappa + 1)\varphi^2, \qquad h^2 = (\kappa + 1)\varphi^2, \tag{1.34}$$

$$Q\xi = 2n\kappa\xi. \tag{1.35}$$

1.13 f-Kenmotsu Manifold

If the fundamental 2-form Φ and the 1-form η are closed, then M is said to be an almost cosympletic manifold (Goldberg and Yano, 1969). A normal almost cosympletic manifold is cosympletic (Blair, 2010). Equivalently, an almost contact metric structure is cosympletic if and only if $\nabla\varphi = 0$. On almost contact metric structure, the conformal transformation is defined by

$$\varphi^* = \varphi, \;\; \xi^* = e^{-\rho}\xi, \;\; \eta^* = e^{\rho}\eta, \;\; g^* = e^{2\rho}\tilde{g},$$

where ρ is a differentiable function. M is said to be a locally conformal almost cosympletic manifold (Olszak, 1989) if every point of M has a neighbourhood $\mathcal{U}$ such that $(\mathcal{U}, \varphi^*, \xi^*, \eta^*, g^*)$ is almost cosympletic for some function ρ on $\mathcal{U}$. A normal locally conformal almost cosympletic manifold (Olszak and Rosca, 1991) is called the f-Kenmotsu manifold. Equivalently, an almost contact metric manifold is called the f-Kenmotsu manifold (Mangione, 2008) if it satisfies

$$(\nabla_X \varphi)(Y) = f(g(\varphi X, Y)\xi - \eta(Y)\varphi X), \tag{1.36}$$

where $f \in C^\infty(M)$ is strictly positive and $df \wedge \eta = 0$ holds. In particular, if $f = constant \equiv \alpha > 0$, then M is called α-Kenmotsu manifold, if $f = 1$ then M is the Kenmotsu manifold (Kenmotsu, 1972) and when $f = 0$ then M becomes cosympletic manifold. An f-Kenmotsu manifold is called regular if $f^2 + \xi f \neq 0$.

The following relations holds in an f-Kenmotsu manifold (Kim et al., 2002; Olszak and Rosca, 1991):

$$\nabla_X \xi = -f\varphi^2 X, \tag{1.37}$$

$$R(X,Y)\xi = f^2(\eta(X)Y - \eta(Y)X) + Y(f)\varphi^2 X - X(f)\varphi^2 Y, \tag{1.38}$$

$$Q\xi = -2nf^2\xi - \xi(f)\xi - (2n-1)gradf, \tag{1.39}$$

for all vector fields X, Y on M.

An f-Kenmotsu manifold M of dimension ≥ 5 is of pointwise constant φ-sectional curvature c if and only if its curvature tensor R is of the form (Olszak, 1989)

$$\begin{aligned} R(X,Y)Z &= \frac{c-3f^2}{4}(g(Y,Z)X - g(X,Z)Y) \\ &+ \frac{c+f^2}{4}(2g(X,\varphi Y)\varphi Z + g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X) \\ &+ (\frac{c+f^2}{4} + \xi f)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi), \end{aligned} \tag{1.40}$$

for all $X, Y, Z \in \chi(M)$. An f-Kenmotsu manifold with pointwise constant φ-sectional curvature c is called f-Kenmotsu space form $M(c)$.

1.14 Some Vector Fields

A vector field V is said to be harmonic vector field if it is a critical point of the energy functional E defined by

$$E(V) = \frac{1}{2}\int ||dV||^2 dM = \frac{n}{2} vol(M, g) + \frac{1}{2}\int_M ||\nabla V||^2 dM$$

on the space χ^1 of all unit vector fields on M. A contact metric manifold whose Reeb vector field is harmonic is called an H-contact manifold. Perrone (2004) proved that *a contact metric manifold is an H-contact manifold, that is ξ is a harmonic vector field, if and only if ξ is an eigenvector of the Ricci operator.* This implies $Q\xi = (Tr.l)\xi$. This is valid for K-contact manifolds, (k, μ)-contact manifolds and unit sphere S^{2n+1} with standard contact metric structure.

A vector field V on a contact metric manifold M is said to be contact if there exists a smooth function $\varrho : M \to \mathbb{R}$ satisfying

$$(\mathcal{L}_V \eta)(Y) = \varrho\eta(Y), \tag{1.41}$$

for all $Y \in \chi(M)$ and if $\varrho = 0$, then the vector field V is called strict.

A smooth vector field X on a Riemannian manifold is said to be a conformal vector field if there exists a smooth function ψ on M that satisfies

$$\mathcal{L}_X g = 2\psi g.$$

We say that X is non-trivial if X is not Killing, that is, $\psi \neq 0$.

A vector field ξ is called torse-forming (Blaga, 2018) if it satisfies

$$\nabla_X \xi = X + \eta(X)\xi, \tag{1.42}$$

for any $X \in \chi(M)$ and 1-form η.

1.15 Ricci-Yamabe Soliton

The theory of geometric flows plays a significant role in understanding the geometric structure in Riemannian geometry. Ricci flow is a well-known geometric flow introduced by Hamilton (1998), who used it to prove a three-dimensional sphere theorem (Hamilton, 1982). The Ricci flow plays a crucial role in forming proof of Thurston's conjecture, including as a special case, the Poincare conjecture. The Ricci soliton on a Riemannian manifold (M, g) are the self-similar solutions to Ricci flow and is defined by

$$\frac{1}{2}\mathcal{L}_V g + S = \lambda g, \tag{1.43}$$

where $\mathcal{L}_V g$ denotes the Lie-derivative of g along potential vector field V, S is the Ricci curvature of M^{2n+1} and λ, a real constant. When the vector field V is the gradient of a smooth function f on M^{2n+1}, that is, $V = \nabla f$, then we say that Ricci soliton is a gradient (For details see (Cao, 2009; Petersen and Wylie, 2009)). According to Petersen (2009), a gradient Ricci soliton is rigid if it is a flat $N \times_\Gamma \mathbb{R}^k$, where N is Einstein and gave certain classification. The notion of almost Ricci soliton was introduced by Pigola et al. (2011) by taking λ as a smooth function in the definition of Ricci soliton (1.43).

To tackle the Yamabe problem on manifolds of positive conformal Yamabe invariant, Hamilton (1998) introduced the geometric flow known as Yamabe flow. The Yamabe soliton is a self-similar solution to the Yamabe flow. On a Riemannian manifold (M, g), a Yamabe soliton is given by

$$\frac{1}{2}\mathcal{L}_V g = (r - \lambda)g, \tag{1.44}$$

where r is the scalar curvature of the manifold and λ, a real constant. The Yamabe soliton preserves the conformal class of the metric but the Ricci soliton does not in general. However, in dimension $n = 2$, both the solitons are similar. If λ is a smooth function in (1.44), then it is called almost Yamabe soliton.

Guler and Crasmareanu (2019) introduced a new type of geometric flow which is a scalar combination of Ricci flow and Yamabe flow under the name Ricci-Yamabe map and define the following:

Definition 1.4. *(Guler and Crasmareanu, 2019) The map* $RY^{(\alpha,\beta,g)} : I \to T_2^s(M)$ *given by:*

$$RY^{(\alpha,\beta,g)} = \frac{\partial g}{\partial t}(t) + 2\alpha S(t) + \beta r(t)g(t),$$

is called the (α, β)*-Ricci-Yamabe map of the Riemannian flow* (M, g)*. If*

$$RY^{(\alpha,\beta,g)} \equiv 0,$$

then $g(.)$ *will be called an* (α, β)*-Ricci-Yamabe flow.*

The Ricci-Yamabe flow can be Riemannian or semi-Riemannian or singular Riemannian flow due to the involvement of scalars α and β. These kind of different choices can be useful in some physical models such as relativity theory. The Ricci-Yamabe soliton emerges as the limit of the solution of Ricci-Yamabe flow.

Definition 1.5. *A Riemannian manifold* $(M^n, g), n > 2$ *is said to admit almost Ricci-Yamabe soliton* $(g, V, \lambda, \alpha, \beta)$ *if there exist smooth function* λ *such that*

$$\mathcal{L}_V g + 2\alpha S = (2\lambda - \beta r)g, \tag{1.45}$$

where $\alpha, \beta \in \mathbb{R}$*.*

Almost Ricci-Yamabe soliton is of particular interest as it generalizes a large group of well-known solitons such as:

1. Ricci almost soliton ($\alpha = 1, \beta = 0$).

2. almost Yamabe soliton ($\alpha = 0, \beta = 1$).

3. Ricci-Bourguignon almost soliton ($\alpha = 1, \beta = -2\rho$).

Also, if λ is constant, then it includes Ricci soliton, Yamabe soliton and Ricci-Bourguignon soliton among others.

If V is a gradient of some smooth function f on M, then the above notion is called gradient almost Ricci-Yamabe soliton and then (1.45) reduces to

$$\nabla^2 f + \alpha S = (\lambda - \frac{1}{2}\beta r)g, \tag{1.46}$$

where $\nabla^2 f$ is the Hessian of f.

The almost Ricci-Yamabe soliton (ARYS) is said to be expending, shrinking or steady if $\lambda < 0, \lambda > 0$ or $\lambda = 0$ respectively. In particular, if λ is constant, then almost Ricci-Yamabe soliton reduces to Ricci-Yamabe soliton.

Extending the notion of Ricci soliton, Cho and Kimura (2009) introduced the η-Ricci soliton which is obtained by perturbing the equation (1.43) by a multiple of a certain $(0, 2)$-tensor field $\eta \otimes \eta$. A more general extension is obtained by Siddiqi and Akyol (2004) and called such soliton as η-Ricci-Yamabe soliton of type (α, β) which is defined as:

$$\mathcal{L}_V g + 2\alpha S + (2\mu - \beta r) + 2\omega\eta \otimes \eta = 0. \tag{1.47}$$

1.16 m-quasi-Einstein Structure

The study of Einstein manifolds and their several generalizations have received a lot of attention in recent years. Extending the notion of the m-Bakry-Emery Ricci tensor, Case (2010) introduced an interesting generalization of gradient Ricci soliton and Einstein manifold. The m-Bakry-Emery Ricci tensor is defined as follows

$$S_f^m = S + \nabla^2 f - \frac{1}{m} df \otimes df,$$

where the integer m satisfies $0 < m \leq \infty$, $\nabla^2 f$ denotes the Hessian form of the smooth function f. The m-Bakery-Emery Ricci tensor arises from the warped product $(M \times N, \bar{g})$ of two Riemannian manifolds (M^n, g) and (N^m, h) with the Riemannian metric $\bar{g} = g + e^{-\frac{2f}{m}} h$. We called a quadruple (g, f, m, λ) on a Riemannian manifold (M, g), m-quasi-Einstein structure if it satisfies the equation

$$S + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g, \tag{1.48}$$

for some $\lambda \in \mathbb{R}$. Notice that for $m = \infty$, Eq. (1.48) gives gradient Ricci soliton and for constant f, it becomes Einstein. The m-quasi-Einstein structure has been deeply studied by Case (2010), Case et al. (2010) and Ghosh (2020a).

Later on Barros-Ribeiro Jr. (2012a) and Limoncu (2010) generalized and studied equation (1.48) independently, by considering a 1-form $V^\flat$ instead of df, which satisfies

$$S + \frac{1}{2}\mathcal{L}_V g - \frac{1}{m} V^\flat \otimes V^\flat = \lambda g, \tag{1.49}$$

where $V^\flat$ is the 1-form associated with the potential vector field V. In particular, if the 1-form $V^\flat$ is closed, we called (1.49), a closed m-quasi-Einstein structure. When $V \equiv 0$, the m-quasi-Einstein structure is said to be trivial, and in this case, the metric becomes an Einstein metric.

Extending the notion of quasi-Einstein structure, Catino (2012) introduced and studied the concept of the generalized quasi-Einstein manifold. A particular case of this was proposed by Barros-Ribeiro Jr. (2014) which is defined as follows:

A Riemannian manifold (M^n, g) is said to be generalized m-quasi-Einstein

(g, f, m, λ) if there exists function $\lambda : M^n \to \mathbb{R}$ such that

$$S + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g. \tag{1.50}$$

Notice that for $m = \infty$, (1.50) reduces to gradient Ricci almost soliton. Also when df is replaced by $V^\flat$, then we called (1.50), generalized m-quasi-Einstein (g, V, m, λ) structure. Moreover, if $V \equiv 0$ then it is said to be trivial.

1.17 Submanifold

Let M and N be smooth manifolds, where $dim(M) \leq dim(N)$, let $F : M \to N$ be a smooth map, and let p be a point in M. We say that F is an immersion at p if the differential map $d_p(F) : T_p(M) \to T_{F(p)}(N)$ is injective, and that F is an immersion if it is an immersion at every p in M.

Definition 1.6. *Suppose $(N, \tilde{g})$ is a Riemannian manifold of dimension m, M is a manifold of dimension n and $\iota : M \to N$ is an immersion. If M is given the induced Riemannian metric $g := \iota * \tilde{g}$, then ι is said to be an isometric immersion. If in addition ι is injective, so that M is an immersed submanifold of N, then M is said to be a Riemannian submanifold of N.*

The geometry of submanifolds in recent decades has become a topic of growing interest for its significant applications in applied mathematics and theoretical physics. The notion of invariant submanifolds can be used to discuss the properties of a non-linear autonomous system. In general, the invariant submanifolds inherit almost all the geometric properties of the ambient manifold. Another important type of submanifold is a totally geodesic submanifold. The significance of this submanifold is that the geodesics of the ambient manifolds remain geodesics in the submanifolds. Moreover, the notion of geodesics plays an important role in the theory of relativity.

Let M be an immersed submanifold of Riemannian manifold $\tilde{M}$ with induced metric g. By $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, we denote the tangent and normal subspaces of M in $\tilde{M}$. Also let ∇ and $\nabla^{\perp}$ denotes the induced connection on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M respectively. Then the Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{1.51}$$

and

$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{1.52}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where σ and A are called the second fundamental form and shape operator of M respectively. They are related by

$$g(A_V X, Y) = g(\sigma(X, Y), V). \tag{1.53}$$

The mean curvature vector H of M^n is defined to be

$$H = \frac{1}{n} Tr(\sigma),$$

where Tr denotes the trace. A submanifold M in a Riemannian manifold is called minimal if its mean curvature vector vanishes identically. The submanifold M is called totally geodesic if $\sigma(X, Y) = 0$ for any $X, Y \in \Gamma(TM)$.

The covariant derivative of the second fundamental form σ is defined by

$$(\tilde{\nabla}_X \sigma)(Y, Z) = \nabla^{\perp}_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{1.54}$$

for all $X, Y, Z \in \Gamma(TM)$, where $\tilde{\nabla}$ is called the Vander-Waerden-Bortolotti connection on M. Then $\tilde{\nabla}\sigma$ is a normal bundle valued tensor of type (0,3) and is called the third fundamental form of M. Whenever $\tilde{\nabla}\sigma = 0$, then M is said to be have parallel second fundamental form. The Gauss equation for the Riemannian curvature R of the submanifold M is given by

$$\tilde{R}(X,Y)Z = R(X,Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X, \tag{1.55}$$

for any $X, Y, Z \in \Gamma(TM)$.

In submanifold theory, obtaining the relationships between the intrinsic invariant and extrinsic invariant has been the primary goal of many geometers in recent decades. Chen invariants were introduced by Chen (1993) to tackle the question raised by Chern's concerning the existence of minimal immersions into a Euclidean space of arbitrary dimension (Chern, 1968).

Suppose L is an r-dimensional subspace of T_xM, $x \in M$, $r \geq 2$ and $\{e_1, ..., e_r\}$ an orthonormal basis of L. The scalar curvature τ of the r-plane section L is given by

$$\tau(L) = \sum_{1 \leq i < j \leq r} K_{ij}, \tag{1.56}$$

where K_{ij} is the sectional curvature of the plane section spanned by e_i and e_j at $x \in M$. Let $\Pi \subset T_xM$ be a 2-plane section and $K(\Pi)$ be the sectional curvature of M for a plane section Π in $T_xM, x \in M$. Then

$$K(\Pi) = \frac{1}{2}[R(e_1, e_2, e_2, e_1) - R(e_1, e_2, e_1, e_2)]. \tag{1.57}$$

The scalar curvature $\tau(x)$ of M at the point x is given by

$$\tau(x) = \sum_{i<j} K_{ij}, \tag{1.58}$$

where $\{e_1, ..., e_m\}$ is an orthonormal basis for T_xM.

Chen's δ-invariant δ_M of a Riemannian manifold M introduced by Chen is

$$\delta_M(x) = \tau(x) - \inf\{K(\Pi) | \Pi \text{ is a plane section } \subset T_xM\}, \tag{1.59}$$

where τ is the scalar curvature of M.

1.18 Lorentzian Manifold

An n-dimensional pseudo-Riemannian (or sometimes called semi-Riemannian) manifold is a pair (M, g), where M is an n-dimensional smooth manifold and g is a symmetric, nondegenerate 2-tensor field on M (called the metric). A pseudo-Riemannian manifold is said to be Riemannian if g has a signature $(+, +, ..., +)$, and is said to be Lorentzian if g has a signature $(-, +, ..., +)$ or $(+, -, ..., -)$.

1.19 Review of Literature

Koufogiorgos (1997) introduced and studied (κ, μ) space forms. The (κ, μ)-space forms are studied by Akbar and Sarkar (2015), De and Samui (2016) and Shashidhar and Nagaraja (2015). Carriazo et al. (2011) introduced the generalized (κ, μ) space form which generalizes the notion of (κ, μ)-space forms. Carriazo and Molina (2011) studied D_α-homothetic deformations of generalized (κ, μ)-space forms and found that deformed spaces are again generalized (κ, μ)-space forms in dimension 3, but not in general. Carriazo et al. (2013) studied generalized (κ, μ)-space forms in contact metric and Trans-Sasakian manifolds. In recent years, many geometers studied generalized (κ, μ)-space forms under several conditions (Shivaprasanna et al., 2014; Faghfouri and Ghaffarzadeh, 2015; Shivaprasanna, 2016; Hui et al., 2018; Kumar and Nagaraja, 2019; Shammukha and Venkatesha, 2019). De and Majhi (2019) studied Q curvature tensor in a generalized Sasakian space form.

A Ricci soliton is a self similar solution to the Ricci flow (Hamilton, 1982). Some applications of Ricci flow are Ricci flow gravity (Graf, 2007), nonlinear reaction-diffusion systems in biology, chemistry and physics (Ivancevic and Ivancevic, 2011), brain surface conformal parametrization with the Ricci flow (Wang et al., 2012) and in the economy (Sandhu et al., 2016). Cao (2006, 2009) and

Petersen and Wylie (2009) introduced and studied gradient Ricci soliton. Cao and Zhou (2010) studied complete shrinking Ricci solitons, Munteanu and Wang (2017) showed that positively curved shrinking Ricci solitons are compact and Wylie (2008) showed that a complete shrinking Ricci solitons have finite fundamental group. Deshmukh et al. (2020) gave a characterization of trivial Ricci solitons. A Ricci solitons with Jacobi-type vector fields were studied by Deshmukh (2012). Cho and Sharma (2010) initiated the study of Ricci solitons in contact geometry. The notion of almost Ricci soliton was introduced by Pigola et al. (2011). Barros et al. (2021) studied the rigidity of the gradient almost Ricci solitons and showed that it is isometric to the Euclidean space $\mathbb{R}^n$ or sphere $\mathbb{S}^n$. Cao et al. (2011), Barros et al. (2013) and Yang and Zhang (2017) obtained several rigidity results.

Chu and Wang (2013) gave scalar curvature estimates for gradient Yamabe solitons. Then, Wang (2016) and Suh and De (2020) studied Yamabe solitons. Shaikh et al. (2021) gave some characterizations of gradient Yamabe solitons. Chaubey et al. (2022) gave complete classification of Yamabe solitons on real hypersurfaces in the complex quadric $Q^m = SO_{m+1}/SO_2SO_m$. Extending Yamabe solitons, Barbosa and Ribeiro (2013) introduced almost Yamabe solitons. Seko and Maeta (2019) gave classification of almost Yamabe solitons in Euclidean spaces. Alkhaldi et al. (2021) gave a characterization of almost Yamabe soliton with conformal vector field.

After Guler and Crasmareanu (2019) introduction of Ricci-Yamabe soliton many geometers such as De et al. (2022), Dey (2020) and Sardar and Sarkar (2022) analyzed Ricci-Yamabe solitons. Siddiqi and De (2022) and Singh and Khatri (2021) studied Ricci-Yamabe soliton in different spacetimes. Siddiqi et al. (2022) consider almost Ricci-Yamabe soliton on static spacetimes.

Barros and Gomes (2017) obtained some triviality of compact m-quasi-Einstein manifolds.Case et al. (2011) studied the properties of quasi-Einstein metrics and proved several rigidity results. Ghosh (2015a, 2019a) studied contact metric manifolds with quasi-Einstein structures (1.48) and (1.49). Recently, Chen (2020) studied quasi-Einstein structure in almost cosympletic manifolds and De et al. (2021) studied quasi-Einstein metric (g, f, m, λ) in the context of three-dimensional cosympletic manifolds. Hu et al. (2015, 2017) studied generalized m-quasi-Einstein metric with restriction on Ricci curvature and scalar curvature. Barros and Ribeiro (2014) obtained characterizations and integral formulae for generalized m-quasi-Einstein metrics. Ghosh (2015b) considered generalized m-quasi-Einstein metric in Sasakian and K-contact manifolds and showed that it is isometric to the unit sphere $\mathbb{S}^{2n+1}$. Barros and Gomes (2013) proved that a compact gradient generalized m-quasi-Einstein metric with constant scalar curvature must be isometric to a standard Euclidean sphere $\mathbb{S}^n$ with the potential function well determined.

The study of the geometry of invariant submanifolds of almost contact manifolds were initiated by Yano and Ishihara (1969). Later on many geometers studied invariant submanifolds of certain classes of almost contact manifolds such as Anitha and Bagewadi (2003), Endo (1986), Kon (1973), De and Majhi (2015b), Shaikh et al. (2016) and Atceken (2021). Chen (1993) obtained an inequality for a Riemannian submanifold M^m of a real space form $\widetilde{M}$ with constant sectional curvature c as

$$\delta_M \leq \frac{m^2(m-2)}{2(m-1)} \parallel H \parallel^2 + \frac{1}{2}(m+1)(m-2)c, \tag{1.60}$$

where H is the mean curvature of the submanifold M^m. Eq. (1.60) is known as first Chen inequality.

Then Chen (1996) gave the inequality for Riemannian submanifold M^m of complex-space-form $\widetilde{M}^n(4c)$ as follows:

$$\delta_M \leq \frac{m^2(m-2)}{2(m-1)} \parallel H \parallel^2 + \frac{1}{2}(m+1)(m-2)c + \frac{3}{2} \parallel P \parallel^2 c - 3\Theta(\pi)c. \tag{1.61}$$

Afterward, many authors obtained Chen's inequalities for different submanifolds in various ambient spaces, like the Sasakian space form (Cioroboin, 2003), generalized space forms (Mihai, 2004; Alegre et al. 2007), Kenmotsu space form (Kumar et al., 2010), Riemannian manifold of quasi-constant curvature (*Ö*zg*ü*r, 2011), Cosympletic space form (Gupta, 2013), quaternionic space forms (Vilcu, 2013), Statistical manifolds (Aytimur et al., 2019; Decu and Haesen, 2022; Lone et al., 2022) and GRW spacetime (Poyraz, 2022).

Qu and Wang (2015) introduced the notion of a special type of a quarter-symmetric connection as a generalization of a semi-symmetric metric connection (Hayden, 1932) and a semi-symmetric non-metric connection (Agashe and Chafle, 1992). They studied the Einstein warped product and multiply warped products with a quarter-symmetric connection (Qu and Wang, 2015). Wang and Zhang (2010) obtained Chen's inequalities for submanifolds of real space forms endowed with a quarter-symmetric connection. Mihai and *Ö*zg*ü*r (2011) obtained the Chen inequalities for submanifolds of complex space forms and Sasakian space forms with a semi-symmetric metric connection. Wang (2019) obtained Chen inequalities for submanifolds of complex space forms and Sasakian space forms with quarter-symmetric connections which improved the results of Mihai and *Ö*zg*ü*r (2011). Sular (2016) obtained Chen inequalities for submanifolds of generalized space forms with a semi-symmetric metric connection. Al-Khaldi et al. (2021) obtained the Chen-Ricci inequalities Lagrangian submanifold in a generalized complex space form and a Legendrian submanifold in a generalized Sasakian space form endowed with the quarter-symmetric connection.

In the last decade, a great deal of work had been done on η-Ricci soliton and η-Yamabe soliton in the framework of Riemannian geometry. Recently, geometric flows are initiated in the investigation of the cosmological model such as perfect fluid spacetime. Blaga (2020) studied η-Ricci and η-Einstein soliton in perfect fluid spacetime and obtained the Poisson equation from the soliton equation when the potential vector field ξ is of gradient type. Kumara and Venkatesha (2019) analyzed Ricci soliton in perfect fluid spacetime with torse-forming vector field. Also, Conformal Ricci soliton in perfect fluid spacetime is studied (Siddiqi and Siddiqui, 2020). Praveena et al. (2021) studied solitons in *Kählerian* space-time manifolds.

Qing and Yuan (2013), Leandro (2021) and Patra and Ghosh (2021) studied the properties of Einstein-type manifolds. The interesting idea of Einstein-type manifolds is characterized in many papers (Catino et al., 2017; Leandro, 2021). Leandro (2021) classified Einstein-type manifold under the assumptions of zero-radial Weyl curvature and harmonic Weyl curvature. As a physical application, Leandro proved that there are no multiple black holes in static vacuum Einstein equation with null cosmological constant having zero radial Weyl curvature and divergence free Weyl tensor of order four. Catino et al. (2017) investigated it under Bach-flat condition. Recently, Patra and Ghosh (2021) considered the Einstein-type equation within the context of contact manifolds. Moreover, an Einstein-type compact contact manifold with zero radial Weyl curvature was considered. The critical point equation, Miao-Tam equation and Fischer-Marsden equation on Kenmotsu and almost Kenmotsu manifold were studied by many authors in Patra (2021), Patra and Ghosh (2018), Wang and Wang (2017) and Chaubey et al. (2021). Kumara et al. (2021) characterized the static perfect fluid space-time metrics on almost Kenmotsu manifolds.

Chapter 2

On (κ, μ)-contact metric manifolds and semiconformal curvature

Chapter 2

On (κ, μ)-contact metric manifolds and semiconformal curvature

This chapter is divided into three main sections. The first section deals with generalized recurrent (κ, μ)-contact metric manifolds. In the second section, generalized (κ, μ)-space forms are considered and the third section is devoted to almost pseudo semiconformally symmetric manifolds.

2.1 On a Class of Generalized Recurrent (κ, μ)-contact Metric Manifolds

After Cartan's (1926) introduction of locally symmetric spaces. Many authors introduced weaker version of symmetric spaces, one of which is a hyper genralized recurrent manifold which is an extension of a generalized recurrent manifold

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M. Khatri, J.P. Singh (2020), On a class of generalized recurrent (κ, μ)-contact metric manifolds, *Commun. Korean Math. Soc.*, **35 (4)**, 1283-1297.

(Dubey, 1979). A Riemannian manifold is said to be hyper generalized recurrent manifold if its curvature tensor R satisfies the condition

$$\nabla R = A \otimes R + B \otimes (g \wedge S), \tag{2.1}$$

where A and B are 1-forms defined in (1.4).

Recently, Venkatesha et al. (2019) extended the notion of hyper generalized recurrent manifolds (resp. quasi generalized recurrent manifolds) to hyper generalized φ-recurrent Sasakian manifolds (resp. quasi generalized φ-recurrent Sasakian manifolds) and obtained interesting results. Continuing this, we studied hyper generalized φ-recurrent (κ, μ)-contact metric manifolds and prove its existence by giving a proper example. Similarly, quasi generalized φ-recurrent (κ, μ)-contact metric manifolds are investigated.

2.1.1 Preliminaries

In a (κ, μ)-contact metric manifold the following properties are true (Blair et al., 1995):

$$h^2 = (\kappa - 1)\varphi^2, \quad \kappa \leq 1, \tag{2.2}$$

$$\nabla_X \xi = -\varphi X - \varphi h X, \quad (\nabla_X \varphi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.3}$$

$$R(X, Y)\xi = \kappa \eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{2.4}$$

$$R(\xi, X)Y = \kappa(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX], \tag{2.5}$$

$$\begin{aligned} S(X, Y) &= [2(n-1) - n\mu]g(X, Y) + [2(n-1) + \mu]g(hX, Y) \\ &\quad + [2(1-n) + n(2\kappa + \mu)]\eta(X)\eta(Y), \end{aligned} \tag{2.6}$$

$$S(X, \xi) = 2n\kappa\eta(X), \tag{2.7}$$

$$r = 2n(2n - 2 + \kappa - n\mu), \tag{2.8}$$

$$S(\varphi X, \varphi Y) = S(X, Y) - 2n\kappa\eta(X)\eta(Y) - 2(2n - 2 + \mu)g(hX, Y), \tag{2.9}$$

for all $X, Y \in \chi(M)$.

Definition 2.1. *A (2n+1)-dimensional (κ, μ)-contact metric manifold is said to be η-Einstein if its Ricci tensor S is of the form*

$$S(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y),$$

for any vector fields X and Y, where α and β are constants. If $\beta = 0$, then the manifold M is an Einstein manifold.

2.1.2 Hyper generalized φ-recurrent (k, μ)-contact metric manifold

Shaikh and Patra (2010) studied hyper generalized recurrent manifolds. Recently, Venkatesha et al. (2019) studied hyper generalized φ-recurrent Sasakian manifold and obtained important results. By observing this, we extended it to (κ, μ)-contact metric manifold. In this subsection, we study hyper generalized φ-recurrent (κ, μ)-contact metric manifold.

Definition 2.2. *A $(2n + 1)$-dimensional (κ, μ)-contact metric manifold is said to be a hyper generalized φ-recurrent if its curvature tensor R satisfies*

$$\varphi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z + B(W)H(X, Y)Z, \tag{2.10}$$

for all vector fields X, Y and Z. Here, A and B are two non-vanishing 1-forms such that $A(X) = g(X, \rho_1)$, $B(X) = g(X, \rho_2)$ and the tensor H is defined by

$$H(X, Y)Z = S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY, \tag{2.11}$$

for all vector fields X, Y and Z. Here, Q is the Ricci operator, ρ_1 and ρ_2 are vector fields associated with 1-forms A and B respectively. If the 1-form B vanishes, then (2.10) reduces to the notion of φ-recurrent manifolds.

Theorem 2.1. *In a hyper generalized φ-recurrent (κ,μ)-contact metric manifold, the 1-forms A and B satisfy the relation*

$$\kappa A(W)+[n(2\kappa-\mu+2)-2]B(W)=0.$$

Proof. Let us consider hyper generalized φ-recurrent (κ,μ)-contact metric manifold. In view of (2.10) and (1.9) we obtain

$$\begin{aligned}-(\nabla_W R)(X,Y)Z+\eta((\nabla_W R)(X,Y)Z)\xi\\ =A(W)R(X,Y)Z+B(W)H(X,Y)Z.\end{aligned} \tag{2.12}$$

Taking an inner product with U in (2.12), we get

$$\begin{aligned}-g((\nabla_W R)(X,Y)Z)+\eta((\nabla_W R)(X,Y)Z)\eta(U)\\ =A(W)g(R(X,Y)Z,U)+B(W)g(H(X,Y)Z,U).\end{aligned} \tag{2.13}$$

Contracting over X and U in (2.12) gives

$$\begin{aligned}-(\nabla_W S)(Y,Z)+\eta((\nabla_W R)(\xi,Y)Z)\\ =[A(W)+(2n-1)B(W)]S(Y,Z)+rB(W)g(Y,Z).\end{aligned} \tag{2.14}$$

Taking $Z=\xi$ in (2.14) and using the fact that $\eta((\nabla_W R)(\xi,Y)\xi)=0$ we obtain

$$-(\nabla_W S)(Y,\xi)=[2n\kappa(A(W)+(2n-1)B(W))+rB(W)]\eta(Y). \tag{2.15}$$

Putting $Y=\xi$ in the above equation gives

$$2n\kappa[A(W)+(2n-1)B(W)]+rB(W)=0. \tag{2.16}$$

Using (2.8) in (2.16), we obtain

$$\kappa A(W)+[n(2k-\mu+2)-2]B(W)=0. \tag{2.17}$$

for any vector field W. This completes the proof. □

Taking $r = 0$ in (2.16), we are in a position to state the following corollary:

Corollary 2.1. *In a hyper generalized φ-recurrent (κ, μ)-contact metric manifold, if the scalar curvature of the manifold vanishes then, either*

1. *1-forms A and B are co-directional, or*
2. *it is $\left(0, \frac{2(n-1)}{n}\right)$-contact metric manifold.*

Let $\{e_i\}_{i=1}^{2n+1}$ be an orthonormal basis of the manifold. Putting $Y = Z = e_i$ in (2.14) and taking summation over $i, 1 \leq i \leq 2n+1$, and using (1.9), (2.3) and (2.7) we obtain

$$-dr(W) = r[A(W) + 4nB(W)]. \tag{2.18}$$

This led us to the following theorem:

Theorem 2.2. *In a hyper generalized φ-recurrent (κ, μ)-contact metric manifold, if the scalar curvature of the manifold is a non-zero constant then, $A(W) + 4nB(W) = 0$, for any vector field W.*

Theorem 2.3. *In a hyper generalized φ-recurrent (κ, μ)-contact metric manifold, the associated vector fields ρ_1 and ρ_2 corresponding to 1-forms A and B satisfy the relation*

$$r\eta(\rho_1) + 2(2n-1)(r - 2n\kappa)\eta(\rho_2) = 0.$$

Proof. Changing X, Y, Z cyclically in (2.13) and using Bianchi's identity we get

$$\begin{aligned} &A(W)g(R(X,Y)Z,U) + A(X)g(R(Y,W)Z,U) + \\ &A(Y)g(R(W,X)Z,U) + B(W)g(H(X,Y)Z,U) + \\ &B(X)g(H(Y,W)Z,U) + B(Y)g(H(W,X)Z,U) = 0. \end{aligned} \tag{2.19}$$

Contracting over Y and Z and using (1.9), we obtain

$$\begin{aligned} &A(W)S(X,U) - A(X)S(W,U) - \kappa g(X,U)A(W) + \kappa g(W,U)A(X) \\ &- \mu g(hW,U)A(X) + B(W)[rg(X,U) + (2n-1)S(X,U)] \\ &+ B(X)[-rg(W,U) - (2n-1)S(W,U)] + B(QX)g(W,U) \\ &- B(QW)g(X,U) + B(X)S(W,U) - B(W)S(X,U) = 0. \end{aligned} \tag{2.20}$$

Again contracting (2.20) over X and U yields

$$\begin{aligned} &(r + 2n\kappa)A(W) - A(QW) + \mu A(hW) \\ &+(4nr - 2r)B(W) - (4n-2)B(QW) = 0. \end{aligned} \tag{2.21}$$

Replacing W by ξ in (2.21) results in

$$r\eta(\rho_1) + 2(2n-1)(r - 2n\kappa)\eta(\rho_2) = 0. \tag{2.22}$$

This completes the proof. □

Theorem 2.4. *A hyper generalized φ-recurrent (κ, μ)-contact metric manifold is an η-Einstein manifold.*

Proof. Since we have

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi). \tag{2.23}$$

Using (2.3) and (2.7) in (2.23) we get

$$(\nabla_W S)(Y, \xi) = -2n\kappa g(\varphi W + \varphi hW, Y) + S(Y, \varphi W + \varphi hW). \tag{2.24}$$

From (2.17) and (2.24) we obtain

$$\begin{aligned} &2n\kappa g(\varphi W + \varphi hW, Y) - S(Y, \varphi W + \varphi hW) \\ &= \Big[2n\kappa\{A(W) + (2n-1)B(W)\} + rB(W)\Big]\eta(Y). \end{aligned} \tag{2.25}$$

Taking $Y = \varphi Y$ in (2.25) gives

$$\begin{aligned} S(Y,W) + S(Y,hW) &= 2n\kappa g(Y,W) + [2n\kappa + 2(2n-2+\mu)]g(Y,hW) \\ &+ 2(2n-2+\mu)(\kappa-1)g(Y,-W+\eta(W)\xi). \end{aligned} \tag{2.26}$$

Using

$$\begin{aligned} S(Y,hW) &= (2n-2-n\mu)g(Y,hW) - (2n-2+\mu)(\kappa-1)g(Y,W) \\ &+ (2n-2+\mu)(\kappa-1)\eta(W)\eta(Y), \end{aligned}$$

and (2.6) in (2.26) led us to the following relation

$$S(Y,W) = \alpha g(Y,W) + \beta\eta(Y)\eta(W), \tag{2.27}$$

where

$$\alpha = \frac{[2(n\kappa+n-1)+\mu(n+2)][2(n-1)-n\mu] - [2(n-1)+\mu][\mu(1-\kappa)+2(n-1)+2\kappa]}{2n\kappa+\mu(n+1)},$$

$$\beta = \frac{[2(n\kappa+n-1)+\mu(n+2)][2(1-n)+n(2\kappa+\mu)] - (\kappa-1)[2(n-1)+\mu]^2}{2n\kappa+\mu(n+1)}.$$

This completes the proof. □

Theorem 2.5. *In a hyper generalized φ-recurrent (κ,μ)-contact metric manifold, the 1-forms A and B satisfy the relation*

$$2n\kappa A(\varphi W) + [r + 2n\kappa(2n-1)]B(\varphi W) = 0.$$

Proof. In view of (1.9), (2.3) and (2.4) we get

$$\begin{aligned} &(\nabla_W R)(X,Y)\xi = \kappa(W+hW,\varphi Y)X - g(W+hW,\varphi X)Y] \\ &+ \mu[g(W+hW,\varphi Y)hX - g(W+hW,\varphi X)hY + \{(1-\kappa)g(W,\varphi X) \\ &+ g(W,h\varphi X)\}\eta(Y)\xi - \{(1-\kappa)g(W,\varphi Y) + g(W,h\varphi Y)\}\eta(X)\xi \\ &+ \mu\eta(W)\{\eta(X)\varphi hY - \eta(Y)\varphi hX\}] + R(X,Y)\varphi W + R(X,Y)\varphi hW. \end{aligned} \tag{2.28}$$

Using (2.28) in (2.12) results in the following relation

$$
\begin{aligned}
&\kappa(W+hW,\varphi Y)\eta(X)-g(W+hW,\varphi Y)\eta(Y)]\xi+\mu[(1-\kappa)g(W,\varphi X)\eta(Y)\\
&+g(W,h\varphi X)\eta(Y)-(1-\kappa)g(W,\varphi Y)\eta(X)-g(W,h\varphi Y)\eta(X)]\xi\\
&+\kappa(Y,\varphi W)\eta(X)-g(X,\varphi W)\eta(Y)+g(Y,\varphi hW)\eta(X)-g(X,\varphi hW)\eta(Y)]\xi\\
&-\kappa(W+hW,\varphi Y)X-g(W+hW,\varphi X)Y]-\mu[g(W+hW,\varphi Y)hX\\
&-g(W+hW,\varphi X)hY+\{(1-\kappa)g(W,\varphi X)+g(W,h\varphi X)\}\eta(Y)\xi\\
&-\{(1-\kappa)g(W,\varphi Y)+g(W,h\varphi Y)\}\eta(X)\xi+\mu\eta(W)\{\eta(X)\varphi hY\\
&-\eta(Y)\varphi hX\}]+R(X,Y)\varphi W+R(X,Y)\varphi hW=A(W)\{\kappa\eta(Y)X\\
&-\eta(X)Y]+\mu[\eta(Y)hX-\eta(X)hY]\}+B(W)\{2n\kappa\eta(Y)X\\
&-\eta(X)Y]+\eta(Y)QX-\eta(X)QY\}.
\end{aligned}
\tag{2.29}
$$

Putting $Y=\xi$ in (2.29) we get

$$
\begin{aligned}
&A(W)[\kappa(X-\eta(X)\xi)+\mu hX]+B(W)[2n\kappa X\\
&-4n\kappa\eta(X)\xi+QX]+\mu^2\eta(W)\varphi hX=0.
\end{aligned}
\tag{2.30}
$$

Taking $W=\varphi W$ and contracting over X in (2.30) gives

$$
2n\kappa A(\varphi W)+[r+2n\kappa(2n-1)]B(\varphi W)=0. \tag{2.31}
$$

This completes the proof. □

2.1.3 Example of hyper generalized φ-recurrent (k,μ)-contact metric manifold

In this subsection, we construct an example of hyper generalized φ-recurrent (k,μ)-contact metric manifold. We consider a 3-dimensional manifold $M^3 = \{(x,y,z)\in\mathbb{R}^3 : x\neq 0\}$ where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Let

$\{E_1, E_2, E_3\}$ be linearly independent vector fields in M^3 which satisfy

$$[E_1, E_2] = 2xE_1, \quad [E_2, E_3] = 0, \quad [E_1, E_3] = 0.$$

Let g be Riemannian metric defined by

$$g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1,$$

$$g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0.$$

Let η be the 1-form defined by

$$\eta(X) = g(X, E_3),$$

for any vector field X. Let φ be (1,1)-tensor field defined by

$$\varphi E_1 = E_2, \quad \varphi E_2 = -E_1, \quad \varphi E_3 = 0.$$

Then we have

$$\eta(E_3) = 1, \quad \varphi^2(X) = -X + \varphi(X)E_3$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Moreover

$$hE_3 = 0, \quad hE_1 = -E_1, \quad hE_2 = E_2.$$

Thus for $E_3 = \xi$, (φ, ξ, η, g) defines a contact metric structure on M^3. Let ∇ be the Riemannian connection of g. Using the Koszul formula we obtain

$$\nabla_{E_1} E_1 = -2xE_2, \quad \nabla_{E_1} E_2 = 2xE_1, \quad \nabla_{E_1} E_3 = 0,$$
$$\nabla_{E_2} E_1 = 0, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_2} E_3 = 0,$$
$$\nabla_{E_3} E_1 = 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0.$$

Thus the metric $M^3(\varphi, \xi, \eta, g)$ under consideration is a (κ, μ)-contact metric manifold. Now, we will show that it is a 3-dimensional hyper generalized φ-recurrent (κ, μ)-contact metric manifold. The non-vanishing components of curvature tensor and Ricci tensor are

$$\begin{aligned} R(E_1, E_2)E_1 &= 4x^2 E_2, \qquad R(E_1, E_2)E_2 = -4x^2 E_1, \\ S(E_1, E_1) &= S(E_2, E_2) = -4x^2. \end{aligned}$$

Since $\{E_1, E_2, E_3\}$ forms the orthonormal basis of the 3-dimensional (κ, μ)-contact metric manifold any vector fields can be expressed as

$$X = a_1 E_1 + b_1 E_2 + c_1 E_3,$$
$$Y = a_2 E_1 + b_2 E_2 + c_2 E_3,$$
$$Z = a_3 E_1 + b_3 E_2 + c_3 E_3.$$

Then,

$$R(X, Y)Z = u_1 E_1 + u_2 E_2, \tag{2.32}$$

where $u_1 = 4x^2 b_3(a_2 b_1 - a_1 b_2)$ and $u_2 = -4x^2 a_3(a_2 b_1 - a_1 b_2)$.
and

$$F(X, Y)Z = v_1 E_1 + v_2 E_2 + v_3 E_3, \tag{2.33}$$

where

$$\begin{aligned} v_1 &= 4x^2[a_1(a_1a_2+b_1b_2)(a_1a_3+b_1b_3+c_1c_3) \\ &+ b_3(a_2b_1-a_1b_2)-a_2(a_1a_2+b_1b_2)(a_2a_3+b_2b_3+c_2c_3)], \end{aligned}$$

$$\begin{aligned} v_2 &= 4x^2[b_1(a_1a_3+b_1b_3+c_1c_3)(a_1a_2+b_1b_2) \\ &- a_3(a_2b_1-a_1b_2)-b_2(a_1a_2+b_1b_2)(a_2a_3+b_2b_3+c_2c_3)] \end{aligned}$$

and

$$\begin{aligned} v_3 &= 4x^2[c_1(a_1a_3+b_1b_3+c_1c_3)(a_1a_2+b_1b_2)-c_1(a_2a_3+b_2b_3)+c_2(a_1a_3+b_1b_3) \\ &- c_2(a_1a_2+b_1b_2)(a_2a_3+b_2b_3+c_2c_3)]. \end{aligned}$$

By virtue of (2.32), we have the following

$$\begin{aligned} (\nabla_{E_1}R)(X,Y)Z &= 8x^3(a_1b_2-a_2b_1)(b_3E_1-a_3E_2), \\ (\nabla_{E_2}R)(X,Y)Z &= 0, \\ (\nabla_{E_3}R)(X,Y)Z &= 0. \end{aligned} \tag{2.34}$$

Form (2.32) one can easily obtain the following

$$\varphi^2(\nabla_{E_i}R)(X,Y)Z = p_iE_1+q_iE_2, \qquad i=1,2,3, \tag{2.35}$$

where $p_1=-8x^3b_3(a_1b_2-a_2b_1), \quad q_1=8x^3a_3(a_1b_2-a_2b_1),$

$p_2=0, \quad q_2=0, \quad p_3=0, \quad q_3=0.$

Let the 1-forms be defined as

$$\begin{aligned} A(E_1) &= \frac{p_1v_2-v_1q_1}{u_1v_2-v_1u_2}, \qquad B(E_1)=\frac{u_1q_1-p_1u_2}{u_1v_2-v_1u_2}, \\ A(E_2) &= 0, \qquad B(E_2)=0, \\ A(E_3) &= 0, \qquad B(E_3)=0, \end{aligned} \tag{2.36}$$

satisfying, $p_1v_2 - v_1q_1 \neq 0$, $u_1v_2 - v_1u_2 \neq 0$, $u_1q_1 - p_1u_2 \neq 0$ and $v_3 = 0$.
In view of (2.32), (2.33) and (2.35) it is easy to show the following relation:

$$\varphi^2(\nabla_{E_i}R)(X,Y)Z) = A(E_i)R(X,Y)Z + B(E_i)F(X,Y)Z, \qquad i = 1,2,3. \tag{2.37}$$

Hence, the metric M^3 under consideration is a 3-dimensional hyper generalized φ-recurrent (κ,μ)-contact metric manifold which is neither φ-symmetric nor φ-recurrent.

We can state the following.

Theorem 2.6. *There exists a 3-dimensional hyper generalized φ-recurrent (κ,μ)-contact metric manifold which is neither φ-symmetric nor φ-recurrent.*

2.1.4 Quasi generalized φ-recurrent (k,μ)-contact metric manifold

Recently, Venkatesha et al. (2019) studied quasi generalized φ-recurrent Sasakian manifolds. A brief study on quasi generalized recurrent manifolds was done by Shaikh and Roy (2010) and obtained interesting results. In this subsection, we will study quasi generalized φ-recurrrent (κ,μ)-contact metric manifolds.

Definition 2.3. *A $(2n+1)$-dimensional (κ,μ)-contact metric manifold is said to be a quasi generalized φ-recurrent if its curvature tensor R satisfies*

$$\varphi^2((\nabla_W R)(X,Y)Z) = D(W)R(X,Y)Z + E(W)F(X,Y)Z, \tag{2.38}$$

for all vector fields X, Y and Z. Here, D and E are two non-vanishing 1-forms such that $D(X) = g(X,\mu_1)$, $E(X) = g(X,\mu_2)$ and the tensor F is define by

$$\begin{aligned} F(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ &+ g(Y,Z)\eta(Y)\xi - g(X,Z)\eta(Y)\xi, \end{aligned} \tag{2.39}$$

for all vector fields X, Y and Z. Here, μ_1 and μ_2 are vector fields associated with 1-forms D and E respectively.

Theorem 2.7. *In a quasi generalized φ-recurrent (κ, μ)-contact metric manifold, the associated 1-forms D and E are related by $\kappa D(W) + 2E(W) = 0$.*

Proof. Consider a quasi generalized φ-recurrent (κ, μ)-contact metric manifold. From (2.38) we get

$$\begin{aligned} -((\nabla_W R)(X,Y)Z) &+ \eta((\nabla_W R)(X,Y)Z)\xi \\ &= D(W)R(X,Y)Z + E(W)F(X,Y)Z. \end{aligned} \tag{2.40}$$

Taking the same steps as in Theorem 2.1, we obtain the relation:

$$\kappa D(W) + 2E(W) = 0. \tag{2.41}$$

This completes the proof. □

Contracting over X in (2.40) gives

$$\begin{aligned} -(\nabla_W S)(Y,Z) &+ \eta((\nabla_W R)(\xi,Y)Z) = D(W)S(Y,Z) \\ &+ [(2n+1)g(Y,Z) + (2n-1)\eta(Y)\eta(Z)]E(W). \end{aligned} \tag{2.42}$$

Putting $Y = Z = e_i$, (2.42) reduce to

$$-dr(W) = rD(W) + 2n(2n+3)E(W). \tag{2.43}$$

We are in a position to state the following:

Theorem 2.8. *In a quasi generalized φ-recurrent (κ, μ)-contact metric manifold, if the scalar curvature is a non-zero constant then*

$$rD(W) + 2n(2n+3)E(W) = 0.$$

Theorem 2.9. *In a quasi generalized φ-recurrent (κ, μ)-contact metric manifold,* *the scalar curvature of the manifold satisfy the relation* $r = \kappa(5+2n^2)]+2(2n-1)]$.

Proof. Changing X, Y, Z cyclically in (2.40) and making use of Bianchi's identity we get

$$D(W)R(X,Y)Z + D(X)R(Y,W)Z + D(Y)R(W,X)Z$$
$$+E(W)F(X,Y)Z + E(X)F(Y,W)Z + E(Y)F(W,X)Z = 0. \qquad (2.44)$$

Contracting over X in (2.44) we get

$$D(W)S(Y,Z) + D(R(Y,W)Z) - D(Y)S(W,Z) + E(W)[(2n+1)g(Y,Z)$$
$$+ (2n-1)\eta(Y)\eta(Z)] + E(Y)g(W,Z) - g(Y,Z)E(W) + \eta(W)\eta(Z)E(X)$$
$$- \eta(Y)\eta(Z)E(W) + g(W,Z)\eta(Y)\eta(\mu_2) - g(Y,Z)\eta(W)\eta(\mu_2)$$
$$- E(Y)[(2n+1)g(W,Z) + (2n+1)\eta(Z)\eta(W)] = 0. \qquad (2.45)$$

Putting $Y = Z = e_i, 1 \leq i \leq 2n+1$ in (2.45) we obtain

$$rD(W) - 2n\kappa D(W) + \mu D(hW) - D(QW) + 2(2n^2+n-1)E(W)$$
$$+ 2(1-2n)\eta(W)\eta(\mu_2) = 0. \qquad (2.46)$$

Replacing W with ξ in (2.46) gives

$$r = \kappa[(5+2n^2) + 2(2n-1)]. \qquad (2.47)$$

This completes the proof. □

Corollary 2.2. *In a quasi generalized φ-recurrent (κ, μ)-contact metric manifold, if $\kappa = 0$ then the scalar curvature is constant.*

Proceeding like in Theorem 2.4, one can easily show that the manifold is an η-Einstein manifold. Hence, we get the following statement:

Theorem 2.10. *A quasi generalized φ-recurrent (κ, μ)-contact metric manifold is an η-Einstein manifold i.e.,*

$$S(Y, W) = \alpha g(Y, W) + \beta \eta(Y)\eta(W),$$

where

$$\alpha = \frac{[2(n\kappa + n - 1) + \mu(n + 2)][2(n - 1) - n\mu] - [2(n - 1) + \mu][\mu(1 - \kappa) + 2(n - 1) + 2\kappa]}{2n\kappa + \mu(n + 1)},$$

$$\beta = \frac{[2(n\kappa + n - 1) + \mu(n + 2)][2(1 - n) + n(2\kappa + \mu)] - (\kappa - 1)[2(n - 1) + \mu]^2}{2n\kappa + \mu(n + 1)}.$$

2.1.5 Example of a quasi generalized φ-recurrent (κ, μ)-contact metric manifold

In this subsection we give an example of a quasi generalized φ-recurrent (κ, μ)-contact metric manifold. We consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : x \neq 0, y \neq 0\}$, where $\{x, y, z\}$ is the standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be the global coordinate frame on M given by

$$E_1 = \frac{\partial}{\partial y}, \quad E_2 = 2xy\frac{\partial}{\partial z} + \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}.$$

Hui (2017) has shown that M is a 3-dimensional (κ, μ)-contact metric manifold with $\kappa = -\frac{1}{y}$ and $\mu = -\frac{1}{y}$. We will show that the manifold M is a 3-dimensional quasi generalized φ-recurrent (κ, μ)-contact metric manifold. Any vector fields X, Y, Z on M can be expressed as

$$X = a_1E_1 + b_1E_2 + c_1E_3,$$

$$Y = a_2E_1 + b_2E_2 + c_2E_3,$$

$$Z = a_3E_1 + b_3E_2 + c_3E_3,$$

where $a_i, b_i, c_i \in \mathbb{R}^+$ (set of positive numbers). Then the Riemannian curvature R becomes

$$R(X,Y)Z = v_1E_1 + v_2E_2, \tag{2.48}$$

where $v_1 = -\frac{2b_3}{y^2}(a_1b_2 - a_2b_1)$ and $v_2 = \frac{2a_3}{y^2}(a_1b_2 - a_2b_1)$.
Also

$$\begin{aligned} F(X,Y)Z &= (b_3u_1 + 2c_3u_2)E_1 + (2c_3u_3 - a_3u_1)E_2 \\ &- 2(a_3u_2 - b_3u_3)E_3, \end{aligned} \tag{2.49}$$

where $\quad u_1 = (a_1b_2 - b_1a_2), \quad u_2 = (a_1c_2 - a_2c_1), \quad u_3 = (b_1c_2 - b_2c_1).$
From (2.48) we obtained

$$(\nabla_{E_1}R)(X,Y)Z = \frac{4}{y^3}(a_1b_2 - a_2b_1)(b_3E_1 - a_3E_2), \tag{2.50}$$

$$(\nabla_{E_2}R)(X,Y)Z = 0, \tag{2.51}$$

$$(\nabla_{E_3}R)(X,Y)Z = 0. \tag{2.52}$$

Making use of (2.50), (2.51) and (2.52) we get the following

$$\varphi^2((\nabla_{E_i}R)(X,Y)Z) = p_iE_1 + q_iE_2, \qquad i = 1,2,3, \tag{2.53}$$

where

$$p_1 = -\frac{4b_3}{y^3}(a_1b_2 - a_2b_1), \quad q_1 = \frac{4a_3}{y^3}(a_1b_2 - a_2b_1),$$
$$p_2 = 0, \quad q_2 = 0, \quad p_3 = 0, \quad q_3 = 0.$$

Let us define 1-forms A and B by

$$\begin{aligned} A(E_1) &= \frac{a_3p_1(2c_3u_2 - b_3u_1) - q_1b_3(b_3u_1 + 2c_3u_2)}{v_1a_3(2c_3u_2 - b_3u_1) - b_3v_3u_2(a_3 + 2c_3)}, \\ B(E_1) &= \frac{b_3(q_1v_1 - p_1v_2)}{v_1a_3(2c_3u_2 - b_3u_1) - b_3v_3u_2(a_3 + 2c_3)}, \\ & A(E_2) = 0, \quad B(E_2) = 0, \\ & A(E_3) = 0, \quad B(E_3) = 0, \end{aligned} \tag{2.54}$$

where $a_3p_1(2c_3u_2 - b_3u_1) - q_1b_3(b_3u_1 + 2c_3u_2) \neq 0$, $b_3(q_1v_1 - p_1v_2) \neq 0$ and $v_1a_3(2c_3u_2 - b_3u_1) - b_3v_3u_2(a_3 + 2c_3) \neq 0$.

Using (2.50), (2.53) and (2.54) one can easily show that

$$\varphi^2((\nabla_{E_i} R)(X, Y)Z) = A(E_i)R(X, Y)Z + B(E_i)F(X, Y)Z, \quad i = 1, 2, 3. \tag{2.55}$$

Hence, the manifold under consideration is a 3-dimensional quasi generalized φ-recurrent (κ, μ)-contact metric manifold. Thus we can state the following.

Theorem 2.11. *There exists a 3-dimensional quasi generalized φ-recurrent (κ, μ)-contact metric manifold which is neither φ-symmetric nor φ-recurrent.*

2.2 On the Geometric Structures of Generalized (κ, μ)-space forms

An almost contact metric manifold $(M^{2n+1}, \varphi, \xi, g, \eta)$ is said to be a generalized (κ, μ)-space form if there exists differentiable functions $f_1, f_2, f_3, f_4, f_5, f_6$ on the manifold whose curvature tensor R is given by

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_5R_5 + f_6R_6, \tag{2.56}$$

J.P. Singh, M. Khatri (2021), On the Geometric Structures of Generalized (k, μ)-space forms, *Facta Univ., Math. Inform.*, **36 (5)**, 1129-1142.

where $R_1, R_2, R_3, R_4, R_5, R_6$ are the following tensors:

$$\begin{aligned}
R_1(X,Y)Z &= g(Y,Z)X - g(X,Z)Y, \\
R_2(X,Y)Z &= g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z, \\
R_3(X,Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi, \\
R_4(X,Y)Z &= g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y, \\
R_5(X,Y)Z &= g(hY,Z)hX - g(hX,Z)hY + g(\varphi hX,Z)\varphi hY - g(\varphi hY,Z)\varphi hX, \\
R_6(X,Y)Z &= \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi,
\end{aligned}$$

for any $X, Y, Z \in \chi(M)$. Here, h is a symmetric tensor given by $2h = \mathcal{L}_\xi\varphi$, where $\mathcal{L}$ is the Lie derivative. In particular, for $f_4 = f_5 = f_6 = 0$ it reduces to the generalized Sasakian space form (Alegre et al., 2004). It is obvious that (κ, μ)-space form is an example of generalized (κ, μ) space form when

$$f_1 = \frac{c+3}{4}, f_2 = \frac{c-1}{4}, f_3 = \frac{c+3}{4} - \kappa, f_4 = 1, f_5 = \frac{1}{2}, f_6 = 1 - \mu$$

are constants.

De and Samui (2016) studied quasi-umbilical hypersurface on (κ, μ)-space forms. A hypersurface $(\widetilde{M}^{2n+1}, \tilde{g})$ of a Riemannian manifold M^{2n+1} is called quasi-umbilical (Chen, 1973) if its second fundamental tensor has the form

$$\sigma(X,Y) = \alpha g(X,Y) + \beta\omega(X)\omega(Y), \tag{2.57}$$

where ω is the 1-form, α, β are scalars and the vector field corresponding to the 1-form ω is a unit vector field. Here, the second fundamental tensor σ is defined by $\sigma(X,Y) = \widetilde{g}(A_\rho X, Y)$, where A is (1,1) tensor and ρ is the unit normal vector field and X, Y are tangent vector fields.

In this section, the geometric structures of generalized (k, μ)-space forms and their quasi-umbilical hypersurface are analyzed. First ξ-$\tilde{Q}$ and conformally flat generalized (k, μ)-space form is investigated and shown that a conformally flat generalized (k, μ)-space form is Sasakian. Next, we prove that a generalized (k, μ)-space form satisfying Ricci pseudosymmetry is η-Einstein. We obtain the condition under which a quasi-umbilical hypersurface of a generalized (k, μ)-space form is a generalized quasi Einstein hypersurface. Also ξ-sectional curvature of a quasi-umbilical hypersurface of generalized (k, μ)-space form is obtained. Finally, the results obtained are verified by constructing an example of a 3-dimensional generalized (k, μ)-space form.

2.2.1 Preliminaries

In a generalized (κ, μ)-space form (M^{2n+1}, g) the following relations hold (Alegre and Blair, 2004):

$$\begin{aligned} R(X,Y)\xi &= (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\} \\ &+ (f_4 - f_6)\{\eta(Y)hX - \eta(X)hY\}, \end{aligned} \tag{2.58}$$

$$\begin{aligned} QX &= (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi \\ &+ ((2n-1)f_4 - f_6)hX, \end{aligned} \tag{2.59}$$

$$r = 2n\{(2n+1)f_1 + 3f_2 - 2f_3\}, \tag{2.60}$$

$$S(\varphi X, \varphi Y) = S(X,Y) - 2n(f_1 - f_3)\eta(X)\eta(Y). \tag{2.61}$$

where, R, S, Q, r are respectively the curvature tensor of type (1,3), the Ricci tensor, the Ricci operator i.e. $g(QX,Y) = S(X,Y)$, for any $X, Y \in \chi(M)$ and the scalar curvature of the manifold respectively.

2.2.2 Flatness of generalized (κ, μ)-space form

De and Samui (2016) studied conformally flat (κ, μ) space form and De and Majhi (2019) analyzed ξ-$\tilde{Q}$ flatness of generalized Sasakian space form. Generalizing the results obtained, in this subsection we studied ξ-$\tilde{Q}$ flat and conformally flat generalized (κ, μ)-space form.

Definition 2.4. *A generalized (κ, μ)-space form (M^{2n+1}, g), is said to be ξ-Q flat if $\tilde{Q}(X, Y)\xi = 0$, for any $X, Y \in \chi(M)$ on M.*

We have, from (1.26)

$$\tilde{Q}(X, Y)\xi = R(X, Y)\xi - \frac{v}{2n}\big[\eta(Y)X - \eta(X)Y\big], \tag{2.62}$$

for any $X, Y \in \chi(M)$. Using (2.58) in (2.62) we get

$$\begin{aligned}\tilde{Q}(X, Y)\xi &= \big(f_1 - f_3 - \frac{v}{2n}\big)\big[\eta(Y)X - \eta(X)Y\big] \\ &+ (f_4 - f_6)[\eta(Y)hX - \eta(X)hY].\end{aligned} \tag{2.63}$$

Suppose non-Sasakian generalized (κ, μ)-space form is $\xi - Q$ flat. Then from (2.63) we get

$$\big(f_1 - f_3 - \frac{v}{2n}\big)\big[\eta(Y)X - \eta(X)Y\big] + (f_4 - f_6)[\eta(Y)hX - \eta(X)hY] = 0. \tag{2.64}$$

Taking $X = \varphi X$ in (2.64), we obtain

$$\Big\{\big(f_1 - f_3 - \frac{v}{2n}\big)\varphi X + (f_4 - f_6)h\varphi X\Big\}\eta(Y) = 0. \tag{2.65}$$

Since $\eta(Y) \neq 0$ and taking the inner product with U in (2.65) gives

$$\big(f_1 - f_3 - \frac{v}{2n}\big)g(\varphi X, U) + (f_4 - f_6)g(\varphi X, hU) = 0. \tag{2.66}$$

Since $g(\varphi X, U) \neq 0$ and $g(\varphi X, hU) \neq 0$, we see that $f_1 - f_3 = \frac{v}{2n}$ and $f_4 = f_6$. Conversely, taking $f_1 - f_3 = \frac{v}{2n}$ and $f_4 = f_6$, and putting these values in (2.63) gives $\tilde{Q}(X,Y)\xi = 0$ and hence M is $\xi - \tilde{Q}$ flat. Therefore we can state the following:

Theorem 2.12. *A non-Sasakian generalized (κ,μ)-space form (M^{2n+1},g), is ξ-Q flat if and only if $f_1 - f_3 = \frac{v}{2n}$ and $f_4 = f_6$.*

In particular, if $v = \frac{r}{2n+1}$ then $\tilde{Q}$ tensor reduces to concircular curvature tensor. Making use of (2.60) in the forgoing equation gives $v = \frac{2n\{(2n+1)f_1+3f_2-2f_3\}}{2n+1}$. In regard to Theorem 2.12, for ξ-concircularly flat we obtain $f_3 = \frac{3f_2}{1-2n}$ and hence we can state the following corollary:

Corollary 2.3. *A non-Sasakian generalized (κ,μ)-space form (M^{2n+1},g), is ξ-concircularly flat if and only if $f_3 = \frac{3f_2}{1-2n}$ and $f_4 = f_6$.*

We can easily see that Theorem 3.1 and Corollary 3.1 obtained by De and Majhi (2019), are particular cases of Theorem 2.12 and Corollary 2.3 respectively for $f_4 = f_5 = f_6 = 0$.

Substituting the values, $f_4 - f_6 = \mu$ and $f_1 - f_3 = \kappa$ in Theorem 2.12, we obtained the following corollary:

Corollary 2.4. *A (κ,μ)-space form (M^{2n+1},g), is ξ-$\tilde{Q}$ flat if and only if $\kappa = \frac{v}{2n}$ and $\mu = 0$.*

Definition 2.5. *A generalized (κ,μ)-space form $(M^{2n+1},g), n > 1$, is said to be conformally flat if $C(X,Y)Z = 0$, for any $X,Y,Z \in \chi(M)$ on M.*

Suppose generalized (κ,μ)-space form is conformally flat. Then from (1.23), we get

$$R(X,Y)Z - \frac{1}{2n-1}\big\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)PX - g(X,Z)PY\big\} + \frac{r}{2n(2n-1)}\big\{g(Y,Z)X - g(X,Z)Y\big\} = 0. \tag{2.67}$$

As a consequence of taking $X = \xi$ in (2.67) and using (1.9), (2.58) and (2.59). Eq.(2.67) becomes

$$\begin{aligned}&(f_1 - f_3)\{g(Y,Z)\xi - \eta(Z)Y\} + (f_4 - f_6)\{g(hY,Z)\xi - \eta(Z)hY\}\\&- \frac{1}{2n-1}\big\{S(Y,Z)\xi - 2n(f_1 - f_3)\eta(Z)Y + 2n(f_1 - f_3)g(Y,Z)\xi\\&- \eta(Z)PY\big\} + \frac{r}{2n(2n-1)}\big\{g(Y,Z)\xi - \eta(Z)Y\big\} = 0. \qquad (2.68)\end{aligned}$$

Putting $Z = \varphi Z$ in (2.68) and making use of (2.58), (2.59) and (2.60) results in the following

$$2(n+1)f_6 g(hY, \varphi Z) = 0. \qquad (2.69)$$

This shows that either $f_6 = 0$ or $\varphi h = 0$. In the second case, from (1.9) we have $h = 0$. Therefore we can state the following:

Theorem 2.13. *If a generalized (κ, μ)-space form $(M^{2n+1}, g), n > 1$, is conformally flat, then either $f_6 = 0$ or M is Sasakian.*

Corollary 2.5. *If a (κ, μ)-space form $(M^{2n+1}, g), n > 1$, is conformally flat, then either $\mu = 1$ or M is Sasakian.*

2.2.3 Pseudosymmetric generalized (κ, μ)-space form

In this subsection certain pseudo symmetry such as Ricci pseudo symmetry and conformal Ricci pseudo symmetry in the context of generalized (κ, μ)-space form are studied. First, we review an important definition

Definition 2.6 (Deszcz, 1992; Shaikh et al., 2015)**.** *A Riemannian manifold $(M, g), n \geq 1$, admitting a $(0, \kappa)$-tensor field T is said to be T-pseudosymmetric if $R \cdot T$ and $D(g, T)$ are linearly dependent, i.e., $R \cdot T = L_T D(g, T)$ holds on the set $U_T = \{x \in M : D(g, T) \neq 0$ at $x\}$, where L_T is some function on U_T.*

In particular, if $R \cdot R = L_R D(g, R)$ and $R \cdot S = L_S D(g, S)$ then the manifold is called pseudosymmetric and Ricci pseudosymmetric respectively. Moreover if

$L_R = 0$ (resp., $L_S = 0$) then pseudosymmetric (resp., Ricci pseudosymmetric) reduces to semisymmetric (resp., Ricci semisymmetric) introduced by Cartan in 1946.

Definition 2.7. *A generalized (κ, μ)-space form (M^{2n+1}, g), is said to be Ricci pseudosymmetric if its Ricci curvature satisfies the following relation,*

$$R \cdot S = f_{S_2} D(g, S),$$

holds on the set $U_{S_2} = \{x \in M : D(g, S) \neq 0 \text{ at } x\}$, where f_{S_2} is some function on U_{S_2}.

Suppose a generalized (κ, μ)-space form (M^{2n+1}, g), is Ricci pseudosymmetric i.e.,

$$R \cdot S = f_{S_2} D(g, S),$$

which can be written as

$$\begin{aligned} S(R(X,Y)U,V) + S(U,R(X,Y)V) = -f_s\big[&S(Y,V)g(X,U) \\ -S(X,V)g(Y,U) + S(U,Y)g(X,V) - S(U,X)g(Y,V)\big] \end{aligned} \tag{2.70}$$

Taking $X = U = \xi$ in (2.70) and using (2.58), (2.59) and (2.61), we get

$$\begin{aligned} &\big(f_3 - f_1 + f_{S_2}\big)S(Y,V) + \big[2n(f_1 - f_3)(f_1 - f_3 - f_{S_2}) - (\kappa - 1)(f_4 \\ &- f_6)((2n-1)f_4 - f_6)\big]g(Y,V) - (\kappa - 1)(f_4 - f_6)\big((2n-1)f_4 \\ &- f_6\big)\eta(Y)\eta(V) + (f_4 - f_6)\big((1-2n)f_3 - 3f_2\big)g(hY,V) = 0. \end{aligned} \tag{2.71}$$

Considering $f_{S_2} \neq f_1 - f_3$ and further taking $(1-2n)f_3 - 3f_2 = 0$ in (2.71), the manifold is η-Einstein. Hence we can state the following:

Theorem 2.14. *A Ricci pseudosymmetric generalized (κ, μ)-space form (M^{2n+1}, g), with $f_{S_2} \neq f_1 - f_3$, is η-Einstein manifold if $f_3 = \frac{3f_2}{1-2n}$.*

If $f_{S_2} = 0$, then Ricci pseudosymmetric generalized (κ, μ)-space form reduces to Ricci semisymmetric generalized (κ, μ)-space form. In view of Theorem (2.14) we obtain the following:

Corollary 2.6. *A Ricci semisymmetric generalized (κ, μ)-space form (M^{2n+1}, g), with $f_1 - f_3 \neq 0$ is η-Einstein manifold if $f_3 = \frac{3f_2}{1-2n}$.*

Definition 2.8. *A generalized (κ, μ)-space form $(M^{2n+1}, g), n > 1$, is said to be conformal Ricci pseudosymmetric if*

$$C \cdot S = f_{S_4} D(g, S),$$

holds on the set $U_{S_4} = \{x \in M : D(g, S) \neq 0 \text{ at } x\}$, where f_{S_4} is any function on U_{S_4}.

Suppose a generalized (κ, μ)-space form is conformal Ricci pseudosymmetric. Then, we have

$$S(C(X,Y)U,V) + S(U, C(X,Y)V) = -f_{S_4}\big[S(Y,V)g(X,U) \\ -S(X,V)g(Y,U) + S(U,Y)g(X,V) - S(U,X)g(Y,V)\big]. \qquad (2.72)$$

Taking $X = U = \xi$ and $f_4 = f_6$ in (2.72) and making use of (1.23),(1.9) and (2.59), we obtain

$$S^2(Y,V) = \big(4nf_1 + 3f_2 - (2n+1)f_3 + 2n(2n-1)f_{S_4}\big)S(Y,V) \\ -(2n-1)f_{S_4}\eta(Y)\eta(V) - \big(2nf_1 + 3f_2 - f_3\big)g(Y,V). \qquad (2.73)$$

Thus, we can state the following:

Theorem 2.15. *If a generalized (κ, μ)-space form $(M^{2n+1}, g), n > 1$, is conformal Ricci pseudosymmetric with $f_4 = f_6$, then the relation(2.73) holds.*

2.2.4 Quasi-umbilical hypersurface of generalized (κ, μ)-space form

Let us consider a quasi-umbilical hypersurface $\widetilde{M}$ of a generalized (κ, μ)-space form. From Gauss, for any vector fields X, Y, Z, W tangent to the hypersurface we have

$$\begin{aligned} R(X,Y,Z,W) &= \widetilde{R}(X,Y,Z,W) - g(\sigma(X,W),\sigma(X,Z)) \\ &+ g(\sigma(X,Z),\sigma(Y,W)), \end{aligned} \tag{2.74}$$

where, $R(X,Y,Z,W) = g(R(X,Y)Z,W)$ and $\widetilde{R}(X,Y,Z,W) = g(\widetilde{R}(X,Y)Z,W)$. Here, σ is the second fundamental tensor of $\widetilde{M}$ given by

$$\sigma(X,Y) = \alpha g(X,Y)\rho + \beta\omega(X)\omega(Y)\rho, \tag{2.75}$$

where, ρ is the only unit normal vector field. Here, ω is the 1-form, the vector field corresponding to the 1-form ω is a unit vector field and α, β are scalars. Using (2.75) in (2.74), we obtain the following result

$$\begin{aligned} &f_1\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big] + f_2\big[g(X,\varphi Z)g(\varphi Y,W) \\ &- g(Y,\varphi Z)g(\varphi X,W) + 2g(X,\varphi Y)g(\varphi Z,W)\big] + f_3\big[\eta(X)\eta(Z)g(Y,W) \\ &- \eta(Y)\eta(Z)g(X,W) + g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W)\big] \\ &+ f_4\big[g(Y,Z)g(hX,W) - g(Y,Z)g(hY,W) + g(hY,Z)g(X,W) \\ &- g(hX,Z)g(Y,W)\big] + f_5\big[g(hY,Z)g(hX,W) - g(hX,Z)g(hY,W) \\ &+ g(\varphi hX,Z)g(\varphi hY,W) - g(\varphi hY,Z)g(\varphi hX,W)\big] + f_6\big[\eta(X)\eta(Z)g(hY,W) \\ &- \eta(Y)\eta(Z)g(hX,W) + g(hX,Z)\eta(Y)\eta(W) - g(hY,Z)\eta(X)\eta(W)\big] \\ &= \widetilde{R}(X,Y,Z,W) - \alpha^2 g(X,W)g(Y,Z) - \alpha\beta g(X,W)\omega(Y)\omega(Z) \\ &- \alpha\beta g(Y,Z)\omega(X)\omega(W) + \alpha^2 g(Y,W)g(X,Z) + \alpha\beta g(Y,W)\omega(X)\omega(Z) \\ &+ \alpha\beta g(X,Z)\omega(Y)\omega(W). \end{aligned} \tag{2.76}$$

Contracting over X and W in (2.76), we obtain

$$\widetilde{S}(Y,Z) = \big(2nf_1 + 3f_2 - f_3 + 2n\alpha^2 + \alpha\beta\big)g(Y,Z)$$
$$- \big(3f_2 + (2n+1)f_3\big)\eta(Y)\eta(Z) + \big((2n-1)f_4 - f_6\big)g(hY,Z)$$
$$+ \alpha\beta(2n-1)\omega(Y)\omega(Z). \tag{2.77}$$

Hence, we can state the following:

Theorem 2.16. *A quasi-umbilical hypersurface of a generalized (κ,μ)-space form is a generalized quasi Einstein hypersurface, provided $f_4 = \frac{f_6}{2n-1}$*

In particular, for a (κ,μ)-space form, the above Theorem 2.16 reduces to the following:

Theorem 2.17 (De and Samui, 2016)**.** *A quasi-umbilical hypersurface of a (κ,μ)-contact space form is a generalized quasi-Einstein hypersurface, provided $\mu = 2-2n$.*

Corollary 2.7. *A quasi-umbilical hypersurface of a generalized Sasakian space form is a generalized quasi-Einstein hypersurface.*

For any vector fields X, Y, the tensor field $K(X,Y) = \widetilde{R}(X,Y,Y,X)$ is called the sectional curvature of $\widetilde{M}$ given by the sectional plane $\{X,Y\}$. The sectional curvature $K(X,\xi)$ of a sectional plane spanned by ξ and vector field X orthogonal to ξ is called the ξ-sectional curvature of $\widetilde{M}$.

Theorem 2.18. *A ξ-sectional curvature of a quasi-umbilical hypersurface of generalized (κ,μ)-space form is given by*

$$K(X,\xi) = \big(f_1 - f_3 + \alpha^2\big)g(\varphi X,\varphi X) + (f_4 - f_6)g(hX,X)$$
$$+\alpha\beta\big[(\omega(\xi))^2 + (\omega(X))^2\big] - 2\alpha\beta\eta(X)\omega(X)\omega(\xi).$$

Proof. Taking $W = X$ and $Z = Y$ in (2.76) results in following

$$
\begin{aligned}
& f_1\big[g(Y,Y)g(X,X) - g(X,Y)g(Y,X)\big] + f_2\big[g(X,\varphi Y)g(\varphi Y,X) \\
& - g(Y,\varphi Y)g(\varphi X,X) + 2g(X,\varphi Y)g(\varphi Y,X)\big] + f_3\big[\eta(X)\eta(Y)g(X,Y) \\
& - \eta(Y)\eta(Y)g(X,X) - g(X,Y)\eta(X)\eta(Y) - g(Y,Y)\eta(X)\eta(X)\big] \\
& + f_4\big[g(Y,Y)g(hX,X) - g(X,Y)g(hY,X) + g(hY,Y)g(X,X) \\
& - g(hX,Y)g(Y,X)\big] + f_5\big[g(hY,Y)g(hX,X) - g(hX,Y)g(hY,X) \\
& + g(\varphi hX,Y)g(\varphi hY,X) - g(\varphi hY,Y)g(\varphi hX,X)\big] + f_6\big[\eta(x)\eta(Y)g(hY,X) \\
& - \eta(Y)\eta(Y)g(hX,X) + g(hX,Y)\eta(Y)\eta(X) - g(hY,Y)\eta(X)\eta(X)\big] \\
& = K(X,Y) - \alpha^2 g(X,X)g(Y,Y) - \alpha\beta g(X,X)\omega(Y)\omega(Y) \\
& - \alpha\beta g(Y,Y)\omega(X)\omega(X) + \alpha^2 g(X,Y)g(X,Y) + \alpha\beta g(X,Y)\omega(X)\omega(Y) \\
& + \alpha\beta g(X,Y)\omega(Y)\omega(X).
\end{aligned}
\tag{2.78}
$$

Putting $Y = \xi$ in (2.78) gives

$$
\begin{aligned}
K(X,\xi) = {} & \big(f_1 - f_3 + \alpha^2\big)g(\varphi X,\varphi X) + (f_4 - f_6)g(hX,X) \\
& + \alpha\beta\big[(\omega(\xi))^2 + (\omega(X))^2\big] - 2\alpha\beta\eta(X)\omega(X)\omega(\xi).
\end{aligned}
$$

This completes the proof. □

2.2.5 Examples of generalized (κ,μ)-space forms

Now we will show the validity of obtained result by considering an example of a generalized (κ,μ)-space form of dimension 3. Koufogiorgos and Tsichlias (2000) constructed an example of generalized (κ,μ)-space of dimension 3 which was later shown by Carriazo et al. (2013) to be a contact metric generalized (κ,μ)-space form $M^3(f_1,0,f_3,f_4,0,0)$ with non-constant f_1, f_3, f_4.

Example 2.1. *Let M^3 be the manifold $M = \{(x_1,x_2,x_3) \in \mathbb{R}^3 | x_3 \neq 0\}$ where*

(x_1, x_2, x_3) *are standard coordinates on* $\mathbb{R}^3$. *Consider the vector fields*

$$e_1 = \frac{\partial}{\partial x_1}, \quad e_2 = -2x_2x_3\frac{\partial}{\partial x_1} + \frac{2x_1}{x_3^2}\frac{\partial}{\partial x_2} - \frac{1}{x_3^2}\frac{\partial}{\partial x_3}, \quad e_3 = \frac{1}{x_3}\frac{\partial}{\partial x_2},$$

are linearly independent at each point of M *and are related by*

$$[e_1, e_2] = \frac{2}{x_3^2}e_3, \quad [e_2, e_3] = 2e_1 + \frac{1}{x_3^3}e_3, \quad [e_3, e_1] = 0.$$

Let g *be the Riemannian metric defined by* $g(e_i, e_j) = \delta_{ij}, i, j = 1, 2, 3$ *and* η *be the 1-form defined by* $\eta(X) = g(X, e_1)$ *for any* X *on* M. *Also, let* φ *be the* $(1, 1)$*-tensor field defined by* $\varphi e_1 = 0, \quad \varphi e_2 = e_3 \quad \varphi e_3 = -e_2$. *Therefore,* (φ, e_1, η, g) *defines a contact metric structure on* M. *Put* $\lambda = \frac{1}{x_3^2}, \kappa = 1 - \frac{1}{x_3^4}$ *and* $\mu = 2(1 - \frac{1}{x_3^2})$, *then symmetric tensor* h *satisfies* $he_1 = 0, he_2 = \lambda e_2, he_3 = -\lambda e_3$. *The non-vanishing components of the Riemannian curvature are as follows:*

$$\begin{aligned}
R(e_1, e_2)e_1 &= -(\kappa + \lambda\mu)e_2, \quad R(e_1, e_2)e_2 = (\kappa + \lambda\mu)e_1, \\
R(e_1, e_3)e_1 &= (-\kappa + \lambda\mu)e_3, \quad R(e-1, e_3)e_3 = (\kappa - \lambda\mu)e_1, \\
R(e_2, e_3)e_2 &= (\kappa + \mu - 2\lambda^3)e_3, \quad R(e_2, e_3)e_3 = -(\kappa + \mu - 2\lambda^3)e_2.
\end{aligned}$$

Therefore, M *is a generalized* (κ, μ)*-space with* κ, μ *not constant. As a contact metric generalized* (κ, μ)*-space is a generalized* (κ, μ)*-space form with* $\kappa = f_1 - f_3$ *and* $\mu = f_4 - f_6$ *(Theorem 4.1 (Carriazo et al., 2013)), the manifold under consideration is a generalized* (κ, μ)*-space form* $M^3(f_1, 0, f_3, f_4, 0, 0)$ *where*

$$\begin{aligned}
f_1 &= -3 + \frac{2}{x_3^2} + \frac{1}{x_3^4} + \frac{2}{x_3^6}, \\
f_3 &= -4 + \frac{2}{x_3^2} + \frac{2}{x_3^4} + \frac{2}{x_3^6} \\
f_4 &= 2(1 - \frac{1}{x_3^2}).
\end{aligned}$$

Next we obtain the non-vanishing components of $\tilde{Q}$*-curvature tensor for arbitrary* *function v as follows:*

$$\tilde{Q}(e_1,e_2)e_1 = -(\kappa+\lambda\mu-\frac{v}{2})e_2, \quad \tilde{Q}(e_1,e_2)e_2 = (\kappa+\lambda\mu-\frac{v}{2})e_1,$$
$$\tilde{Q}(e_1,e_3)e_1 = (-\kappa+\lambda\mu+\frac{v}{2})e_3, \quad \tilde{Q}(e_1,e_3)e_3 = (\kappa-\lambda\mu-\frac{v}{2})e_1,$$
$$\tilde{Q}(e_2,e_3)e_2 = (\kappa+\mu-2\lambda^3+\frac{v}{2})e_3, \quad \tilde{Q}(e_2,e_3)e_3 = -(\kappa+\mu-2\lambda^3+\frac{v}{2})e_2.$$

From the above equations we see that $\tilde{Q}(X,Y)e_1 = 0$ for all X, Y on M if and only if $v = 2(1-\frac{1}{x_3^4})$ and $x_3^2 = 1$. Hence, Theorem 2.12 is verified.

Example 2.2. *Alegre et al. (2004) showed that the warped product $\mathbb{R}\times_f \mathbb{C}^m$ with*

$$f_1 = -\frac{(f')^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f},$$

is a generalized Sasakian space form. Since every generalized Sasakian space form is a particular case of generalized (κ,μ)-space form, $\mathbb{R}\times_f \mathbb{C}^m$ with f_1, f_2, f_3 define as above and $f_4 = f_5 = f_6 = 0$ is a generalized (κ,μ)-space form.

2.3 On almost pseudo semiconformally symmetric manifold

A Riemannian manifold (M^n, g) of dimension $n \geq 4$ is said to be pseudo semiconformally symmetric (Kim, 2017) if its semiconformal curvature tensor P of type $(0,4)$ satisfies the relation

$$\begin{aligned}(\nabla_E P)(X,Y,W,V) &= 2A(E)P(X,Y,W,V) + A(X)P(E,Y,W,V) \\ &+ A(Y)P(X,E,W,V) + A(W)P(X,Y,E,V) \\ &+ A(V)P(X,Y,W,E). \end{aligned} \tag{2.79}$$

J.P. Singh, M. Khatri (2020), On almost pseudo semiconformally symmetric manifold, *Differ. Geom.-Dyn. Syst.*, **22**, 233-253.

for all vector fields X, Y, W, V and E on M. Extending the notion of pseudo semiconformally symmetric manifold we introduced a type of non-flat Riemannian manifold $(M^n, g), (n \geq 4)$ whose semiconformal curvature tensor P of type $(0,4)$ satisfies the condition

$$\begin{aligned}(\nabla_E P)(X,Y,W,V) &= [A(E)+B(E)]P(X,Y,W,V) + A(X)P(E,Y,W,V) \\ &+ A(Y)P(X,E,W,V) + A(W)P(X,Y,E,V) \\ &+ A(V)R(X,Y,W,E), \end{aligned} \tag{2.80}$$

where A and B are non-zero 1-forms and are called the associated 1-forms.

2.3.1 Preliminaries

In this subsection, we will derive some formulas, which we will be using in the study of $A(PSCS)_n$ throughout this subsection. Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold where $1 \leq i \leq n$.
Now from equation (1.25), we have

$$\sum_{i=1}^{n} P(X,Y,e_i,e_i) = 0 = \sum_{i=1}^{n} P(e_i,e_i,X,Y), \tag{2.81}$$

and,

$$\sum_{i=1}^{n} P(e_i,Y,W,e_i) = \sum_{i=1}^{n} P(Y,e_i,e_i,W) = -\frac{\{a+(n-2)b\}r}{(n-2)} g(Y,W), \tag{2.82}$$

where, $r = \sum_{i=1}^{n} S(e_i,e_i)$ is the scalar curvature.

Making use of equation (1.25) we obtain the following relations:

$$\begin{aligned}
&(i) \quad P(X,Y,W,V) = -P(Y,X,W,V),\\
&(ii) \quad P(X,Y,W,V) = -P(X,Y,V,W),\\
&(iii) \quad P(X,Y,W,V) = P(W,V,X,Y),\\
&(iv) \quad P(X,Y,W,V) + P(Y,W,X,V) + P(W,X,Y,V) = 0. \qquad (2.83)
\end{aligned}$$

2.3.2 An $A(PSCS)_n, (n \geq 4)$ with non-zero constant scalar curvature and Codazzi type of Ricci tensor

Theorem 2.19. *In* $A(PSCS)_n, (n \geq 4)$ *the scalar curvature is a non-zero constant if and only if* $(4+n)A(E) + nB(E) = 0,$ *provided* $[a + (n-2)b] \neq 0.$

Proof. Taking the covariant derivative of equation (1.25) with respect to E we get,

$$\begin{aligned}
a(\nabla_E R)(X,Y,W,V) &= (\nabla_E P)(X,Y,W,V) + \frac{a}{(n-2)}\Big\{(\nabla_E S)(Y,W)g(X,V)\\
&- (\nabla_E S)(X,W)g(Y,V) + (\nabla_E S)(X,V)g(Y,W)\\
&- (\nabla_E S)(Y,V)g(X,W)\Big\} + \frac{b\,dr(E)}{(n-1)}\Big\{g(Y,W)g(X,V)\\
&- g(X,W)g(Y,V)\Big\}. \qquad (2.84)
\end{aligned}$$

Inserting equation (2.80) in equation (2.84) we obtain,

$$\begin{aligned}
a(\nabla_E R)(X,Y,W,V) &= [A(E) + B(E)]P(X,Y,W,V) + A(X)P(E,Y,W,V)\\
&+ A(Y)P(X,E,W,V) + A(W)P(X,Y,E,V)\\
&+ A(V)R(X,Y,W,E) + \frac{a}{(n-2)}\Big\{(\nabla_E S)(Y,W)g(X,V)\\
&- (\nabla_E S)(X,W)g(Y,V) + (\nabla_E S)(X,V)g(Y,W)\\
&- (\nabla_E S)(Y,V)g(X,W)\Big\} + \frac{b\,dr(E)}{(n-1)}\Big\{g(Y,W)g(X,V)\\
&- g(X,W)g(Y,V)\Big\}. \qquad (2.85)
\end{aligned}$$

Putting $X = V = e_i, (i = 1, 2, ..., n)$ and $\lambda = \dfrac{\{a + (n-2)b\}r}{(n-2)}$ in equation (2.85), we obtain

$$\begin{aligned} a(\nabla_E S)(Y,W) &= [A(E)+B(E)]\Big[-\lambda r g(Y,W)\Big] + A(\tilde{P}(E,Y)W) \\ &+ A(Y)\Big[-\lambda r g(E,W)\Big] + A(W)\Big[-\lambda r g(Y,E)\Big] - A(\tilde{P}(W,E)Y) \\ &+ \frac{a}{(n-2)}\Big[n(\nabla_E S)(Y,W) - (\nabla_E S)(W,Y) + dr(E)g(Y,W) \\ &- (\nabla_E S)(Y,W)\Big] + b\, dr(E)g(Y,W). \end{aligned} \tag{2.86}$$

Contracting over Y and W in equation (2.86), the above equation reduces to

$$n[a + (n-2)b]\, dr(E) = [a + (n-2)b]r[(4+n)A(E) + nB(E)]. \tag{2.87}$$

Assuming $[a + (n-2)b] \neq 0$, then equation (2.87) reduces to

$$n\, dr(E) = r[(4+n)A(E) + nB(E)]. \tag{2.88}$$

Clearly if $[(4+n)A(E) + nB(E)] = 0$ then r is a non-zero constant.
Conversely, if r is a non-zero constant then $[(4+n)A(E) + nB(E)] = 0$.
This completes the proof. □

Theorem 2.20. *In $A(PSCS)_n$, if the semiconformal curvature tensor P satisfies Bianchi's second identity then $A(PSCS)_n$ reduces to a pseudo semiconformally symmetric manifold, provided $[a + (n-2)b] \neq 0$ and $r \neq 0$.*

Proof. Suppose that the semiconformal tensor P in $A(PSCS)_n$ satisfies Bianchi's second identity. Then making use of equation (2.80), we get

$$\begin{aligned} &[B(E) - A(E)]P(X,Y,W,V) + [B(X) - A(X)]P(Y,E,W,V) \\ &\qquad + [B(Y) - A(Y)]P(E,X,W,V) = 0. \end{aligned} \tag{2.89}$$

Let $Q(E) = B(E) - A(E)$ and ρ_1 be a basic vector such that

$$g(E, \rho_1) = Q(E), \tag{2.90}$$

for all E. Equation (2.89) with the help of equation (2.90) may be written as

$$Q(E)P(X,Y,W,V) + Q(X)P(Y,E,W,V) + Q(Y)P(E,X,W,V) = 0. \quad (2.91)$$

Putting $X = V = e_i$ in equation (2.91), the above equation reduces to

$$\begin{aligned} Q(E)\Big\{-\frac{[a+(n-2)b]r}{(n-2)}g(Y,W)\Big\} + Q(\tilde{P}(Y,E)W) \\ - Q(Y)\Big\{-\frac{[a+(n-2)b]r}{(n-2)}g(E,W)\Big\} = 0, \end{aligned} \quad (2.92)$$

and contracting over Y and W, we infer

$$[a+(n-2)b]rQ(E) = 0. \quad (2.93)$$

Suppose $r \neq 0$ and $[a+(n-2)b] \neq 0$ in above equation implies $Q(E) = 0$. This completes the proof. □

Theorem 2.21. *If $A(PSCS)_n$ satisfies Bianchi's second identity then the scalar curvature is constant provided $[a+(n-2)b] \neq 0$.*

Proof. Suppose $A(PSCS)_n$ satisfies Bianchi's second identity. Then, from equation (1.25), we obtain

$$\begin{aligned} &\frac{a}{(n-2)}\Big\{(\nabla_E S)(Y,W)g(X,V) - (\nabla_E S)(X,W)g(Y,V) + (\nabla_E S)(X,V)g(Y,W) \\ &- (\nabla_E S)(Y,V)g(X,W) + (\nabla_X S)(E,W)g(Y,V) - (\nabla_X S)(Y,W)g(E,V) \\ &+ (\nabla_X S)(Y,V)g(E,W) - (\nabla_X S)(E,V)g(Y,W) + (\nabla_Y S)(X,W)g(E,V) \\ &- (\nabla_Y S)(E,W)g(X,V) - (\nabla_Y S)(X,V)g(E,W) + (\nabla_Y S)(E,V)g(X,W)\Big\} \\ &+ \frac{b}{(n-1)}\Big\{dr(E)\{g(Y,W)g(X,V) - g(X,W)g(Y,V)\} \\ &+ dr(X)\{g(E,W)g(Y,V) - g(Y,W)g(E,V)\} \\ &+ dr(Y)\{g(X,W)g(E,V) - g(E,W)g(X,V)\}\Big\} = 0. \end{aligned} \quad (2.94)$$

Contracting equation (2.94) over Y and W, the equation reduces to

$$\frac{a}{(n-2)}\left[\frac{1}{2}dr(E)g(X,V)+(n-2)(\nabla_E S)(X,V)+(2-n)(\nabla_X S)(E,V)\right.$$
$$\left.-\frac{1}{2}dr(X)g(E,V)-(\nabla_E S)(X,V)+(\nabla_X S)(E,V)\right]+bg(X,V)dr(E)$$
$$-bg(E,V)dr(X)+\frac{b}{(n-1)}\left[dr(X)g(E,V)-dr(E)g(X,V)\right]=0. \quad (2.95)$$

Substituting $X = V = e_i$ in equation (2.95) yields

$$[a+(n-2)b]\,dr(E)=0. \quad (2.96)$$

This completes the proof. □

2.3.3 Ricci Symmetric $A(PSCS)_n, (n \geq 4)$ and Ricci-recurrent $A(PSCS)_n, (n \geq 4)$.

Theorem 2.22. *In a Ricci symmetric $A(PSCS)_n, (n \geq 4)$, Bianchi's second identity holds for semiconformal curvature tensor.*

Proof. Since $A(PSCS)_n$ is Ricci symmetric, the Ricci tensor S satisfies the condition

$$\nabla S = 0$$

and $dr = 0$.
Using this, we have

$$(\nabla_E P)(X,Y,W,V) = a(\nabla_E R)(X,Y,W,V).$$

Hence,

$$(\nabla_E P)(X,Y,W,V)+(\nabla_X P)(Y,E,W,V)+(\nabla_Y P)(E,X,W,V)=$$
$$a[(\nabla_E R)(X,Y,W,V)+(\nabla_X R)(Y,E,W,V)+(\nabla_Y R)(E,X,W,V)], \quad (2.97)$$

implies,

$$(\nabla_E P)(X, Y, W, V) + (\nabla_X P)(Y, E, W, V) + (\nabla_Y P)(E, X, W, V) = 0. \qquad (2.98)$$

Hence, the theorem is proved. □

Theorem 2.23. *In a Ricci symmetric $A(PSCS)_n, (n \geq 4)$ the vector fields corresponding to the 1-forms A and B are in opposite direction, provided $r \neq 0$ and $[a + (n-2)b] \neq 0$.*

Proof. Contracting equation (2.80) over E, we get

$$\begin{aligned}&(div\tilde{P})(X, Y)W = A(\tilde{P}(X, Y)W) + B(\tilde{P}(X, Y)W)\\&- A(X)\left\{\frac{[a + (n-2)b]r}{(n-2)}\right\}g(Y, W)\\&+ A(Y)\left\{\frac{[a + (n-2)b]r}{(n-2)}\right\}g(X, W) + A(\tilde{P}(X, Y)W).\end{aligned} \qquad (2.99)$$

Moreover we have,

$$\begin{aligned}&(div\tilde{P})(X, Y)W = \frac{a(n-3)}{(n-2)}\left\{(\nabla_X S)(Y, W) - (\nabla_Y S)(X, W)\right\}\\&-\left\{\frac{[a(n-1) + b(n-2)]}{2(n-1)(n-2)}\right\}\left\{dr(X)g(Y, W) - dr(Y)g(X, W)\right\}.\end{aligned} \qquad (2.100)$$

Combining equations (2.99) and (2.100), the above equations reduces to

$$\begin{aligned}&A(\tilde{P}(X, Y)W) + B(\tilde{P}(X, Y)W) - A(X)\left\{\frac{[a + (n-2)b]r}{(n-2)}\right\}\\&g(Y, W) + A(Y)\left\{\frac{[a + (n-2)b]r}{(n-2)}\right\}g(X, W) + A(\tilde{P}(X, Y)W)\\&= \frac{a(n-3)}{(n-2)}\left\{(\nabla_X S)(Y, W) - (\nabla_Y S)(X, W)\right\}\\&- \left\{\frac{[a(n-1) + b(n-2)]}{2(n-1)(n-2)}\right\}\left\{dr(X)g(Y, W) - dr(Y)g(X, W)\right\}.\end{aligned} \qquad (2.101)$$

Suppose the manifold is Ricci symmetric, then equation (2.101) becomes

$$2A(\tilde{P}(X,Y)W) + B(\tilde{P}(X,Y)W) - A(X)\left\{\frac{[a+(n-2)b]r}{(n-2)}\right\}g(Y,W)$$
$$+ A(Y)\left\{\frac{[a+(n-2)b]r}{(n-2)}\right\}g(X,W) = 0. \tag{2.102}$$

Inserting $Y = W = e_i$ in equation (2.102) and taking summation over $1 \leq i \leq n$, we obtain

$$[a+(n-2)b]r[(n+1)A(X) + B(X)] = 0. \tag{2.103}$$

If $r \neq 0$ and $[a+(n-2)b] \neq 0$, then above equation gives $B(X) = -(n+1)A(X)$. Therefore, this led to the statement of the above theorem. □

Corollary 2.8. *In a Ricci symmetric $A(PSCS)_n, (n \geq 4)$ the scalar curvature vanishes if $[(n+1)A(X) + B(X)] \neq 0$, provided $[a+(n-2)b] \neq 0$.*

Theorem 2.24. *In a Ricci-recurrent $A(PSCS)_n, (n \geq 4)$, if the scalar curvature is non-zero and $[a+(n-2)b] \neq 0$, then $\tilde{H}(E) = 3A(E) + B(E)$, for all E.*

Proof. Equation (1.25) making use of (2.80) results in the following

$$[A(E)+B(E)]P(X,Y,W,V) + A(X)P(E,Y,W,V) + A(Y)P(X,E,W,V)$$
$$+ A(W)P(X,Y,E,V) + A(V)R(X,Y,W,E) = a(\nabla_E R)(X,Y,W,V)$$
$$- \frac{a}{(n-2)}\Big\{(\nabla_E S)(Y,W)g(X,V) - (\nabla_E S)(X,W)g(Y,V)$$
$$+ (\nabla_E S)(X,V)g(Y,W) - (\nabla_E S)(Y,V)g(X,W)\Big\}$$
$$- \frac{b\,dr(E)}{(n-1)}\Big\{g(Y,W)g(X,V) - g(X,W)g(Y,V)\Big\}. \tag{2.104}$$

Now, contracting the above equation yields

$$dr(E) = r\tilde{H}(E). \tag{2.105}$$

The use of equation (2.105) in equation (2.104) gives

$$\begin{aligned}
&[A(E)+B(E)]P(X,Y,W,V)+A(X)P(E,Y,W,V)+A(Y)P(X,E,W,V)\\
&+A(W)P(X,Y,E,V)+A(V)R(X,Y,W,E)=a(\nabla_E R)(X,Y,W,V)\\
&-\frac{a}{(n-2)}\Big\{S(Y,W)g(X,V)-S(X,W)g(Y,V)\\
&+S(X,V)g(Y,W)-S(Y,V)g(X,W)\Big\}H(E)\\
&-\frac{br\tilde{H}(E)}{(n-1)}\Big\{g(Y,W)g(X,V)-g(X,W)g(Y,V)\Big\}. \qquad (2.106)
\end{aligned}$$

Putting $X=V=e_i$ in equation (2.106), we get

$$\begin{aligned}
&[A(E)+B(E)]\Big\{-\frac{[a+(n-2)b]r}{(n-2)}\Big\}g(Y,W)+A(\tilde{P}(E,Y)W)\\
&-A(Y)\Big\{\frac{[a+(n-2)b]r}{(n-2)}\Big\}g(E,W)-A(W)\Big\{\frac{[a+(n-2)b]r}{(n-2)}\Big\}g(Y,E)\\
&-A(\tilde{P}(W,E)Y)=-r\Big\{\frac{[a+(n-2)b]}{(n-2)}\Big\}g(Y,W)\tilde{H}(E). \qquad (2.107)
\end{aligned}$$

Moreover, inserting $Y=W=e_i$ in equation (2.107), the above equation becomes

$$[(n+4)A(E)+nB(E)]=n\tilde{H}(E). \qquad (2.108)$$

Similarly, taking $E=Y=e_i$ in equation (2.107) gives,

$$(1+n)A(W)+B(W)=\tilde{H}(W), \qquad (2.109)$$

and replacing $W=E$ in the above equation, we get

$$(1+n)A(E)+B(E)=\tilde{H}(E). \qquad (2.110)$$

Again, contracting the equation (2.107) over E and W, we infer

$$(n+1)A(Y)+B(Y)=\tilde{H}(Y). \qquad (2.111)$$

Substituting $Y = E$ in equation (2.111) gives

$$(1 + n)A(E) + B(E) = \tilde{H}(E). \tag{2.112}$$

Combining equations (2.108),(2.110) and (2.112), we obtain

$$\tilde{H}(E) = 3A(E) + B(E). \tag{2.113}$$

Hence, $\tilde{H}(E) = 3A(E) + B(E)$ provided $r \neq 0$ and $[a + (n - 2)b] \neq 0$. □

2.3.4 Decomposition of $A(PSCS)_n, (n \geq 4)$

A Riemannian manifold (M^n, g) is said to be decomposable or a product manifold (Schouten, 1954) if it can be written as $M_1^p \times M_2^{n-p}$ for $2 \leq p \leq (n-2)$, that is, in some coordinate neighborhood of the Riemannian manifold (M^n, g) the metric can be expressed as

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ab}dx^a dx^b + g^*_{\alpha\beta}dx^\alpha dx^\beta, \tag{2.114}$$

where $\bar{g}_{ab}$ are functions of $x^1, x^2, ..., x^p$ denoted by $\bar{x}$ and $g^*_{\alpha\beta}$ are functions of $x^{p+1}, x^{p+2}, ..., x^n$ denoted by x^* : $a, b, c, ...$run from 1 to p and $\alpha, \beta, \gamma, ...,$ run from $p+1$ to n. In (2.114), $\bar{g}_{ab}$ and $g^*_{\alpha\beta}$ are the matrices of $M_1^p(p \geq 2)$ and $M_2^{n-p}(n-p \geq 2)$ respectively, which are called the components of the decomposable manifold $M^n = M_1^p \times M_2^{n-p}(2 \leq p \leq n-2)$.

We will assume throughout this section that all objects indicated by a 'bar' belong to M_1 and all objects indicated by a 'star' belongs to M_2.

Let $\bar{E}, \bar{X}, \bar{Y}, \bar{W}, \bar{V} \in \chi(M_1)$ and $E^*, X^*, Y^*, W^*, V^* \in \chi(M_2)$. Then in a decomposable Riemannian manifold $M^n = M_1^p \times M_2^{n-p}(2 \leq p \leq n-2)$, the following relations hold

$$R(E^*,\bar{X},\bar{Y},\bar{W}) = 0 = R(\bar{E},X^*,\bar{Y},W^*) = R(\bar{E},X^*,Y^*,W^*),$$
$$(\nabla_{E^*}R)(\bar{X},\bar{Y},\bar{W},\bar{V}) = 0 = (\nabla_{\bar{E}}R)(\bar{X},Y^*,\bar{W},V^*) = (\nabla_{E^*}R)(\bar{X},Y^*,\bar{W},V^*),$$
$$R(\bar{E},\bar{X},\bar{Y},\bar{W}) = \bar{R}(\bar{E},\bar{X},\bar{Y},\bar{W}); R(E^*,X^*,Y^*,W^*) = R^*(E^*,X^*,Y^*,W^*),$$
$$S(\bar{E},\bar{X}) = \bar{S}(\bar{E},\bar{X}); S(E^*,X^*) = S^*(E^*,X^*),$$
$$(\nabla_{\bar{E}}S)(\bar{X},\bar{Y}) = (\bar{\nabla}_{\bar{E}}S)(\bar{X},\bar{Y}); (\nabla_{E^*}S)(X^*,Y^*) = (\nabla^*_{E^*}S)(X^*,Y^*), \tag{2.115}$$

where $\bar{r}$,r^* and r are scalar curvature of M_1,M_2 and M respectively and are related as $r = \bar{r} + r^*$. Also $S(\bar{E},X^*) = 0$ and $g(\bar{E},X^*) = 0$.

Theorem 2.25. *Let an $A(PSCS)_n$ be a decomposable space such that $M^n = M_1^p \times M_2^{n-p}$ for $(2 \leq p \leq n-2)$, then the following holds:*
i) In the case of $A = B = 0$ on M_2, the manifold M_2 is Ricci symmetric and scalar curvature r^* is constant in M_2, provided $d\bar{r}(E^*) = 0$ and $\frac{a(n-p-2)}{(n-2)} \neq \frac{bp(n-p)}{(n-1)}$.
ii) when M_1 is semiconformally flat, then M_1 is an Einstein manifold.

Proof. Let us consider a Riemannian manifold (M^n, g) which is a decomposable $A(PSCS)_n$, then

$$M^n = M_1^p \times M_2^{n-p}(2 \leq p \leq n-2).$$

Now from equation (1.25), we obtain

$$
\begin{aligned}
P(X^*, \bar{Y}, \bar{W}, \bar{V}) &= 0 = P(\bar{X}, Y^*, W^*, V^*) \\
&= P(\bar{X}, Y^*, \bar{W}, \bar{V}) = P(\bar{X}, \bar{Y}, W^*, \bar{V}); \\
P(X^*, \bar{Y}, \bar{W}, V^*) &= -\frac{a}{(n-2)}\Big[S(\bar{Y}, \bar{W})g(X^*, W^*) + S(X^*, V^*)g(\bar{Y}, \bar{W})\Big] \\
&- \frac{rb}{(n-1)}\Big[g(\bar{Y}, \bar{W})g(X^*, V^*)\Big]; \\
P(X^*, Y^*, \bar{W}, \bar{V}) &= 0 = P(\bar{X}, \bar{Y}, W^*, V^*); \\
P(X^*, \bar{Y}, W*, \bar{V}) &= \frac{a}{(n-2)}\Big[S(\bar{Y}, \bar{V})g(X^*, W^*) + S(X^*, W^*)g(\bar{Y}, \bar{V})\Big] \\
&+ \frac{rb}{(n-1)}\Big[g(\bar{Y}, \bar{V})g(X^*, W^*)\Big].
\end{aligned}
$$

Further simplifying the above equation, we get

$$
\begin{aligned}
(\nabla_{\bar{E}}P)(\bar{X}, \bar{Y}, \bar{W}, \bar{V}) &= [A(\bar{E}) + B(\bar{E})]P(\bar{X}, \bar{Y}, \bar{W}, \bar{V}) \\
&+A(\bar{X})P(\bar{E}, \bar{Y}, \bar{W}, \bar{V}) + A(\bar{Y})P(\bar{X}, \bar{E}, \bar{W}, \bar{V}) \\
&+A(\bar{W})P(\bar{X}, \bar{Y}, \bar{E}, \bar{V}) + A(\bar{V})P(\bar{X}, \bar{Y}, \bar{W}, \bar{E}). \qquad (2.116)
\end{aligned}
$$

Putting $\bar{X} = X^*$ in equation (2.116) gives

$$A(X^*)P(\bar{E}, \bar{Y}, \bar{W}, \bar{V}) = 0. \qquad (2.117)$$

Also, inserting $\bar{E} = E^*$ in equation (2.116), we have

$$[A(E^*) + B(E^*)]P(\bar{X}, \bar{Y}, \bar{W}, \bar{V}) = 0. \qquad (2.118)$$

Similarly inserting $\bar{E} = E^*$ and $\bar{X} = X^*$ in equation (2.116), we infer

$$A(\bar{W})P(X^*, \bar{Y}, E^*, \bar{V}) + A(\bar{V})P(X^*, \bar{Y}, \bar{W}, E^*) = 0. \qquad (2.119)$$

Putting $\bar{E} = E^*$ and $\bar{W} = W^*$ in equation (2.116), we get

$$A(\bar{X})P(E^*, \bar{Y}, W^*, \bar{V}) + A(\bar{Y})P(\bar{X}, E^*, W^*, \bar{V}) = 0. \qquad (2.120)$$

And, taking $\bar{X} = X^*, \bar{Y} = Y^*$ and $\bar{W} = W^*$ in equation (2.116) results in

$$A(X^*)P(\bar{E}, Y^*, W^*, \bar{V}) + A(Y^*)P(X^*, \bar{E}, W^*, \bar{V}) = 0. \tag{2.121}$$

Substituting $\bar{Y} = Y^*, \bar{W} = W^*$ and $\bar{V} = V^*$ in equation (2.116), we have

$$A(W^*)P(\bar{X}, Y^*, \bar{E}, V^*) + A(V^*)P(\bar{X}, Y^*, W^*, \bar{E}) = 0. \tag{2.122}$$

Moreover, using equation (1.25) gives

$$\begin{aligned}(\nabla_{E^*}P)(X^*, Y^*, W^*, V^*) &= [A(E^*) + B(E^*)]P(X^*, Y^*, W^*, V^*)\\ &+A(X^*)P(E^*, Y^*, W^*, V^*) + A(Y^*)P(X^*, E^*, W^*, V^*)\\ &+A(W^*)P(X^*, Y^*, E^*, V^*) + A(V^*)P(X^*, Y^*, W^*, E^*).\end{aligned} \tag{2.123}$$

From equation (2.123), we obtain

$$[A(\bar{E} + B(\bar{E})]P(X^*, Y^*, W^*, V^*) = 0, \tag{2.124}$$

and,

$$A(\bar{X})P(E^*, Y^*, W^*, V^*) = 0. \tag{2.125}$$

Putting $\bar{E} = E^*, \bar{X} = X^*$ and $\bar{V} = V^*$ in equation (2.116) gives

$$\begin{aligned}(\nabla_{E^*}P)(X^*, \bar{Y}, \bar{W}, V^*) &= [A(E^*) + B(E^*)]P(X^*, \bar{Y}, \bar{W}, V^*)\\ &+A(X^*)P(E^*, \bar{Y}, \bar{W}, V^*) + A(V^*)P(X^*, \bar{Y}, \bar{W}, E^*).\end{aligned} \tag{2.126}$$

Similarly, putting $E^* = \bar{E}, X^* = \bar{X}$ and $V^* = \bar{V}$ in equation (2.123) gives

$$\begin{aligned}(\nabla_{\bar{E}}P)(\bar{X}, Y^*, W^*, \bar{V}) &= [A(\bar{E}) + B(\bar{E})]P(\bar{X}, Y^*, W^*, \bar{V})\\ &+A(\bar{X})P(\bar{E}, Y^*, W^*, \bar{V}) + A(\bar{V})P(\bar{X}, Y^*, W^*, \bar{E}).\end{aligned} \tag{2.127}$$

In regard to equations (2.117) and (2.118), we have the following two cases:

$$i) \quad A = B = 0 \,\text{on}\, M_2.$$

$$ii) \quad M_1 \,\text{is semiconformally flat.}$$

First, we consider the case (i). Then, equation (2.126) becomes

$$(\nabla_{E^*}P)(X^*,\bar{Y},\bar{W},V^*) = 0, \tag{2.128}$$

implies,

$$a(\nabla_{E^*}R)(X^*,\bar{Y},\bar{W},V^*) - \frac{a}{(n-2)}(\nabla_{E^*}S)(X^*,V^*)g(\bar{Y},\bar{W}) - \frac{b\,dr(E^*)}{(n-1)}g(\bar{Y},\bar{W})g(X^*,V^*) = 0. \tag{2.129}$$

Now, Putting $\bar{Y} = \bar{W} = \bar{e}_\alpha, 1 \leq \alpha \leq p$ in equation (2.129), we get

$$\frac{a(n-p-2)}{(n-2)}(\nabla_{E^*}S)(X^*,V^*) - \frac{b\,dr(E^*)}{(n-1)}pg(X^*,V^*) = 0. \tag{2.130}$$

Also, taking $X^* = V^* = e_i^*, p+1 \leq i \leq n$ in equation (2.130) gives

$$\frac{a(n-p-2)}{(n-2)}\,dr^*(E^*) - \frac{bp(n-p)}{(n-1)}\,dr(E^*) = 0. \tag{2.131}$$

If possible let $d\bar{r}(E^*) = 0$. The equation (2.131) becomes

$$\left[\frac{a(n-p-2)}{(n-2)} - \frac{bp(n-p)}{(n-1)}\right]dr^*(E^*) = 0. \tag{2.132}$$

Thus r^* is constant in M_2 provided, $\frac{a(n-p-2)}{(n-2)} \neq \frac{bp(n-p)}{(n-1)}$. Then from equation (2.130), we get

$$(\nabla_{E^*}S)(X^*,V^*) = 0.$$

Therefore, M_2 is Ricci symmetric.

Secondly, we will consider case (ii). Since M_1 is semiconformally flat, we get

$$aR(\bar{X},\bar{Y},\bar{W},\bar{V}) - \frac{a}{(n-2)}\Big[S(\bar{Y},\bar{W})g(\bar{X},\bar{V}) - S(\bar{X},\bar{W})g(\bar{Y},\bar{V}) + S(\bar{X},\bar{V})g(\bar{Y},\bar{W}) - S(\bar{Y},\bar{V})g(\bar{X},\bar{W})\Big] - \frac{br}{(n-1)}\Big[g(\bar{Y},\bar{W})g(\bar{X},\bar{V}) - g(\bar{X},\bar{W})g(\bar{Y},\bar{V})\Big] = 0. \quad (2.133)$$

Putting $\bar{X} = \bar{V} = \bar{e}_\alpha$ in equation (2.133), the above equation becomes

$$S(\bar{Y},\bar{W}) = \left[\frac{a\bar{r}(n-1) + br(p-1)(n-2)}{a(n-p-2)}\right]g(\bar{Y},\bar{W}). \quad (2.134)$$

Therefore, M_1 is an Einstein manifold.

Hence, this completes the theorem. □

Theorem 2.26. *Let an $A(PSCS)_n$ be a decomposable space such that $M^n = M_1^p \times M_2^{n-p}$ for $(2 \le p \le n-2)$, then the following holds:*

i) In the case of $A = B = 0$ on M_1, the manifold M_1 is Ricci symmetric and scalar curvature $\bar{r}$ is constant in M_1, provided $dr^*(\bar{E}) = 0$ and $\frac{a(p-2)}{(n-2)} \neq \frac{bp(n-p)}{(n-1)}$.

ii) when M_2 is semiconformally flat, then M_2 is an Einstein manifold.

Proof. Making use of equations (2.124) and (2.125), we get the following two cases:

$i)$ $A = B = 0$ on M_1.

$ii)$ M_2 is semiconformally flat.

Proceeding in a similar manner as in Theorem 6.1,

Hence, we will obtain the required result. □

Corollary 2.9. *If $A(PSCS)_n$ is a decomposable space such that $M^n = M_1^p \times M_2^{n-p}$ for $(2 \le p \le n-2)$, then one of the decomposed manifold is semiconformally flat while on other manifold both the associate 1-form A and B vanishes.*

2.3.5 Examples of $A(PSCS)_4$

In this subsection, we have constructed two examples of an $A(PSCS)_4$ on coordinate space $\mathbb{R}^4$ (with coordinates(x^1, x^2, x^3, x^4)) and obtain all the non-vanishing components of the curvature tensor, the Ricci tensor, the scalar curvature and the semiconformal curvature tensor along with its covariant derivatives. Then we verified the relation (2.80).

Example 2.3. *Let us consider a Riemannian metric g defined on 4-dimensional manifold $M^4 = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : x^1 \neq -1\}$ given by*

$$ds^2 = (x^1 + 1)(x^4)^2(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (dx^4)^2. \tag{2.135}$$

A similar Riemannian metric g is given by De and Gazi (2009).

Then the covariant and contravariant components of the metric are as follows

$$\begin{aligned} g_{11} = (x^1 + 1)(x^4)^2, g_{12} = g_{21} = 1, g_{33} = g_{44} = 1 \\ g^{11} = 0, g^{12} = g^{21} = 1, g^{33} = g^{44} = 1, g^{22} = -(x^1 + 1)(x^4)^2 \end{aligned} \tag{2.136}$$

All non-vanishing components of the Christoffel symbols and the curvature tensor in the considered metric are as follows:

$$\begin{aligned} \Gamma^4_{11} = -(x^1 + 1)(x^4), \Gamma^2_{11} &= \frac{1}{2}(x^4)^2, \Gamma^2_{14} = (x^1 + 1)(x^4) \\ R_{1441} &= (x^1 + 1) \end{aligned} \tag{2.137}$$

From equations (2.136) and (2.137), the non-vanishing components of Ricci tensor are

$$S_{11} = x^1 + 1. \tag{2.138}$$

The scalar curvature of the metric considered is given by,

$$r = 0. \tag{2.139}$$

The only non-vanishing components of the semiconformal curvature tensor are

$$P_{1441} = \frac{a}{2}(x^1 + 1) \neq 0. \tag{2.140}$$

Clearly, the only non-vanishing term of $\nabla_l P_{hijk}$ *are*

$$\nabla_1 P_{1441} = \frac{a}{2} \neq 0. \tag{2.141}$$

In term of the local coordinate system, let us define the components of the 1-form A and B as

$$A_i = \begin{cases} \dfrac{1}{6(x^1+1)} \, for \, i = 1 \\ 0, \;\; otherwise \end{cases}$$

and,

$$B_i = \begin{cases} \dfrac{1}{2(x^1+1)} \, for \, i = 1 \\ 0 \;\; , otherwise \end{cases} \tag{2.142}$$

at any point in M^4*.*

In (M^4, g) *the considered 1-form reduces the equation (2.80) in the following equations*

$$\nabla_1 P_{1441} = (3A_1 + B_1)P_{1441} + A_4 P_{1141} + A_4 P_{1411}. \tag{2.143}$$

$$\nabla_4 P_{1141} = [A_4 + B_4]P_{1141} + A_1 P_{4141} + A_1 P_{1441} + A_4 P_{1141} + A_1 P_{1144}. \tag{2.144}$$

$$\nabla_4 P_{1411} = [A_4 + B_4]P_{1411} + A_1 P_{4411} + A_4 P_{1411} + A_1 P_{1441} + A_1 P_{1414}. \tag{2.145}$$

In all other cases excluding (2.143),(2.144), and (2.145), the relation (2.80) either holds trivially or the components of each term vanish identically.

By (2.142), we get

$$\begin{aligned} RHS\ of\,(2.143) &= (3A_1 + B_1)P_{1441} + A_4P_{1141} + A_4P_{1411} \\ &= \left[\frac{3}{6(x^1+1)} + \frac{1}{2(x^1+1)}\right]\frac{a}{2}(x^1+1) \\ &= \frac{a}{4} + \frac{a}{4} \\ &= \frac{a}{2} \\ &= \nabla_1 P_{1441} \\ &= LHS\ of\,(2.143). \end{aligned} \tag{2.146}$$

By proceeding in a similar manner, it can be shown that the equations (2.144) and (2.145) are also true.
Thus, (M^4, g) *is an* $A(PSCS)_4$.

Example 2.4. *Let us consider a Riemannian metric g defined on 4-dimensional manifold* $M^4 = (x^1, x^2, x^3, x^4) \in \mathbb{R}^4$ *given by*

$$ds^2 = (1+2q)[(dx^1)^2 + (dx^2)^2] + (dx^3)^2 + (dx^4)^2, \tag{2.147}$$

where $q = \dfrac{e^{x^1}}{k^2}$, *where k is a non-zero constant.*
Then the covariant and contravariant components of the metric are as follows:

$$\begin{aligned} g_{11} = g_{22} = 1+2q,\ g_{33} = g_{44} = 1 \\ g^{11} = g^{22} = \frac{1}{1+2q},\ g^{33} = g^{44} = 1 \end{aligned} \tag{2.148}$$

All the non-vanishing components of the Christoffel symbols and the curvature tensor in the considered metric are

$$\begin{aligned} \Gamma^1_{11} = \Gamma^2_{12} = \frac{q}{1+2q},\ \Gamma^1_{22} = -\frac{q}{1+2q} \\ R_{1221} = \frac{q}{1+2q} \end{aligned} \tag{2.149}$$

By (2.148) and (2.149), the non-vanishing components of the Ricci tensor are

$$S_{11} = \frac{q}{(1+2q)^2}. \tag{2.150}$$

The Scalar curvature is given by

$$\begin{aligned} r = g^{ij}S_{ij} &= g^{11}S_{11} + g^{22}S_{22} + g^{33}S_{33} + g^{44}S_{44} \\ &= \frac{q}{(1+2q)^3}. \end{aligned} \tag{2.151}$$

The only non-vanishing components of semiconformal curvature tensors are

$$P_{1221} = \frac{q}{1+2q}\left\{\frac{a}{2} - \frac{b}{3}\right\}. \tag{2.152}$$

From equation (2.152), it can be shown that only non-zero terms of $\nabla_l P_{hijk}$ are

$$\nabla_1 P_{1221} = \frac{1}{(1+2q)^2}\left\{\frac{a}{2} - \frac{b}{3}\right\}, \tag{2.153}$$

and all other components of $\nabla_l P_{hijk}$ vanish identically.

In terms of the local coordinate system, let us consider the components of the 1-form A and B as

$$A_i = \begin{cases} \dfrac{1}{6q(1+2q)} \, for\, i = 1 \\ 0, \;\; otherwise \end{cases}$$

and,

$$B_i = \begin{cases} \dfrac{1}{2q(1+2q)} \, for\, i = 1 \\ 0 \;\; , otherwise \end{cases} \tag{2.154}$$

at any point in M^4.

In (M^4, g), the considered 1-form reduces equation (2.80) into the following equations

$$\nabla_1 P_{1221} = (3A_1 + B_1)P_{1221} + A_2 P_{1121} + A_2 P_{1211}. \tag{2.155}$$

$$\nabla_2 P_{1121} = (A_2 + B_2)P_{1121} + A_1 P_{2121} + A_1 P_{1221} + A_2 P_{1121} + A_1 P_{1122}. \quad (2.156)$$

$$\nabla_2 P_{1211} = [A_2 + B_2]P_{1211} + A_1 P_{2211} + A_2 P_{1211} + A_1 P_{1221} + A_1 P_{1212}. \quad (2.157)$$

The relation (2.80) either holds trivially or the components of each term vanishes identically excluding the above cases.
By (2.155) we get

$$\begin{aligned} RHS\ of\,(2.155) &= (3A_1 + B_1)P_{1221} + A_2 P_{1121} + A_2 P_{1211}. \\ &= \left[\frac{3}{6q(1+2q)} + \frac{1}{2q(1+2q)}\right]\frac{q}{(1+2q)}\left\{\frac{a}{2} - \frac{b}{3}\right\} \\ &= \frac{1}{(1+2q)^2}\left\{\frac{a}{2} - \frac{b}{3}\right\} \\ &= \nabla_1 P_{1221} \\ &= LHS\ of\,(2.155). \end{aligned} \quad (2.158)$$

By proceeding similarly it can be shown that the equations (2.156) and (2.157) also holds.
Thus, (M^4, g) *is an* $A(PSCS)_4$.

Chapter 3

Properties of Generalized m-quasi-Einstein Structure

Chapter 3

Properties of Generalized m-quasi-Einstein Structure

This chapter is divided into two sections. First section is devoted to the study of generalized m-quasi-Einstein metric on certain almost contact manifolds and in the section, almost Kenmotsu manifolds admitting generalized m-quasi-Einstein structure are considered.

3.1 Generalized m-quasi-Einstein metric on certain almost contact manifolds

Firstly, we will give some examples of generalized m-quasi-Einstein structures.

Example 3.1. *On a standard unit sphere* $(\mathbb{S}^n, g_0), n \geq 2$, *considering the function* $f = -m\ In(\tau - \frac{h_v}{n})$, *where* τ *is a real parameter lying in* $(1/n, +\infty)$ *and* h_v *is some height function. Then considering* $\lambda = (n-1) - m\frac{\tau - u}{u}$, *we find that* $(\mathbb{S}^n, g_0)$ *admits generalized m-quasi-Einstein metric. For details, see (Barros and Ribeiro, 2014).*

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Example 3.2. *On the Euclidean space* $(\mathbb{R}^n, g_0), n \geq 2$ *together with function* $f = -m\ In(\tau + |x|^2)$, *where* τ *is a positive real perimeter and* $|x|$ *is the Euclidean norm of* x, *we see that* $u = e^{-\frac{f}{m}} = \tau + |x|^2$ *and considering* $\lambda = -2\frac{m}{u}$, *it admits generalized m-quasi-Einstein structure (Barros and Ribeiro, 2014).*

Next, we will construct an example in a warped product manifold. Let us consider $M = \mathbb{R} \times_\sigma N^{n-1}$ with the product metric $g = dt^2 + \sigma^2(t)g_0$, where g_0 is a fixed metric in N^{n-1} and σ is a positive function on $\mathbb{R}$.

Example 3.3. *For a positive* $m \in \mathbb{R}$, *let us assume,*

$$f(x,t) = f(t) = m(t - e^t), \qquad \sigma(t) = e^{-t}$$

Inserting the value of σ *in Eq. 2.3, 2.4 (Wang, 2011) together with the assumption that* N^{n-1} *is a Ricci flat manifold we get*

$$S + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g,$$

where $\lambda = e^t(e^t + 2 - m) - n$. *Hence* M *admits generalized m-quasi-Einstein metric.*

Example 3.4. *Consider a Hyperbolic space* $\mathbb{H}^n(-1) \subset \mathbb{R}^{n+1} : \langle x, x\rangle_0 = -1,$. *Now, consider a height function* $h_v : \mathbb{H}^n(-1) \to \mathbb{R}$ *given by* $h_v(X) = \langle x, v\rangle_0$ *for a fixed point* $v \in \mathbb{H}^n(-1)$. *Let us assume* $u = e^{-\frac{f}{m}} = \tau + h_v, \quad \tau > -1$, *then* $\mathbb{H}^n(-1)$ *admits generalized m-quasi-Einstein metric for* $\lambda = -(n-1) - m\frac{\tau - u}{u}$. *For details, see (Barros and Ribeiro, 2014).*

Ghosh (2019a) on H-contact manifold proved, "*Let* $M^{2n+1}(\varphi, \xi, \eta, g)$ *be an* H*-contact manifold. If* g *represents an m-quasi-Einstein metric with non-zero potential vector field* V *collinear with* ξ, *then* M *is* K*-contact and* η*-Einstein.*" Generalizing this we prove the following result.

Theorem 3.1. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be an H-contact manifold. If g represents a generalized m-quasi-Einstein metric with non-zero potential vector field V collinear with ξ, then M is K-contact and η-Einstein. Moreover, λ is constant.*

Proof. A potential vector field V collinear with Reeb vector field ξ implies $V = \sigma\xi$, for some smooth function σ on M. Differentiating this along any $X \in \chi(M)$ we get

$$\nabla_X V = X(\sigma)\xi - \sigma(\varphi X + \varphi hX). \tag{3.1}$$

In consequence of (3.1), Eq. (1.50) reduces to the following

$$\begin{aligned} &X(\sigma)\eta(Y) + Y(\sigma)\eta(X) - 2\sigma g(\varphi hX, Y) \\ &+2S(X,Y) - \frac{2}{m}\sigma^2\eta(X)\eta(Y) = 2\lambda g(X,Y), \end{aligned} \tag{3.2}$$

for any $X, Y \in \chi(M)$. Replacing X and Y by ξ in (3.2) and using (1.13) yields

$$\xi\sigma + Tr.\ell - \frac{\sigma^2}{m} = \lambda. \tag{3.3}$$

Putting $Y = \xi$ in (3.2) and using (3.3) we obtain

$$Q\xi - (Tr.\ell)\xi = -\frac{1}{2}\{D\sigma - (\xi\sigma)\xi\}. \tag{3.4}$$

Moreover, contracting (3.2) we obtain the following result

$$\xi\sigma + r - \frac{\sigma^2}{m} = (2n+1)\lambda. \tag{3.5}$$

By hypothesis, H-contactness implies ξ is an eigenvector of the Ricci operator at each point of M i.e. $Q\xi = (Tr.\ell)\xi$. Making use of this in (3.4), we get $D\sigma = (\xi\sigma)\xi$. By Lemma 1 (Patra, 2021), σ is constant on M. Then (3.2) reduces to

$$QX = -\sigma h\varphi X + \frac{\sigma^2}{m}\eta(X)\xi + \lambda X, \tag{3.6}$$

for any $X \in \chi(M)$. Differentiating (3.6) along arbitrary $Y \in \chi(M)$ and using (1.11) we obtain

$$\begin{aligned}(\nabla_Y Q)X &= -\sigma(\nabla_Y h\varphi)X - \frac{\sigma^2}{m}[g(X, \varphi Y + \varphi hY)\xi \\ &+ \eta(X)(\varphi X + \varphi hY)] + (Y\lambda)X. \end{aligned} \tag{3.7}$$

Contracting (3.7) over Y and making use of (1.12) gives

$$\frac{1}{2}Xr = -\sigma(divh\varphi)X + (X\lambda). \tag{3.8}$$

Recalling that for any contact metric manifold $div(\varphi h)X = 2n\eta(X) - g(Q\xi, X)$. By hypothesis, since $Q\xi = Tr.\ell\xi$, we get $div(\varphi h)X = (2n - Tr.\ell)\eta(X)$. Applying this in the forgoing eq. (3.8) infers

$$\frac{1}{2}Xr = \sigma(2n - Tr.\ell)\eta(X) + (X\lambda). \tag{3.9}$$

Also differentiating (3.5) along $X \in \chi(M)$ gives $Xr = (2n+1)(X\lambda)$. Using this in (3.9) and replacing X by φX gives $g(\varphi X, D\lambda) = 0$, which implies $D\lambda = (\xi\lambda)\xi$. Then by Lemma 1 (Patra, 2021), we have λ is constant and hence $Xr = 0$ i.e. r is constant on M. In consequence of this (3.9) reduces to $\sigma(2n - Tr.\ell) = 0$. Thus either $\sigma = 0$ or $Tr.\ell = 2n$. Since V is non-zero implies $\sigma \neq 0$. Hence, $Tr.\ell = 2n$ which implies the manifold is K-contact. From (3.6) we see that m is η-Einstein i.e. $QX = \lambda X + \frac{\sigma^2}{m}\eta(X)\xi$, where $\frac{\sigma^2}{m} = \lambda - 2n$. This completes the proof. □

Boyer and Galicki (2001) studied Einstein K-contact and η-Einstein K-contact manifolds. In particular, they proved that a *compact Einstein K-contact is Sasakian*. This is also true for compact η-Einstein ($S = \alpha g + \beta\eta \otimes \eta$ for constant α, β) K-contact with $\alpha > -2$. These results are also valid if one relaxes compactness by completeness (Sharma, 2008). Because of the above theorem and the Boyer-Galicki result, we can state the following:

Corollary 3.1. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a complete H-contact manifold. If g ad-*

mits shrinking generalized m-quasi-Einstein metric with non-zero potential vector field V collinear with ξ then M is compact Sasakian and η-Einstein.

Replacing H-contactness by a compact contact metric manifold and generalizing Theorem 3 (Rovenski and Patra, 2021) we prove the following result.

Theorem 3.2. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a complete contact metric manifold. If g admits a generalized m-quasi-Einstein metric with non-zero potential vector field collinear with ξ and $||\nabla(\sigma^2) - \frac{4}{3m}\sigma^2 V + 2(2n-1)\sigma\xi\lambda||_g \in L^1(M, g)$ then M is K-contact and η-Einstein.*

Proof. By our assumption $V = \sigma\xi$ and hence Eq. (3.1)-(3.5) are valid. Making use of (3.1) generalized m-quasi-Einstein equation becomes

$$QX + \frac{1}{2}[g(X, D\sigma)\xi + \eta(X)D\sigma] + \sigma h\varphi X = \lambda X + \frac{\sigma^2}{m}\eta(X)\xi. \tag{3.10}$$

Differentiate (3.10) along arbitrary $Y \in \chi(M)$ then contracting the obtained result along Y and taking $X = \xi$ together with $div(\varphi h)\xi = ||h||^2$ we get

$$\frac{1}{2}\{\xi r + \xi(\xi\sigma) + divD\sigma\} - \sigma||h||^2 = \frac{2}{m}\sigma(\xi\sigma) + \xi\lambda. \tag{3.11}$$

Differentiating (3.5) along ξ yields

$$\xi r = (2n+1)(\xi\lambda) + \frac{2\sigma}{m}(\xi\sigma) - \xi(\xi\sigma). \tag{3.12}$$

Using convention $divD\sigma = -\Delta\sigma$ and combining (3.11) and (3.12) we obtain

$$\frac{1}{2}\Delta\sigma + \sigma||h||^2 + \frac{\sigma}{m}(\xi\sigma) = \frac{1}{2}(2n-1)(\xi\lambda). \tag{3.13}$$

In contact metric manifold $div\xi = 0$ and hence $g(D\sigma, \xi) = \xi\sigma = divV$. Now contracting the well-known formula $\nabla_X(\sigma^2 V) = X(\sigma^2)V + \sigma^2(\nabla_X V)$ over X

gives

$$div(\sigma^2 V) = g(\nabla\sigma^2, V) + \sigma^2 divV = 3\sigma^2\xi(\sigma). \tag{3.14}$$

Multiplying (3.13) by σ and using (3.14) and $(\Delta\sigma)\sigma = \frac{1}{2}\Delta(\sigma^2)+||\Delta\sigma||^2$ we obtain the following relation

$$div(\nabla(\sigma^2) - \frac{4}{3m}\sigma^2 V + 2(2n-1)\sigma\xi\lambda) = 4\sigma^2||h||^2 + 2||\nabla\sigma||^2, \tag{3.15}$$

Here we have used the fact that $div(\xi\lambda) = \lambda div\xi + \xi(\lambda)$. Applying Proposition 1 (Caminha et al., 2010), the foregoing equation (3.15) infers

$$2\sigma^2||h||^2 + ||\nabla\sigma||^2 = 0. \tag{3.16}$$

This implies $\nabla\sigma = 0$ and $h = 0$, hence M is K-contact and σ is constant. Moreover, from (3.10) it is η-Einstein. This completes the proof. □

Using a similar argument as in Corollary 3.1, we can state the following:

Corollary 3.2. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a complete contact metric manifold. If g admits shrinking generalized m-quasi-Einstein metric with non-zero potential vector field collinear with ξ and $||\nabla(\sigma^2) - \frac{4}{3m}\sigma^2 V + 2(2n-1)\sigma\xi\lambda||_g \in L^1(M,g)$ then M is compact Sasakian and η-Einstein.*

Theorem 3.3. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a complete K-contact manifold. If g admits a closed generalized m-quasi-Einstein metric whose potential vector field is contact then M is compact, Einstein and Sasakian. Moreover, V is strict and λ is constant.*

Proof. Taking the exterior derivative of (1.41) and by properties of Lie-derivative we obtain

$$\begin{aligned}(\mathcal{L}_V d\eta)(X,Y) &= d(\mathcal{L}_V\eta)(X,Y) \\ &= \frac{1}{2}[X(\varrho)\eta(Y) - Y(\varrho)\eta(X)] + \varrho d\eta(X,Y),\end{aligned} \tag{3.17}$$

for any $X, Y \in \chi(M)$. Taking the Lie-derivative of $d\eta(X,Y) = g(X, \varphi Y)$ along V and using (3.17) gives

$$(\mathcal{L}_V \varphi)Y = \frac{1}{2}[D\varrho\eta(Y) - Y(\varrho)\xi] + \varrho\varphi Y \\ -\frac{2}{m}V^b(\varphi Y)V + 2Q\varphi Y - 2\lambda\varphi Y. \tag{3.18}$$

Replacing Y by ξ in a generalized m-quasi-Einstein equation becomes

$$(\mathcal{L}_V g)(X,\xi) = \frac{2}{m}V^b(X)\eta(V) - 4n\eta(X) + 2\lambda\eta(X), \tag{3.19}$$

for any $X \in \chi(M)$. Combining the forgoing equation and (1.41) on the Lie-derivative of $\eta(X) = g(X,\xi)$ yields

$$g(X, \mathcal{L}_V\xi) = (\varrho + 4n - 2\lambda)\eta(X) - \frac{2}{m}V^b(X)\eta(V), \tag{3.20}$$

for all $X \in \chi(M)$. Replacing Y by ξ in (3.18) and making use of the fact that $\varphi\xi = 0$ implies $(\mathcal{L}_V\varphi)\xi = 0$ we obtain $D\varrho = \xi(\varrho)\xi$. By Lemma 1 (Patra, 2021), we see that ϱ is constant. As a consequence of this (3.18) becomes

$$(\mathcal{L}_V \varphi)Y = \varrho\varphi Y - \frac{2}{m}V^b(\varphi Y)V + 2Q\varphi Y - 2\lambda\varphi Y. \tag{3.21}$$

On the other hand, taking Lie-derivative of $g(\xi,\xi) = 1$ and using (3.19) we get

$$\lambda = 2n + \varrho - \frac{1}{m}\eta(V)\eta(V). \tag{3.22}$$

Now taking Lie-derivative of (1.9) along V we obtain

$$(\mathcal{L}_V\varphi)\varphi X + \varphi(\mathcal{L}_V\varphi)X = (\mathcal{L}_V\eta)(X)\xi + \eta(X)\mathcal{L}_V\xi, \tag{3.23}$$

for all $X \in \chi(M)$. Making use of (1.41), (3.20) and (3.21) in (3.23) infers

$$(2\lambda - \varrho)X + \frac{1}{m}[V^b(X)V - V^b(\varphi X)\varphi V] \\ -QX + \varphi Q\varphi X - \lambda\eta(X)\xi = 0. \tag{3.24}$$

Replacing X by ξ in (3.24) and inserting (3.22) we get $\eta(V)[V - \eta(V)\xi] = 0$ which implies $V = \eta(V)\xi$ or $\eta(V) = 0$ i.e. $V = 0$. Assume $V \neq 0$, then taking derivative of $V = \eta(V)\xi$ along arbitrary $X \in \chi(M)$ and using (1.27) gives $\nabla_X V = g(\nabla_X V, \xi) - \eta(V)\varphi X$, which implies

$$dV^b(X,Y) = 2\eta(V)g(X,\varphi Y) + g(\nabla_X V, \xi)\eta(Y) - g(\nabla_Y V, \xi)\eta(X).$$

Replacing X by φX and Y by φY in the forgoing equation and using the fact that V^b is closed we get $\eta(V)d\eta(X,Y) = 0$. Since $d\eta$ is non-vanishing everywhere on M implies $\eta(V) = 0$, a contradiction. Hence $V = 0$, consequently M is Einstein i.e. $QX = \lambda X$. Making use of this in (3.24) shows $\varrho = 0$. Then (3.22) implies M is Einstein with Einstein constant $2n$. Suppose M is complete. Since M is complete Einstein by Myer's theorem (Myers, 1935) it is compact. Finally, applying the Boyer-Gallicki (2001) theorem we can conclude that M is Sasakian. This completes the proof. □

Finally, we studied the generalized m-quasi-Einstein metric in the framework of 3-dimensional normal almost contact metric manifold and prove the following result.

Theorem 3.4. *If a 3-dimensional normal almost contact metric manifold with β =constant admits a generalized m-quasi-Einstein metric whose non-zero potential vector field is collinear with ξ then M^3 is either η-Einstein, β-Kenmotsu or locally the product of a Kähler manifold and an interval or unit circle S^1.*

Proof. In a 3-dimensional Riemannian manifold the curvature tensor is given by (Blair, 2010)

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X \\ & -g(QX,Z)Y - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{3.25}$$

By our hypothesis, $V = \sigma\xi$, for some smooth σ. Differentiating this and using

(1.16), the generalized m-quasi-Einstein equation becomes

$$QX = (\sigma\beta + \frac{\sigma^2}{m})\eta(X)\xi + (\lambda - \sigma\beta)X - \frac{1}{2}[\eta(X)D\sigma + (X\sigma)\xi]. \tag{3.26}$$

Inserting (3.26) in (3.25) and replacing Z by ξ gives

$$\begin{aligned} R(X,Y)\xi = \frac{1}{2}[(Y\sigma)\eta(X)\xi - (X\sigma)\eta(Y)\xi] + \frac{1}{2}[(X\sigma)Y - (Y\sigma)X] \\ +(\frac{\sigma^2}{m} - \frac{\xi\sigma}{2} + 2\lambda - \sigma\beta - \frac{r}{2})[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{3.27}$$

Replacing X by φX and Y by φY in (3.27) we obtain

$$\varphi X(\sigma)\varphi Y = \varphi Y(\sigma)\varphi X. \tag{3.28}$$

Taking $X = D\sigma$ in (3.28) gives $\varphi Y(\sigma)\varphi D\sigma = 0$ which implies $D\sigma = \xi(\sigma)\xi$. Differentiating forgoing equation along any $X \in \chi(M)$ infers

$$(\nabla_X D\sigma) = X(\xi\sigma)\xi - \alpha(\xi\sigma)\varphi X + \beta[X - \eta(X)\xi](\xi\sigma). \tag{3.29}$$

Making use of the fact that $g(\nabla_X D\sigma, Y) = g(\nabla_Y D\sigma, X)$ from (3.29) we get

$$X(\xi\sigma)\eta(Y) - Y(\xi\sigma)\eta(X) - 2\alpha(\xi\sigma)g(\varphi X, Y) = 0. \tag{3.30}$$

Choosing $X, Y \perp \xi$ above equation reduces to $\alpha(\xi\sigma) = 0$. Therefore, either $\alpha = 0$ or $\xi\sigma = 0$. If $\alpha = 0$ then M is either β-Kenmotsu (for $\beta \neq 0$) or cosympletic manifold (for $\beta = 0$). Assuming the next case when $\xi\sigma = 0$, implies $D\sigma = 0$ and hence σ is constant. In consequence, from (3.26) we see that M is η-Einstein. This completes the proof. □

Replacing X by ξ in (3.26) and differentiating it along any $Y \in \chi(M)$ results in

$$\begin{aligned} (\nabla_Y Q)\xi = (\lambda + \frac{\sigma^2}{m})\nabla_Y\xi + Y(\lambda + \frac{\sigma^2}{m})\xi \\ -\frac{1}{2}[(\nabla_Y D\sigma) + Y(\xi\sigma)\xi + (\xi\sigma)(\nabla_Y\xi)]. \end{aligned} \tag{3.31}$$

Contracting the foregoing (3.31) yields

$$\frac{1}{2}\xi r = 2\beta(\lambda + \frac{\sigma^2}{m}) + \xi\lambda + \frac{2\sigma}{m}(\xi\sigma) - \frac{1}{2}[\Delta\sigma + \xi(\xi\sigma) + 2\beta(\xi\sigma)]. \qquad (3.32)$$

Contracting (3.26) and then differentiating the obtained result by ξ and finally inserting it in (3.32) we obtain

$$\frac{1}{2}\Delta\sigma = (\xi\lambda) + (\frac{\sigma}{m} + \beta)(\xi\sigma) + 2\beta(\lambda + \frac{\sigma^2}{m} + \alpha\sigma). \qquad (3.33)$$

For the case when $\alpha = 0$ and β a non-zero constant, M is β-Kenmotsu manifold. In a β-Kenmotsu manifold we have $Q\xi = -2\beta^2\xi$. Replacing X by ξ in (3.26) and using the forgoing equation along with $D\sigma = (\xi\sigma)\xi$ infers

$$\xi\sigma = \lambda + \frac{\sigma^2}{m} + 2\beta^2. \qquad (3.34)$$

Making use of the fact $\Delta\sigma = div(D\sigma) = \xi(\xi\sigma) + 2\beta(\xi\sigma)$ and inserting (3.34) we get

$$\Delta\sigma = \xi\lambda + 2(\beta + \frac{\sigma}{m})(\xi\sigma). \qquad (3.35)$$

Combining (3.35) and (3.33) infers

$$\xi\lambda = -4\beta(\lambda + \frac{\sigma^2}{m}). \qquad (3.36)$$

Now, for the second case when σ is constant, Eq. (3.33) gives

$$\xi\lambda = -2\beta(\lambda + \frac{\sigma^2}{m} + \alpha\sigma). \qquad (3.37)$$

Choosing λ as constant, Eq. (3.36) implies either $\beta = 0$ or $\lambda = -\frac{\sigma^2}{m}$. Assume $\beta \neq 0$ then σ is constant. Therefore, inserting the value of λ in (3.34) shows $\beta = 0$, a contradiction. Hence, $\beta = 0$ and M is cosympletic. In the second case, (1.13) implies either $\beta = 0$ or $\lambda + \frac{\sigma^2}{m} + \alpha\sigma = 0$. Fix $\beta \neq 0$ then it is obvious that α is a non-zero constant. Therefore M is α-Sasakian manifold and hence has constant scalar curvature. Hence we can state the following:

Corollary 3.3. *If a 3-dimensional normal almost contact metric manifold with β =constant admits m-quasi-Einstein metric whose potential vector field is collinear with ξ then M^3 is locally the product of a Kähler manifold and an interval or unit circle S^1 or has constant scalar curvature. Moreover, σ is constant.*

3.2 Generalized m-quasi-Einstein structure in almost Kenmotsu manifolds

The goal of this section is to analyze the generalized m-quasi-Einstein structure in the context of almost Kenmotsu manifolds. Firstly we showed that a complete Kenmotsu manifold admitting a generalized m-quasi-Einstein structure (g, f, m, λ) is locally isometric to a hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or a warped product $\widetilde{M} \times_\gamma \mathbb{R}$ under certain conditions. Next, we proved that a $(\kappa, \mu)'$-almost Kenmotsu manifold with $h' \neq 0$ admitting a closed generalized m-quasi-Einstein metric is locally isometric to some warped product spaces. Finally, generalized m-quasi-Einstein metric (g, f, m, λ) in almost Kenmotsu 3-H-manifold is considered and proved that either it is locally isometric to the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$.

3.2.1 On Kenmotsu manifold

Firstly, we construct some examples of the Kenmotsu manifold admitting generalized m-quasi-Einstein metric.

Example 3.5. *Let (N, J, g_0) be a Kähler manifold of dimension $2n$. Consider the warped product $(M, g) = (\mathbb{R} \times_\sigma N, dt^2 + \sigma^2 g_0)$, where t is the coordinate on $\mathbb{R}$. We set $\eta = dt, \xi = \frac{\partial}{\partial t}$ and the tensor field φ is defined on $\mathbb{R} \times_\sigma N$ by $\varphi X = JX$ for vector field X on N and $\varphi X = 0$ if X is tangent to $\mathbb{R}$. Then the warped product $\mathbb{R} \times_\sigma N, \sigma^2 = ce^{2t}$ with the structure (φ, ξ, η, g) is a Kenmotsu manifold*

(Kenmotsu, 1972). In particular, if we take $N = \mathbb{CH}^{2n}$*, then* N *being Einstein, the Ricci tensor of* M *becomes* $S = -2ng$*. Further we define a smooth function* $f(t) = ke^t, k > 0$*. Then it is easy to verify that* (M, f, g, λ) *is a generalized* m*-quasi-Einstein with* $\lambda = \frac{ke^t}{m}(m - ke^t) - 2n$ *on* $\mathbb{R} \times_\sigma \mathbb{CH}^{2n}$*.*

Similarly, a large group of examples of generalized m-quasi-Einstein metric on the Kenmotsu manifold can be constructed by taking different potential functions on the warped product.

Example 3.6. *Consider the warped product* $\mathbb{R} \times_\sigma \mathbb{H}^n$ *with metric* $g = dt^2 + \sigma^2 g_0$ *where* g_0 *is the standard metric on the hyperbolic space* $\mathbb{H}^n$ *(Ghosh, 2019b). Let* $\sigma(t) = cosht$*, then the warped product becomes Einstein manifold with Ricci tensor* $S = -ng$ *and it admits a generalized* m*-quasi-Einstein structure* $(\mathbb{R} \times_\sigma \mathbb{H}^n, f, g, \lambda)$ *with* $f(x,t) = sinht$ *and* $\lambda(x,t) = sinht - \frac{cosh^2 t}{m} - n$*.*

Example 3.7. *Let* $M^{2n+1} = \mathbb{R} \times_{cosht} \mathbb{CH}^{2n}$ *with metric* $g = dt^2 + (cosh^2 t) g_0$*, where* g_0 *is the standard metric on the complex hyperbolic space* $\mathbb{CH}^{2n}$ *(Ghosh, 2019b). Then* M^{2n+1} *becomes Einstein manifold with the Ricci tensor* $S^M = -2ng$ *(see Lemma 1.1 (Pigola et al., 2011)). Consider a function* $f(x,t) = sinht$*, then* $(M^{2n+1}, f, g, \lambda)$ *is a generalized* m*-quasi-Einstein structure if* $\lambda = sinht - \frac{cosh^2 t}{m} - 2n$*.*

Next, we state and proved the following result:

Theorem 3.5. *If the metric of a Kenmotsu manifold* $M^{2n+1}(\varphi, \xi, \eta, g)$ *represents a generalized* m*-quasi-Einstein structure* (g, f, m, λ)*, then it is* η*-Einstein, provided* $1 + \frac{\xi(f)}{m} \neq 0$*. Moreover, if* M^{2n+1} *is complete and Reeb vector field* ξ *leaves the scalar curvature invariant, then we have*

1. *If* f *has a critical point, then* M *is isometric to the hyperbolic space* $\mathbb{H}^{2n+1}(-1)$*.*

2. If f is without critical points, then M is isometric to the warped product $\widetilde{M} \times_\gamma \mathbb{R}$ of a complete Riemannian manifold $\widetilde{M}^{2n}$ and the real line $\mathbb{R}$ with warped function $\gamma : \mathbb{R} \to \mathbb{R}$ such that $\ddot{\gamma} - \gamma = 0, \gamma > 0$.

Proof. From (1.50), we have

$$\nabla_X Df = \lambda X + \frac{1}{m} g(X, Df)Df - QX. \tag{3.38}$$

Taking the covariant derivative of (3.38) along arbitrary vector field Y we get

$$\begin{aligned} \nabla_Y \nabla_X Df = (Y\lambda)X + \lambda(\nabla_Y X) + \frac{1}{m}\{g(X, \nabla_Y Df)Df \\ +g(X, Df)(\nabla_Y Df)\} - (\nabla_Y Q)X - Q(\nabla_Y X). \end{aligned} \tag{3.39}$$

Making use of (3.38) and (3.39) in the relation $R(X,Y)Df = \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df$ we obtain

$$\begin{aligned} R(X,Y)Df = (X\lambda)Y - (Y\lambda)X + (\nabla_Y Q)X - (\nabla_X Q)Y \\ +\frac{\lambda}{m}[g(Y, Df)X - g(X, Df)Y] + \frac{1}{m}[g(X, Df)QY - g(Y, Df)QX]. \end{aligned} \tag{3.40}$$

Taking an inner product of (3.40) with ξ and using (1.30) yields

$$\begin{aligned} g(R(X,Y)Df, \xi) = (X\lambda)\eta(Y) - (Y\lambda)\eta(X) + g((\nabla_Y Q)\xi, X) \\ -g((\nabla_X Q)\xi, Y) + \frac{(\lambda + 2n)}{m}[g(Y, Df)\eta(X) - g(X, Df)\eta(Y)]. \end{aligned} \tag{3.41}$$

Taking an inner product of (1.29) with Df and inserting it in the last equation (3.41) we obtain

$$\begin{aligned} (X\lambda)\eta(Y) - (Y\lambda)\eta(X) + g((\nabla_Y Q)\xi, X) - g((\nabla_X Q)\xi, Y) \\ +\frac{(\lambda + 2n + m)}{m}[g(Y, Df)\eta(X) - g(X, Df)\eta(Y)] = 0. \end{aligned} \tag{3.42}$$

Replacing Y by ξ in (3.42) and making use of the relation $(\nabla_\xi Q)Y = -2QY - 4nY$

(see Lemma 2 (Ghosh, 2019b)) we get

$$\sigma Df - mD\lambda = \{\sigma(\xi f) - m(\xi\lambda)\}\xi, \tag{3.43}$$

where $\sigma = m + \lambda + 2n$. Contracting (3.40) along arbitrary vector field X gives

$$\frac{(m-1)}{m}S(Y, Df) = \frac{1}{2}(Yr) - 2n(Y\lambda) + \frac{1}{m}(2n\lambda - r)g(Y, Df). \tag{3.44}$$

Replacing Y by ξ and using (1.30) in (3.44) we get

$$\frac{1}{m}(2n\sigma - 4n^2 - r - 2n)(\xi f) - 2n(\xi\lambda) + \frac{1}{2}(\xi r) = 0. \tag{3.45}$$

Also on the Kenmotsu manifold, we have $\xi r = -2(r + 2n(2n+1))$ (Lemma 2 (Ghosh, 2019b)). Inserting this in the last equation infer

$$\frac{2n}{m}[\sigma(\xi f) - m(\xi\lambda)] = \{r + 2n(2n+1)\}\{1 + \frac{(\xi f)}{m}\}. \tag{3.46}$$

Replacing Y by ξ in (3.40) and using the relation $R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X$, we obtain

$$\begin{aligned}\frac{1}{m}g(X, \sigma Df - mD\lambda)\xi = {} & \frac{(\sigma - 2n)}{m}(\xi f)X \\ & -(\xi\lambda)X - (1 + \frac{\xi f}{m})QX - 2nX.\end{aligned} \tag{3.47}$$

Combining (3.43), (3.46) and (3.47) we obtain the following relation

$$(1 + \frac{\xi f}{m})(\frac{r}{2n} + 2n + 1)\eta(X)\xi = (1 + \frac{\xi f}{m})\{(\frac{r}{2n} + 1)X - QX\}. \tag{3.48}$$

If possible take $1 + \frac{\xi f}{m} \neq 0$. Then from the last equation we get

$$QX = (1 + \frac{r}{2n})X - (\frac{r}{2n} + 2n + 1)\eta(X)\xi, \tag{3.49}$$

for any vector field X on M. Therefore, M is η-Einstein.

Suppose that ξ leaves the scalar curvature r invariant i.e., $\xi r = 0$. Consequently, $r = -2n(2n+1)$. By virtue of this in (3.49) we get $QX = -2nX$, i.e.,

M is Einstein. Inserting $r = -2n(2n+1)$ in (3.46) gives $\sigma(\xi f) - m(\xi\lambda) = 0$ and hence (3.43) implies $D\lambda = \frac{\sigma}{m}Df$. Now we consider a function $u = e^{-\frac{f}{m}}$ on M. Then it follows $Du = -\frac{u}{m}Df$. Taking covariant derivative of the forgoing expression along arbitrary vector field X we get

$$\nabla_X Df - \frac{1}{m}g(X, Df)Df = -\frac{m}{u}\nabla_X Du. \tag{3.50}$$

Using (3.50) along with the fact that $QX = -2nX$, (3.38) yields

$$\nabla_X Du = -\frac{(\lambda + 2n)u}{m}X. \tag{3.51}$$

Also we have $(\lambda+m+2n)Df = mD\lambda$, simplifying it gives $D(\lambda u) = -(m+2n)Du$ which implies $\lambda u = -(m+2n)u + k$, k is a constant. Inserting the forgoing relations in (3.51) we get

$$\nabla_X Du = (u - \frac{k}{m})X.$$

Applying Kanai's theorem (Kanai, 1983), we conclude that if f has a critical point then M is isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or if f is without critical points then M is isometric to the warped product $\widetilde{M} \times_\gamma \mathbb{R}$ of a complete Riemannian manifold $\widetilde{M}^{2n}$ and the real line $\mathbb{R}$ with warped function $\gamma : \mathbb{R} \to \mathbb{R}$ such that $\ddot{\gamma} - \gamma = 0, \gamma > 0$. □

Remark 3.1. *Suppose* $1+\frac{\xi f}{m} = 0$ *in some open set* $\mathcal{O}$ *of* M. *Then* $\xi f = -m$, *since the Kenmotsu manifold is locally isometric to the warped product* $(-\epsilon, \epsilon) \times_{ce^t} N$, *where* N *is a Kähler manifold of dimension* $2n$ *and* $(-\epsilon, \epsilon)$ *is an open interval (Kenmotsu, 1972). Using the local parametrization:* $\xi = \frac{\partial}{\partial t}$ *then we have* $\frac{\partial f}{\partial t} = -m$ *hence the potential function is* $f = -mt, t > 0$.

Theorem 3.6. *If a Kenmotsu manifold admits a non-trivial generalized m-quasi-Einstein structure* (g, V, m, λ) *whose potential vector field is pointwise collinear with the Reeb vector field* ξ *then it is* η*-Einstein.*

Proof. Suppose potential vector field V is pointwise collinear with the Reeb vector field ξ then $V = F\xi$, where F is a smooth function. Differentiating covariantly along arbitrary vector field X of $V = F\xi$ and using (1.27) we get

$$\nabla_X V = (XF)\xi + F(-\varphi^2 X - \varphi hX). \tag{3.52}$$

Inserting (3.52) in (1.50) gives

$$S(X,Y) + \frac{1}{2}[(XF)\eta(Y) + (YF)\eta(X)] + Fg(h'X,Y) \\ -(\frac{F^2}{m} + F)\eta(X)\eta(Y) = (\lambda - F)g(X,Y), \tag{3.53}$$

for all vector fields X, Y. Replacing X, Y by ξ in (3.53) and using (1.9) we get $\xi F = \lambda + 2n + \frac{F^2}{m}$. Further taking Y as ξ and using the last expression in (3.53) we obtain

$$XF = (\lambda + \frac{F^2}{m} + 2n)\eta(X). \tag{3.54}$$

Contracting (3.53) and inserting in the above equation (3.54), yields

$$r = 2n(\lambda - F - 1). \tag{3.55}$$

In consequence of (3.54) and (3.55) in (3.53) gives

$$QX = (\frac{r}{2n} + 1)X - (\frac{r}{2n} + 2n + 1)\eta(X)\xi, \tag{3.56}$$

for any vector field X. Thus manifold is η-Einstein. This completes the proof. □

Suppose F is constant, then (3.54) gives $\lambda = -2n - \frac{F^2}{m}$. This in (3.55) implies r is constant. Hence $\xi r = 0$ which implies $r = -2n(2n+1)$. Inserting the values of r and λ in (3.55) gives $F = -m$ which further implies $\lambda = -m - 2n$. Hence we can state the following:

Corollary 3.4. *If a Kenmotsu manifold admits a non-trivial generalized m-quasi-Einstein structure (g, V, m, λ) whose potential vector is a constant multiple of Reeb* *vector field* ξ *then it is Einstein i.e.,* $QX = -2nX$ *with* $\lambda = -m - 2n$.

3.2.2 On almost Kenmotsu manifolds

Lemma 3.1 (Wang and Liu, 2016a). *Let* $M^{2n+1}(\varphi, \xi, \eta, g)$ *be a generalized* $(\kappa, \mu)'$*-almost Kenmotsu manifold with* $h' \neq 0$. *For* $n > 1$, *the Ricci operator* Q *of* M *can be expressed as*

$$QX = -2nX + 2n(\kappa + 1)\eta(X)\xi - [\mu - 2(n-1)h']X,$$

for any vector field X *on* M. *Further, if* κ *and* μ *are constants and* $n \geq 1$, *then* $\mu = -2$ *and hence*

$$QX = -2nX + 2n(\kappa + 1)\eta(X)\xi - 2nh'X, \tag{3.57}$$

for any vector field X *on* M. *In both cases, the scalar curvature of* M *is* $2n(\kappa - 2n)$.

Proposition 3.1. *There does not exist a generalized* m*-quasi-Einstein structure with* $\varphi V = 0$ *in* $(\kappa, \mu)'$*-almost Kenmotsu manifold with* $h' \neq 0$.

Proof. By hypothesis we have $\varphi V = 0$. Operating this with φ gives $V = \eta(V)\xi$ i.e., $V = F\xi$ where F is a smooth function. Taking the covariant derivative along arbitrary vector field X of the last equation and inserting it in (1.49) we obtain

$$\begin{aligned} S(X,Y) + \frac{1}{2}[(XF)\eta(Y) + (YF)\eta(X)] + Fg(h'X, Y) \\ -(\frac{F^2}{m} + F)\eta(X)\eta(Y) = (\lambda - F)g(X,Y), \end{aligned} \tag{3.58}$$

Replacing X by ξ in (3.58) yields

$$\frac{1}{2}(YF) = [\lambda + \frac{F^2}{m} - 2n\kappa - \frac{1}{2}(\xi F)]\eta(Y), \tag{3.59}$$

for any vector field X on M. Contracting (1.35) and using Lemma 3.1, we get

$$\xi F = (2n+1)\lambda - 2n(\kappa - 2n) + \frac{F^2}{m} - 2nF. \tag{3.60}$$

Replacing Y by ξ in (3.59) and combining it with (3.60) gives $F = \lambda + 2n$. Inserting (3.59) in (3.58) and using it in Lemma 3.1, we obtain

$$\{2\lambda + \frac{F^2}{m} - F - 2n\kappa + 2n - (\xi F)\}\eta(X)\eta(Y) + \lambda g(h'X, Y) = 0. \tag{3.61}$$

Replacing X by $h'X$ in (3.61) implies $\lambda(\kappa+1)g(\varphi X, \varphi Y) = 0$. Since $h' \neq 0$ and $\kappa < -1$, we get $\lambda = 0$ and using it in $F = \lambda + 2n$ gives $F = 2n$. In a consequence of this, in (3.60) we get $\kappa = \frac{2n}{m}$, a contradiction. This completes the proof. □

Now using the above Lemmas and proposition we proved the following:

Theorem 3.7. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a $(\kappa, \mu)'$-almost Kenmotsu manifold with $h' \neq 0$. If g admits a closed generalized m-quasi-Einstein metric then we get one of the following:*

1. *M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*
2. *M^{2n+1} is locally isometric to the warped product*

$$\mathbb{H}^{n+1}(\alpha) \times_f \mathbb{R}^n, \qquad \mathbb{B}^{n+1}(\alpha') \times_{f'} \mathbb{R}^n$$

where $\mathbb{H}^{n+1}(\alpha)$ is the hyperbolic space of constant curvature $\alpha = -1 - \frac{2m}{n} - \frac{m^2}{n^2}$, $\mathbb{B}^{n+1}(\alpha')$ is a space of constant curvature $\alpha' = -1 + \frac{2m}{n} - \frac{m^2}{n^2}$, $f = ce^{(1-\frac{m}{n})t}$ and $f' = c'e^{(1+\frac{m}{n})t}$ where c, c' are positive constants.

Proof. Since $V^\flat$ is closed, Eq. (1.49) implies

$$\nabla_X V = \lambda X + \frac{1}{m} g(X, V)V - QX. \tag{3.62}$$

Making use of the relation $R(X,Y)V = \nabla_X \nabla_Y V - \nabla_Y \nabla_X V - \nabla_{[X,Y]}V$ in (3.62) we get

$$R(X,Y)V = (X\lambda)Y - (Y\lambda)X + (\nabla_Y Q)X - (\nabla_X Q)Y$$
$$+\frac{\lambda}{m}\{g(Y,V)X - g(X,V)Y\} + \frac{1}{m}\{g(X,V)QY - g(Y,V)QX\}. \tag{3.63}$$

Taking an inner product of (3.63) with ξ and using Lemma 4.4 (Patra et al., 2020) we obtain

$$g(R(X,Y)V,\xi) = (X\lambda)\eta(Y) - (Y\lambda)\eta(X) + g(Q\varphi hY, X)$$
$$-g(Q\varphi hX, Y) + \frac{(\lambda - 2n\kappa)}{m}\{g(Y,V)\eta(X) - g(X,V)\eta(Y)\}. \tag{3.64}$$

Contracting (3.63) and making use of the fact that scalar curvature is constant yields

$$\frac{(m-1)}{m}S(Y,V) = -2n(Y\lambda) + \frac{1}{m}(2n\lambda - r)g(Y,V). \tag{3.65}$$

Taking an inner product of (1.33) with V and inserting it in (3.64) we get

$$(\xi\lambda)\xi - D\lambda - \frac{1}{m}\{\lambda - (2n+m)\kappa\}\varphi^2 V + 2h'V = 0. \tag{3.66}$$

Operating by φ in (3.66) yields

$$\frac{1}{m}\{\lambda - (2n+m)\kappa\}\varphi V - \varphi D\lambda + 2\varphi h'V = 0. \tag{3.67}$$

Making use of the second equation in Lemma 3.1 in (3.65) and operating the obtained expression by φ we get

$$\{2n\lambda - r + 2n(m-1)\}\varphi V - 2nm\varphi D\lambda + 2n(m-1)\varphi h'V = 0. \tag{3.68}$$

Combining (3.67) and (3.68) we get

$$[2n(\lambda + m - 1) - r - \frac{n(m-1)}{m}\{\lambda - (2n+m)\kappa\}]\varphi V - n(1+m)\varphi D\lambda = 0,$$

implies

$$[2n(\lambda+m-1)-r-\frac{n(m-1)}{m}\{\lambda-(2n+m)\kappa\}]V-n(1+m)D\lambda\in\mathbb{R}\xi.$$

Therefore we can write

$$D\lambda=\alpha V+s\xi, \tag{3.69}$$

where

$$\alpha=\frac{1}{n(m+1)}[2n(\lambda+m-1)-r-\frac{n(m-1)}{m}\{\lambda-(2n+m)\kappa\}]$$

and s is a smooth function on M. Inserting (3.69) in (3.66) gives

$$(\xi\lambda)\xi-\alpha V-s\xi-\frac{1}{m}\{\lambda-(2n+m)\kappa\}\varphi^2V+2h'V=0. \tag{3.70}$$

Operating (3.70) by h' we get

$$\frac{1}{m}\{\lambda-(2n+m)\kappa-\alpha m\}h'V+2(\kappa+1)\varphi^2V=0.$$

Inserting the last equation in (3.70) we obtain

$$\begin{aligned}4(\kappa+1)\varphi^2V=&\frac{1}{m}[\lambda-(2n+m)\kappa-\alpha m][(\xi\lambda)\xi\\&-\alpha V-s\xi-\frac{1}{m}\{\lambda-(2n+m)\kappa\}\varphi^2V],\end{aligned} \tag{3.71}$$

then operating (3.71) by φ and using Proposition 3.1, we get

$$[\lambda-(2n+m)\kappa-\alpha m]^2+4m^2(\kappa+1)=0, \tag{3.72}$$

implies λ is constant. Replacing Y by ξ in (3.65) and taking λ as constant, gives

$$[\lambda-\frac{r}{2n}-\kappa(m-1)]\eta(V)=0. \tag{3.73}$$

So we get either $\eta(V)=0$ or $\lambda-\frac{r}{2n}-\kappa(m-1)=0$.

Case-I: Suppose $\eta(V)=0$. Then taking covariant derivative along ξ and using

(3.62) gives $\lambda = 2n\kappa$. Inserting this in (3.72) we get $\kappa = -2$. Without loss of generality, we may choose $\nu = 1$. As a consequence of this in Theorem 5.1 (Pastore and Saltarelli, 2011) we get

$$R(X_\nu, Y_\nu)Z_\nu = -4[g(Y_\nu, Z_\nu)X_\nu - g(X_\nu, Z_\nu)Y_\nu],$$

$$R(X_{-\nu}, Y_{-\nu})Z_{-\nu} = 0,$$

for any $X_\nu, Y_\nu, Z_\nu \in [\nu]'$ and $X_{-\nu}, Y_{-\nu}, Z_{-\nu} \in [-\nu]'$. Making use of the fact that $\mu = -2$ it follows from Proposition 4.1 (Dileo and Pastore, 2009) and Proposition 4.3 (Dileo and Pastore, 2009) that $K(X, \xi) = -4$ for any $X \in [\nu]'$ and $K(X, \xi) = 0$ for any $X \in [-\nu]'$. As shown by Dileo and Pastore (2009), the distribution $[\xi] \oplus [\nu]'$ is integrable with totally geodesic leaves and the distribution $[-\nu]'$ is integrable with total umbilical leaves by $H = -(1 - \nu)\xi$, where H is the mean curvature vector field for the leaves of $[-\nu]'$ immersed in M^{2n+1}. Taking $\nu = 1$, then the two distribution $[\xi] \oplus [\nu]'$ and $[-\nu]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Hence M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Case-II: If $\lambda - \frac{r}{2n} - \kappa(m - 1) = 0$, then inserting the value of scalar curvature from Lemma 3.1 gives $\lambda = m\kappa - 2n$. Using this in (3.72) implies $\kappa = -1 - \frac{m^2}{n^2}$. By applying Dileo-Pastore (2009) result we complete the proof. □

Remark 3.2. *When $V = Df$, it is clear that $V^\flat$ is closed. Therefore if the non-normal $(\kappa, \mu)'$-almost Kenmotsu manifold admits a generalized m-quasi-Einstein structure (g, f, m, λ) then we get similar results as in Theorem 3.7. In a particular case of Theorem 3.7, for $m = \infty$ we easily obtain Theorem 3.1 (Wang, 2016).*

Let $\mathcal{U}_1$ be the open subset of a 3-dimensional almost Kenmotsu manifold M^3 such that $h \neq 0$ and $\mathcal{U}_2$ the open subset of M^3 which is defined by $\mathcal{U}_2 = \{p \in M^3 : h = 0$ in a neighbourhood of $p\}$. Therefore $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open and dense subset of M^3 and there exists a local orthonormal basis $\{e_1 = e, e_2 = \varphi e, e_3 = \xi\}$

of three smooth unit eigenvectors of h for any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$. On $\mathcal{U}_1$ we may set $he_1 = \vartheta e_1$ and $he_2 = -\vartheta e_2$, where ϑ is a positive function.

Lemma 3.2 (Cho, 2014). *On $\mathcal{U}_1$ we have*

$$
\begin{aligned}
&\nabla_\xi \xi = 0, && \nabla_\xi e = a\varphi e, && \nabla_\xi \varphi e = -ae,\\
&\nabla_e \xi = e - \vartheta\varphi e, && \nabla_e e = -\xi - b\varphi e, && \nabla_e \varphi e = \vartheta\xi + be,\\
&\nabla_{\varphi e} \xi = -\vartheta e + \varphi e, && \nabla_{\varphi e} e = \vartheta\xi + c\varphi e, && \nabla_{\varphi e}\varphi e = -\xi - ce,
\end{aligned}
$$

where a, b, c are smooth functions.

From Lemma 3.2, the poisson brackets for $\{e_1 = e, e_2 = \varphi e, e_3 = \xi\}$ are as follows:

$$
[e_3, e_1] = (a + \vartheta)e_2 - e_1, [e_1, e_2] = be_1 - ce_2, [e_2, e_3] = (a - \vartheta)e_1 + e_2. \quad (3.74)
$$

Then the expression for the Ricci operator are as follows:

Lemma 3.3. *The Ricci operator Q with respect to the local basis $\{\xi, e, \varphi e\}$ on $\mathcal{U}_1$ can be written as*

$$
\begin{aligned}
Q\xi &= -2(\vartheta^2 + 1)\xi - (\varphi e(\vartheta) + 2\vartheta b)e - (e(\vartheta) + 2\vartheta c)\varphi e,\\
Qe &= -(\varphi e(\vartheta) + 2\vartheta b)\xi - (A + 2\vartheta a)e + (\xi(\vartheta) + 2\vartheta)\varphi e,\\
Q\varphi e &= -(e(\vartheta) + 2\vartheta c)\xi + (\xi(\vartheta) + 2\vartheta)e - (A - 2\vartheta a)\varphi e,
\end{aligned}
$$

where we set $A = e(c) + b^2 + c^2 + \varphi e(b) + 2$ for simplicity.

Now we state and prove the following:

Theorem 3.8. *If a 3-dimensional almost Kenmotsu 3-H-manifold with $h' \neq 0$ admits a generalized m-quasi-Einstein (g, f, m, λ) structure whose potential function is constant along the Reeb vector field, then it is Einstein or is locally isometric to a non-unimodular Lie group with a left-invariant almost Kenmotsu structure.*

Proof. For an almost Kenmotsu 3-H-manifold from Lemma 3.3, we have

$$e(\vartheta) = -2\vartheta c, \qquad \varphi e(\vartheta) = -\vartheta b. \tag{3.75}$$

By our assumption, since the potential function is constant along the Reeb vector field, we can write

$$Df = f_1 e + f_2 \varphi e, \tag{3.76}$$

for smooth functions $f_1 = f(e)$ and $f_2 = \varphi e(f)$. Substituting $X = \xi$ in (3.38) and using Lemma 3.2, Lemma 3.3 and (3.76) gives

$$\begin{cases} \xi f_1 - a f_2 = 0, \\ a f_1 + \xi(f_2) = 0, \\ \lambda = 2(\vartheta^2 + 1). \end{cases} \tag{3.77}$$

Again, putting $X = e$ in (3.38) and then using Lemma 3.2, Lemma 3.3 and (3.76) gives

$$\begin{cases} e(f_1) + b f_2 = \lambda + \frac{f_1^2}{m} - A - 2\vartheta a, \\ \vartheta f_2 - f_1 = 0, \\ e(f_2) - b f_1 = \frac{f_1 f_2}{m} - \xi(\vartheta) - 2\vartheta. \end{cases} \tag{3.78}$$

Similarly, for $X = \varphi e$, we get

$$\begin{cases} \varphi e(f_1) - c f_2 = \frac{f_1 f_2}{m} - \xi(\vartheta) - 2\vartheta, \\ \varphi e(f_2) + c f_1 = \lambda + \frac{f_2^2}{m} + A - 2a\vartheta, \\ \vartheta f_1 - f_2 = 0. \end{cases} \tag{3.79}$$

Comparing the second argument of (3.78) and the third argument of (3.79), we get $(\vartheta^2 - 1) f_2 = 0$. If $f_2 = 0$, then the third argument of (3.79) implies $f_1 = 0$, then (3.76) gives $Df = 0$, that is, f is constant.

For the case $f_2 \neq 0$, we have $\vartheta = 1$. In consequence, the second argument of (3.78) and the third argument of (3.79) gives $f_1 = f_2$. Moreover, taking $\vartheta = 1$ in (3.75), we get $b = c = 0$. Also, first and second equation of (3.77) gives $a = 0$ when $f_1 = f_2$. Inserting the above values in (3.74), we get

$$[e_3, e_1] = e_2 - e_1, [e_1, e_2] = 0, [e_2, e_3] = -e_1 + e_2.$$

Using Milnor's result (Milnor, 1976), we can conclude that M^3 is locally isometric to a non-unimodular Lie group with a left-invariant almost Kenmotsu structure. This completes the proof. □

In consequence of Theorem 3.8, we can state the following corollary.

Corollary 3.5. *If a 3-dimensional almost Kenmotsu* 3*-H-manifold admits a non-trivial generalized* m*-quasi-Einstein* (g, f, m, λ) *structure whose potential function is constant along the Reeb vector field, then it is locally isometric to either the hyperbolic space* $\mathbb{H}^3(-1)$ *or the Riemannian product* $\mathbb{H}^2(-4) \times \mathbb{R}$.

Proof. We shall divide the proof into two cases:

Case-I: When $h = 0$, then M is a Kenmotsu manifold. Then we have

$$QX = (\frac{r}{2} + 1)X - (\frac{r}{2} + 3)\eta(X)\xi. \tag{3.80}$$

By assumption, $\xi f = 0$. Replacing $X = \xi$ in (3.38) then taking the inner product with ξ gives $\lambda = -2n$ under our assumptions. In consequence, (1.29) gives $\xi r = 0$. Since $\xi r = -2(r+6)$, we get $r = -6$ which reduces (3.80) to $QX = -2X$. Clearly, M^3 is conformally flat.

Case-II: When $h \neq 0$, then by Theorem 3.8, we have $a = b = c = 0$. From Lemma 3.3, we see that $r = -2(\vartheta^2 + 1) - 2A$. Making use of the fact that $a = b = c = 0$ implies $r = -8$. It is easy to see that M^3 is conformally flat.

Applying Wang's theorem (Theorem 1.6 (Wang, 2017)), we can conclude that

M^3 is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$. □

Corollary 3.6. *If a 3-dimensional almost Kenmotsu* 3*-H-manifold admits a non-trivial m-quasi-Einstein (g, f, m, λ) structure whose potential function is constant along the Reeb vector field, then either it is locally isometric to the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$.*

Next, we constructed an example of almost Kenmotsu manifold admitting a generalized m-quasi-Einstein structure.

Example 3.8. *Let $(N, J, \bar{g})$ be a strictly almost Kähler Einstein manifold. We set $\eta = dt$, $\xi = \frac{\partial}{\partial t}$ and the tensor field φ is defined on $\mathbb{R} \times_f N$ by $\varphi X = JX$ for vector field X on N and $\varphi X = 0$ if X is tangent to $\mathbb{R}$. Consider a metric $g = g_0 + \sigma^2 \bar{g}$, where $\sigma^2 = ce^{2t}$, g_0 is the Euclidean metric on $\mathbb{R}$ and c is a positive constant. Then it is easy to verify that the warped product $\mathbb{R} \times_\sigma N, \sigma^2 = ce^{2t}$, with the structure (φ, ξ, η, g) is an almost Kenmotsu manifold (Dileo and Pastore, 2007). Since N is Einstein $S = -2ng$. We define a smooth function $f(x, t) = t^2$. then it is easy to verify that the warped product $\mathbb{R} \times_\sigma N, \sigma^2 = ce^{2t}$ admits a generalized m-quasi-Einstein structure (g, f, m, λ) with $\lambda = \frac{2}{m}(m(1-n) - 2t^2)$.*

Chapter 4

Characterization of Almost Ricci-Yamabe Solitons

Chapter 4

Characterization of Almost Ricci-Yamabe Solitons

This chapter is divided into two sections. First section is devoted to the study of almost Ricci-Yamabe soliton on certain almost contact metric manifolds and in the section, we considered almost Ricci-Yamabe soliton in the context of almost Kenmotsu manifolds.

4.1 Almost Ricci-Yamabe Soliton on Contact Metric Manifolds

4.1.1 Almost (α, β)-Ricci-Yamabe solitons with $V = \sigma\xi$

Ghosh (2014) obtained a result for contact metric manifold with potential vector field collinear with the Reeb vector field. Motivated by this study, we extended it to an almost (α, β)-Ricci-Yamabe soliton. We prove the following:

Theorem 4.1. *Let $M^{(2n+1)}(\varphi, \xi, \eta, g)$ be a complete contact metric manifold where the Reeb vector field ξ is an eigenvector of the Ricci operator at each point of M. If g admits an almost (α, β)-Ricci-Yamabe soliton with $\alpha \neq 0$ and non-*

zero potential vector field collinear with the Reeb vector field ξ, then M is compact Einstein Sasakian and the potential vector field is a constant multiple of the Reeb vector field ξ.

Proof. Suppose the potential vector field is collinear with the Reeb vector field i.e., $V = \sigma\xi$, where σ is a non-zero function on M. Differentiating it along arbitrary vector field X gives

$$\nabla_X V = (X\sigma)\xi - \sigma(\varphi X + \varphi h X). \tag{4.1}$$

Using this in (1.45) and simplifying we obtain

$$\begin{aligned}(X\sigma)\eta(Y) + (Y\sigma)\eta(X) - 2\sigma g(\varphi hX, Y)\\ +2\alpha S(X,Y) = (2\lambda - \beta r)g(X,Y).\end{aligned} \tag{4.2}$$

Taking $X = Y = \xi$ in (4.2) yields

$$\xi\sigma + 2\alpha Tr\ell = 2\lambda - \beta r. \tag{4.3}$$

Replacing Y by ξ in (4.2) gives

$$D\sigma + (\xi\sigma)\xi + 2\alpha Q\xi = (2\lambda - \beta r)\xi. \tag{4.4}$$

Suppose that the Reeb vector field ξ is an eigenvector of the Ricci operator at each point of M, then $Q\xi = (Trl)\xi$. Using this in the forgoing equation along with (4.3) gives, $D\sigma = (\xi\sigma)\xi$. Differentiating it along with vector field X yields

$$\nabla_X D\sigma = X(\xi\sigma)\xi - (\xi\sigma)(\varphi X + \varphi h X). \tag{4.5}$$

Making use of the Poincare lemma in (4.5), we obtain

$$X(\xi\sigma)\eta(Y) - Y(\xi\sigma)\eta(X) + 2(\xi\sigma)d\eta(X,Y) = 0. \tag{4.6}$$

Choosing $X, Y \perp \xi$ and using the fact that $d\eta \neq 0$ in (4.6), we see that $\xi\sigma = 0$.

Hence, $D\sigma = 0$ i.e. σ is a constant. Then (4.2) becomes,

$$2\alpha QY + 2\sigma h\varphi Y = (2\lambda - \beta r)Y. \tag{4.7}$$

Contracting (4.7) and using the fact that $Trh\varphi = 0$, we get

$$[2\alpha + (2n+1)\beta]r = 2(2n+1)\lambda. \tag{4.8}$$

Differentiating (4.7) along arbitrary vector field X gives

$$2\alpha(\nabla_X Q)Y + 2\sigma(\nabla_X h\varphi)Y = 2(X\lambda)Y - \beta(Xr)Y. \tag{4.9}$$

Contracting (4.9) and using the fact that in contact metric manifold, $div(h\varphi)Y = g(Q\xi, Y) - 2n\eta(Y)$, in the forgoing equation result in the following

$$(\alpha + \beta)(Yr) + 2\sigma[Tr\ell - 2n]\eta(Y) - 2(Y\lambda) = 0. \tag{4.10}$$

Taking $Y \perp \xi$ and using (4.8) in (4.10) gives $\alpha = 0$ or $Yr = 0$. Assuming $\alpha \neq 0$ and replacing Y by $\varphi^2 Y$ shows $Dr = (\xi r)\xi$. Differentiating along arbitrary vector field X gives, $\nabla_X Dr = X(\xi r)\xi - (\xi r)(\varphi X + \varphi hX)$. Applying Poincare lemma, the forgoing equation yields

$$X(\xi r)\eta(Y) - Y(\xi r)\eta(X) - (\xi r)d\eta(X, Y) = 0. \tag{4.11}$$

Choosing $X, Y \perp \xi$, it follows that $\xi r = 0$. Hence, $Dr = 0$ i.e. r is constant. Then (4.8) implies λ is constant and consequently from (4.3), $Tr\ell$ is constant. In view of (4.10) we get $Tr\ell = 2n$ i.e. $h = 0$. Hence manifold is K-contact and then from (4.7), it is Einstein provided $\alpha \neq 0$. Suppose M is complete, then making use of results in Sharma (2008) and Boyer and Galicki (2001), we see that the manifold is compact Einstein Sasakian. This completes the proof. □

From (4.3) we get, $2\alpha Tr\ell = (2\lambda - \beta r)$. Using this in (4.4) gives

$$2\alpha[Q\xi - (Tr\ell)\xi] + D\sigma + (\xi\sigma)\xi = 0. \tag{4.12}$$

Making use of result by Perrone (2004) and (4.12), we can state the following

Corollary 4.1. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a contact metric manifold such that g represents an almost (α, β)-Ricci-Yamabe soliton with $\alpha \neq 0$. Then M is an H-contact metric manifold if and only if the potential vector field is a constant multiple of the Reeb vector field ξ.*

In consequence of Theorem 4.1, considering a particular case when potential vector field V is the Reeb vector field ξ, we can easily prove the following:

Corollary 4.2. *There does not exist almost Ricci-Yamabe soliton in a non-Sasakian (k, μ)-contact metric manifold whose potential vector field is the Reeb vector field ξ.*

4.1.2 Almost Ricci-Yamabe soliton on K-contact Manifold

Sharma (2008) proved that if a compact K-contact metric is a gradient Ricci soliton then it is Einstein Sasakian. Extending this for gradient Ricci almost soliton, Ghosh (2014) proved that compact K-contact metric is Einstein Sasakian and isometric to a unit sphere $\mathbb{S}^{2n+1}$. However, this result is also true if one relax the hypothesis compactness to completeness (Patra, 2021). In this section we consider the gradient almost Ricci-Yamabe soliton and extend these results and prove

Theorem 4.2. *If a K-contact manifold $M^{(2n+1)}(\varphi, \xi, \eta, g)$ admits a gradient almost Ricci-Yamabe soliton with $\alpha \neq 0$, then it is Einstein with constant scalar curvature $r = 2n(2n+1)$. Further, if M is complete, then it is compact Sasakian and isometric to a unit sphere $\mathbb{S}^{2n+1}$.*

Proof. A gradient almost Ricci-Yamabe soliton is given by

$$\nabla_X Df + 2\alpha QX = (2\lambda - \beta r)X. \tag{4.13}$$

Taking the covarient differentiation of (4.13) along arbitrary vector field Y yields

$$\begin{aligned} &\nabla_Y \nabla_X Df + 2\alpha(\nabla_Y Q)X + 2\alpha Q(\nabla_Y X) \\ &= 2(Y\lambda)X - \beta(Yr)X + (2\lambda - \beta r)(YX). \end{aligned} \tag{4.14}$$

Since $R(X, Y)Df = \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_[X, Y]Df$, then in consequence of (4.14) we get

$$\begin{aligned} R(X, Y)Df &= 2[(X\lambda)Y - (Y\lambda)X] - \beta[(Xr)Y - (Yr)X] \\ &- 2\alpha[(\nabla_X Q)Y - (\nabla_Y Q)X]. \end{aligned} \tag{4.15}$$

Differentiating (1.13) along vector field Y and using (1.14) gives

$$(\nabla_X Q)\xi = Q\varphi X - 2n\varphi X. \tag{4.16}$$

Taking the inner product of (4.15) with ξ and replacing Y by ξ and using the fact that $g(R(X, Y)Df, \xi) = -g(R(X, Y)\xi, Df)$ along with $R(X, \xi)\xi = X - \eta(X)\xi$ and (4.16), Eq. (4.15) reduces to $X(f + 2\lambda - \beta r) = \xi(f + 2\lambda - \beta r)\eta(X)$, which can be written as $d(f + 2\lambda - \beta r) = \xi(f + 2\lambda - \beta r)\eta$. Then operating the last equation by d and using Poincare lemma i.e., $d^2 = 0$ we get $d\xi(f + 2\lambda - \beta r) \wedge \eta + \xi(f + 2\lambda - \beta r)d\eta = 0$. Taking the wedge product of forgoing equation with η and using the fact that $\eta \wedge \eta = 0$ yields $\xi(f + 2\lambda - \beta r)d\eta \wedge \eta = 0$. Therefore $\xi(f + 2\lambda - \beta r) = 0$ on M as $d\eta$ is non-vanishing everywhere on M, consequently, $D(f + 2\lambda - \beta r) = 0$. Hence $f + 2\lambda - \beta r$ is constant on M.

Taking Lie differentiation of (4.13) along ξ and noting $\mathcal{L}_\xi Q = 0$ (as ξ is Killing) we obtain

$$\mathcal{L}_\xi(\nabla_X Df) + 2\alpha Q(\mathcal{L}_\xi X) = 2(\xi\lambda)X - \beta(\xi r)X + (2\lambda - \beta r)\mathcal{L}_\xi X. \tag{4.17}$$

Lie differentiating Df along ξ and using (1.11) yields

$$\mathcal{L}_\xi Df = [\xi, Df] = \nabla_\xi Df - \nabla_{Df}\xi = (2\lambda - \beta r)\xi - 4n\alpha\xi + \varphi Df. \tag{4.18}$$

Differentiating covariantly (4.18) along vector field Y and using (1.11) we obtain

$$\nabla_Y \mathcal{L}_\xi Df = 2(Y\lambda)\xi - \beta(Yr)\xi + 4n\alpha\varphi Y + (\nabla_Y\varphi)Df - 2\alpha\varphi QY \tag{4.19}$$

According to Yano (1970), we have the commutative formula

$$\mathcal{L}_V\nabla_Y X - \nabla_Y\mathcal{L}_V X - \nabla_{[V,Y]}X = (\mathcal{L}_V\nabla)(Y,X). \tag{4.20}$$

Setting $V = \xi$ and $X = Df$ in (4.20) and noting $\mathcal{L}_\xi\nabla = 0$ and using (4.17)-(4.19) yields

$$[2(\xi\lambda) - \beta(\xi r)]g(X,Y) - Y(2\lambda - \beta r)\eta(X) - 4n\alpha g(\varphi Y, X) \\ + g((\nabla_Y\varphi)X, Df) + 2\alpha g(\varphi QY, X) = 0. \tag{4.21}$$

Replacing X by φX and Y by φY along with well known formula

$$(\nabla_Y\varphi)X + (\nabla_{\varphi Y}\varphi)\varphi X = 2g(Y,X)\xi - \eta(X)(Y + \eta(Y)\xi)$$

we get

$$2\xi(f + 2\lambda - \beta r)g(X,Y) - Y(f + 2\lambda - \beta r)\eta(X) \\ - \xi(f + 2\lambda - \beta r)\eta(X)\eta(Y) + 2\alpha g(Q\varphi Y, X) \\ + 2\alpha g(\varphi QY, X) - 8n\alpha g(\varphi Y, X) = 0. \tag{4.22}$$

Suppose $\alpha \neq 0$. Since $f + 2\lambda - \beta r$ is constant Eq. (4.22) reduces to

$$Q\varphi X + \varphi QX = 4n\varphi X, \tag{4.23}$$

for any $X \in \chi(M)$. □

Taking an inner product of (4.15) along with $f + 2\lambda - \beta r = constant$ yields

$$g((\nabla_Y Q)X - (\nabla_X Q)Y, Df) = 0. \tag{4.24}$$

Let $\{e_i, \varphi e_i, \xi; i = 1, 2, ...n\}$ be an orthonormal $\varphi-$basis of M such that $Qe_i =$

$\sigma_i e_i$. Using this in (4.23) we get $Q\varphi e_i = (4n - \sigma_i)\varphi e_i$. Then the scalar curvature is given by

$$r = g(Q\xi, \xi) + \sum_{i=1}^{n}[g(Qe_i.e_i) + g(Q\varphi e_i, \varphi e_i)] = 2n(2n+1).$$

Replacing X by ξ in (4.24) and using (4.16) yields $Q\varphi Df - 2n\varphi Df = 0$. In consequence of this in (4.23), it reduces to $\varphi QDf = 2n\varphi Df$. Operating last equation with φ and using (1.13) gives $QDf = 2nDf$. Then taking covarient derivative results in

$$(\nabla_X Q)Df - 2\alpha Q^2 X + (2\lambda - \beta r + 4n\alpha)QX - 2n(2\lambda - \beta r)X = 0. \quad (4.25)$$

Since $r = 2n(2n+1)$ is constant, then $divQ = \frac{1}{2}dr = 0$. Making use of this and contracting (4.25) we obtain $||Q||^2 = 2nr$. As a consequence of this with $r = 2n(2n+1)$, we can easily see that $||Q - \frac{r}{2n+1}I||^2 = 0$ i.e., the length of the symmetric tensor $Q - \frac{r}{2n+1}I$ vanish, we must have $QX = 2nX$. Thus M is Einstein with Einstein constant $2n$. Suppose M is complete, then by the result of Sharma (2008) we can conclude that M is compact. Applying Boyer-Galicki (2001) we conclude that it is Sasakian. Also, Eq. (4.13) can be rewritten as $\nabla_X Df = -\rho X$, where $\rho = 4\alpha n + \beta r - 2\lambda$, then by Obata's theorem (1962) it is isometric to a unit sphere $\mathbb{S}^{2n+1}$. This completes the proof.

4.1.3 Almost Ricci-Yamabe soliton on (k, μ)-contact metric manifold

Theorem 4.3. *If a non-Sasakian (k, μ)-contact metric manifold $M^{(2n+1)}(\varphi, \xi, \eta, g)$ admits a gradient almost Ricci-Yamabe soliton with $\alpha \neq 0$, then M^3 is flat and the soliton vector field is homothetic, and for $n > 1$, M is locally isometric to $\mathbb{E}^{n+1} \times \mathbb{S}^n(4)$ and the soliton vector field is tangential to the Euclidean factor $\mathbb{E}^{n+1}$.*

Proof. Making use of $R(X,Y)Df = \nabla_X\nabla_Y Df - \nabla_Y\nabla_X Df - \nabla_{[X,Y]}Df$ and (4.13), we get

$$R(X,Y)Df = 2\alpha[(\nabla_Y Q)X - (\nabla_X Q)Y] + 2[(X\lambda)Y - (Y\lambda)X]. \tag{4.26}$$

Taking the covariant derivative of (1.22) and using it in (4.26) yields

$$\begin{aligned} R(X,Y)Df &= 2\alpha\{[2(n-1)+\mu][2(1-k)g(Y,\varphi X)\xi \\ &+ \eta(X)\{h(\varphi Y + \varphi hY\} - \eta(Y)\{h(\varphi X + \varphi hX\} + \mu\eta(X)\varphi hY \\ &- \mu\eta(Y)\varphi hX] + [2(1-n) + n(2k+\mu)]\{2g(Y,\varphi X)\xi \\ &- (\varphi Y + \varphi hY)\eta(X) + (\varphi X + \varphi hX)\eta(Y)\}\} + 2[(X\lambda)Y - (Y\lambda)X]. \end{aligned} \tag{4.27}$$

Taking the inner product of (4.27) with ξ gives

$$\begin{aligned} g(R(X,Y)Df,\xi) &= 4\alpha(\mu + 2k - k\mu + n\mu)g(Y,\varphi X) \\ &+ 2[(X\lambda)Y - (Y\lambda)X]. \end{aligned} \tag{4.28}$$

Taking the inner product of (1.21) with Df, we get

$$\begin{aligned} g(R(X,Y)\xi, Df) = k[\eta(Y)g(X,Df) - \eta(X)g(Y,Df)] \\ +\mu[\eta(Y)g(hX,Df) - \eta(X)g(hY,Df)]. \end{aligned} \tag{4.29}$$

Combining (4.28) and (4.29) we get

$$\begin{aligned} k[\eta(Y)g(X,Df) - \eta(X)g(Y,Df)] \\ +\mu[\eta(Y)g(hX,Df) - \eta(X)g(hY,Df)] \\ +4\alpha(\mu + 2k - k\mu + n\mu)g(Y,\varphi X) \\ +2[(X\lambda)\eta(Y) - (Y\lambda)\eta(X)] = 0. \end{aligned} \tag{4.30}$$

Taking $X = \varphi X$ and $Y = \varphi Y$ and using the fact that $R(\varphi X, \varphi Y)\xi = 0$, Eq.

(4.30) for $\alpha \neq 0$ reduces to

$$k = \frac{\mu(1+n)}{\mu - 2}. \tag{4.31}$$

Replacing $Y = \xi$ in (4.30) gives

$$(k + \mu h)Df + 2(D\lambda) - [k(\xi f) + 2(\xi\lambda)]\xi = 0. \tag{4.32}$$

As a consequence of (1.22), replacing X with Df and simplifying we obtain

$$QDf = -4n(D\lambda). \tag{4.33}$$

Making use of (4.33) in (4.32) gives

$$2n(k + \mu h)Df - QDf - 2n[k(\xi f) + 2(\xi\lambda)]\xi = 0. \tag{4.34}$$

Taking an inner product of (4.34) with ξ we get, $k(\xi f) + 2(\xi\lambda) = 0$ and using this in forgoing equation

$$2n(k + \mu h)Df = QDf. \tag{4.35}$$

Differentiating (4.35) and simplifying, we obtain

$$(2n\mu^2 - \mu[2(n-1) + \mu])\varphi hDf - 2n\mu h(2\lambda - \beta r - 4n\alpha k)\xi = 0. \tag{4.36}$$

Taking inner product of (4.36) with ξ gives, $\mu h(2\lambda - \beta r - 4n\alpha k) = 0$, and using it in (4.36)

$$(2n\mu^2 - \mu[2(n-1) + \mu])\varphi hDf = 0. \tag{4.37}$$

Operating h in the above equation and using (1.20), we get

$$(k-1)\mu[2(n-1) + \mu - 2n\mu]\varphi Df = 0. \tag{4.38}$$

We get the following cases:

Case-I: For $\mu = 0$. In consequence, equation (4.31) gives $k = 0$. Hence,

$R(X,Y)\xi = 0$.

Now Blair (1977) proved that a $(2n+1)$-dimensional contact metric manifold satisfying $R(X,Y)\xi = 0$ is locally isometric to $\mathbb{E}^{n+1} \times \mathbb{S}^n(4)$ for $n > 1$ and flat if $n = 1$.

Therefore, we conclude that the manifold under consideration is locally isometric to $\mathbb{E}^{n+1} \times \mathbb{S}^n(4)$ for $n > 1$ and flat if $n = 1$.

Case-II: For $\varphi Df = 0$. Operating φ on both sides gives $Df = (\xi f)\xi$. Differentiating along arbitrary vector field X gives

$$\nabla_X Df = X(\xi f)\xi - (\xi f)(\varphi X + \varphi hX). \tag{4.39}$$

Applying Poincare lemma in the above equation yields

$$X(\xi f)\eta(Y) - Y(\xi f)\eta(X) + (\xi f)d\eta(X,Y) = 0. \tag{4.40}$$

Taking $X, Y \perp \xi$ and since $d\eta$ is nowhere vanishing on M, it follows $\xi f = 0$. Hence $Df = 0$ i.e., f is constant. Then from (4.13) we see that M is Einstein (i.e., $2\alpha QY = (2\lambda - \beta r)Y$). Taking a trace of the last equation yields $2\alpha r = (2n+1)(2\lambda - \beta r)$. Also, replacing Y by ξ in the second last equation and using the previous equation results in $QY = 2nkY$. Consequently the scalar curvature is $r = 2nk(2n+1)$. Now proceeding similarly as in Theorem 4.1 of Ghosh (2014), we also find that for $n = 1$, M is locally flat (as $\mu = 0$ and $k = 0$ consequently $R(X,Y)\xi = 0$), using $\mu = 2(1-n)$ in (4.31) we see that $k = n - \frac{1}{n} > 1$, a contraction. Since M^3 is flat and λ is constant in view of (4.13) we see that the vector field is homothetic.

Case-III: For $2(n-1) + \mu - 2n\mu = 0$ implies $\mu = \frac{2(1-n)}{1-2n}$.

Using this value of μ in the expression of k in (4.31), we get $k = \frac{1}{n} - n$.

Making us of (4.41) in (4.35) yields

$$[2(1-n)+n(2k+\mu)](Df-(\xi f)\xi)+[2n\mu-2(n-1)-\mu]hDf=0. \quad (4.41)$$

Inserting $\mu=\frac{2(1-n)}{1-2n}$ and $k=\frac{1}{n}-n$ in (4.41), we obtain $Df=(\xi f)\xi$. Then proceeding similarly as in Case II we obtain a similar conclusion. Since $QX = 2nkX$, taking covarient differentiation gives $\nabla Q=0$ and consequently (4.26) reduces to

$$R(X,Y)Df=2[(X\lambda)Y-(Y\lambda)X].$$

Since $R(X,Y)\xi=0$ and taking the inner product of forgoing equation with ξ and replacing Y by ξ gives $X\lambda=(\xi\lambda)\eta(X)$. Similarly as above we can easily see that λ is constant and consequently $R(X,Y)Df=0$ i.e., Df is tangent to the flat factor $\mathbb{E}^{n+1}$. This completes the proof.

□

Example 4.1. *Finally, we construct an example for verifying the obtained result. Replacing $\alpha=0$ and $\beta=x, x\neq 0$ in an example of (k,μ)-spaces given by Boeckx (2000), we obtain a non-Sasakian (k,μ)-contact metric manifold with $k=1-\frac{x^4}{16}$ and $\mu=2+\frac{x^2}{2}$. We consider a 5-dimensional manifold $M=\{(x_1,x_2,x_3,x_4,x_5)\in \mathbb{R}^5 : x_i\neq 0, i=1,2,..,5\}$ where (x_1,x_2,x_3,x_4,x_5) are standard coordinates in $\mathbb{R}^5$. Let $\{e_1,e_2,e_3,e_4,e_5\}$ be a linearly independent global frame on M such that*

$$[e_5,e_1]=0, \quad [e_5,e_2]=0, \quad [e_5,e_3]=\frac{x^2}{2}e_1, \quad [e_5,e_4]=\frac{x^2}{2}e_2,$$

$$[e_1,e_2]=0, \qquad [e_1,e_3]=-xe_2+2e_5, \qquad [e_1,e_4]=0,$$

$$[e_2,e_3]=xe_1, \qquad [e_2,e_4]=2e_5, \qquad [e_3,e_4]=-xe_3.$$

Let g be the Riemannian metric defined by

$$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1,$$

$$g(e_4,e_4)=g(e_5,e_5)=1, \quad g(e_i,e_j)=0, i\neq j.$$

Let η be the 1-form defined by $\eta(X) = g(X, e_5)$ for any $X \in \chi(M)$. Also, let φ be (1,1) tensor field defined by $\varphi e_1 = e_3, \varphi e_2 = e_4, \varphi e_3 = -e_1, \varphi e_4 = -e_2, \varphi e_5 = 0$. Then for $\xi = e_5$, (φ, ξ, g, η) defines a contact metric structure on M. Let ∇ be Levi-Civita connection on M. Then using the Koszul formula we calculate

$$\nabla_{e_1} e_5 = \rho e_3, \quad \nabla_{e_2} e_5 = \rho e_4, \quad \nabla_{e_3} e_5 = -(\rho+2)e_1, \quad \nabla_{e_4} e_5 = -(\rho+2)e_2,$$

$$\nabla_{e_5} e_1 = \rho e_3, \quad \nabla_{e_5} e_2 = \rho e_4, \quad \nabla_{e_5} e_3 = \rho e_1, \quad \nabla_{e_5} e_4 = \rho e_2,$$

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -\rho e_5, \quad \nabla_{e_1} e_4 = 0,$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = 0, \quad \nabla_{e_2} e_4 = -\rho e_5,$$

$$\nabla_{e_3} e_1 = x e_2 - (\rho+2)e_5, \quad \nabla_{e_3} e_2 = -x e_1, \quad \nabla_{e_3} e_3 = x e_4, \quad \nabla_{e_3} e_4 = -x e_3,$$

$$\nabla_{e_4} e_1 = 0, \quad \nabla_{e_4} e_2 = -(\rho+2)e_5, \quad \nabla_{e_4} e_3 = 0, \quad \nabla_{e_4} e_4 = 0,$$

where $\rho = (\frac{x^2}{4} - 1)$. Moreover using (1.11) in the above expressions gives $he_1 = -(\rho+1)e_1$, $he_2 = -(\rho+1)e_2$, $he_3 = -(\rho+3)e_3$, $he_4 = -(\rho+3)e_4$, $he_5 = 0$. From the above it can be easily seen that $M^5(\varphi, \eta, \xi, g)$ is a non-Sasakian (k, μ)-contact metric manifold.

The non-vanishing components of Riemannian curvature on M are as follows

$$R(e_1, e_2)e_4 = -\rho^2 e_3 + 2\rho e_5, \quad R(e_1, e_2)e_5 = -2\rho e_4, \quad R(e_1, e_3)e_1 = -\rho(\rho+2)e_3 - 2\rho e_3,$$

$$R(e_1, e_3)e_2 = -2\rho e_4, \quad R(e_1, e_3)e_3 = -\rho(\rho+1)e_1 - 2\rho e_1, \quad R(e_1, e_3)e_4 = -2\rho e_2,$$

$$R(e_1, e_4)e_2 = -\rho(\rho+2)e_3, \quad R(e_1, e_4)e_3 = -\rho(\rho+2)e_2, \quad R(e_1, e_5)e_1 = -\rho^2 e_5,$$

$$R(e_1, e_5)e_5 = -\rho^2 e_1, \quad R(e_2, e_3)e_1 = -\rho(\rho+2)e_4, \quad R(e_2, e_3)e_4 = -\rho(\rho+2)e_1,$$

$$R(e_2, e_4)e_1 = -2\rho e_3, \quad R(e_2, e_4)e_2 = -\rho(\rho+2)e_2 - 2\rho e_2, \quad R(e_2, e_4)e_3 = -2\rho e_1,$$

$$R(e_2, e_4)e_4 = -\rho(\rho+2)e_2 - 2\rho e_2, \quad R(e_2, e_5)e_2 = -\rho^2 e_5, \quad R(e_2, e_5)e_5 = \rho^2 e_2,$$

$$R(e_3, e_4)e_1 = x^2 e_2 - (\rho+2)^2 e_2, \quad R(e_3, e_4)e_2 = (\rho+2)^2 e_1 - x^2 e_1, \quad R(e_3, e_4)e_3 = x^2 e_4,$$

$$R(e_3,e_4)e_4 = -x^2e_3,\ \ R(e_3,e_5)e_3 = -\rho(\rho+2)e_5 - \frac{x^2}{2}\rho e_5,\ \ R(e_3,e_5)e_5 = \rho(\rho+2)e_3 + \frac{x^2}{2}\rho e_3,$$

$$R(e_4,e_5)e_4 = -\rho(\rho+2)e_5 - \frac{x^2}{2}\rho e_5,\quad R(e_4,e_5)e_5 = \rho(\rho+2)e_4 + \frac{x^2}{2}\rho e_4.$$

The non-vanishing components of Ricci curvature are

$$S(e_1,e_1) = S(e_2,e_2) = \frac{x^4}{16} - 2,\quad S(e_3,e_3) = S(e_4,e_4) = \frac{x^4}{8} - 2x^2 + 2,$$

$$S(e_5,e_5) = \frac{3x^4}{8} - x^2 - 2.$$

The scalar curvature on M is $r = x^4 - 5x^2 - 2$. Clearly, one can see that for $V = e_5$, the metric g under consideration does not satisfy (1.45). Thus, Corollary 4.2 is verified.

4.2 Almost Ricci-Yamabe soliton on Almost Kenmotsu Manifolds

In this section, we examine ARYS within the framework of certain classes of almost Kenmotsu manifolds. Firstly, we prove that a complete Kenmotsu manifold, admitting ARYS with $\alpha \neq 0$ is locally isometric to hyperbolic space $\mathbb{H}^{2n+1}(-1)$ when Reeb vector field leaves the scalar curvature invariant. Secondly, we show that ARYS on the Kenmotsu manifold reduces to Ricci-Yamabe soliton under the certain conditions on the soliton function. Next, it is proved that if a $(\kappa,\mu)'$-almost Kenmotsu manifold with $h' \neq 0$ admits gradient ARYS then either it is locally isometric to $\mathbb{H}^{2n+1}(-4) \times \mathbb{R}^n$ or potential vector field is pointwise collinear with the Reeb vector field. Moreover, 3-dimensional non-Kenmotsu almost Kenmotsu manifolds admitting gradient ARYS are considered. Several examples have been constructed of ARYS on different classes of warped product spaces.

4.2.1 On Normal almost Kenmotsu manifold

In this section, we deal with a normal almost Kenmotsu manifold, that is, Kenmotsu manifold admitting ARYS and gradient ARYS. Firstly, we give some examples of gradient ARYS.

Example 4.2. *Let (N, J, g_0) be a Kähler manifold of dimension $2n$. Consider the warped product $(M, g) = (\mathbb{R} \times_\sigma N, dt^2 + \sigma^2 g_0)$, where t is the coordinate on $\mathbb{R}$. We set $\eta = dt$, $\xi = \frac{\partial}{\partial t}$ and (1,1) tensor field φ by $\varphi X = JX$ for vector field X on N and $\varphi X = 0$ if X is tangent to $\mathbb{R}$. The above warped product with the structure (φ, ξ, η, g) is a Kenmotsu manifold (Kenmotsu, 1972). In particular, if we take $N = \mathbb{CH}^{2n}$, then N being Einstein, the Ricci tensor of M becomes $S^M = -2ng$. Then it is easy to verify that (M, f, g, λ) is an ARYS for $f(x, t) = ke^t, k > 0$ and $\lambda(x, t) = -2n\alpha - n\beta(2n + 1) + ke^t$.*

Therefore, a large number of examples can be constructed by considering different potential functions f on warped product spaces. Next, we constructed an example by using Kanai's result (Kanai, 1983).

Example 4.3. *Let N^{2n} be a complete Einstein Kähler manifold with $S^N = -(2n - 1)g_0$. Now consider the warped product $M^{2n+1} = \mathbb{R} \times_{cosht} N^{2n}$ with the metric $g = dt^2 + (cosht)^2 g_0$. Then by using the result by Kanai (1983), there exists a function f on M without critical points satisfying $\nabla^2 f = -fg$. Then it is easy to see that $(M, g, \nabla f, \lambda)$ is an ARYS for $\lambda = -2n\alpha - f - n\beta(2n + 1)$.*

Ghosh (2011) initiated the study of Ricci soliton in Kenmotsu 3-manifold. He later studied the gradient almost Ricci soliton in the Kenmotsu manifold and obtained Theorem 3 (Ghosh, 2019b). Here, we generalized these results for ARYS and prove them.

Theorem 4.4. *If the metric of a Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ admits a gradient ARYS with $\alpha \neq 0$, then it is η-Einstein. Moreover, if M is complete and*

ξ leaves the scalar curvature invariant then it is locally isometric to Hyperbolic space $\mathbb{H}^{2n+1}(-1)$. Also, λ can be expressed locally as $\lambda = Acosht + Bsinht - 2n\alpha - n\beta(2n+1)$, where A, B are constants on M.

Proof. Suppose the metric g of Kenmtosu manifold admits gradient Ricci-Yamabe soliton, then from (1.46) we have

$$\nabla_X Df = \sigma X - \alpha QX, \tag{4.42}$$

for any vector field X on M and $\sigma = \lambda - \frac{\beta r}{2}$ is a smooth function on M. Taking an inner product of (4.42) along arbitrary vector field Y, we obtain:

$$\nabla_Y \nabla_X Df = (Y\sigma)X + \sigma(\nabla_Y X) - \alpha(\nabla_Y Q)X - \alpha Q(\nabla_Y X). \tag{4.43}$$

Making use of (4.43) in the well-known formula $R(X,Y)Df = \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df$ yields

$$R(X,Y)Df = (X\sigma)Y - (Y\sigma)X - \alpha[(\nabla_X Q)Y - (\nabla_Y Q)X]. \tag{4.44}$$

Taking a covariant derivative of (1.30) and using (1.27), we get $(\nabla_X Q)\xi = -2n(X - \eta(X)\xi)$. Because of this the inner product of (4.44) with ξ gives

$$g(R(X,Y)Df, \xi) = (X\sigma)\eta(Y) - (Y\sigma)\eta(X). \tag{4.45}$$

Now, taking an inner product of (1.29) with Df yields

$$g(R(X,Y)\xi, Df) = (Yf)\eta(X) - (Xf)\eta(Y). \tag{4.46}$$

Combining (4.45) and (4.46) and replacing Y by ξ in the obtain relations we obtain

$$d(\sigma - f) = \xi(\sigma - f)\eta, \tag{4.47}$$

where d is the exterior derivative. This means that $\sigma - f$ is invariant along the

distribution $\mathcal{D}$ (i.e., $\mathcal{D} = ker\eta$) hence $\sigma - f$ is constant for all $X \in \mathcal{D}$.
Contracting (4.44) infer

$$S(Y, Df) = -2n(Y\sigma) + \frac{\alpha}{2}(Yr), \tag{4.48}$$

for any vector field Y on M. Replacing Y by ξ in (4.44) and taking an inner product with Y gives

$$g(R(X,\xi)Df, Y) = (X\sigma)\eta(Y) - (\xi\sigma)g(X,Y) - \alpha S(X,Y) + 2n\alpha g(X,Y). \tag{4.49}$$

As a consequence of (1.27) and (1.29) in (4.49), we get

$$[(Xf) - (X\sigma)]\eta(Y) + \xi(\sigma - f)g(X,Y) + \alpha S(X,Y) + 2n\alpha g(X,Y) = 0. \tag{4.50}$$

Contracting (4.50) over X gives

$$2n\xi(\sigma - f) + \alpha[r + 2n(2n+1)] = 0. \tag{4.51}$$

Replacing Y by ξ in (4.48) and making use of (4.51) and (1.30), we see that $\xi r = -2(r + 2n(2n+1))$, for $\alpha \neq 0$. In consequence, (4.51) in (4.47) gives

$$d(\sigma - f) = -\alpha(\frac{r}{2n} + 2n + 1)\eta. \tag{4.52}$$

Applying Poincare lemma and using the fact that $d\eta = 0$ on (4.52), we obtain $-\alpha dr \wedge \eta = 0$, and making use of the value of ξr we have

$$Dr = -2(r + 2n(2n+1))\xi. \tag{4.53}$$

Taking an inner product of (4.52) with vector field X, then inserting it along with (4.51) in (4.50) we get

$$QX = (\frac{r}{2n} + 1)X - (\frac{r}{2n} + 2n + 1)\eta(X)\xi, \tag{4.54}$$

for any vector field X on M. Therefore, M is η-Einstein.

Suppose that $\xi r = 0$, i.e., ξ leaves the scalar curvature invariant. In consequence of this we get $r = -2n(2n+1)$, a constant. Inserting this in (4.54) implies $QX = -2nX$ i.e., M is Einstein. Now suppose that M is complete. As r is constant, (4.52) gives $Df = D\lambda$. In consequence of this, (4.42) becomes

$$\nabla_X D\lambda = (\lambda + k)X, \tag{4.55}$$

where $k = n(2\alpha + \beta(2n+1))$. Applying Tashiro's theorem (Tashiro, 1965), we see that it is locally isometric to hyperbolic space $\mathbb{H}^{2n+1}(-1)$. Replacing X by ξ and taking inner product with ξ, (4.55) gives $\xi(\xi\lambda) = \lambda + k$. But as we know that a Kenmotsu manifold is locally isometric to the warped product $(-\epsilon, \epsilon) \times_{ce^t} N$, where N is a Kähler manifold of dimension $2n$ and $(-\epsilon, \epsilon)$ is an open interval. Using the local parametrization: $\xi = \frac{\partial}{\partial t}$ (where t is the coordinate on $(-\epsilon, \epsilon)$) we get from (4.55)

$$\frac{\partial^2 \lambda}{\partial t^2} = \lambda + 2n\alpha + n\beta(2n+1)$$

Its solution can be exhibited as $\lambda = Acosht + Bsinht - 2n\alpha - n\beta(2n+1)$, where A, B are constants on M. This completes the proof. □

Lemma 4.1. *If the metric of a Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)(n > 1)$ admits ARYS then*

1. $\xi(\xi\lambda) + \xi\lambda = 2(2n\alpha + \lambda + n\beta(2n+1))$.

2. $D\lambda = (\xi\lambda)\xi + \beta\{(r + 2n(2n+1))\xi + \frac{Dr}{2}\}$.

Proof. Taking the covariant derivative of (1.45) along arbitrary vector field X, we get

$$(\nabla_X \mathcal{L}_V g)(Y, Z) = 2(X\sigma)g(Y, Z) - 2\alpha(\nabla_X S)(Y, Z), \tag{4.56}$$

where $\sigma = \lambda - \frac{\beta r}{2}$. We know the following commutative formula (Yano, 1970):

$$\begin{aligned}&(\mathcal{L}_V \nabla_X g - \nabla_X \mathcal{L}_V g - \nabla_{[V,X]} g)(Y, Z)\\ &= -g((\mathcal{L}_V \nabla)(X, Y), Z) - g((\mathcal{L}_V \nabla)(X, Z), Y),\end{aligned} \tag{4.57}$$

for all vector fields X, Y, Z on M. Since g is parallel with respect to Levi-Civita connection ∇, the above relation becomes:

$$(\nabla_X \mathcal{L}_V g)(Y, Z) = g((\mathcal{L}_V \nabla)(X, Y), Z) + g((\mathcal{L}_V \nabla)(X, Z), Y). \tag{4.58}$$

We know that $\mathcal{L}_V \nabla$ is a symmetric tensor of type (1,2) and so it follows from (4.58) that

$$2g((\mathcal{L}_V \nabla)(X, Y), Z) = (\nabla_X \mathcal{L}_V g)(Y, Z) + (\nabla_Y \mathcal{L}_V g)(Z, X) - (\nabla_Z \mathcal{L}_V g)(X, Y) \tag{4.59}$$

Inserting (4.56) in (4.59), then replacing Y by ξ we obtain

$$(\mathcal{L}_V \nabla)(X, \xi) = 2\alpha QX + (4n\alpha + \xi\sigma)X + g(X, D\sigma)\xi - \eta(X)D\sigma. \tag{4.60}$$

Taking the covariant derivative of (4.60) along arbitrary vector field Y gives

$$\begin{aligned}
&(\nabla_Y \mathcal{L}_V \nabla)(X, \xi) + (\mathcal{L}_V \nabla)(X, Y) - \eta(Y)(\mathcal{L}_V \nabla)(X, \xi) \\
&= 2\alpha(\nabla_Y Q)X + Y(\xi\sigma)X + g(X, \nabla_Y D\sigma)\xi \\
&- g(X, D\sigma)\varphi^2 Y + g(X, \varphi^2 Y)D\sigma - \eta(X)(\nabla_Y D\sigma).
\end{aligned} \tag{4.61}$$

Making use of this in the formula (Yano, 1970)

$$(\mathcal{L}_V R)(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z)$$

we obtain

$$\begin{aligned}
&(\mathcal{L}_V R)(X, Y)\xi = 2\alpha\{(\nabla_X Q)Y - (\nabla_Y Q)X\} + X(\xi\sigma)Y - Y(\xi\sigma)X \\
&+ g(Y, D\sigma)X - g(X, D\sigma)Y + \eta(X)\nabla_Y D\sigma - \eta(Y)\nabla_X D\sigma \\
&+ 2\alpha\{\eta(X)QY - \eta(Y)QX\} + (4\alpha n + \xi\sigma)\{\eta(X)Y - \eta(Y)X\}.
\end{aligned} \tag{4.62}$$

Now differentiating $\xi\sigma = g(\xi, D\sigma)$ along vector field X and using (1.27) we get

$$X(\xi\sigma) = g(X, D\sigma) - (\xi\sigma)\eta(X) + g(\nabla_X D\sigma, \xi). \tag{4.63}$$

Replacing Y by ξ in (1.45), then inserting it in the Lie-derivative of (1.29) yields

$$(\mathcal{L}_V R)(X,Y)\xi + R(X,Y)\mathcal{L}_V\xi = g(X,\mathcal{L}_V\xi)Y$$
$$-g(Y,\mathcal{L}_V\xi)X + 2(\sigma + 2\alpha n)\{\eta(X)Y - \eta(Y)X\}. \tag{4.64}$$

Combining (4.62), (4.63) and (4.64), we obtain

$$\begin{aligned}&g(X,\mathcal{L}_V\xi)Y - g(Y,\mathcal{L}_V\xi)X - R(X,Y)\mathcal{L}_V\xi\\ &= 2\alpha\{(\nabla_X Q)Y - (\nabla_Y Q)X + \eta(X)QY\\ &- \eta(Y)QX\} + g(\nabla_X D\sigma,\xi)Y - g(\nabla_Y D\sigma,\xi)X\\ &+ \eta(X)\nabla_Y D\sigma - \eta(Y)\nabla_X D\sigma - 2\sigma\{\eta(X)Y - \eta(Y)X\}.\end{aligned} \tag{4.65}$$

Replacing X and Y by φX and φY in (4.65) then contracting the obtained equation and using Lemma 4.2 (Ghosh, 2020b) results in

$$S(Y,\mathcal{L}_V\xi) + 2ng(Y,\mathcal{L}_V\xi) = \alpha(Yr) + 2\alpha(r + 4n^2 + 2n)\eta(Y) - g(\nabla_\xi D\lambda, \varphi^2 Y).$$

Contracting (4.65) and combining it with the forgoing equation yields

$$2(n-1)g(\nabla_\xi D\sigma, Y) + \xi(\xi\sigma)\eta(Y) + \eta(Y) div D\sigma = 4n(2n\alpha + \sigma)\eta(Y). \tag{4.66}$$

Replacing Y by ξ in (4.66), we get $(2n-1)\xi(\xi\sigma) + divD\sigma = 4n(2n\alpha + \sigma)$. In view of this in (4.66) infer $g(\nabla_\xi D\sigma, X) = \xi(\xi\sigma)\eta(X)$ for $n > 1$. Now taking ξ instead of Y in (4.65) and making use of the above relations we obtain

$$\nabla_X D\sigma = -2(2n\alpha + \sigma)\varphi^2 X + \xi(\xi\sigma)\varphi^2 X + \xi(\xi\sigma)\eta(X)\xi. \tag{4.67}$$

As a consequence of (4.67), the expression of the curvature tensor is as follows:

$$R(X,Y)D\sigma = 2(Y\sigma)\varphi^2 X - 2(X\sigma)\varphi^2 Y + Y(\xi(\xi\sigma))X$$
$$-X(\xi(\xi\sigma))Y + 2(\xi(\xi\sigma) - \sigma - 2n\alpha)\{\eta(Y)X - \eta(X)Y\}. \tag{4.68}$$

Replacing Y by ξ in (4.68), then inserting the obtained equation back in (4.68)

gives

$$R(X,Y)D\sigma = (X\sigma)Y - (Y\sigma)X - 2\{(X\sigma)\eta(Y)\xi$$
$$- (Y\sigma)\eta(X)\xi\} + \{\xi(\xi(\xi\sigma)) - \xi\sigma\}\{\eta(Y)X - \eta(X)Y\}$$
$$+ 2\{\xi(\xi\sigma) - \sigma - 2n\alpha\}\{\eta(Y)X - \eta(X)Y\}. \tag{4.69}$$

Replacing X and Y by φX and φY in (4.69) and then contracting the obtained result yields

$$S(Y, D\sigma) = -2ng(Y, D\sigma).$$

As a consequence of this in the contraction of (4.69) and further replacing Y by φY in the obtained expression yields $\varphi D\sigma = 0$. Differentiating this along vector field X and inserting it in (4.67) along with the fact that $\sigma = \lambda - \frac{\beta r}{2}$ and (1.27) gives

$$\xi(\xi\lambda) + \xi\lambda = 2(2n\alpha + \lambda + n\beta(2n+1)). \tag{4.70}$$

This completes the proof. □

Theorem 4.5. *Let $M^{2n+1}(\varphi, \xi, \eta, g)(n > 1)$ be a Kenmotsu manifold whose metric represents an ARYS. If $Hess_\lambda(\xi, \xi)$ is constant along Reeb vector field then it reduces to Ricci-Yamabe soliton with $\lambda = -2n\alpha - n\beta(2n+1)$.*

Proof. By hypothesis, $Hess_\lambda(\xi, \xi)$ is constant along the Reeb vector field ξ i.e., $\xi(\xi(\xi\lambda)) = 0$ implies $\xi(\xi\lambda)$ is constant along ξ. In view of this in the covariant derivative of first relation in Lemma (4.1) along ξ, we get $\xi(\xi\lambda) = 2(\xi\lambda)$. Again differentiating this along ξ yields $\xi\lambda = 0$, that is, λ is constant along ξ. Making use of this in first relation of Lemma (4.1) gives $\lambda = -2n\alpha - n\beta(2n+1)$. Therefore ARYS reduces to Ricci-Yamabe soliton. This completes the proof. □

Remark 4.1. *The above Theorem 4.5 is a generalization of Theorem 4.1 in Ghosh (2020b), where he obtained the condition under which a Ricci almost soli-*

ton reduces to an expanding Ricci soliton with $\lambda = -2n$. It is easy to see that for $\alpha = 1$ and $\beta = 0$ Theorem 4.1 (Ghosh, 2020b) can be obtained from Theorem 4.5. Moreover, the first condition of Theorem 4.1 (Ghosh, 2020b) is also true for this choice of scalars.

A Kenmotsu manifold is said to be η-Einstein if there exists smooth functions a and b such that

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{4.71}$$

for all vector field X, Y on M. If $b = 0$, then M becomes an Einstein manifold.

Theorem 4.6. *If the metric of an η-Einstein Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)(n > 1)$ admits Ricci-Yamabe soliton with $\alpha \neq 0$ then it is Einstein with constant scalar curvature $r = -2n(2n+1)$, provided $2\alpha + n\beta \neq 0$.*

Proof. Replacing Y by ξ in (4.71) and using (1.30), we get $a + b = -2n$. Then contracting (4.71) gives $r = (2n+1)a + b$. In view of this (4.71) becomes

$$S(X, Y) = (\frac{r}{2n} + 1)g(X, Y) - (\frac{r}{2n} + 2n + 1)\eta(X)\eta(Y), \tag{4.72}$$

for any vector field X, Y on M. Making use of (4.72) in (1.45) yields

$$(\mathcal{L}_V g)(Y, Z) = \{2\lambda - \beta r - 2\alpha(\frac{r}{2n} + 1)\}g(Y, Z) + 2\alpha\{(2n+1) + \frac{r}{2n}\}\eta(Y)\eta(Z) \tag{4.73}$$

Taking the covariant derivative of (4.73) along arbitrary vector field X we obtain

$$\begin{aligned}(\nabla_X \mathcal{L}_V g)(Y, Z) &= -(\frac{\alpha}{n} + \beta)(Xr)g(Y, Z) + \frac{\alpha}{n}(Xr)\eta(Y)\eta(Z) \\ &+2\alpha(\frac{r}{2n} + 2n + 1)\{g(X, Y)\eta(Z) + g(X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)\}.\end{aligned} \tag{4.74}$$

Making use of (4.74) in (4.59) yields

$$\begin{aligned}2(\mathcal{L}_V\nabla)(X,Y) = &-(\frac{\alpha}{n}+\beta)\{(Xr)Y+(Yr)X-g(X,Y)Dr\}\\ &+\frac{\alpha}{n}\{(Xr)\eta(Y)\xi+(Yr)\eta(X)\xi-\eta(X)\eta(Y)Dr\}\\ &+4\alpha(2n+1+\frac{r}{2n})\{g(X,Y)\xi-\eta(X)\eta(Y)\xi\}. \end{aligned}\tag{4.75}$$

Setting $X=Y=e_i$ where $e_i: i=1,2,..,2n+1$ is an orthonormal frame in (4.75) and summing over i, we get

$$\begin{aligned}2\sum_{i=1}^{2n+1}\varepsilon_i(\mathcal{L}_V\nabla)(e_i,e_i) = &-\{\beta(1-2n)+\frac{\alpha}{n}(1-n)\}Dr\\ &+\frac{2\alpha}{n}(\xi r)\xi+8n\alpha(\frac{r}{2n}+2n+1)\xi. \end{aligned}\tag{4.76}$$

Now taking the covariant derivative of (4.73) give

$$(\nabla_X\mathcal{L}_Vg)(Y,Z)=-\beta(Xr)g(Y,Z)-2\alpha(\nabla_XS)(Y,Z),$$

which on contracting yields $2\sum_{i=1}^{2n+1}\varepsilon_i(\mathcal{L}_V\nabla)(e_i,e_i)=\beta(2n-1)Dr$. In consequence of this in (4.76), we obtain

$$2\alpha(n-1)Dr+2\alpha(\xi r)\xi+8n\alpha(\frac{r}{2n}+2n+1)\xi=0. \tag{4.77}$$

Taking an inner product of (4.77) with ξ, we get $\xi r=-2(r+2n(2n+1))$ for $\alpha\neq 0$. In view of this, (4.77) yields $Dr=(\xi r)\xi$ for $n>1$. Then, replacing Y by ξ in (4.75) results in

$$2(\mathcal{L}_V\nabla)(X,\xi)=-\beta(Xr)\xi+(\frac{\alpha}{n}+\beta)(\xi r)\varphi^2X. \tag{4.78}$$

Taking the covariant derivate of (4.78), then inserting it in Yano's result (Yano, 1970):

$$(\mathcal{L}_VR)(X,Y)Z=(\nabla_X\mathcal{L}_V\nabla)(Y,Z)-(\nabla_Y\mathcal{L}_V\nabla)(X,Z)$$

, for $Z=\xi$ yields

$$2(\mathcal{L}_V R)(X,Y)\xi = -\beta\{X(Yr)\xi - Y(Xr)\xi\} - \beta\{(Xr)\varphi^2 Y - (Yr)\varphi^2 X\}$$
$$+(\frac{\alpha}{n}+\beta)\{X(\xi r)\varphi^2 Y - Y(\xi r)\varphi^2 X + 2(\xi r)\{\eta(Y)X - \eta(X)Y\}\}. \quad (4.79)$$

Contracting (4.79) over X gives $(\mathcal{L}_V S)(Y,\xi) = -n\beta(\xi r)\eta(Y)$. In consequence of this in the Lie-derivative of (1.30), we obtain

$$(\frac{r}{2n}+1)g(Y,\mathcal{L}_V\xi) - (\frac{r}{2n}+2n+1)\eta(Y)\eta(\mathcal{L}_V\xi) =$$
$$n\beta(\xi r)\eta(Y) - 2n(2\lambda - \beta r + 4\alpha n)\eta(Y) - 2ng(Y,\mathcal{L}_V\xi). \quad (4.80)$$

Replacing Y by ξ in (4.80) then inserting back in (4.80) gives $\lambda = -2\alpha n - n\beta(2n+1)$. In view of this in (4.73) we get $\eta(\mathcal{L}_V\xi) = -\frac{\beta}{2}(r+2n(2n+1))$. In consequence of this and $Dr = (\xi r)\xi$ in the Lie-derivative of $S(X,\xi) = -2n\eta(X)$ we get

$$(r+2n(2n+1))\{2\mathcal{L}_V\xi - \frac{\beta}{2}(\xi r)\xi\} = 0. \quad (4.81)$$

Thus we get either $r = -2n(2n+1)$ in this case M is Einstein or $\mathcal{L}_V\xi = \frac{\beta}{4}(\xi r)\xi$. Suppose $r \neq -2n(2n+1)$ in some open set $\mathcal{O}$ on M. Then using (1.27) and (4.81) implies

$$\nabla_\xi V = V - \eta(V)\xi - \frac{\beta}{4}(\xi r)\xi. \quad (4.82)$$

Taking $Y=\xi$ in the commutative formula $(\mathcal{L}_V\nabla)(X,Y) = \nabla_X\nabla_Y V - \nabla_{\nabla_X Y}V + R(V,X)Y$ and using (4.82), (4.81) and (4.78), we obtain

$$(2\alpha + n\beta)(\xi r)\varphi^2 X = 0,$$

for any vector field X on $\mathcal{O}$. This shows that $\xi r = 0$, that is, $r = -2n(2n+1)$, a contradiction. This completes the proof. □

In particular, if we take scalar $\alpha = 1$ and $\beta = 0$ then in regard to the Theorem 4.6, we can state the following:

Corollary 4.3 (Ghosh, 2013). *If the metric of an η-Einstein Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g), n > 1$ is a Ricci soliton then it is Einstein and the soliton is expanding.*

For the case $\alpha = 1$ and $\beta = -2\rho$, we can state the following:

Corollary 4.4. *If the metric of an η-Einstein Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)(n > 1)$ admits ρ-Einstein soliton then it is Einstein with constant scalar curvature $r = -2n(2n+1)$, provided $\rho \neq \frac{1}{n}$.*

It is known that the warped product $\mathbb{R} \times_{ce^t} V(k)$ where $V(k)$ is a Kähler manifold of constant holomorphic sectional curvature of dimension $2n$ admits Kenmotsu structure (Kenmotsu, 1972)). Moreover, its sectional tensor is given by (Bishop and O'Neill, 1969)

$$\begin{aligned}
R(X,Y)Z &= H\{g(Y,Z)X - g(X,Z)Y\} + (H+1)\{g(X,Z)\eta(Y)\xi \\
&- g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,\varphi Z)\varphi Y \\
&- g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\}.
\end{aligned}$$

Contracting the above equation we see that

$$S(X,Y) = 2\{(n-1)H - 1\}g(X,Y) - 2(n-1)(H+1)\eta(X)\eta(Y),$$

that is, it is η-Einstein. Now making use of Theorem (4.6) we get $H = -1$. Hence, we can state the following:

Corollary 4.5. *If the metric of the warped product $\mathbb{R} \times_{ce^t} V(k)$, $(n > 1)$ is a Ricci-Yamabe soliton with $\alpha \neq 0$ then it is of constant curvature -1, provided $2\alpha + n\beta \neq 0$.*

4.2.2 On Non-Normal almost Kenmotsu manifolds

First, we give some examples of almost Kenmotsu manifold admitting ARYS.

Example 4.4. *Let $(N, J, \bar{g})$ be a strictly almost Kähler Einstein manifold. We take $\eta = dt$, $\xi = \frac{\partial}{\partial t}$ and (1,1)-tensor φ by $\varphi X = JX$ for vector field X on N and $\varphi X = 0$ if X is tangent to $\mathbb{R}$. Then it is known that $(M, g) = (\mathbb{R} \times_{ce^t} N, g_0 + ce^{2t}\bar{g})$ together with the structure (φ, ξ, η, g) is an almost Kenmotsu manifold (Kenmotsu, 1972). Also since N is Einstein, we see $S^M = -2ng$. We define a smooth function $f(x, t) = t^2$ then it is easy to see that (M, g, f, λ) is an ARYS for $\lambda(x, t) = -2n\alpha - n\beta(2n + 1) + 2$.*

We can also construct an example of ARYS in almost Kenmotsu manifold constructed by Barbosa-Ribeiro (2013).

Example 4.5. *On the warped product $M = \mathbb{R} \times_{\sigma(t)} \mathbb{H}^{2n}$ consider the metric $g = dt^2 + \sigma^2(t)g_0$, where g_0 is the standard metric on the hyperbolic space $\mathbb{H}^{2n}$. Then by Algere et al. (2004) result, it is easy to see that it is almost Kenmotsu manifold. Let $\sigma(t) = cosht$ and $f(x, t) = sinht$ then (M, g, f, λ) is an ARYS with $\lambda = sinht - n\beta(2n + 1) - 2n\alpha$.*

Here, we consider the gradient almost Ricci-Yamabe soliton in the context of $(\kappa, \mu)'$-almost Kenmotsu manifold and generalized Theorem 3.5 (Dey, 2020) and Theorem 3.1 (Wang, 2016). We state and prove the following:

Theorem 4.7. *If $M^{2n+1}(\varphi, \xi, \eta, g)$ be a $(\kappa, \mu)'$-almost Kenmotsu manifold with $h' \neq 0$ admitting gradient ARYS then either M is locally isometric to $\mathbb{H}^{2n+1}(-4) \times \mathbb{R}^n$ or potential vector field is pointwise collinear with the Reeb vector field.*

Proof. Suppose that $(\kappa, \mu)'$-almost Kenmotsu manifold admits gradient ARYS then Eq. (4.42)-(4.44) is valid. Taking an inner product of (4.44) with ξ and inserting Lemma 3.1, we obtain

$$g(R(X, Y)Df, \xi) = (X\lambda)\eta(Y) - (Y\lambda)\eta(X) - \alpha\{g(Qh'Y, X) - g(Qh'X, Y)\}. \tag{4.83}$$

Taking an inner product of (1.21) with Df, then inserting it in (4.83) and replacing X by ξ gives

$$-(\xi\lambda)\xi + D\lambda = \kappa\{(\xi f)\xi - Df\} - \mu h' Df. \tag{4.84}$$

Contracting (4.44) over X, we get $QDf = -2nD\lambda$. As a consequence of this in Lemma 3.1 gives

$$D\lambda - Df + (\kappa + 1)(\xi f)\xi = h' Df. \tag{4.85}$$

Combining (4.84) and (4.85), we obtain

$$\kappa\{(\xi f)\xi - Df\} + D\lambda + (\xi\lambda)\xi - 2Df + 2(\kappa + 1)(\xi f)\xi = 0. \tag{4.86}$$

Operating the forgoing equation by φ yields

$$\varphi D\lambda - (\kappa + 2)\varphi Df = 0,$$

implies,

$$D\lambda - (\kappa + 2)Df \in \mathbb{R}\xi.$$

Therefore, we can write $D\lambda = (\kappa + 2)Df + s\xi$, where s is a smooth function. In view of this in (4.84) infer

$$2(\kappa + 1)Df + (s - \xi\lambda - \kappa\xi f)\xi = 2h' Df. \tag{4.87}$$

Operating (4.87) by h' and then inserting the obtained expression in (4.87) gives

$$(\kappa + 2)\varphi Df = 0.$$

Thus we have either $\kappa = -2$ or $Df = (\xi f)\xi$.

Suppose $\kappa = -2$. Then without loss of generality, we may choose $\nu = 1$. Then we have from Theorem 5.1 (Pastore and Saltarelli, 2011) we get

$$R(X_\nu, Y_\nu)Z_\nu = -4[g(Y_\nu, Z_\nu)X_\nu - g(X_\nu, Z_\nu)Y_\nu],$$

$$R(X_{-\nu}, Y_{-\nu})Z_{-\nu} = 0,$$

for any $X_\nu, Y_\nu, Z_\nu \in [\nu]'$ and $X_{-\nu}, Y_{-\nu}, Z_{-\nu} \in [-\nu]'$. As a consequence of this, Proposition 4.1 and Proposition 4.3 (Pastore and Saltarelli, 2011) along with $\nu = 1$ show that it is locally isometric to $\mathbb{H}^{2n+1}(-4) \times \mathbb{R}^n$. This completes the proof. □

Suppose $\kappa \neq -2$. Then in regard to Theorem 4.7, we have $V = Df = (\xi f)\xi$. Take $F = \xi f$ and taking the covariant derivative of $V = F\xi$ along arbitrary vector field X we get

$$\nabla_X V = (XF)\xi + F(-\varphi^2 X + h'X).$$

Making use of this in (1.45) yields

$$\begin{aligned}(XF)\eta(Y) + (YF)\eta(X) + 2Fg(X, Y) - 2F\eta(X)\eta(Y) \\ +2Fg(h'X, Y) = (2\lambda - \beta r)g(X, Y) - 2\alpha S(X, Y).\end{aligned} \tag{4.88}$$

Replacing Y by ξ in (4.88) gives

$$XF = (2\lambda - \beta r - 4n\alpha\kappa - \xi F)\eta(X), \tag{4.89}$$

for any vector field X on M. Contracting (4.88) then inserting it in (4.89) and replacing X by ξ in the obtained expression we obtain

$$F = \lambda - \frac{\beta r}{2} + 2n\alpha. \tag{4.90}$$

Inserting (4.89) in (4.88) and comparing it with Lemma 3.1 yields $(F - 2n\alpha)(\kappa + 1)\varphi^2 X = 0$ for any X on M. As $\kappa < -1$, we see that $F = 2n\alpha$, in view of this in (4.90) implies $\lambda = \frac{\beta r}{2}$ i.e., a constant. Therefore, M reduces to gradient Ricci-Yamabe soliton. Hence using Corollary 3.7 (Dey, 2020) we can state the following:

Corollary 4.6. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a non-Kenmotsu $(\kappa, \mu)'$-almost Ken-*

motsu manifold with $\kappa \neq -2$ admitting a gradient ARYS then

1. *The potential vector field V is a constant multiple of ξ.*
2. *V is a strict infinitisimal contact transformation.*
3. *V leaves h' invariant.*

Consider a generalized $(\kappa, \mu)'$-almost Kenmotsu manifold of dimension three with $\kappa < -1$. If we assume that κ is invariant along ξ, then from (Proposition 3.2 (Pastore and Saltarelli, 2011)) we have $\xi(\kappa) = -2(\kappa + 1)(\mu + 2)$ implies $\mu = -2$. Moreover, from (Lemma 3.3 (Saltarelli, 2015)), we have $h'(grad\mu) = grad\kappa - \xi(\kappa)\xi$ which implies κ is constant under our assumption. Therefore M^3 becomes a $(\kappa, -2)'$-almost Kenmotsu manifold. By applying Theorem 4.7, we can conclude the following:

Corollary 4.7. *Let $M^3(\varphi, \eta, \xi, g)$ be a generalized non-Kenmotsu $(\kappa, \mu)'$-almost Kenmotsu manifold with $\kappa < -1$ invariant along the Reeb vector field admitting gradient ARYS then it is either locally isometric to the product space $\mathbb{H}^{2n+1}(-4) \times \mathbb{R}^n$ or potential vector field is pointwise collinear with the Reeb vector field.*

4.2.3 On 3-dimensional Almost Kenmotsu manifolds

Suppose that the non-trivial potential vector field of gradient ARY soliton $(\alpha \neq 0)$ is orthogonal to the Reeb vector field ξ, then we can write $V = f_1 e + f_2 \varphi e$, where f_1, f_2 are smooth functions. Replacing X by ξ in (4.42) and making use of Lemmas 3.2 and 3.3, we get

$$\begin{cases} -2\alpha(\vartheta^2 + 1) = \lambda - \frac{\beta r}{2}, \\ \varphi e(\vartheta) = -2b\vartheta, \\ e(\vartheta) = -2c\vartheta. \end{cases} \tag{4.91}$$

Similarly, taking $X = e$ in (4.42) gives

$$\begin{cases} \vartheta f_1 - f_2 = 0, \\ e(f_1) + bf_2 - \alpha(A + 2a\vartheta) = \lambda - \frac{\beta r}{2}, \\ e(f_2) - bf_1 + \alpha(\xi(\vartheta) + 2\vartheta) = 0. \end{cases} \tag{4.92}$$

Also for $X = \varphi e$, we obtain

$$\begin{cases} \vartheta f_1 - f_2 = 0, \\ \varphi e(f_1) - cf_2 + \alpha(\xi(\vartheta) + 2\vartheta) = 0, \\ cf_1 + \varphi e(f_2) - \alpha(A - 2a\vartheta) = \lambda - \frac{\beta r}{2}. \end{cases} \tag{4.93}$$

Comparing the first arguments of (4.92) and (4.93), we see that $(\vartheta^2 - 1)f_1 = 0$. If $f_1 = 0$, then from first argument of (4.93) we get $f_2 = 0$, which further implies $V = 0$, a contradiction. Therefore, we must have $\vartheta = 1$. As a consequence of this in second and third statement of (4.91) yields $b = c = 0$ and first argument of (4.92) gives $f_1 = f_2$.

Combining the first equation of (4.91) with the second eqn. (4.92) and third eqn. (4.93), then making use of the fact that $\vartheta = 1$ yields

$$e(f_1) - \varphi e(f_1) = cf_1 - bf_1 + 4\alpha a = 0, \tag{4.94}$$

where we use $f_1 = f_2$. Similarly from (4.92) and (4.93), one can get

$$\varphi e(f_1) - cf_1 = e(f_1) - bf_1. \tag{4.95}$$

Making use of (4.94) and (4.95), together with $b = c = 0$ gives $a = 0$. In consequence, Eq. (3.74) becomes

$$[\xi, e] = \varphi e - e, \quad [e, \varphi e] = 0, \quad [\varphi e, \xi] = -e + \varphi e.$$

Using Milnor's result (Milnor, 1976) we can conclude that M^3 is locally isometric to a non-unimodular Lie group with a left-invariant almost Kenmotsu structure. Moreover, it is obvious that $\nabla_\xi h = 0$ and it is conformally flat with constant scalar curvature $r = -8$. Now making use of Wang's result (Wang, 2017) which state that, *"An almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = 0$ is conformally flat with constant scalar curvature if and only if it is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$"* we can state the following:

Theorem 4.8. *If a 3-dimensional non-Kenmotsu almost Kenmotsu manifold admits a gradient ARYS ($\alpha \neq 0$) whose non-trivial potential vector field is orthogonal to the Reeb vector field, then it is locally isometric to the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$.*

Chapter 5

On Invariant Submanifolds and Chen's Inequalities

Chapter 5

On Invariant Submanifolds and Chen's Inequalities

This chapter is divided into two sections. The first section involved the study of invariant submanifolds of the f-Kenmotsu manifold and in the second section, we obtained Chen's inequalities for submanifolds of generalized Sasakian-space-forms endowed with the quarter-symmetric connection.

5.1 Invariant Submanifolds of f-Kenmotsu Manifolds

Definition 5.1. *A submanifold M of an f-Kenmotsu manifold $\tilde{M}$ is said to be invariant if the structure vector field ξ is tangent to M at every point of M and φX is tangent to M for any vector field X tangent to M at every point of M, that is, $\varphi(TM) \subset TM$ at every point of M.*

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It is easy to see that for invariant submanifolds of f-Kenmotsu manifolds, we have

$$\sigma(X, \xi) = 0, \tag{5.1}$$

for any $X \in \Gamma(TM)$.

Proposition 5.1. *Let M be an invariant submanifold of an f-Kenmotsu manifold $\tilde{M}$. Then the following relations hold:*

$$\nabla_X \xi = -f\varphi^2 X, \tag{5.2}$$

$$\varphi\sigma(X, Y) = \sigma(\varphi X, Y) = \sigma(X, \varphi Y), \tag{5.3}$$

$$(\nabla_X \varphi)Y = f(g(\varphi X, Y)\xi - \eta(Y)\varphi X), \tag{5.4}$$

$$R(X, Y)\xi = f^2(\eta(X)Y - \eta(Y)X) + \varphi^2(Yf) - \varphi^2(Xf), \tag{5.5}$$

where ∇, σ and R denote the induced Levi-Civita connection, shape operator and Riemannian curvature tensor of M, respectively.

Proof. By using (1.36), (1.37), (1.38), (1.51), (1.55) and (5.1) we can directly compute the required results. □

Thus we can state the following:

Lemma 5.1. *An invariant submanifold M of f-Kenmotsu manifold $\tilde{M}$ is again an f-Kenmotsu manifold.*

Lemma 5.2. *Any invariant submanifold M of f-Kenmotsu manifold $\tilde{M}$ is a minimal submanifold.*

Proof. Since an invariant submanifold M of f-Kenmotsu manifold $\tilde{M}$ is again f-Kenmotsu manifold, M is of odd dimension, say $(2n+1)$-dimension. Consider an orthonormal basis $e_1, ..., e_{2n+1}$ of M such that $e_{n+t} = \varphi e_t (t = 1, ..., n), e_{2n+1} = \xi$. Then by (5.1) and (5.4) we have

$$\sigma(\varphi e_i, \varphi e_i) = \varphi^2 \sigma(e_i, e_i) = -\sigma(e_i, e_i).$$

Now,

$$Tr(\sigma) = \sum_{i=1}^{2n+1} (\sigma(e_i, e_i) + \sigma(\varphi e_i, \varphi e_i)) + \sigma(\xi, \xi) = 0.$$

This shows that M is a minimal submanifold. □

Now $\tilde{R} \cdot \sigma$ is given by

$$(\tilde{R}(X,Y) \cdot \sigma)(Z,U) = R^{\perp}(X,Y)\sigma(Z,U) - \sigma(R(X,Y)Z,U) - \sigma(Z, R(X,Y)U), \tag{5.6}$$

for all $X, Y, Z, U \in \Gamma(TM)$, where

$$R^{\perp}(X,Y) = [\nabla^{\perp}_X, \nabla^{\perp}_Y] - \nabla^{\perp}_{[X,Y]}.$$

If $\tilde{R} \cdot \sigma = 0$, then the submanifold is called semiparallel. Arslan et al. ((1990) defined and studied submanifolds satisfying the condition $\tilde{R}(X,Y) \cdot \tilde{\nabla}\sigma = 0$ for all $X, Y \in \Gamma(TM)$ and called it as 2-semiparallel. We can write

$$(\tilde{R}(X,Y) \cdot \tilde{\nabla}\sigma)(Z,U,V) = R^{\perp}(X,Y)(\tilde{\nabla}\sigma)(Z,U,V) - (\tilde{\nabla}\sigma)(R(X,Y)Z,U,V) - (\tilde{\nabla}\sigma)(Z,R(X,Y)U,V) - (\tilde{\nabla}\sigma)(Z,U,R(X,Y)V), \tag{5.7}$$

for all $X, Y, Z, U, V \in \Gamma(TM)$ and $(\tilde{\nabla}\sigma)(Z,U,V) = (\tilde{\nabla}_Z \sigma)(U,V)$.

For a $(0,k)$-type tensor field T, $k \geq 1$ and a $(0,2)$-type tensor field G on a Riemannian manifold M. $D(G,T)$-tensor field is defined by (Atceken and Uygun, 2021)

$$D(G,T)(X,Y,...T_k; X,Y) = -T((X \wedge_G Y)X, Y, ...T_k) .... - T(X, Y, ...T_{k-1}, (X \wedge_G Y)T_k) \tag{5.8}$$

for all $X, Y, ...T_k, X, Y \in \Gamma(TM)$, where

$$(X \wedge_G Y)Z = G(Y,Z)X - G(X,Z)Y.$$

Definition 5.2 (Atceken, 2021; Atceken et al., 2020)**.** *Let M be a submanifold of a Riemannian manifold $(\tilde{M}, g)$. If there exist functions L_1, L_2, L_3 and L_4 on $\tilde{M}$ such that*

$$\tilde{R} \cdot \sigma = L_1 D(g, \sigma), \tag{5.9}$$

$$\tilde{R} \cdot \tilde{\nabla}\sigma = L_2 D(g, \tilde{\nabla}\sigma), \tag{5.10}$$

$$\tilde{R} \cdot \sigma = L_3 D(S, \sigma), \tag{5.11}$$

$$\tilde{R} \cdot \tilde{\nabla}\sigma = L_4 D(S, \tilde{\nabla}\sigma), \tag{5.12}$$

then M is, respectively, pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci-generalized pseudoparallel submanifold. In particular, if $L_1 = 0$ or $L_3 = 0$ (resp., $L_2 = 0$ or $L_4 = 0$), then M is said to be semiparallel (resp., 2-semiparallel).

Next, we give some characterization theorems for totally geodesic submanifolds of f-Kenmotsu manifolds.

Theorem 5.1. *Let M be an invariant submanifold of an f-Kenmotsu manifold $\tilde{M}$. Then M is totally geodesic if and only if the second fundamental form is parallel, provided M is non-cosympletic.*

Proof. Suppose that the second fundamental form σ is parallel, then form (1.54) we have

$$(\tilde{\nabla}_X \sigma)(Y,Z) = \nabla_X^{\perp}\sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) = 0. \tag{5.13}$$

Replacing Z by ξ in (5.13) and using (1.9), (5.1) and (5.2) gives

$$f\sigma(X,Y) = 0.$$

As M is non-cosympletic, $f \neq 0$. Hence $\sigma = 0$. This completes the proof. $\square$

Theorem 5.2. *Let M be an invariant submanifold of an f-Kenmotsu manifold $\tilde{M}$. If M is a pseudoparallel submanifold then it is totally geodesic or the function L_1 satisfies $L_1 = -(f^2 + \xi f)$.*

Proof. Consider that M is pseudoparallel, then from (5.9) we have

$$(\tilde{R}(X,Y)\cdot\sigma)(Z,U) = L_1 D(g,\sigma)(Z,U;X,Y), \tag{5.14}$$

for all $X, Y, Z, U \in \Gamma(TM)$. Making use of (5.6) and (5.8) in (5.14) gives

$$\begin{aligned} R^{\perp}(X,Y)\sigma(Z,U) - \sigma(R(X,Y)Z,U) - \sigma(Z,R(X,Y)U) \\ = -L_1\{\sigma((X\wedge_g Y)Z,U) + \sigma(Z,(X\wedge_g Y)U)\}. \end{aligned} \tag{5.15}$$

Substituting X and Z by ξ and using (5.1) in (5.15) we get

$$(L_1 + f^2 + \xi f)\sigma(X,Y) = 0,$$

for all $X, Y \in \Gamma(TM)$. This completes the proof. $\square$

Corollary 5.1. *Let M be an invariant submanifold of a regular f-Kenmotsu manifold $\tilde{M}$. Then M is totally geodesic if and only if it is semiparallel.*

Theorem 5.3. *Let M be an invariant submanifold of an f-Kenmotsu manifold $\tilde{M}$. If M is 2-pseudoparallel submanifold then it is either totally geodesic or cosympletic or the function L_2 satisfies $L_2 = -(f^2 + \xi f)$.*

Proof. Under our assumption, from (5.10) we have

$$(\tilde{R}(X,Y)\cdot\tilde{\nabla}\sigma)(U,V,Z) = L_2 D(g,\tilde{\nabla}\sigma)(U,V,Z;X,Y), \tag{5.16}$$

for all $X, Y, Z, U, V \in \Gamma(TM)$. Making use of (5.6) and (5.8) in (5.16) gives

$$\begin{aligned} &R^{\perp}(X,Y)(\tilde{\nabla}_U\sigma)(V,Z) - (\tilde{\nabla}_{R(X,Y)U}\sigma)(V,Z) - (\tilde{\nabla}_U\sigma)(R(X,Y)V,Z) \\ &- (\tilde{\nabla}_U\sigma)(V,R(X,Y)Z) = -L_2\{(\tilde{\nabla}_{(X\wedge_g Y)U}\sigma)(V,Z) \\ &+ (\tilde{\nabla}_U\sigma)((X\wedge_g Y)V,Z) + (\tilde{\nabla}_U\sigma)(V,(X\wedge_g Y)Z)\}. \end{aligned} \tag{5.17}$$

Putting $X = V = \xi$ in (5.17) yields

$$\begin{aligned} &R^{\perp}(\xi,Y)(\tilde{\nabla}_U\sigma)(\xi,Z) - (\tilde{\nabla}_{R(\xi,Y)U}\sigma)(\xi,Z) - (\tilde{\nabla}_U\sigma)(R(\xi,Y)\xi,Z) \\ &- (\tilde{\nabla}_U\sigma)(\xi,R(\xi,Y)Z) = -L_2\{(\tilde{\nabla}_{(\xi\wedge_g Y)U}\sigma)(\xi,Z) \\ &+ (\tilde{\nabla}_U\sigma)((\xi\wedge_g Y)\xi,Z) + (\tilde{\nabla}_U\sigma)(\xi,(\xi\wedge_g Y)Z)\}. \end{aligned} \tag{5.18}$$

Computing each term of (5.18) individually and using (1.9), (1.54), (5.1), (5.2) and (5.5), we obtain the following:

$$\begin{aligned} (\tilde{\nabla}_{(\xi\wedge_g Y)U}\sigma)(\xi,Z) &= f\sigma(\varphi^2(\xi\wedge_g Y)U,Z) \\ &= f\eta(U)\sigma(Y,Z). \end{aligned} \tag{5.19}$$

$$\begin{aligned} (\tilde{\nabla}_U\sigma)((\xi\wedge_g Y)\xi,Z) &= (\tilde{\nabla}_U\sigma)(\eta(Y)\xi - Y,Z) \\ &= -\sigma(\nabla_U\xi,Z)\eta(Y) - (\tilde{\nabla}_U\sigma)(Y,Z) \\ &= f\sigma(\varphi^2 U,Z)\eta(Y) - (\tilde{\nabla}_U\sigma)(Y,Z) \\ &= -f\sigma(U,Z)\eta(Y) - (\tilde{\nabla}_U\sigma)(Y,Z). \end{aligned} \tag{5.20}$$

$$\begin{aligned} (\tilde{\nabla}_U\sigma)(\xi,(\xi\wedge_g Y)Z) &= -\sigma(\nabla_U\xi, g(Y,Z)\xi - \eta(Z)Y) \\ &= f\sigma(U,Y)\eta(Z). \end{aligned} \tag{5.21}$$

$$\begin{aligned} R^{\perp}(\xi,Y)(\tilde{\nabla}_U\sigma)(\xi,Z) &= -R^{\perp}(\xi,Y\sigma(\nabla_U\xi,Z) \\ &= -fR^{\perp}(\xi,Y)\sigma(U,Z). \end{aligned} \tag{5.22}$$

$$\begin{aligned}(\tilde{\nabla}_{R(\xi,Y)U}\sigma)(\xi,Z) &= -\sigma(\nabla_{R(\xi,Y)U}\xi,Z)\\ &= f\sigma(\varphi^2R(\xi,Y)U,Z)\\ &= -f^3\sigma(Y,Z)\eta(U) - f\sigma(gradf,Z)g(\varphi^2U,Y) - f(Uf)\sigma(Y,Z).\end{aligned} \tag{5.23}$$

$$\begin{aligned}(\tilde{\nabla}_U\sigma)(R(\xi,Y)\xi,Z) &= f^2(\tilde{\nabla}_U\sigma)(Y,Z) - f^2\eta(Y)(\tilde{\nabla}_U\sigma)(\xi,Z)\\ &\quad - (\xi f)(\tilde{\nabla}_U\sigma)(\varphi^2Y,Z).\end{aligned} \tag{5.24}$$

$$\begin{aligned}(\tilde{\nabla}_U\sigma)(\xi,R(\xi,Y)Z) &= -f^2\sigma(\nabla_U\xi,Y) - \sigma(\nabla_U\xi,gradf)g(\varphi^2Z,Y)\\ &\quad + (Zf)\sigma(\nabla_U\xi,\varphi^2Y).\end{aligned} \tag{5.25}$$

Combining (5.18)-(5.25) and then replacing Z by ξ in the forgoing equation we lead

$$f(f^2 + \xi f + L_2)\sigma(Y,U) = 0,$$

for all $Y,U \in \Gamma(TM)$. This completes the proof. □

Corollary 5.2. *Let M be an invariant submanifold of a regular f-Kenmotsu manifold. If M is 2-semiparallel, then it is totally geodesic or cosympletic.*

Theorem 5.4. *Let M be an invariant submanifold of a regular f-Kenmotsu manifold $\tilde{M}$. If M is a Ricci-generalized pseudoparallel submanifold then M is either totally geodesic or the function L_3 satisfies $L_3 = \frac{1}{2n}$.*

Proof. Suppose that M is a Ricci-generalized pseudoparallel submanifold then from (5.11) becomes

$$(\tilde{R}(X,Y)\cdot\sigma)(Z,U) = L_3D(S,\sigma)(Z,U;X,Y), \tag{5.26}$$

for all $X, Y, Z, U \in \Gamma(TM)$. Making use of (5.6) and (5.8) in (5.26) gives

$$R^{\perp}(X,Y)\sigma(Z,U) - \sigma(R(X,Y)Z,U) - \sigma(Z,R(X,Y)U)$$
$$= -L_3\{\sigma((X \wedge_S Y)Z,U) + \sigma(Z,(X \wedge_S Y)U)\}. \quad (5.27)$$

Inserting $X = U = \xi$ in (5.27) and then using (5.1) and (5.5) we obtain

$$(1 - 2nL_3)(f^2 + \xi f)\sigma(Z,Y) = 0,$$

for all vector fields Z, Y. This completes the proof. □

Theorem 5.5. *Let M be an invariant submanifold of a regular f-Kenmotsu manifold $\tilde{M}$. If M is a 2-generalized Ricci pseudoparallel submanifold then M is either totally geodesic or cosympletic or the function L_4 satisfies $L_4 = \frac{1}{2n}$.*

Proof. By hypothesis, from (5.12) we have

$$(\tilde{R}(X,Y)\cdot\tilde{\nabla}\sigma)(U,V,Z) = L_4 D(S,\tilde{\nabla}\sigma)(U,V,Z;X,Y), \quad (5.28)$$

for all vector fields X, Y, Z, U, V. Making use of (5.6) and (5.8) in (5.28) gives

$$R^{\perp}(X,Y)(\tilde{\nabla}_U\sigma)(V,Z) - (\tilde{\nabla}_{R(X,Y)U}\sigma)(V,Z) - (\tilde{\nabla}_U\sigma)(R(X,Y)V,Z)$$
$$- (\tilde{\nabla}_U\sigma)(V,R(X,Y)Z) = -L_4\{(\tilde{\nabla}_{(X\wedge_S Y)U}\sigma)(V,Z)$$
$$+ (\tilde{\nabla}_U\sigma)((X \wedge_S Y)V,Z) + (\tilde{\nabla}_U\sigma)(V,(X \wedge_S Y)Z)\}. \quad (5.29)$$

Replacing $X = V = \xi$ in (5.29) gives

$$R^{\perp}(\xi,Y)(\tilde{\nabla}_U\sigma)(\xi,Z) - (\tilde{\nabla}_{R(\xi,Y)U}\sigma)(\xi,Z) - (\tilde{\nabla}_U\sigma)(R(\xi,Y)\xi,Z)$$
$$- (\tilde{\nabla}_U\sigma)(\xi,R(\xi,Y)Z) = -L_4\{(\tilde{\nabla}_{(\xi\wedge_S Y)U}\sigma)(\xi,Z)$$
$$+ (\tilde{\nabla}_U\sigma)((\xi \wedge_S Y)\xi,Z) + (\tilde{\nabla}_U\sigma)(\xi,(\xi \wedge_S Y)Z)\}. \quad (5.30)$$

Computing each terms separately gives

$$
\begin{aligned}
(\tilde{\nabla}_{(\xi\wedge_S Y)U}\sigma)(\xi, Z) &= -\sigma(\nabla_{(\xi\wedge_S Y)U}\xi, Z)\\
&= f\sigma(\varphi^2(\xi\wedge_S Y)U, Z)\\
&= fS(\xi, U)\sigma(Y, Z). \qquad (5.31)
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{\nabla}_U\sigma)((\xi\wedge_S Y)\xi, Z) &= (\tilde{\nabla}_U\sigma)(S(Y,\xi)\xi - S(\xi,\xi)Y, Z)\\
&= S(Y,\xi)(\tilde{\nabla}_U\sigma)(\xi, Z) - S(\xi,\xi)(\tilde{\nabla}_U\sigma)(Y, Z). \qquad (5.32)
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{\nabla}_U\sigma)(\xi, (\xi\wedge_S Y)Z) &= (\tilde{\nabla}_U\sigma)(\xi, S(Y,Z)\xi - S(\xi,Z)Y)\\
&= -S(\xi, Z)(\tilde{\nabla}_U\sigma)(\xi, Y). \qquad (5.33)
\end{aligned}
$$

Making use of (5.22)-(5.25) and (5.31)-(5.33) in (5.30) then replacing Z by ξ we get

$$f(1-2nL_4)(f^2+\xi f)\sigma(Y,U) = 0,$$

for all $Y, U \in \Gamma(TM)$. This completes the proof. □

5.1.1 3-dimensional invariant submanifold of f-kenmotsu manifold

Lemma 5.3. *Let M be an invariant submanifold of f-Kenmotsu manifold $\tilde{M}$, then there exists the distributions D and $D^{\perp}$ such that*

$$TM = D \oplus D^{\perp} \oplus <\xi>, \quad \varphi(D) \subset D^{\perp} and \quad \varphi(D^{\perp}) \subset D.$$

Proof. The proof is similar to the proof of Lemma 4.1 (Chaubey et al., 2022) and Proposition 6.1 (Shaikh et al., 2016). □

Theorem 5.6. *A 3-dimensional submanifold M of f-Kenmotsu manifold $\tilde{M}$ is* *totally geodesic if and only if it is invariant.*

Proof. Suppose that a 3-dimensional submanifold M of f-Kenmotsu manifold $\tilde{M}$ is invariant, then form (5.3) for $X_1, Y_1 \in D$ we have

$$\varphi\sigma(X_1, Y_1) = \sigma(\varphi X_1, Y_1) = \sigma(X_1, \varphi Y_1). \tag{5.34}$$

Operating (5.34) by φ and using (1.9) gives

$$\varphi\sigma(X_1, \varphi Y_1) = \varphi^2\sigma(X_1, Y_1) = -\sigma(X_1, Y_1) + \eta(\sigma(X_1, Y_1))\xi. \tag{5.35}$$

Since $\sigma(X_1, Y_1) \subset T^{\perp}M$, $\sigma(X_1, Y_1)$ is orthogonal to $\xi \in TM$. In consequence, from (5.34) and (5.35) we get

$$\sigma(\varphi X_1, \varphi Y_1) = \sigma(X_2, Y_2) = -\sigma(X_1, Y_1), \tag{5.36}$$

where $X_2 = \varphi X_1, Y_2 = \varphi Y_1 \in D^{\perp}$. Now for any $X_1, Y_1 \in D$ and $X_2, Y_2 \in D^{\perp}$ we see that

$$\sigma(X_1 + X_2 + \xi, Y_1) = \sigma(X_1, Y_1) + \sigma(X_2, Y_1) + \sigma(\xi, Y_1),$$
$$\sigma(X_1 + X_2 + \xi, Y_2) = \sigma(X_1, Y_2) + \sigma(X_2, Y_2) + \sigma(\xi, Y_2),$$
$$\sigma(X_1 + X_2 + \xi, \xi) = \sigma(X_1, \xi) + \sigma(X_2, \xi) + \sigma(\xi, \xi).$$

In view of the above equations and (5.22), we can write

$$\sigma(X_1 + X_2 + \xi, Y_1 + Y_2 + \xi) = \sigma(X_2, Y_1) + \sigma(X_1, Y_2). \tag{5.37}$$

Taking $U, V \in TM$ as $U = X_1 + X_2 + \xi$ and $V = Y_1 + Y_2 + \xi$, (5.37) becomes

$$\sigma(U, V) = \sigma(X_2, Y_1) + \sigma(X_1, Y_2).$$

Operating the last equation by φ then using (5.34) and (5.36) yields

$$\varphi\sigma(U, V) = \sigma(X_2, \varphi Y_1) + \sigma(X_1, \varphi Y_2) = 0,$$

Again operating by φ gives $\sigma(U,V)=0$, for any vector fields U,V. Therefore, M is totally geodesic

The proof of the converse part is similar to the proof of Theorem 4.6 (Chaubey et al., 2022). This completes the proof. □

5.1.2 η-Ricci soliton on invariant submanifolds of f-Kenmotsu manifolds

Let M be an invariant submanifold of an f-Kenmotsu manifold. Consider the equation

$$\frac{1}{2}(\mathcal{L}_\xi g)(X,Y)+S(X,Y)+\lambda g(X,Y)+\omega\eta(X)\eta(Y)=0, \tag{5.38}$$

for any $X,Y\in\Gamma(TM)$, where $\mathcal{L}_\xi g$ is the Lie-derivative of g along ξ, S is the Ricci tensor of g, and λ and ω are real constants. The data (g,ξ,λ,ω) satisfies (5.38) is called η-Ricci soliton on M (Cho and Kimura, 2009). In particular, if $\omega=0$ then it is called Ricci soliton (Hamilton, 1998) and it is expanding, steady or shrinking according to $\lambda>0,\lambda=0$ or $\lambda<0$ (Chow et al., 2006).

Making use of (5.2), we can write

$$\begin{aligned}(\mathcal{L}_\xi g)(X,Y)&=g(\nabla_X\xi,Y)+g(X,\nabla_Y\xi)\\&=2f\{g(X,Y)-\eta(X)\eta(Y)\}.\end{aligned}$$

In consequence of the last expression in (5.38) gives

$$S(X,Y)=-(f+\lambda)g(X,Y)+(f-\omega)\eta(X)\eta(Y), \tag{5.39}$$

for any vector fields X,Y on M. Thus we can state the following:

Theorem 5.7. *If (g,ξ,λ,ω) is an η-Ricci soliton on an invariant submanifold M of an f-Kenmotsu manifold $\tilde{M}$, then M is η-Einstein.*

In particular for $\omega=0$ we have

Corollary 5.3. *If (g, ξ, λ) is a Ricci soliton on an invariant submanifold M of f-Kenmotsu manifold $\tilde{M}$, then M is η-Einstein.*

Replacing $X = Y = \xi$ in (5.39) and using (5.5), we get

$$\lambda = 2n(f^2 + \xi f) - \omega.$$

Theorem 5.8. *Let an invariant submanifold M of f-Kenmotsu manifold $\tilde{M}$ admit an η-Ricci soliton (g, ξ, λ, μ) then $\lambda = 2n(f^2 + \xi f) - \omega$.*

Suppose $\xi = \frac{\partial}{\partial t}$, then for Ricci soliton (g, ξ, λ) on M, λ assumes the form $\lambda = 2n(f^2 + \frac{\partial f}{\partial t})$. Obvious that $f = \frac{1}{t+F}$ is the general solution of $f^2 + \frac{\partial f}{\partial t} = 0$, for some function F(independent of t), provided $t + F \neq 0$. Thus, we can write

Corollary 5.4. *Let an invariant submanifold M of an f-Kenmotsu manifold $\tilde{M}$ admits a Ricci soliton (g, ξ, λ). Then the soliton (g, λ, ξ) is shrinking, expanding or steady if $f < \frac{1}{t+F}$, $f > \frac{1}{t+F}$ or $f = \frac{1}{t+F}$, respectively.*

Corollary 5.5. *Let an invariant submanifold M of Kenmotsu manifold $\tilde{M}$ admit a Ricci soliton (g, ξ, λ). Then M is η-Einstein and the soliton (g, ξ, λ) is expanding.*

Corollary 5.6. *Let an invariant submanifold M of a cosympletic manifold $\tilde{M}$ admit a Ricci soliton (g, ξ, λ). Then M is Einstein and the soliton (g, ξ, λ) is steady.*

Suppose that invariant submanifold M of an f-Kenmotsu manifold $\bar{M}$ admits an η-Ricci soliton. Let the Reeb vector field ξ of M is the gradient of some smooth function ψ, that is, $\xi = grad\psi$. Then equation (5.38) becomes

$$\frac{1}{2}\{g(\nabla_X grad\psi, Y) + g(X, \nabla_Y grad\psi)\} + S(X, Y) + \lambda g(X, Y) + \omega\eta(X)\eta(Y) = 0.$$

Contracting the above equation over X and Y, we find

$$\nabla^2\psi = \Psi, \tag{5.40}$$

where $\Psi = -(\tau + (2n+1)\lambda + \omega)$ and ∇^2 denotes the Laplacian operator of g and τ represents the scalar curvature of M.

A smooth function ψ on a Riemannian manifold M is said to satisfy the Poisson's equation if it satisfies the partial differential equation (5.40) for some smooth function Ψ on M. Particularly if we choose $\Psi = 0$, then the above Poisson equation reduces to the Laplace equation, and ψ is said to be harmonic.
By considering the above facts, we can state the following:

Theorem 5.9. *Let an invariant submanifold M of an f-Kenmotsu manifold $\tilde{M}$ admits an η-Ricci soliton $(g, \xi, \lambda, \omega)$. If the Reeb vector field of M is the gradient of some smooth function ψ, then ψ satisfies Poisson's equation (5.40).*

Theorem 5.10. *Let an invariant submanifold M of an f-Kenmotsu manifold $\tilde{M}$ admits an η-Ricci soliton $(g, \xi, \lambda, \omega)$. If the Reeb vector field of M is the gradient of some smooth function ψ, then ψ satisfies the Laplace equation if and only if $\lambda = -\frac{\tau+\omega}{2n+1}$.*

Remark 5.1. *To study celestial mechanics, Pierre-Simon de Laplace used the Laplace operator. Laplacian represents the flux density of the gradient flow of a function. Laplacian occurs in differential equations that describes many physical phenomena, such as electrical and gravitational potentials, the diffusion equation for heat and fluid flow, wave propagation, quantum mechanics, Hodge theory, de Rham cohomology, image processing and computer version.*

5.1.3 Invariant submanifold of f-Kenmotsu space forms

Let M be a $(2n+1)$-dimensional invariant submanifold of f-Kenmotsu space form $\tilde{M}^{2m+1}(c)$. We consider an orthonormal basis $e_1, ..., e_{2n+1}$ of M such that $e_{n+t} = \varphi e_t (t = 1, ..., n), e_{2n+1} = \xi$. Then, using (1.40), (1.55) and (5.1) the expression for the curvature tensor of M is given by

$$\begin{aligned} R(X,Y,Z,U) &= \frac{c-3f^2}{4}(g(Y,Z)g(X,U)-g(X,Z)g(Y,U)) \\ &+ \frac{c+f^2}{4}(2g(X,\varphi Y)g(\varphi Z,U)+g(X,\varphi Z)g(\varphi Y,U) \\ &- g(Y,\varphi Z)g(\varphi X,U))+(\frac{c+f^2}{4}+\xi f)(\eta(X)\eta(Z)g(Y,U) \\ &- \eta(Y)\eta(Z)g(X,U)+g(X,Z)\eta(Y)\eta(U)-g(Y,Z)\eta(X)\eta(U)) \\ &+ g(\sigma(X,U),\sigma(Y,Z))-g(\sigma(X,Z),\sigma(Y,U)), \qquad (5.41) \end{aligned}$$

for all $X,Y,Z,U \in \Gamma(TM)$. From (5.41), the Ricci curvature and scalar curvature of M is given by

$$\begin{aligned} S(Y,Z) = \frac{c-3f^2}{2}ng(Y,Z)+\frac{3(c+f^2)}{4}g(\varphi Y,\varphi Z) \\ +(\frac{c+f^2}{4}+\xi f)((1-2n)\eta(Y)\eta(Z)-g(Y,Z)) \\ -\sum_i g(\sigma(e_i,Z),\sigma(Y,e_i)). \qquad (5.42) \end{aligned}$$

and,

$$\tau = n(n+1)c-n(1+3n)f^2-4n\xi f-||\sigma||^2, \qquad (5.43)$$

where, $||\sigma||^2=\sum_{i,j=1}^n g(\sigma(e_i,e_j),\sigma(e_i,e_j))$.

Now, for $1 \leq i,j \leq n, n+1 \leq \alpha \leq 2p$, where $p=m-n$ we define

$$\sigma_{ij}^{\alpha} = g(\sigma(e_i,e_j),e_\alpha), \qquad (5.44)$$

and its derivatives as

$$\sigma_{ijk}^{\alpha} = \tilde{\nabla}_{e_k}\sigma_{ij}^{\alpha} = g((\tilde{\nabla}_{e_k}\sigma)(e_i,e_j),e_\alpha), \qquad (5.45)$$

and

$$\sigma_{ijkl}^{\alpha} = \tilde{\nabla}_{e_l}\tilde{\nabla}_{e_k}\sigma_{ij}^{\alpha} = g((\tilde{\nabla}_{e_l}\tilde{\nabla}_{e_k}\sigma)((e_i,e_j),e_\alpha). \qquad (5.46)$$

Yildiz and Murathan (2009) obtain

$$\frac{1}{2}\Delta(||\sigma||^2) = \sum_{i,j,k=1}^{n} g((\tilde{\nabla}_{e_k}\tilde{\nabla}_{e_k}\sigma)(e_i, e_j), \sigma(e_i, e_j)) + ||\tilde{\nabla}\sigma||^2, \tag{5.47}$$

where,

$$||\sigma||^2 = \sum_{i,j,k=1}^{n} \sum_{\alpha=n+1}^{2p} (\sigma_{ij}^{\alpha})^2,$$

$$||\tilde{\nabla}\sigma||^2 = \sum_{i,j,k=1}^{n} \sum_{\alpha=n+1}^{2p} (\sigma_{ijkk}^{\alpha})^2,$$

where Δ denotes the Laplacian operator of g. For Ricci-generalized pseudoparallel submanifold for $L_3 = \frac{1}{2n}$, from (5.26) we have

$$\tilde{R}(e_l, e_k) \cdot \sigma = \frac{1}{2n}(e_l \wedge_S e_k) \cdot \sigma. \tag{5.48}$$

Also we have

$$(\tilde{R}(e_l, e_k) \cdot \sigma)(e_i, e_j) = (\tilde{\nabla}_{e_l}\tilde{\nabla}_{e_k}\sigma)(e_i, e_j) - (\tilde{\nabla}_{e_k}\tilde{\nabla}_{e_l}\sigma)(e_i, e_j). \tag{5.49}$$

Making use of (5.1) and (5.8) we get

$$\begin{aligned}((e_l \wedge_S e_k)\sigma)(e_i, e_j) &= -S(e_k, e_i)\sigma(e_l, e_j) + S(e_l, e_i)\sigma(e_k, e_j) \\ &\quad -S(e_k, e_j)\sigma(e_i, e_l) + S(e_l, e_j)\sigma(e_i, e_k).\end{aligned} \tag{5.50}$$

Combining (5.48), (5.49) and (5.50) gives

$$\begin{aligned}(\tilde{\nabla}_{e_l}\tilde{\nabla}_{e_k}\sigma)(e_i, e_j) - (\tilde{\nabla}_{e_k}\tilde{\nabla}_{e_l}\sigma)(e_i, e_j) &= \frac{1}{2n}[-S(e_k, e_i)\sigma(e_l, e_j) \\ +S(e_l, e_i)\sigma(e_k, e_j) - S(e_k, e_j)\sigma(e_i, e_l) &+ S(e_l, e_j)\sigma(e_i, e_k)].\end{aligned} \tag{5.51}$$

Now using (5.51) we compute

$$\begin{aligned} g((\tilde{\nabla}_{e_k}\tilde{\nabla}_{e_k}\sigma)(e_i,e_j),\sigma(e_i,e_j)) &= g((\tilde{\nabla}_{e_k}\tilde{\nabla}_{e_i}\sigma)(e_k,e_j),\sigma(e_i,e_j)) \\ &= g((\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_k}\sigma)(e_k,e_j),\sigma(e_i,e_j)) \\ &+ \frac{1}{2n}[S(e_k,e_k)g(\sigma(e_i,e_j),\sigma(e_i,e_j)) \\ &- S(e_i,e_k)g(\sigma(e_k,e_j),\sigma(e_i,e_j)) \\ &+ S(e_k,e_j)g(\sigma(e_i,e_k),\sigma(e_i,e_j)) \\ &- S(e_i,e_j)g(\sigma(e_k,e_k),\sigma(e_i,e_j))]. \end{aligned} \tag{5.52}$$

Inserting (5.52) in (5.47) we get

$$\begin{aligned} &\frac{1}{2}\Delta(||\sigma||^2) - ||\tilde{\nabla}\sigma||^2 = \sum_{i,j,k=1}^{n}\{g((\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_k}\sigma)(e_k,e_j),\sigma(e_i,e_j)) \\ &+ \frac{1}{2n}[S(e_k,e_k)g(\sigma(e_i,e_j),\sigma(e_i,e_j)) - S(e_i,e_k)g(\sigma(e_k,e_j),\sigma(e_i,e_j)) \\ &+ S(e_k,e_j)g(\sigma(e_i,e_k),\sigma(e_i,e_j)) - S(e_i,e_j)g(\sigma(e_k,e_k),\sigma(e_i,e_j))]\}. \end{aligned} \tag{5.53}$$

Now we compute each term separately.

$$\sum_{i,j,k=1}^{n} S(e_k,e_k)g(\sigma(e_i,e_j),\sigma(e_i,e_j)) = \tau||\sigma||^2. \tag{5.54}$$

$$\begin{aligned} &\sum_{i,j,k=1}^{n} S(e_i,e_k)g(\sigma(e_k,e_j),\sigma(e_i,e_j)) = \sum_{i,j,k=1}^{n}\sum_{\alpha=n+1}^{2p} S(e_i,e_k)g(A_\alpha e_k, A_\alpha e_i) \\ &= \sum_{i,j,k=1}^{n}\sum_{\alpha=n+1}^{2p}[(\frac{c(n+1)+f^2(1-n)}{2} - 1)g(e_i,e_k)g(A_\alpha e_k, A_\alpha e_i) \\ &- (\frac{(n+1)(c+f^2)}{2} + (2n-1)\xi f)\eta(e_k)\eta(e_j)g(A_\alpha e_k, A_\alpha e_i) \\ &- g(A_\alpha e_k, A_\alpha e_i)g(A_\alpha e_k, A_\alpha e_i)]. \end{aligned} \tag{5.55}$$

$$\sum_{i,j,k=1}^{n} S(e_i,e_j)g(\sigma(e_k,e_k),\sigma(e_i,e_j)) = \sum_{i,j,k=1}^{n}\sum_{\alpha=n+1}^{2p} S(e_i,e_j)TrA_\alpha g(A_\alpha e_i,e_j). \tag{5.56}$$

Combining (5.53)-(5.56) and using the minimality we get

$$\frac{1}{2}\Delta(||\sigma||^2) = \sum_{i,j,k=1}^{n} g((\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_j}\sigma)(e_k, e_k), \sigma(e_i, e_j)) + \frac{1}{2n}\tau||\sigma||^2 + ||\tilde{\nabla}\sigma||^2. \quad (5.57)$$

Take $H^\alpha = \sum_{k=1}^{n} \sigma_{kk}^\alpha$ then (5.57) becomes

$$\frac{1}{2}\Delta(||\sigma||^2) = \sum_{i,j,k=1}^{n} \sum_{\alpha=n+1}^{2p} \sigma_{ij}^\alpha(\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_j}H^\alpha) + \frac{1}{2n}\tau||\sigma||^2 + ||\tilde{\nabla}\sigma||^2.$$

Then by Lemma 5.2, last equation becomes

$$\frac{1}{2}\Delta(||\sigma||^2) = \frac{1}{2n}\tau||\sigma||^2 + ||\tilde{\nabla}\sigma||^2. \quad (5.58)$$

Hence, we can state the following.

Theorem 5.11. *Let M be a $(2n+1)$-dimensional invariant submanifold of f-Kenmotsu space form $\tilde{M}^{2m+1}(c)$. If M is a Ricci-generalized pseudoparallel submanifold with $L_3 = \frac{1}{2n}$, then we obtain the following relation:*

$$\frac{1}{2}\Delta(||\sigma||^2) = \frac{1}{2n}\tau||\sigma||^2 + ||\tilde{\nabla}\sigma||^2.$$

5.1.4 Examples of an invariant submanifold of f-Kenmotsu manifolds

Example 5.1. *Consider a 5-dimensional manifold $\tilde{M} = \{(x_1, x_2, x_3, x_4, t) \in \mathbb{R}^5 : t \neq 0\}$ where (x_1, x_2, x_3, x_4, t) are the standard coordinates in $\mathbb{R}^5$. Now let $\{e_1, e_2, e_3, e_4, e_5\}$ be a linearly independent global frame on $\tilde{M}$ given by*

$$e_1 = e^{t^2}\frac{\partial}{\partial x_1},\ e_2 = e^{t^2}\frac{\partial}{\partial x_2},\ e_3 = e^{t^2}\frac{\partial}{\partial x_3},\ e_4 = e^{t^2}\frac{\partial}{\partial x_4},\ e_5 = \frac{\partial}{\partial t}.$$

Let g be a Riemannian metric on $\tilde{M}$ define as

$$g(e_i, e_j) = \begin{cases} 1, & \textit{if} \quad i = j, \\ 0, & \textit{if} \quad i \neq j, \quad \textit{where} \quad 1 \leq i, j \leq 5. \end{cases}$$

Set $e_5 = \xi$*, then we see that* $\eta(e_5) = 1$ *and* $\eta(e_i) = 0$ *for* $i = 1, 2, 3, 4$.
Also, we define (1,1)-tensor φ *as*

$$\varphi(e_1) = -e_2, \quad \varphi(e_2) = e_1, \quad \varphi(e_3) = -e_4, \quad \varphi(e_4) = e_3, \quad \varphi(e_5) = 0.$$

As a consequence of the above equations its easy to see that $\varphi^2 X = -X + \eta(X)\xi$ *and* $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$*, for any vector fields* X, Y *on* $\tilde{M}$*. Clearly,* $\tilde{M}(\varphi, \xi, g, \eta)$ *forms an almost contact metric manifold. Let* $\tilde{\nabla}$ *be the Levi-Civita connection with respect to the metric* g*. Then we have*

$$[e_i, e_j] = \begin{cases} -2te_i, & \textit{if} \quad i = 1, 2, 3, 4; j = 5, \\ 0, & \textit{otherwise.} \end{cases} \tag{5.59}$$

Making use of the Koszul formula and (5.59), we obtained the following

$$\tilde{\nabla}_{e_1} e_1 = 2te_5, \tilde{\nabla}_{e_1} e_2 = 0, \tilde{\nabla}_{e_1} e_3 = 0, \tilde{\nabla}_{e_1} e_4 = 0, \tilde{\nabla}_{e_1} e_5 = -2te_1,$$
$$\tilde{\nabla}_{e_2} e_2 = 2te_5, \tilde{\nabla}_{e_2} e_3 = 0, \tilde{\nabla}_{e_2} e_4 = 0, \tilde{\nabla}_{e_2} e_5 = -2te_2, \tilde{\nabla}_{e_3} e_3 = 2te_5,$$
$$\tilde{\nabla}_{e_3} e_4 = 0, \tilde{\nabla}_{e_3} e_5 = -2te_3, \tilde{\nabla}_{e_4} e_5 = -2te_4, \tilde{\nabla}_{e_4} e_4 = 2te_5, \tilde{\nabla}_{e_5} e_5 = 0.$$

The above relations imply that the manifold satisfies $\tilde{\nabla}_X \xi = -f\varphi^2 X$*, for* $\xi = e_5$ *and* $f = -2t$*. Therefore,* $\tilde{M}$ *is an* f*-Kenmotsu manifold with* $f = -2t$*. Moreover,* $\tilde{M}$ *is regular as* $f^2 + \xi f \neq 0$*.*

Let M *be a subset of* $\tilde{M}$ *and consider the isometric immersion* $\pi : M \to \tilde{M}$ *defined by* $\pi(x_1, x_3, t) = (x_1, 0, x_3, 0, t)$*. Clearly,* $M = \{(x_1, x_3, t) \in \mathbb{R}^3 : (x_1, x_3, t) \neq 0\}$ *is a 3-dimensional submanifold of* $\tilde{M}$*, where* (x_1, x_3, t) *are standard coordinates in* $\mathbb{R}^3$*. We choose the vector fields*

$$e_1 = e^{t^2} \frac{\partial}{\partial x_1}, \quad e_3 = e^{t^2} \frac{\partial}{\partial x_3}, \quad e_5 = \frac{\partial}{\partial t}.$$

We define g_1 *such that* $\{e_1, e_3, e_5\}$ *is an orthonormal basis of* M *as follows:*

$$g_1(e_i, e_j) = \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j, \quad \text{where} \quad i, j = 1, 3, 5. \end{cases}$$

Set $\xi = e_5$. *Then define 1-form* η_1 *and (1,1)-tensor field* φ_1 *as* $\eta_1(\cdot) = g_1(\cdot, e_5)$ *and* $\varphi_1(e_1) = -e_3, \varphi_1(e_3) = e_1, \varphi_1(e_5) = 0$.
Making use of the above relations, it is easy to see that

$$\eta_1(e_5) = 1, \varphi_1^2(X) = -X + \eta_1(X)e_5,$$
$$g_1(\varphi_1 X, \varphi_1 Y) = g_1(X, Y) - \eta_1(X)\eta_1(Y),$$

for vector fields X, Y *on* M. *Clearly,* $M(\eta_1, g_1, e_5, \varphi_1)$ *is an invariant submanifold of* $\tilde{M}$. *Let* ∇ *be the Levi-Civita connection induced by the metric* g_1, *then by using Koszul formula, we derive the following:*

$$\nabla_{e_1} e_1 = 2te_5, \nabla_{e_1} e_3 = 0, \nabla_{e_1} e_5 = -2te_1,$$
$$\nabla_{e_3} e_3 = 2te_5, \nabla_{e_3} e_5 = -2te_3, \nabla_{e_5} e_5 = 0.$$

One can see that $M(g_1, \eta_1, \varphi_1, e_5)$ *forms a 3-dimensional* f*-Kenmotsu manifold with* $f = -2t$. *Thus, Lemma 5.1 is verified.*

Let σ *be the second fundamental form, then using (1.51) and the above relations we obtained*

$$\sigma(X, Y) = 0,$$

for any vector field X, Y *on* M. *Thus,* M *is a totally geodesic submanifold of* $\tilde{M}$. *Hence, Theorem 5.6 and Corollary 5.1, 5.2 are verified.*

Example 5.2. *Let* $\mathbb{R}^n$ *be an n-dimensional real number space. Define* $M^5 = \{(x_1, x_2, x_3, x_4, x_5) : x_i \in \mathbb{R}, i = 1, 2, ..., 5 \text{ and } x_3 \neq 0\}$. *Let* $\{e_1, e_2, e_3, e_4, e_5\}$ *be* *a set of linearly independent vector fields of* M^5 *given by*

$$e_1 = e^{-e^{x_3}}\frac{\partial}{\partial x_1}, e_2 = e^{-e^{x_3}}\frac{\partial}{\partial x_2}, e_3 = \frac{\partial}{\partial x_3}, e_4 = e^{-e^{x_3}}\frac{\partial}{\partial x_4}, e_5 = e^{-e^{x_3}}\frac{\partial}{\partial x_5}.$$

Let g *be the associated metric of* M^5 *which is define as*

$$g(e_i, e_j) = \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j, \quad \text{where} \quad 1 \leq i, j \leq 5. \end{cases}$$

Also, we define (1,1)-tensor field φ *of* M^5 *as*

$$\varphi(e_1) = -e_2, \quad \varphi(e_2) = e_1, \quad \varphi(e_3) = 0, \quad \varphi(e_4) = -e_5, \quad \varphi(e_5) = e_4.$$

By the linearity property of g *and* φ*, we can easily show that the following relations* $\varphi^2 e_i = -e_i + \eta(e_i)e_3, g(e_i, e_3) = \eta(e_i)$ *and* $g(\varphi e_i, \varphi e_j) = g(e_i, e_j) - \eta(e_i)\eta(e_j)$ *holds for* $i, j = 1, 2, 3, 4, 5$ *and* $\xi = e_3$*. Thus,* $M^5(g, \varphi, \eta, \xi = e_3)$ *is an almost contact metric manifold.*

Now by simple computation, we get the following relation:

$$[e_i, e_j] = \begin{cases} e^{x_3} e_i, & \text{if} \quad i = 1, 2, 4, 5; j = 3, \\ 0, & \text{otherwise.} \end{cases}$$

Let ∇ *denote the Levi-Civita connection, then by using Koszul's formula and above relations, we can obtain the following:*

$$\nabla_{e_1} e_1 = -e^{x_3} e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = e^{x_3} e_1, \nabla_{e_1} e_4 = 0,$$
$$\nabla_{e_1} e_5 = 0, \nabla_{e_2} e_2 = -e^{x_3} e_3, \nabla_{e_2} e_3 = e^{x_3} e_2, \nabla_{e_2} e_4 = 0,$$
$$\nabla_{e_2} e_5 = 0, \nabla_{e_3} e_3 = 0, \nabla_{e_3} e_4 = -e^{x_3} e_4, \nabla_{e_3} e_5 = -e^{x_3} e_5,$$
$$\nabla_{e_4} e_4 = -e^{x_3} e_3, \nabla e_4 e_5 = 0, \nabla_{e_5} e_5 = -e^{x_3} e_3.$$

Clearly, from the above relations, we can see that $M^5(g, \varphi, \xi, \eta)$ *is an* f*-Kenmotsu manifold for* $\xi = e_3$ *and* $f = e^{x_3}$*. Moreover,* M^5 *is a regular* f*-Kenmotsu mani-*

fold as $f^2 - \xi f = e^{x_3} \neq 0.$

Let M^3 *be a subset of* M^5*. Now consider an isometric immersion* $\pi : M^3 \to M^5$ *define as* $\pi(x_1, x_2, x_3) = (x_1, x_2, x_3, 0, 0)$ *where* (x_1, x_2, x_3) *is the standard coordinates in* $\mathbb{R}^3$*. Clearly,* $M^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ and } (x_1, x_2, x_3) \neq 0\}$ *is a submanifold of* M^5*. Let* $\{e_1, e_2, e_3\}$ *be the basis of* M^3 *which is define as*

$$e_1 = e^{-e^{x_3}} \frac{\partial}{\partial x_1}, e_2 = e^{-e^{x_3}} \frac{\partial}{\partial x_2}, e_3 = \frac{\partial}{\partial x_3}.$$

Let us define the associate metric g_1 *of* M^3 *as*

$$g_1(e_i, e_j) = \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j, \quad \text{where} \quad 1 \leq i, j \leq 3. \end{cases}$$

Also, 1-from η_1 *and (1,1)-tensor field* φ_1 *are define as follows:*

$$\varphi_1(e_1) = -e_2, \quad \varphi_1(e_2) = e_1, \quad \varphi_1(e_3) = 0 \quad \text{and} \quad \eta_1(\cdot) = g_1(\cdot, e_3)$$

Making use of the above relations, it is easy to see that

$$\eta_1(e_3) = 1, \varphi_1^2(X) = -X + \eta_1(X)e_3,$$
$$g_1(\varphi_1 X, \varphi_1 Y) = g_1(X, Y) - \eta_1(X)\eta_1(Y),$$

for any vector fields X, Y *on* M^3*. Clearly,* $M^3(\eta_1, g_1, e_3, \varphi_1)$ *is an invariant submanifold of* M^5*. Let* ∇ *be the Levi-Civita connection induced by the metric* g_1*, then by using the Koszul formula, we derive the following:*

$$\nabla_{e_1} e_1 = -e^{x_3} e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = e^{x_3} e_1,$$
$$\nabla_{e_2} e_2 = -e^{x_3} e_3, \nabla_{e_2} e_3 = e^{x_3} e_2, \nabla_{e_3} e_3 = 0.$$

It is obvious that $M^3(g_1, \varphi_1, \eta_1, e_3)$ *is also an* f*-Kenmotsu manifold for* $f = e^{x_3}$*. Thus, Lemma 5.1 is verified.*

Let σ be the second fundamental form, then using (1.51) and the above relations we obtained

$$\sigma(X, Y) = 0,$$

for any vector field X, Y on M^3. This shows that the 3-dimensional invariant submanifold of an f-Kenmotsu manifold is totally geodesic. Hence, the statement of Theorem 5.6 is verified. Also, we can show that Corollary 5.1 and Corollary 5.2 holds on M^3.

5.2 Improved Chen's Inequalities for Submanifolds of Generalized Sasakian-space-forms

Let M^m be an m-dimensional submanifold of a $(m+p)$-dimensional Riemannian manifold $\widetilde{M}^{m+p}$ endowed with the quarter-symmetric connection $\overline{\nabla}$ and the Levi-Civita connection $\overline{\widehat{\nabla}}$. Let ∇ and $\widehat{\nabla}$ denote the induced quarter-symmetric connection and the induced Levi-Civita connection on the submanifold M. The Gauss formula with respect to ∇ and $\widehat{\nabla}$ can be written as

$$\overline{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + h(X_1, X_2), \quad X_1, X_2 \in \Gamma(TM)$$
$$\overline{\widehat{\nabla}}_{X_1} X_2 = \widehat{\nabla}_{X_1} X_2 + \widehat{h}(X_1, X_2), \quad X_1, X_2 \in \Gamma(TM)$$

where h and $\widehat{h}$ are the second fundamental forms associated with the quarter-symmetric connection ∇ and the Levi-Civita connection $\widehat{\nabla}$ respectively, and are related as follows:

$$h(X_1, X_2) = \widehat{h}(X_1, X_2) - \psi_2 g(X_1, X_2) P^{\perp}, \tag{5.60}$$

Y. Li, M. Khatri, J.P. Singh, S.K. Chaubey (2022), Improved Chen's Inequalities for Submanifolds of Generalized Sasakian-space-forms, *Axioms*, **11(7)**, 324.

where $P^{\perp}$ is the normal component of the vector field P on M. If P^T represents that tangent component of the vector field P on M, then $P = P^T + P^{\perp}$.

The curvature tensor $\overline{R}$ with respect to the quarter-symmetric connection $\overline{\nabla}$ on $\widetilde{M}^{m+p}$ can be expressed as (Qu and Wang, 2015):

$$\overline{R}(X_1, X_2, X_3, X_4) = \widehat{\widehat{R}}(X_1, X_2, X_3, X_4) + \psi_1\beta_1(X_1, X_3)g(X_2, X_3)$$
$$-\psi_1\beta_1(X_2, X_3)g(X_1, X_4) + \psi_2 g(X_1, X_3)\beta_1(X_2, X_4) - \psi_2 g(X_2, X_3)\beta_1(X_1, X_4)$$
$$+\psi_2(\psi_1 - \psi_2)g(X_1, X_3)\beta_2(X_2, X_4) - \psi_2(\psi_1 - \psi_2)g(X_2, X_3)\beta_2(X_1, X_4) \quad (5.61)$$

where β_1 and β_2 are symmetric $(0, 2)$-tensor field defined as

$$\beta_1(X_1, X_2) = (\widehat{\widehat{\nabla}}_{X_1}\Lambda)(X_2) - \psi_1\Lambda(X_1)\Lambda(X_2) + \frac{\psi_2}{2}g(X_1, X_2)\Lambda(P),$$

and

$$\beta_2(X_1, X_2) = \frac{\Lambda(P)}{2}g(X_1, X_2) + \Lambda(X_1)\Lambda(X_2).$$

Moreover, we assume that $tr(\beta_1) = \lambda$ and $tr(\beta_2) = \mu$.

Let R and $\widehat{R}$ be the curvature tensors of ∇ and $\widehat{\nabla}$ respectively, then the Gauss equation with respect to the quarter-symmetric connection is as follows (Qu and Wang, 2015):

$$\overline{R}(X_1, X_2, X_3, X_4) = R(X_1, X_2, X_3, X_4) - g(h(X_1, X_4), h(X_2, X_3))$$
$$+g(h(X_2, X_4), h(X_1, X_3)) + (\psi_1 - \psi_2)g(h(X_2, X_3), P)g(X_1, X_4)$$
$$+(\psi_2 - \psi_1)g(h(X_1, X_3), P)g(X_2, X_4). \quad (5.62)$$

Let $\{e_1, \ldots, e_m\}$ and $\{e_{m+1}, \ldots, e_{m+p}\}$ be an orthonormal frame of T_xM and $T_x^{\perp}M$ at the point $x \in M$, then the mean curvature vector of M associated to ∇ is $H = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_j)$. Similarly, the mean curvature vector of M associated to $\widehat{\nabla}$ is $\widehat{H} = \frac{1}{m}\sum_{i=1}^{m} \widehat{h}(e_i, e_j)$. Also, the squared length of h is $\| h \|^2 = \sum_{i,j=1}^{m} g(h(e_i, e_j), h(e_i, e_j))$.

First, we recall the well-known lemma obtained by Chen (1993), which is as follows:

Lemma 5.4. *If* $a_1, ..., a_m, a_{m+1}$ *are* $m+1$ $(m \geq 2)$ *real numbers such that*

$$\Big(\sum_{i=1}^{m} a_i\Big)^2 = (m-1)\Big(\sum_{i=1}^{m} a_i^2 + a_{m+1}\Big),$$

then $2a_1a_2 \geq a_{m+1}$, *with equality holding if and only if* $a_1 + a_2 = a_3 = ... = a_m$.

Let M^m be a submanifold of a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ of dimension $(2n+1)$. For any tangent vector field X_1 on M, we can write $\varphi X_1 = \mathcal{T} X_1 + \mathcal{F} X_1$, where $\mathcal{T} X_1$ is the tangential component and $\mathcal{F} X_1$ is the normal component of φX_1. The squared norm of $\mathcal{T}$ at $x \in M$ is defined as

$$\parallel \mathcal{T} \parallel^2 = \sum_{i,j=1}^{m} g^2(\varphi e_i, e_j), \tag{5.63}$$

where $\{e_1, ..., e_m\}$ is any orthonormal basis of the tangent space T_xM and decomposing the structural vector field $\xi = \xi^T + \xi^{\perp}$, where ξ^T and $\xi^{\perp}$ denotes the tangential and normal components of ξ. Moreover, we set $\Theta^2(\Pi) = g^2(\mathcal{T} e_1, e_2) = g^2(\varphi e_1, e_2)$, where $\{e_1, e_2\}$ is the orthonormal basis of 2-plane section Π.

Theorem 5.12. *Let* $M^m, m \geq 3$ *be an m-dimensional submanifold of a* $(2n+1)$*-dimensional generalized Sasakian space form* $\widetilde{M}(f_1, f_2, f_3)$ *endowed with a quarter-symmetric connection* $\overline{\nabla}$, *then*

$$\begin{aligned}
\tau(x) - K(\Pi) \leq{}& (m-2)\Big(\frac{m^2}{2(m-1)} \parallel H \parallel^2 + (m+1)\frac{f_1}{2}\Big) \\
&+\Big(3 \parallel \mathcal{T} \parallel^2 - 6\Theta^2(\Pi)\Big)\frac{f_2}{2} + \Big(\parallel \xi_\Pi \parallel^2 - (m-1) \parallel \xi^T \parallel^2 \Big) f_3 \\
&+\frac{(\psi_1+\psi_2)}{2}\Big(tr(\beta_1 \mid_\Pi) - \lambda(m-1)\Big) + \frac{\psi_2(\psi_1-\psi_2)}{2}\Big(tr(\beta_2 \mid_\Pi) \\
&-\mu(m-1)\Big) + \frac{(\psi_1-\psi_2)}{2}\Big(\Lambda(tr(h \mid_\Pi)) - m(m-1)\Lambda(H)\Big),
\end{aligned}$$

where Π *is a 2-plane section* T_xM, $x \in M$.

If in addition, P is a tangent vector field on M^m, then $H = \widehat{H}$ and the equality case holds at a point $x \in M$ if and only if there exists an orthonormal basis $\{e_1, ..., e_m\}$ of T_xM and an orthonormal basis $\{e_{m+1}, ..., e_{2n+1}\}$ of $T_x^{\perp}M$ such that the shape operators of M in $\widetilde{M}(f_1, f_2, f_3)$ at x have the following forms:

$$A_{e_{m+1}} = \begin{pmatrix} h_{11}^{m+1} & 0 & 0 & \dots & 0 \\ 0 & h_{22}^{m+1} & 0 & \dots & 0 \\ 0 & 0 & h_{11}^{m+1} + h_{22}^{m+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & h_{11}^{m+1} + h_{22}^{m+1} \end{pmatrix}$$

and

$$A_{e_r} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \dots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, m+2 \leq r \leq 2n+1$$

Proof. Let $x \in M$ and $\{e_1, e_2, ..., e_m\}, \{e_{m+1}, ..., e_{2n+1}\}$ be an orthonormal basis of T_xM and $T_x^{\perp}M$, respectively, then from (5.62), (1.58), (5.61) and (1.19) we get

$$\begin{aligned} 2\tau(x) = m^2 \parallel H \parallel^2 - \parallel h \parallel^2 + m(m-1)f_1 + 3f_2 \parallel \mathcal{T} \parallel^2 \\ -2(m-1)f_3 \parallel \xi^T \parallel^2 - (\psi_1 + \psi_2)\lambda(m-1) \\ -\psi_2(\psi_1 - \psi_2)\mu(m-1) - m(m-1)(\psi_1 - \psi_2)\Lambda(H). \end{aligned} \tag{5.64}$$

We set,

$$\begin{aligned} c = 2\tau(x) - \tfrac{m^2(m-2)}{m-1} \parallel H \parallel^2 - m(m-1)f_1 - 3f_2 \parallel \mathcal{T} \parallel^2 \\ +2(m-1)f_3 \parallel \xi^T \parallel^2 + (\psi_1 + \psi_2)\lambda(m-1) \\ +\psi_2(\psi_1 - \psi_2)\mu(m-1) + m(m-1)(\psi_1 - \psi_2)\Lambda(H), \end{aligned} \tag{5.65}$$

then (5.64) becomes

$$m^2 \parallel H \parallel^2 = (m-1)\big(\parallel h \parallel^2 + c\big). \tag{5.66}$$

For a chosen orthonormal basis, (5.66) can be written as:

$$\Big(\sum_{i=1}^{m} h_{ii}^{m+1}\Big)^2 = (m-1)\Big[\sum_{i=1}^{m}(h_{ii}^{m+1})^2 + \sum_{i\neq j}(h_{ij}^{m+1})^2 + \sum_{r=m+2}^{2n+1}\sum_{i,j=1}^{m}(h_{ij}^{r})^2 + c\Big],$$

then using Lemma (Zhang et al., 2014), we have

$$2h_{11}^{m+1}h_{22}^{m+1} \geq \sum_{i\neq j}(h_{ij}^{m+1})^2 + \sum_{r=m+2}^{2n+1}\sum_{i,j=1}^{m}(h_{ij}^{r})^2 + c. \tag{5.67}$$

Now, let $\Pi = span\{e_1, e_2\}$, then from (5.62), (5.61) and (1.19) we get

$$\begin{aligned} R(e_1,e_2,e_2,e_1) = \sum_{r=m+1}^{2n+1} & [h_{11}^r h_{22}^r - (h_{12}^r)^2] - (\psi_1 - \psi_2) g(h(e_2,e_2),P) \\ & + f_1 + 3f_2 g^2(\varphi e_1, e_2) - f_3(\eta^2(e_1) + \eta^2(e_2)) \\ & -\psi_1\beta_1(e_2,e_2) - \psi_2\beta_1(e_1,e_1) - \psi_2(\psi_1-\psi_2)\beta_2(e_1,e_1). \end{aligned} \tag{5.68}$$

and

$$\begin{aligned} R(e_1,e_2,e_1,e_2) = \sum_{r=m+1}^{2n+1} & [(h_{12}^r)^2 - h_{11}^r h_{22}^r] + (\psi_1 - \psi_2) g(h(e_1,e_1),P) \\ & - f_1 - 3f_2 g^2(\varphi e_1, e_2) + f_3(\eta^2(e_1) + \eta^2(e_2)) \\ & +\psi_1\beta_1(e_1,e_1) + \psi_2\beta_1(e_2,e_2) + \psi_2(\psi_1-\psi_2)\beta_2(e_2,e_2). \end{aligned} \tag{5.69}$$

Making use of (5.68) and (5.69) in (1.57), we obtain

$$\begin{aligned} K(\Pi) = \sum_{r=m+1}^{2n+1} & [h_{11}^r h_{22}^r - (h_{12}^r)^2] - \frac{(\psi_1 - \psi_2)}{2}\Lambda(tr(h\mid_\Pi)) \\ & + f_1 + 3f_2\Theta^2(\Pi) - f_3(\parallel \xi_\Pi \parallel^2) \\ & -\frac{\psi_1}{2}tr(\beta_1 \mid \Pi) - \frac{\psi_2}{2}tr(\beta_1\mid_\Pi) - \frac{\psi_2}{2}(\psi_1-\psi_2)tr(\beta_2\mid_\Pi). \end{aligned} \tag{5.70}$$

Combining (5.64) and (5.70) gives

$$\begin{aligned}\tau(x) - K(\Pi) = (m-2)\Big(&\frac{m^2}{2(m-1)} \parallel H \parallel^2 + (m+1)\frac{f_1}{2}\Big)\\
+\Big(3 \parallel \mathcal{T} \parallel^2 - 6\Theta^2(\Pi)\Big)\frac{f_2}{2} &+ \Big(\parallel \xi_\Pi \parallel^2 - (m-1) \parallel \xi^T \parallel^2 \Big) f_3\\
+\frac{(\psi_1+\psi_2)}{2}\Big(tr(\beta_1 \mid_\Pi) - \lambda(m-1)\Big) &+ \frac{\psi_2(\psi_1-\psi_2)}{2}\Big(tr(\beta_2 \mid_\Pi)\\
-\mu(m-1)\Big) &+ \frac{(\psi_1-\psi_2)}{2}\Big(\Lambda(tr(h \mid_\Pi)) - m(m-1)\Lambda(H)\Big)\\
+\sum_{r=m+1}^{2n+1}\Big[\sum_{1\le i<j\le m} h^r_{ii}h^r_{jj} - h^r_{11}h^r_{22} &- \sum_{1\le i<j\le m}(h^r_{ij})^2 + (h^r_{12})^2\Big]. \qquad (5.71)\end{aligned}$$

Making use of Lemma 2.4 (Zhang et al. 2014), we have

$$\sum_{r=m+1}^{2n+1}\Big[\sum_{1\le i<j\le m} h^r_{ii}h^r_{jj} - h^r_{11}h^r_{22} - \sum_{1\le i<j\le m}(h^r_{ij})^2 + (h^r_{12})^2\Big] \le \frac{m^2(m-2)}{2(m-1)} \parallel H \parallel^2 \qquad (5.72)$$

In view of the last expression in (5.71), we obtain

$$\begin{aligned}\tau(x) - K(\Pi) \le (m-2)\Big(&\frac{m^2}{2(m-1)} \parallel H \parallel^2 + (m+1)\frac{f_1}{2}\Big)\\
+\Big(3 \parallel \mathcal{T} \parallel^2 - 6\Theta^2(\Pi)\Big)\frac{f_2}{2} &+ \Big(\parallel \xi_\Pi \parallel^2 - (m-1) \parallel \xi^T \parallel^2 \Big) f_3\\
+\frac{(\psi_1+\psi_2)}{2}\Big(tr(\beta_1 \mid_\Pi) - \lambda(m-1)\Big) &+ \frac{\psi_2(\psi_1-\psi_2)}{2}\Big(tr(\beta_2 \mid_\Pi)\\
-\mu(m-1)\Big) &+ \frac{(\psi_1-\psi_2)}{2}\Big(\Lambda(tr(h \mid_\Pi)) - m(m-1)\Lambda(H)\Big). \qquad (5.73)\end{aligned}$$

Now, if P is a tangent vector field on M, then (5.60) implies $h = \widehat{h}$ and $H = \widehat{H}$. If the equality case (5.73) holds at a point $x \in M$, then the equality cases of (5.67) and (5.72) hold, which gives

$$h^{m+1}_{11} = h^{m+1}_{22} = h^{m+1}_{33} = \cdots = h^{m+1}_{mm}$$

$$h^{m+1}_{1j} = h^{m+1}_{2j} = 0, j > 2$$

$$h^r_{11} + h^r_{22} = 0, r = m+2, \ldots, 2n+1$$

$$h^r_{ij} = 0, i \neq j, r = m+1, \ldots, 2n+1$$

$$h^{m+1}_{ij} = 0, i \neq j, i, j > 2$$

So choosing a suitable orthonormal basis, the shape operators take the desired forms. □

Corollary 5.7. *Under the same arguments as in Theorem 5.12,*

1. *If the structure vector field ξ is tangent to M, we have*

$$\begin{aligned}\tau(x)-K(\Pi)\leq (m-2)\Big(\frac{m^2}{2(m-1)}\parallel H\parallel^2+(m+1)\frac{f_1}{2}\Big)\\ +\Big(3\parallel \mathcal{T}\parallel^2-6\Theta^2(\Pi)\Big)\frac{f_2}{2}+\Big(\parallel \xi_\Pi\parallel^2-(m-1)\Big)f_3\\ +\frac{(\psi_1+\psi_2)}{2}\Big(tr(\beta_1\mid_\Pi)-\lambda(m-1)\Big)+\frac{\psi_2(\psi_1-\psi_2)}{2}\Big(tr(\beta_2\mid_\Pi)\\ -\mu(m-1)\Big)+\frac{(\psi_1-\psi_2)}{2}\Big(\Lambda(tr(h\mid_\Pi))-m(m-1)\Lambda(H)\Big).\end{aligned}$$

2. *If the structure vector field ξ is normal to M, we have*

$$\begin{aligned}\tau(x)-K(\Pi)\leq (m-2)\Big(\frac{m^2}{2(m-1)}\parallel H\parallel^2+(m+1)\frac{f_1}{2}\Big)\\ +\Big(3\parallel \mathcal{T}\parallel^2-6\Theta^2(\Pi)\Big)\frac{f_2}{2}++\frac{(\psi_1+\psi_2)}{2}\Big(tr(\beta_1\mid_\Pi)\\ -\lambda(m-1)\Big)+\frac{\psi_2(\psi_1-\psi_2)}{2}\Big(tr(\beta_2\mid_\Pi)\\ -\mu(m-1)\Big)+\frac{(\psi_1-\psi_2)}{2}\Big(\Lambda(tr(h\mid_\Pi))-m(m-1)\Lambda(H)\Big).\end{aligned}$$

Remark 5.2. *It should be noted that Theorem 5.12 generalizes the Theorem 6 obtained by Wang (2019). Moreover, taking different values of $f_i, i=1,2,3$, we can obtain similar inequalities as Theorem 5.12 for the Kenmotsu space form and Cosympletic space form endowed with certain types of connections by restricting the values of $\psi_i, i=1,2$.*

Remark 5.3. *If in Theorem 5.12, we take $\psi_1=\psi_2=1$ then we obtained Theorem 5.1 (Sular, 2016).*

Corollary 5.8. *Let $M^m, m\geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian space form $\widetilde{M}(f_1,f_2,f_3)$ endowed with a semi-*

symmetric non-metric connection, then

$$\tau(x) - K(\Pi) \leq (m-2)\Big(\frac{m^2}{2(m-1)} \parallel H \parallel^2 + (m+1)\frac{f_1}{2}\Big)$$
$$+\Big(3 \parallel \mathcal{T} \parallel^2 - 6\Theta^2(\Pi)\Big)\frac{f_2}{2} + \Big(\parallel \xi_\Pi \parallel^2 - (m-1) \parallel \xi^T \parallel^2 \Big) f_3$$
$$+\frac{1}{2}\Big(tr(\beta_1 \mid_\Pi) - \lambda(m-1)\Big) + \frac{1}{2}\Big(\Lambda(tr(h \mid_\Pi)) - m(m-1)\Lambda(H)\Big),$$

where Π *is a 2-plane section* T_xM, $x \in M$.

For an integer $k \geq 0$ we denote by $S(m,k)$ the set of k-tuples $(m_1, \ldots, m_k)$ of integers ≥ 2 satisfying $m_1 < m$ and $m_1, \ldots, m_k \leq m$. Also, let $S(m)$ be the set of unordered k-tuples with $k \geq 0$ for a fixed m. Then, for each k-tuples $(m_1, \ldots, m_k) \in S(m)$, Chen introduced a Riemannian invariant $\delta(m_1, \ldots, m_k)$ as follows (Chen, 1995)

$$\delta(m_1, \ldots, m_k)(x) = \tau(x) - inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \tag{5.74}$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_xM such that $\dim L_j = m_j, j \in \{1, \ldots, k\}$. For simplicity, we set

$$\Psi_1(L_j) = \sum_{1\leq i<j\leq r} g^2(\mathcal{T}e_i, e_j), \quad \Psi_2(L_j) = \sum_{1\leq i<j\leq r} [g(\xi^T, e_i)^2 + g(\xi^T, e_j)^2]$$
$$\Psi_3(L_j) = \sum_{1\leq i<j\leq r} [\beta_1(e_i, e_i) + \beta_1(e_j, e_j)], \quad \Psi_4(L_j) = \sum_{1\leq i<j\leq r} [\beta_2(e_i, e_i) + \beta_2(e_j, e_j)]$$
$$\Psi_5(L_j) = \sum_{1\leq i<j\leq r} \Lambda(h(e_i, e_i) + h(e_j, e_j))$$

As the generalization of Theorem 5.12, we state and prove the following results using the methods used by Zhang et al. (2014).

Theorem 5.13. *Let* $M^m, m \geq 3$ *be an* m*-dimensional submanifold of a* $(2n+1)$*-dimensional generalized Sasakian space form* $\widetilde{M}(f_1, f_2, f_3)$ *endowed with a quarter-*

symmetric connection $\overline{\nabla}$, then

$$\delta(m_1,\ldots,m_k) \leq b(m_1,\ldots,m_k) \parallel H \parallel^2 + a(m_1,\ldots,m_k) f_1$$
$$+3f_2\Big(\frac{\parallel \mathcal{T} \parallel^2}{2} - \sum_{j=1}^{k} \Psi_1(L_j)\Big) - f_3\Big((m-1) \parallel \xi^T \parallel^2 - \sum_{j=1}^{k} \Psi_2(L_j)\Big)$$
$$-\frac{(\psi_1+\psi_2)}{2}\Big((m-1)\lambda - \sum_{j=1}^{k} \Psi_3(L_j)\Big) - \frac{\psi_2}{2}(\psi_1-\psi_2)\Big((m-1)\mu$$
$$-\sum_{j=1}^{k} \Psi_4(L_j)\Big) + \frac{(\psi_1-\psi_2)}{2}\Big(m(m-1)\Lambda(H) - \sum_{j=1}^{k} \Psi_5(L_j)\Big),$$

for any k-tuples $(m_1,\ldots,m_k) \in S(m)$. If P is a tangent vector field on M, the equality case holds at $x \in M^m$ if and only if there exists an orthonormal basis $\{e_1,...,e_m\}$ of T_xM and an orthonormal basis $\{e_{m+1},...,e_{2n+1}\}$ of $T_x^{\perp}M$ such that the shape operators of M in $\widetilde{M}(f_1,f_2,f_3)$ at x have the following forms:

$$A_{e_{m+1}} = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_m \end{pmatrix}, A_{e_r} = \begin{pmatrix} A_1^r & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & A_k^r & 0 \\ 0 & \ldots & 0 & \varsigma_r I \end{pmatrix}, r = m+2,\ldots,2n+1,$$

where $a_1,\ldots,a_m$ satisfy

$$a_1 + \cdots + a_{m_1} = \cdots = a_{m_1+\cdots+m_{k-1}+1} + \cdots + a_{m_1+\cdots+m_k+1} = \cdots = a_m$$

and each A_j^r is a symmetric $m_j \times m_j$ submatrix satisfying $tr(A_1^r) = \cdots = tr(A_k^r) = \varsigma_r$, I is an identity matrix.

Proof. Choose an orthonormal basis $\{e_1,...,e_m\}$ of T_xM and an orthonormal basis $\{e_{m+1},...,e_{2n+1}\}$ of $T_x^{\perp}M$ such that mean curvature vector H is in the direction of the normal vector to e_{m+1}. We set

$$a_i = h_{ii}^{m+1}, \qquad i = 1, \dots, m$$

$$b_1 = a_1, b_2 = a_2 + \cdots + a_{m_1}, b_3 = a_{m_1+1} + \cdots + a_{m_1+m_2}, \dots,$$

$$b_{k+1} = a_{m_1+\cdots+m_{k-1}+1} + \cdots + a_{m_1+\cdots+m_{k-1}+m_k}, \dots, b_{\gamma+1} = a_m,$$

and consider the following sets

$$D_1 = \{1, \dots, m_1\}, \quad D_2 = \{m_1+1, \dots, m_1+m_2\}, \dots,$$

$$D_k = \{(m_1 + \cdots + m_{k-1}) + 1, \dots, (m_1 + \cdots + m_{k-1}) + m_k\}.$$

Let $L_1, \dots, L_k$ be mutually orthogonal subspace of T_xM with $\dim L_j = m_j$, defined by

$$L_j = Span\{e_{m_1+\cdots+m_{j-1}+1}, \dots, e_{m_1+\cdots+m_j}\}, \quad j = 1, \dots, k.$$

From (5.62), (1.56), (5.61) and (1.19), we get

$$\begin{aligned} \tau(L_j) = \frac{m_j(m_j-1)}{2} f_1 + 3f_2\Psi_1(L_j) - f_3\Psi_2(L_j) \\ -\frac{(\psi_1+\psi_2)}{2}\Psi_3(L_j) - \frac{\psi_2}{2}(\psi_1-\psi_2)\Psi_4(L_j) - \frac{(\psi_1-\psi_2)}{2}\Psi_5(L_j) \\ + \sum_{r=m+1}^{2n+1} \sum_{\alpha_j<\beta_j} \big[h^r_{\alpha_j\alpha_j} h^r_{\beta_j\beta_j} - (h_{\alpha_j\beta_j})^2\big]. \end{aligned} \tag{5.75}$$

We set

$$\begin{aligned} \varepsilon = 2\tau - 2b(m_1, \dots, m_k) \parallel H \parallel^2 - m(m-1)f_1 - 3f_2 \parallel \mathcal{T} \parallel^2 \\ +2(m-1)f_3 \parallel \xi^T \parallel^2 + (\psi_1+\psi_2)\lambda(m-1) \\ +\psi_2(\psi_1-\psi_2)\mu(m-1) + m(m-1)(\psi_1-\psi_2)\Lambda(H), \end{aligned} \tag{5.76}$$

where,

$$b(m_1,\ldots,m_k) = \frac{m^2\Big(m+k-1-\sum_{j=1}^{k} m_j\Big)}{2\Big(m+k-\sum_{j=1}^{k} m_j\Big)},$$

for each $(m_1,\ldots,m_k) \in S(m)$.

Also, let $\gamma = m+k-\sum_{j=1}^{k} m_j$. Then in view of this and (5.76), Eq. (5.64) becomes

$$m^2 \parallel H \parallel^2 = (\parallel h \parallel^2 + \varepsilon)\gamma,$$

which can be written as

$$\Big(\sum_{i=1}^{\gamma+1} b_i\Big)^2 = \gamma\Big[\varepsilon + \sum_{i=1}^{\gamma+1} b_i^2 + \sum_{i\neq j}(h_{ij}^{m+1})^2 + \sum_{r=m+2}^{2n+1}\sum_{i,j=1}^{m}(h_{ij}^r)^2 \\ -2\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1} - \cdots - 2\sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k}\Big], \tag{5.77}$$

where $\alpha_j, \beta_j \in D_j$ for all $j = 1,\ldots,k$.

Now applying Lemma 2.3 (Zhang et al., 2014) in (5.77), we obtain

$$\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1} + \cdots + \sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k} \geq \\ \frac{1}{2}\Big[\varepsilon + \sum_{i\neq j}(h_{ij}^{m+1})^2 + \sum_{r=m+2}^{2n+1}\sum_{i,j=1}^{m}(h_{ij}^r)^2\Big],$$

which further implies

$$\sum_{j=1}^{k}\sum_{r=m+1}^{2n+1}\sum_{\alpha_j<\beta_j}\Big[h_{\alpha_j\alpha_j}^r h_{\beta_j\beta_j}^r - (h_{\alpha_j\beta_j}^r)^2\Big] \geq \frac{\varepsilon}{2} \\ +\frac{1}{2}\sum_{r=m+1}^{2n+1}\sum_{(\alpha,\beta)\notin D^2}(h_{\alpha\beta}^r)^2 + \sum_{r=m+2}^{2n+1}\sum_{\alpha_j\in D_j}(h_{\alpha_j\alpha_j}^r)^2 \leq \frac{\varepsilon}{2}, \tag{5.78}$$

where $D^2 = (D_1 \times D_1) \cup \cdots \cup (D_k \times D_k)$. Combining (5.64), (5.75) and (5.78)

gives

$$\tau-\sum_{j=1}^{k}\tau(L_j)\leq b(m_1,\ldots,m_k)\parallel H\parallel^2+a(m_1,\ldots,m_k)f_1$$
$$+3f_2\Big(\frac{\parallel \mathcal{T}\parallel^2}{2}-\sum_{j=1}^{k}\Psi_1(L_j)\Big)-f_3\Big((m-1)\parallel \xi^T\parallel^2-\sum_{j=1}^{k}\Psi_2(L_j)\Big)$$
$$-\frac{(\psi_1+\psi_2)}{2}\Big((m-1)\lambda-\sum_{j=1}^{k}\Psi_3(L_j)\Big)-\frac{\psi_2}{2}(\psi_1-\psi_2)\Big((m-1)\mu$$
$$-\sum_{j=1}^{k}\Psi_4(L_j)\Big)+\frac{(\psi_1-\psi_2)}{2}\Big(m(m-1)\Lambda(H)-\sum_{j=1}^{k}\Psi_5(L_j)\Big),\qquad(5.79)$$

where, $a(m_1,\ldots,m_k)=\frac{1}{2}\Big[m(m-1)-\sum_{j=1}^{k}m_j(m_j-1)\Big]$.

The equality case (5.79) at a point $x\in M$ holds if and only if all the previous inequalities hold, thus, the shape operators take the desired forms. □

Remark 5.4. *Restricting the values of $f_i, i=1,2,3$ and ψ_i for $i=1,2$, we can obtain similar bounds as Theorem 5.13 for certain contact space forms endowed with certain connections.*

Theorem 5.14. *Let $M^m, m\geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian space form $\widetilde{M}(f_1,f_2,f_3)$ endowed with a quarter-symmetric connection $\overline{\nabla}$, then*

(i) For each unit vector X_1 in T_xM, we have

$$Ric(X_1)\leq(m-1)f_1+3f_2\textstyle\sum_{j=2}^{m}g^2(\varphi X_1,e_j)+f_3\big((2-m)\eta^2(X_1)-\parallel\xi^T\parallel^2\big)$$
$$+[\psi_1+(1-m)\psi_2]\beta_1(X_1,X_1)-\psi_1\lambda+\psi_2(\psi_1-\psi_2)(1-m)\beta_2(X_1,X_1)$$
$$-(\psi_1-\psi_2)[m\Lambda(H)-\Lambda(h(X_1,X_1))]+\tfrac{m^2}{4}\parallel H\parallel^2.\qquad(5.80)$$

(ii) If $H(x)=0$, then a unit tangent vector X_1 at x satisfies the equality case of (5.80) if and only if $X_1\in\mathcal{M}(x)=\{X_1\in T_xM\mid h(X_1,X_2)=0,\forall X_2\in$

$T_xM\}$.

(iii) The equality of (5.80) holds for all unit tangent vectors at x if and only if either

1. $m \neq 2, h^r_{ij} = 0, i, j = 1, 2..., m.r = m+1, \ldots, 2n+1,$ *or*

2. $m = 2, h^r_{11} = h^r_{22}, h^r_{12} = 0, r = 3, \ldots, 2n+1.$

Proof. Choosing the orthonormal basis $\{e_1, \ldots, e_m\}$ such that $e_1 = X_1$, where $X_1 \in T_xM$ is a unit tangent vector at the point x on M. In view of (5.62), (5.61) and (1.19) then proceeding similarly as the proof of Theorem 4 (Wang, 2019), one can easily obtained the desire results. □

By choosing an orthonormal frame $\{e_1, \ldots, e_k\}$ of L such that $e_1 = X_1$, a unit tangent vector, Chen (1995) defined the k-Ricci curvature of L at X_1 by

$$Ric_L(X_1) = K_{12} + K_{13} + \cdots + K_{1k}. \tag{5.81}$$

For an integer $k, 2 \leq k \leq m$, the Riemannian invariant Θ_k on M is defined by

$$\Theta_k(x) = \frac{1}{k-1} inf\{Ric_L(X_1) \mid L, X_1\}, x \in M$$

where L runs over all k-plane sections in T_xM and X_1 runs over all unit vectors in L. From Zhang et al. (2014), we have

$$\tau(x) \geq \frac{m(m-1)}{2}\Theta_k(x). \tag{5.82}$$

Let us choose $\{e_1, \ldots, e_m\}$ and $\{e_{m+1}, \ldots, e_{2n+1}\}$ as an orthonormal basis of T_xM and $T^\perp_x M, x \in M$, respectively, where e_{m+1} is parallel to the mean curvature vector H. In addition, let $\{e_1, \ldots, e_m\}$ diagonalize the shape operator $A_{e_{m+1}}$.

Then,

$$A_{e_{m+1}} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_m \end{pmatrix}$$

and

$$A_{e_r} = h_{ij}^r, \quad i,j = 1,\dots,m, \qquad r = m+2,\dots,2n+1, \qquad trA_{e_r} = 0. \tag{5.83}$$

In consequence of the above assumptions, Eq. (5.64) can be written as follows:

$$\begin{aligned} m^2 \parallel H \parallel^2 = 2\tau + \sum_{i=1}^{m} a_i^2 + \sum_{r=m+2}^{2n+1} \sum_{i,j=1}^{m} (h_{ij}^r)^2 - m(m-1)f_1 \\ -3f_2 \parallel \mathcal{T} \parallel^2 + 2(m-1)f_3 \parallel \xi^T \parallel^2 + (\psi_1 + \psi_2)\lambda(m-1) \\ +\psi_2(\psi_1 - \psi_2)\mu(m-1) + m(m-1)(\psi_1 - \psi_2)\Lambda(H). \end{aligned} \tag{5.84}$$

Using the Cauchy-Schwartz inequality we have

$$\sum_{i=1}^{m} a_i^2 \geq m \parallel H \parallel^2 . \tag{5.85}$$

Combining (5.82), (5.84) and (5.84), we are able to state the following:

Theorem 5.15. *Let $M^m, m \geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a quarter-symmetric connection $\overline{\nabla}$, then for any integer $k, 2 \leq k \leq m$ and any point $x \in M$, we have*

$$\begin{aligned} \parallel H \parallel^2 (x) \geq \Theta_k(x) - f_1 - \frac{3f_2}{m(m-1)} \parallel \mathcal{T} \parallel^2 + \frac{2f_3}{m} \parallel \xi^T \parallel^2 \\ + \frac{\lambda}{m}(\psi_1 + \psi_2) + \frac{\mu}{m}\psi_2(\psi_1 - \psi_2) + (\psi_1 - \psi_2)\Lambda(H). \end{aligned}$$

As a particular case of Theorem 5.15, we obtained Theorem 6.2 (Sular, 2016) which is as follows:

Corollary 5.9 (Sular, 2016)**.** *Let $M^m, m \geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a semi-symmetric metric connection, then for any integer $k, 2 \leq k \leq m$ and any point $x \in M$, we have*

$$\| H \|^2 (x) \geq \Theta_k(x) - f_1 - \frac{3f_2}{m(m-1)} \| \mathcal{T} \|^2 + \frac{2f_3}{m} \| \xi^T \|^2 + \frac{2\lambda}{m}.$$

Corollary 5.10. *Let $M^m, m \geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a semi-symmetric non-metric connection, then for any integer $k, 2 \leq k \leq m$ and any point $x \in M$, we have*

$$\| H \|^2 (x) \geq \Theta_k(x) - f_1 - \frac{3f_2}{m(m-1)} \| \mathcal{T} \|^2 + \frac{2f_3}{m} \| \xi^T \|^2 + \frac{\lambda}{m} + \Lambda(H).$$

Remark 5.5. *Restricting function $f_i, i = 1, 2, 3$, we can easily obtain similar inequality in the case of Sasakian, Kenmotsu and Cosympletic space forms.*

5.2.1 Some Applications

The notion of slant submanifolds in almost contact geometry was introduced by Lotta (1996). A submanifold M of an almost contact metric manifold $(\widetilde{M}, \varphi, \xi, \eta, g)$ tangent to the structure vector field ξ is said to be a contact slant submanifold if, for any point $x \in M$ and any vector $X_1 \in T_xM$ linearly independent on ξ_x, the angle between the vector φX_1 and the tangent space T_xM is constant. This angle is known as the slant angle of M. The concept of slant submanifold is further generalized as follows:

Definition 5.3 (Alqahtani et al., 2017)**.** *A submanifold M of an almost contact metric manifold M is called a bi-slant submanifold, whenever we have*

1. $TM = \mathcal{D}_{\theta_1} \oplus \mathcal{D}_{\theta_2} \oplus \xi$

2. $\varphi\mathcal{D}_{\theta_1} \perp \mathcal{D}_{\theta_2}$ *and* $\varphi\mathcal{D}_{\theta_2} \perp \mathcal{D}_{\theta_1}$.

3. For $i = 1, 2$, *the distribution* D_i *is slant with slant angle* θ_i.

Now, as a consequence of Theorem 5.12, we can state the following:

Theorem 5.16. *Let* M *be a* $(m = 2d_1+2d_2+1)$*-dimensional bi-slant submanifold of a* $(2n+1)$*-dimensional generalized Sasakian space form* $\widetilde{M}(f_1, f_2, f_3)$ *endowed with a quarter-symmetric connection* $\overline{\nabla}$*, then we have*

$$\begin{aligned}\tau(x) - K(\Pi) \leq (m-2)\Big(\frac{m^2}{2(m-1)} \parallel H \parallel^2 +(m+1)\frac{f_1}{2}\Big)\\ +3\Big((d_1-1)cos^2\theta_1 + d_2cos^2\theta_2\Big)\frac{f_2}{2} - (m-1)f_3\\ +\frac{(\psi_1+\psi_2)}{2}\Big(tr(\beta_1 \mid_\Pi) - \lambda(m-1)\Big) + \frac{\psi_2(\psi_1-\psi_2)}{2}\Big(tr(\beta_2 \mid_\Pi)\\ -\mu(m-1)\Big) + \frac{(\psi_1-\psi_2)}{2}\Big(\Lambda(tr(h \mid_\Pi)) - m(m-1)\Lambda(H)\Big),\end{aligned}$$

for any plane Π *invariant by* $\mathcal{T}$ *and tangent to slant distribution* $\mathcal{D}_{\theta_1}$ *and*

$$\begin{aligned}\tau(x) - K(\Pi) \leq (m-2)\Big(\frac{m^2}{2(m-1)} \parallel H \parallel^2 +(m+1)\frac{f_1}{2}\Big)\\ +3\Big(d_1cos^2\theta_1 + (d_2-1)cos^2\theta_2\Big)\frac{f_2}{2} - (m-1)f_3\\ +\frac{(\psi_1+\psi_2)}{2}\Big(tr(\beta_1 \mid_\Pi) - \lambda(m-1)\Big) + \frac{\psi_2(\psi_1-\psi_2)}{2}\Big(tr(\beta_2 \mid_\Pi)\\ -\mu(m-1)\Big) + \frac{(\psi_1-\psi_2)}{2}\Big(\Lambda(tr(h \mid_\Pi)) - m(m-1)\Lambda(H)\Big),\end{aligned}$$

for any plane Π *invariant by* $\mathcal{T}$ *and tangent to slant distribution* $\mathcal{D}_{\theta_2}$*. Moreover, the ideal case is the same as Theorem 5.12.*

Proof. Let M be a bi-slant submanifold of a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ of dimension $(m = 2d_1 + 2d_2 + 1)$ and let $\{e_1, \ldots, e_m = \xi\}$ be an orthonormal frame of tangent space T_xM at a point $x \in M$, such that

$$\begin{aligned}e_1, e_2 = sec\theta_1\mathcal{T}e_1, \ldots, e_{2d_1-1}, e_{2d_1} = sec\theta_1\mathcal{T}e_{2d_1-1}, e_{2d_1+1}, e_{2d_1+2}\\ = sec\theta_2\mathcal{T}e_{2d_1+1}, \ldots, e_{2d_1+2d_2-1}, e_{2d_1+2d_2} = sec\theta_2\mathcal{T}e_{2d_1+2d_2-1}, e_{2d_1+2d_2+1} = \xi,\end{aligned}$$

which gives

$$g^2(\varphi e_{i+1}, e_i) = \begin{cases} cos^2\theta_1, & \text{for} \quad i = 1, 2, \ldots, 2d_1 - 1 \\ cos^2\theta_2, & \text{for} \quad i = 2d_1 + 1, \ldots, 2d_1 + 2d_2 - 1. \end{cases}$$

Thus we have

$$\| \mathcal{T} \|^2 = 2\{d_1 cos^2\theta_1 + d_2 cos^2\theta_2\}$$

Making use of the above facts in Theorem 5.12, the proof is straight forward. □

In a similar manner Theorems 5.13, 5.14 and 5.15 can be stated for bi-slant submanifold of a generalized Sasakian space form. Moreover, restricting the values of $\theta_i, i = 1, 2$, similar results can be obtained for a large class of submanifolds such as slant, semi-slant, hemi-slant, semi-invariant submanifolds. Also, by taking different values of $f_i, i = 1, 2, 3$ we can derive similar inequalities for Sasakian, Kenmotsu and Cosympletic space forms.

Chapter 6

Some Results on Spacetime

Chapter 6

Some Results on Spacetime

6.1 On Ricci-Yamabe soliton and Geometrical Structure in a Perfect Fluid Spacetime

As Ricci-Yamabe soliton is a scalar combination of Ricci and Yamabe soliton, it is fruitful to study it in the context of perfect fluid spacetime and obtain results that generalize the previously known results in perfect fluid spacetime.

6.1.1 Geometrical structure of perfect fluid spacetime with torse-forming vector field

According to Einstein's field equation, the energy-momentum tensor describes the curvature of spacetime and hence plays a crucial role in the theory of relativity. The spacetime of general relativity is regarded as a connected four dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g with signature $(-, +, +, +)$. A spacetime is said to be a perfect fluid spacetime if the Ricci tensor

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is of the form:

$$S = ag + b\eta \otimes \eta, \tag{6.1}$$

where a, b are scalars and η is non-zero 1-form.
The general form of energy-momentum tensor T for a perfect fluid is (O'Neill, 1983)

$$T(X,Y) = \rho g(X,Y) + (\sigma + \rho)\eta(X)\eta(Y), \tag{6.2}$$

for any $X, Y \in \chi(M)$, where σ is the energy density, ρ is the isotropic pressure, g is the metric tensor of Minkowski spacetime, $\eta(X) = -g(X, \xi)$ is 1-form, equivalent to unit vector ξ and $g(\xi, \xi) = -1$. If $\rho = \rho(\sigma)$ then perfect fluid spacetime is called isentropic (Hawking and Ellis, 1973) and if $\sigma = 3\rho$ then it is a radiation fluid.

Einstein's field equation (O'Neill, 1983) governing the perfect fluid motion is defined as:

$$S(X,Y) + (\lambda - \frac{r}{2})g(X,Y) = kT(X,Y), \tag{6.3}$$

for any $X, Y \in \chi(M)$, where λ is the cosmological constant, $k(\approx 8\pi G$, where G is universal Gravitational constant) is the gravitational constant.
Combining (6.2) and (6.3) we obtain

$$S(X,Y) = -(\lambda - \frac{r}{2} + k\rho)g(X,Y) + k(\sigma + \rho)\eta(X)\eta(Y). \tag{6.4}$$

Taking trace of (6.4), the scalar curvature becomes $r = 4\lambda + k(\sigma - 3\rho)$, using in (6.4) we infer

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{6.5}$$

where $a = \lambda + \frac{k(\sigma-\rho)}{2}$ and $b = k(\sigma + \rho)$.

Lemma 6.1 (Blaga, 2018; Venkatesha and Kumara, 2019; Siddiqi and Siddiqui, 2020)**.** *In perfect fluid spacetime with torse-forming vector field ξ, the following relations hold:*

$$\begin{aligned}
\eta(\nabla_\xi \xi) &= 0, \quad \nabla_\xi \xi = 0, \\
(\nabla_X \eta)(Y) &= g(X,Y) + \eta(X)\eta(Y), \\
R(X,Y)\xi &= \eta(Y)X - \eta(X)Y, \\
(\mathcal{L}_\xi g)(X,Y) &= 2[g(X,Y) + \eta(X)\eta(Y)], \\
R(X,\xi)\xi &= -X - \eta(X)\xi.
\end{aligned}$$

Let (M^4, g) be a semiconformally flat perfect fluid spacetime with torse-forming vector field ξ. As $P = 0$, we have $divP = 0$ where "div" is the divergent. Since r is constant, implies $X(r) = 0$ for any $X \in \chi(M)$. From (1.25) for $divP = 0$ we obtain

$$k(\sigma + \rho)[\eta(Y)X - \eta(X)Y] = 0. \tag{6.6}$$

As $k \neq 0$, in this case the equation of state $\rho + \sigma = 0$ emerges. This is the characteristic equation of state for dark energy in the universe and corresponds to the cosmological constant (Stephani et al., 2003). Essentially, as density cannot be negative, the pressure ρ must be negative which is useful in explaining the observed accelerated expansion of the universe problem.

Making use of $\rho = -\sigma$ in (6.5) and (1.25) gives

$$R(X,Y)Z = \frac{1}{3\alpha}(3\alpha - 4\beta)(\lambda + k\sigma)[g(Y,Z)X - g(X,Z)Y]. \tag{6.7}$$

Therefore, spacetime has constant curvature. As de-Sitter space is a Lorentzian manifold of constant curvature with implied negative pressure driving cosmic inflation (Schmidt, 1993) we can state the following:

Theorem 6.1. *If perfect fluid spacetime with torse-forming vector field ξ is semi-*

conformally flat, then the spacetime represents de-Sitter space, provided $\alpha \neq 0$.

We know that manifold of constant curvature is Einstein. Also from (6.7) we easily see that $R \cdot R = 0$. A perfect fluid spacetime satisfying $R \cdot R = 0$ and $R \cdot S = 0$ are called semi-symmetric and Ricci semi-symmetric respectively. A semi-symmetric implies Ricci semi-symmetric but conversely not true.

Proposition 6.1. *A semiconformally flat perfect fluid spacetime with torse-forming vector field* ξ *is*
i) Einstein.
ii) semi-symmetric and Ricci semi-symmetric.

According to Karchar (1992), a Lorentzian manifold is called infinitesimal spatially isotropic relative to timelike unit vector field ρ if its curvature tensor R satisfies relations

$$R(X, Y)Z = l[g(Y, Z)X - g(X, Z)Y],$$

for all $X, Y, Z \in \rho^{\perp}$ and

$$R(X, \rho)\rho = mX,$$

for all $X \in \rho^{\perp}$, where l, m are real-valued functions on the manifold.

Let $\xi^{\perp}$ denote the 3-dimensional distribution in a semiconformally flat perfect fluid spacetime orthogonal to torse-forming vector field ξ, then from (6.7) we get

$$R(X, Y)Z = \frac{1}{3\alpha}(3\alpha - 4\beta)(\lambda + k\sigma)[g(Y, Z)X - g(X, Z)Y], \tag{6.8}$$

for all $X, Y, Z \in \xi^{\perp}$. Also taking $Y = Z = \xi$ in (6.8) gives

$$R(X, \rho)\rho = -\frac{1}{3\alpha}(3\alpha - 4\beta)(\lambda + k\sigma)X, \tag{6.9}$$

for every $X \in \xi^{\perp}$. Hence we can state the following:

Theorem 6.2. *A semiconformally flat perfect fluid spacetime with $a \neq 0$ and torse-forming vector field ξ is infinitesimally spatially isotropic relative to unit vector field ξ.*

Theorem 6.3. *Let (M^4, g) be a general relativistic perfect fluid spacetime with torse-forming vector field ξ.*
1. If $P(\xi, \cdot) \cdot S = 0$ then $\rho = -\sigma$ or $\rho = \frac{3\alpha(\lambda-1)+\beta(4\lambda+k\sigma)}{3k(\alpha-\beta)}$.
2. If $S(\xi, \cdot) \cdot P = 0$ then $\rho = \frac{\lambda}{k}$ or $\rho = \frac{3\alpha(\lambda-1)+\beta(4\lambda+k\sigma)}{3k(\alpha-\beta)}$.

Proof. 1. Suppose perfect fluid spacetime with torse-forming vector field ξ satisfies $P(\xi, X) \cdot S(U, V) = 0$, implies

$$S(P(\xi, X)U, V) + S(U, P(\xi, X)V) = 0, \tag{6.10}$$

for all $X, U, V \in \chi(M)$. Inserting (6.5) and (1.25) in (6.10) results in

$$\begin{aligned} &- 2\alpha k(\sigma + \rho)(\lambda + \frac{k}{2}(\sigma - \rho))\eta(X)\eta(U)\eta(V) + k(\sigma + \rho)(\alpha - \frac{\beta r}{3} - \\ &\alpha(\lambda - k\rho))[-g(X, U)\eta(V) - 2\eta(X)\eta(U)\eta(V) - g(X, V)\eta(U)] \\ &+ 2\alpha k^2(\sigma + \rho)^2\eta(X)\eta(U)\eta(V) = 0. \end{aligned} \tag{6.11}$$

Replacing U by ξ in (6.11) we obtain that either $\rho = -\sigma$ or $\rho = \frac{3\alpha(\lambda-1)+\beta(4\lambda+k\sigma)}{3k(\alpha-\beta)}$.
2. Suppose perfect fluid spacetime satisfies $S(\xi, X) \cdot P(U, V)W = 0$, implies

$$\begin{aligned} &S(X, P(U, V)W)\xi - S(\xi, P(U, V)W)X + S(X, U)P(\xi, V)W \\ &- S(\xi, U)P(X, V)W + S(X, V)P(U, \xi)W - S(\xi, V)P(U, X)W \\ &+ S(X, W)P(U, V)\xi - S(\xi, W)P(U, V)X = 0, \end{aligned} \tag{6.12}$$

for all $X, U, V, W \in \chi(M)$.
Taking $V = W = \xi$ in (6.12) and using (6.5) and (1.25), we obtain the following relation

$$(\lambda - k\rho)(\alpha - \frac{\beta r}{3} - \alpha(\lambda - k\rho))[g(X, U) + \eta(X)\eta(U)] = 0.$$

Thus either $\rho = \frac{\lambda}{k}$ or $\rho = \frac{3\alpha(\lambda-1)+\beta(4\lambda+k\sigma)}{3k(\alpha-\beta)}$.

This completes the proof. □

6.1.2 Ricci-Yamabe soliton in a perfect fluid spacetime

In this subsection, we study Ricci-Yamabe soliton in the framework of perfect fluid spacetime admitting a torse-forming vector field ξ.

Taking potential vector field, $V = \xi$ in (1.45) and using Lemma 6.1 we obtain

$$\alpha S(X,Y) = [\mu - \frac{\beta r}{2} - 1]g(X,Y) - \eta(X)\eta(Y). \tag{6.13}$$

Inserting $X = Y = \xi$ in (6.13) yields

$$\mu = \lambda(\alpha + 2\beta) + \frac{\beta k}{2}(\sigma - 3\rho) - \frac{\alpha k}{2}(\sigma + 3\rho). \tag{6.14}$$

Hence we can state the following:

Theorem 6.4. *If a perfect fluid spacetime with torse-forming vector field ξ admits Ricci-Yamabe soliton $(g, \xi, \mu, \alpha, \beta)$, then the Ricci-Yamabe soliton is expanding, steady or shrinking according to as $\lambda > \frac{k}{2(\alpha+2\beta)}\{\alpha(\sigma + 3\rho) - \beta(\sigma - 3\rho)\}, \lambda = \frac{k}{2(\alpha+2\beta)}\{\alpha(\sigma+3\rho) - \beta(\sigma-3\rho)\}$ or $\lambda < \frac{k}{2(\alpha+2\beta)}\{\alpha(\sigma+3\rho) - \beta(\sigma-3\rho)\}$ respectively, provided $\alpha + 2\beta \neq 0$.*

Remark 6.1. *Now we will look at some of the particular cases of Theorem 6.4. If a perfect fluid spacetime with torse-forming vctor field ξ admits:*

1. Ricci soliton $(\alpha = 1, \beta = 0)$, then the Ricci soliton is expanding, steady or shrinking according as $\lambda > \frac{k}{2}(\sigma+3\rho), \lambda = \frac{k}{2}(\sigma+3\rho)$ or $\lambda < \frac{k}{2}(\sigma+3\rho)$ respectively. This was shown by Venkatesha and Kumara (2019).

2. Yamabe soliton $(\alpha = 0, \beta = 2)$, then the Yamabe soliton is expanding, steady or shrinking according as $\lambda > \frac{k}{4}(3\rho-\sigma), \lambda = \frac{k}{4}(3\rho-\sigma)$ or $\lambda < \frac{k}{4}(3\rho-\sigma)$ respectively.

3. Einstein soliton $(\alpha = 1, \beta = -1)$, then $\mu = -\lambda - k\sigma$ implies Einstein soliton is expanding if $\lambda < -k\sigma$, steady if $\lambda = -k\sigma$ and shrinking if $\lambda > -k\sigma$.

Theorem 6.5. *If a perfect fluid spacetime with torse-forming vector field ξ admits Ricci-Yamabe soliton $(g, V, \mu, \alpha, \beta)$, then either every perfect fluid spacetime with torse-forming vector field ξ is a spacetime with the equal associated scalar or the Ricci-Yamabe soliton is expanding, steady or shrinking according to as Theorem 6.4.*

Proof. Inserting (6.5) in (1.45) we get

$$(\mathcal{L}_V g)(X,Y) = 2(\mu - \frac{\beta r}{2} - a\alpha)g(X,Y) - 2\alpha b\eta(X)\eta(Y). \tag{6.15}$$

Taking Lie-differentiation of (6.5) and using it in (6.15) yields

$$\begin{aligned}(\mathcal{L}_V S)(X,Y) &= 2a(\mu - \frac{\beta r}{2} - a\alpha)g(X,Y) - 2a\alpha b\eta(X)\eta(Y) \\ &+ b[(\mathcal{L}_V \eta)(X)\eta(Y) + (\mathcal{L}_V \eta)(Y)\eta(X)].\end{aligned} \tag{6.16}$$

Differentiating covariantly (6.5) along vector field Z and using Lemma 6.1 infer

$$(\nabla_Z S)(X,Y) = b[g(Z,X)\eta(Y) + g(Z,Y)\eta(X) + 2\eta(X)\eta(Y)\eta(Z)]. \tag{6.17}$$

According to Yano (1970), we have the following commutative formula:

$$\begin{aligned}(\mathcal{L}_V \nabla_Z g &- \nabla_Z \mathcal{L}_V g - \nabla_{[V,Z]}(X,Y) \\ &= -g((\mathcal{L}_V \nabla)(Z,X),Y) - g((\mathcal{L}_V \nabla)(Z,Y),X).\end{aligned} \tag{6.18}$$

Combining (1.45) and (6.18) we obtain

$$g((\mathcal{L}_V \nabla)(X,Y),Z) = (\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z). \tag{6.19}$$

Inserting (6.17) in (6.19), we get the form

$$(\mathcal{L}_V \nabla)(X,Y) = -2b[g(X,Y)\xi + \eta(X)\eta(Y)\xi]. \tag{6.20}$$

Again consider the commutative formula given by Yano (1970):

$$(\mathcal{L}_V R)(X,Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y,Z) - (\nabla_Y \mathcal{L}_V \nabla)(X,Z). \tag{6.21}$$

Taking covariant differentiation of (6.20) and using it in (6.21), yields

$$\begin{aligned}(\mathcal{L}_V R)(X,Y)Z &= 2b[g(X,Z)Y - g(Y,Z)X \\ &+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X]. \qquad (6.22)\end{aligned}$$

Contracting (6.22) with respect to X gives

$$(\mathcal{L}_V S)(Y,Z) = -6b[g(Y,Z) + \eta(Y)\eta(Z)]. \qquad (6.23)$$

Putting $Y = Z = \xi$ in (6.23), we have

$$(\mathcal{L}_V S)(\xi,\xi) = 0. \qquad (6.24)$$

Inserting $X = Y = \xi$ in (6.16) we obtain

$$-2a(\mu - \frac{\beta r}{2} - a\alpha) - 2a\alpha b + 2b(\mathcal{L}_V \eta)(\xi) = 0. \qquad (6.25)$$

Also, taking $X = \xi$ in (6.15) infer

$$(\mathcal{L}_V g)(X,\xi) = [2(\mu - \frac{\beta r}{2} - a\alpha) + 2\alpha b]\eta(X). \qquad (6.26)$$

Taking Lie-differentiation of $\eta(X) = g(X,\xi)$ and using it in (6.26) give us the relation:

$$(\mathcal{L}_V \eta)(X) - g(\mathcal{L}_V \xi, X) - [2(\mu - \frac{\beta r}{2} - a\alpha) + 2b\alpha]\eta(X) = 0. \qquad (6.27)$$

Again, taking Lie-differentiation of $g(\xi,\xi) = -1$ along V and using (6.16) gives

$$\eta(\mathcal{L}_V \xi) = \mu - \frac{\beta r}{2} - a\alpha + \alpha b. \qquad (6.28)$$

Making use of (6.28) and (6.25) and substituting the values of a and b we obtain the following relation:

$$[2\lambda - k(\sigma + 3\rho)][\mu - \lambda(\alpha + 2\beta) + \frac{\beta k}{2}(\sigma - 3\rho) - \frac{\alpha k}{2}(\sigma + 3\rho)] = 0. \qquad (6.29)$$

Thus we see that either $\lambda = \frac{k}{2}(\sigma+3\rho)$ or $\mu = \lambda(\alpha+2\beta)+\frac{\beta k}{2}(\sigma-3\rho)-\frac{\alpha k}{2}(\sigma+3\rho)$. We obtain the following two cases:

Case-I: If $\lambda \neq \frac{k}{2}(\sigma+3\rho)$, then $\mu = \lambda(\alpha+2\beta)+\frac{\beta k}{2}(\sigma-3\rho)-\frac{\alpha k}{2}(\sigma+3\rho)$. In this case Ricci-Yamabe soliton is expanding, steady or shrinking accordingly as Theorem 6.4.

Case-II: If $\lambda = \frac{k}{2}(\sigma+3\rho)$ and $\mu \neq \lambda(\alpha+2\beta)+\frac{\beta k}{2}(\sigma-3\rho)-\frac{\alpha k}{2}(\sigma+3\rho)$, implies $\mu \neq 3\beta k(\sigma+3\rho)$. Then we get

$$S(X,Y) = k(\sigma+\rho)[g(X,Y)+\eta(X)\eta(Y)], \tag{6.30}$$

i.e. perfect fluid spacetime is a spacetime with equal associated scalar constant. This completes the proof. □

Taking $X = Y = \xi$ in (6.20) yields

$$(\mathcal{L}_V\nabla)(\xi,\xi) = 0. \tag{6.31}$$

Using the commutative formula:

$$(\mathcal{L}_V\nabla)(X,Y) = \nabla_X\nabla_Y V - \nabla_{\nabla_X Y}V + R(V,X)Y. \tag{6.32}$$

Replacing X, Y by ξ in (6.32) and using (6.31) gives

$$\nabla_\xi\nabla_\xi V - \nabla_{\nabla_\xi \xi}V + R(V,\xi)\xi = 0. \tag{6.33}$$

Since ξ is torse-forming vector field, $\nabla_\xi \xi = 0$ then (6.33) becomes

$$\nabla_\xi\nabla_\xi V + R(V,\xi)\xi = 0. \tag{6.34}$$

This implies that potential vector field V is a Jacobi vector field along the geodesic of ξ. Hence we can state the following:

Theorem 6.6. *If a perfect fluid spacetime with torse-forming vector field ξ admits a Ricci-Yamabe soliton $(V, g, \mu, \alpha, \beta)$, then the potential vector field V is a Jacobi* *vector field along the geodesics of* ξ*.*

6.1.3 η-Ricci-Yamabe soliton in a perfect fluid spacetime

In this section we consider η-Ricci-Yamabe soliton in the context of perfect fluid spacetime admitting torse-forming vector field ξ and obtain the Poisson equation.

Writing explicitly the Lie derivative $\mathcal{L}_\xi g$ and taking potential vector $V = \xi$ in (1.47) we get

$$[g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi)] + 2\alpha S(X, Y)$$
$$+(2\mu - \beta r)g(X, Y) + 2\omega\eta(X)\eta(Y) = 0, \tag{6.35}$$

for any $X, Y \in \chi(M)$. Contracting (6.35) yields

$$div(\xi) + \alpha r + (\mu - \frac{\beta r}{2})dim(M) = \omega. \tag{6.36}$$

Let (M^4, g) be a general relativistic perfect fluid spacetime and $(g, \xi, \mu, \omega, \alpha, \beta)$ be an η-Ricci-Yamabe soliton in M. From (1.47) and (6.5) we get

$$\frac{1}{2}[g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi)] + (a\alpha + \mu - \frac{\beta r}{2})g(X, Y)$$
$$+(\alpha b + \omega)\eta(X)\eta(Y) = 0. \tag{6.37}$$

Consider $\{e_i\}_{1\leq i\leq 4}$ an orthonormal frame field and let $\xi = \Sigma_{i=1}^4 \xi^i e_i$, we have $\Sigma_{i=1}^4 \epsilon_{ii}(\xi^i)^2 = -1$ and $\eta(e_i) = \epsilon_{ii}\xi^i$.

Multiplying (6.37) by ϵ_{ii} and summing over i for $X = Y = e_i$ we obtain

$$4\mu - \omega = (2\beta - \alpha)r - div(\xi). \tag{6.38}$$

Taking $X = Y = \xi$ in (6.37) gives

$$\omega - \mu = \alpha(a - b) - \frac{\beta r}{2}. \tag{6.39}$$

Therefore,

$$\mu = (2\beta - \alpha)\lambda + \frac{\beta k}{2}(\sigma - 3\rho) - \frac{\alpha k}{2}(\sigma - \rho) - \frac{div(\xi)}{3} \tag{6.40}$$

$$\omega = -\alpha k(\sigma + \rho) - \frac{div(\xi)}{3} \tag{6.41}$$

Hence we can state the following:

Theorem 6.7. *Let (M, g) be a 4-dimensional pseudo-Riemannian manifold and let η be the g-dual 1-form of the gradient vector field $\xi = grad(f)$ with $g(\xi, \xi) = -1$. If (1.47) defines an η-Ricci-Yamabe soliton in M, then the Poisson equation satisfies by f is*

$$\Delta(f) = -3[\omega + \alpha k(\sigma + \rho)].$$

In view of (1.47), taking $\alpha = 0$ and $\beta = 1$ it gives η-Yamabe soliton. Thus we can state the following:

Corollary 6.1. *Let (M, g) be a 4-dimensional pseudo-Riemannian manifold and let η be the g-dual 1-form of the gradient vector field $\xi = grad(f)$ with $g(\xi, \xi) = -1$. If (1.47) defines an η-Yamabe soliton in M, then the Poisson equation satisfies by f is*

$$\Delta(f) = -3\omega.$$

Remark 6.2. *Now we look at some of the particular cases of Theorem 6.7. Under similar hypothesis as in Theorem 6.7, if g admits:*
1. η-Ricci soliton $(\alpha = 1, \beta = 0)$, then the Poisson equation satisfies by f is $\Delta(f) = -3[\omega + k(\sigma + \rho)]$.
2. η-Einstein soliton $(\alpha = 1, \beta = -1)$, then the Poisson equation becomes $\Delta(f) = -3[\omega + k(\sigma + \rho)]$. This result was shown by Blaga (2018).

Example 6.1. *An η-Ricci-Yamabe soliton $(g, \xi, \mu, \omega, \alpha, \beta)$ in a radiation fluid is* *given by*

$$\mu = (4\beta - \alpha)\lambda - \alpha kp - \frac{div(\xi)}{3}$$
$$\omega = -4\alpha kp - \frac{div(\xi)}{3}$$

From this example, we deduce that Ricci-Yamabe soliton in radiation fluid is steady if $p = \frac{(\alpha-4\beta)\lambda}{3\alpha k}$, *expanding if* $p > \frac{(\alpha-4\beta)\lambda}{3\alpha k}$ *and shrinking if* $p < \frac{(\alpha-4\beta)\lambda}{3\alpha k}$ *for* $\alpha \neq 0$.

Example 6.2. *In this section, we constructed a non-trivial example of a perfect fluid spacetime admitting* η*-Ricci-Yamabe soliton in a 4-dimensional pseudo-Riemannian manifold. Let* $M = \{(x, y, z, t) \in \mathbb{R}^4; t \neq 0\}$, *where* (x, y, z, t) *are the standard coordinates of* $\mathbb{R}^4$. *Consider a Lorentzian metric* g *on* M *is given by*

$$ds^2 = e^{2t}[dx^2 + dy^2 + dz^2] - dt^2. \tag{6.42}$$

The non-vanishing components of the Christoffel symbol, the curvature tensor and Ricci tensor are

$$\Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = e^{2t}, \Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = 1,$$
$$R_{1441} = R_{2442} = R_{3443} = e^{2t}, R_{1221} = R_{1331} = R_{2332} = -e^{4t},$$
$$S_{11} = S_{22} = S_{33} = -3e^{2t}, S_{44} = 3.$$

Therefore, the scalar curvature of the manifold is $r = -12$. *Thus,* (M^4, g) *is a perfect fluid spacetime whose isotropic pressure and energy density are* $\rho = \frac{1}{k}(\lambda + 3)$ *and* $\sigma = -\frac{1}{k}(\lambda + 3)$ *respectively.*
Let η *be the 1-form defined by* $\eta(Z) = -g(Z, t)$ *for any* $Z \in \chi(M)$. *Take* $\xi = t$. *Replacing* $V = \xi$ *in (1.47) and using* $(\mathcal{L}_\xi g)(X, Y) = 2[g(X, Y) + \eta(X)\eta(Y)]$ *we see that the soliton equation becomes*

$$2[g_{ii} + \eta_i \otimes \eta_i] + 2\alpha S_{ii} + (2\mu - \beta r)g_{ii} + 2\omega\eta_i \otimes \eta_i = 0, \tag{6.43}$$

for all $i \in \{1, 2, 3, 4\}$*. Thus the data* $(\xi, g, \mu, \omega, \alpha, \beta)$ *is a* η*-Ricci-Yamabe soliton on* (M^4, g) *where* $\mu = 3\alpha - 4\beta - 1$ *and* $\omega = -1$*, which is expanding if* $3\alpha - 4\beta > 1$*, shrinking if* $3\alpha - 4\beta < 1$ *and steady if* $3\alpha - 4\beta = 1$*.*

6.2 Einstein-type metric on Almost Kenmotsu manifolds

In general relativity, obtaining the global solutions to Einstein field equations has been an important topic for both Mathematics and Physics. One such special solution is the static space-time which is closely connected to the general relativity's cosmic no-hair conjecture (Boucher et al., 1984)). Recently, the authors Leandro (2021), Qing and Yuan (2013) and Patra and Ghosh (2021) studied a generalized version of static space that contains several well circulated critical point equations that occur as solutions of the Euler-Lagrange equations on a compact manifold for curvature functionals.

Definition 6.1 (Patra and Ghosh, 2021)**.** *A smooth Riemannian manifold* $(\mathbf{M}^n, g)$ *is named an Einstein-type manifold if* $\psi : \mathbf{M}^n \to \mathbb{R}$ *solves*

$$\psi Ric = \nabla^2 \psi + \sigma g, \tag{6.44}$$

where ψ *is a non-constant smooth function. Here,* σ*,* Ric *and* $\nabla^2\psi$ *indicate a smooth function, the Ricci tensor and the Hessian of* ψ*, respectively. Moreover, contracting (6.44) yields*

$$r\psi = \Delta\psi + n\sigma, \tag{6.45}$$

where $\Delta\psi$ *being the Laplacian of* ψ *and* r *denotes the scalar curvature.*

As highlighted by authors Patra and Ghosh (2021) and Leandro (20210, the above stated two equations generalize numerous fascinating geometric equations such as the static perfect fluid equation (Leandro and Solórzano, 2019; Coutinho et al., 2019; Masood-ul-Alam, 1987), Miao-Tam equation (Miao and Tam, 2009, 2011; Barros et al., 2015) and critical point equation (Baltazar, 2017; Ghosh and Patra, 2017; Qing and Yuan, 2013), Einstein equation (Hwang et al., 2016) and static vacuum equation (Ambrozio, 2017) with null and non-null cosmological constant, respectively.

Also, we recall the results obtained by Kanai (1983).

Lemma 6.2. *Suppose that (M, g) is a complete Riemannian manifold of dimension $n(\geq 2)$ and that $k < 0$. Then there is a non-trivial function f on M with a critical point which satisfies*

$$Hessf + kfg = 0$$

if and only if (M, g) is the simply connected complete Riemannian manifold $(\mathbb{H}^n, -(1/k)g_0)$ of constant curvature k, where g_0 is the canonical metric on the hyperbolic space of constant curvature -1.

Lemma 6.3. *Let (M, g) and k be as Lemma 6.2. Then there is a function f on M without critical points which satisfies*

$$Hessf + kfg = 0$$

if and only if (M, g) is the warped product $(\widetilde{M}, \widetilde{g})_\xi \times (\mathbb{R}, g_0)$ of a complete Riemannian manifold $(\widetilde{M}, \widetilde{g})$ and the real line $(\mathbb{R}, g_0)$ warped by a function $\xi : \mathbb{R} \to \mathbb{R}$ such that $\ddot{\xi} + k\xi = 0, \xi > 0$, where g_0 denotes the canonical metric on $\mathbb{R}$; $g_0 = dt^2$.

6.2.1 Kenmotsu manifolds satisfying Einstein-type equations

Before proceeding to the main result, we construct an example of a Kenmotsu manifold admitting a non-trivial smooth function ψ which is the solution of the equation (6.44).

Example 6.3. *Let* (N^{2n}, J, g_0) *be a Kähler manifold and the WP* $(\mathbf{M}, g) = (\mathbb{R} \times_{\bar{\sigma}} N, dt^2 + \bar{\sigma}^2 g_0)$*. If we set* $\eta = dt, \xi = \frac{\partial}{\partial t}$ *and the tensor field* φ *is defined on* $\mathbb{R} \times_{\bar{\sigma}} N$ *by* $\varphi X = JX$ *for any* X *on* N *and* $\varphi X = 0$ *if* X *is tangent to* $\mathbb{R}$*, then the WP* $\mathbb{R} \times_{\bar{\sigma}} N, \bar{\sigma}^2 = ce^{2t}$ *with the structure* (φ, ξ, η, g) *is a Kenmotsu manifold (Kenmotsu, 1972). Specifically, if we set* $N = \mathbb{CH}^{2n}$*, then* N *is Einstein and the Ricci tensor of* $\mathbf{M}$ *becomes* $Ric = -2ng$*. Further, we set a smooth function as* $\psi(t) = ke^t, k > 0$*. Hence, it is very easy to verify that* $\psi(t)$ *solves the equation (6.44) for* $\sigma = -(2n+1)ke^t$*.*

Next, we establish the following:

Theorem 6.8. *If* (g, ψ) *is a non-trivial solution of equation (6.44) in a Kenmotsu manifold* $(\mathbf{M}^{2n+1}, \varphi, \xi, \eta, g)$*, then it is* η*-Einstein manifold, provided* $\xi\psi \neq \psi$*. Moreover, if* $\mathbf{M}$ *is complete and the Reeb vector field leaves the scalar curvature invariant, then we have*

1. *If* ψ *has a critical point which satisfies (6.44), then* M *is isometric to the hyperbolic space* $\mathbb{H}^{2n+1}(-1)$*.*

2. *If* ψ *is without critical points which satisfies (6.44), then* M *is isometric to the warped product* $\widetilde{M} \times_\gamma \mathbb{R}$ *of a complete Riemannian manifold* $\widetilde{M}^{2n}$ *and the real line* $\mathbb{R}$ *with warped function* $\gamma : \mathbb{R} \to \mathbb{R}$ *such that* $\ddot{\gamma} - \gamma = 0, \gamma > 0$*.*

Proof. Executing the covariant derivative of (6.44) along Y, we obtain

$$\nabla_Y \nabla_X D\psi = (Y\psi)QX + \psi(\nabla_Y Q)X - (Y\sigma)X. \tag{6.46}$$

As a consequence of (6.46), we get the curvature tensor as follows:

$$\begin{aligned} R(X,Y)D\psi = (X\psi)QY - (Y\psi)QX + \psi\{(\nabla_X Q)Y \\ -(\nabla_Y Q)X\} + (Y\sigma)X - (X\sigma)Y, \end{aligned} \tag{6.47}$$

for any X, Y on $\mathbf{M}$. Executing the covariant derivative of (1.30) and using (1.27) gives

$$(\nabla_X Q)\xi = -QX - 2nX. \tag{6.48}$$

Now taking an inner product of (6.47) with ξ and inserting the last expression along with (1.30) we obtain

$$g(R(X,Y)D\psi,\xi) = 2n\{(Y\psi)\eta(X) - (X\psi)\eta(Y)\} + (Y\sigma)\eta(X) - (X\sigma)\eta(Y) \tag{6.49}$$

Taking an inner product of (1.29) with $D\psi$, then combining it with (6.49) gives

$$(2n+1)\{D\psi - (\xi\psi)\xi\} + D\sigma - (\xi\sigma)\xi = 0. \tag{6.50}$$

Contracting (6.47) infers

$$4nD\sigma - \psi Dr - 2rD\psi = 0. \tag{6.51}$$

Taking the trace of (6.48) and then using it in the inner product of (6.51) with ξ, we acquire

$$4n(\xi\sigma) + 2\psi(r + 2n(2n+1)) - 2r(\xi\psi) = 0. \tag{6.52}$$

Replacing Y by ξ in (6.47), then taking an inner product with Y gives

$$g(R(X,\xi)D\psi, Y) = -2n(X\psi)\eta(Y) - (\xi\psi)Ric(X,Y)$$
$$+\psi\{Ric(X,Y) + 2ng(X,Y)\} + (\xi\sigma)g(X,Y) - (X\sigma)\eta(Y). \tag{6.53}$$

As a consequence of taking an inner product of (1.29) with $D\psi$ and combining it with (6.53) we get

$$g(X,(2n+1)D\psi + D\sigma)\eta(Y) - (2n\psi + \xi\psi + \xi\sigma)g(X,Y) = (\psi - \xi\psi)Ric(X,Y) \tag{6.54}$$

Combining (6.51), (6.52) and (6.54) gives

$$(\psi-\xi\psi)\{(\frac{r}{2n}+1)X-(\frac{r}{2n}+2n+1)\eta(X)\xi\}=(\psi-\xi\psi)QX, \tag{6.55}$$

for any X on $\mathbf{M}$. Hence, $\mathbf{M}$ is η-Einstein or $\xi\psi=\psi$.

Let ξ leave the scalar curvature r invariant, i.e., $\xi r=0$ implies $r=-2n(2n+1)$. In view of this (6.55) gives $QX=-2nX$. Utilizing the fact that r is constant in (6.51), we get $\sigma=-(2n+1)\psi+k$, where k indicates a constant. In consequence of last equation and $QX=-2nX$ in (6.44) infer

$$\nabla_X D\psi=(\psi-k)X.$$

By applying Kanai's theorems (Kanai, 1983), that is, lemma 6.2 and lemma 6.3 we can conclude that if ψ has a critical point which satisfies (6.44), then M is isometric to the hyperbolic space $\mathbb{H}^{2n+1}(1)$ or if ψ is without critical points which satisfies (6.44), then M is isometric to the warped product $\widetilde{M}\times_\gamma\mathbb{R}$ of a complete Riemannian manifold $\widetilde{M}^{2n}$ and the real line $\mathbb{R}$ with warped function $\gamma:\mathbb{R}\to\mathbb{R}$ such that $\ddot{\gamma}-\gamma=0, \gamma>0$. This completes the proof. □

Remark 6.3. *From (6.55), we see that either $\boldsymbol{M}$ is η-Einstein or $\psi-\xi\psi=0$. Suppose $\psi-\xi\psi=0$, then since Kenmotsu maifold is locally isometric to the warped product $(-\epsilon,\epsilon)\times_{ce^t}N$, where N is a Kähler manifold of dimension $2n$ and $(-\epsilon,\epsilon)$ is an open interval (Kenmotsu, 1972). Using the local parametrization: $\xi=\frac{\partial}{\partial t}$ (where t is the coordinate on $(-\epsilon,\epsilon)$) we get*

$$\frac{\partial\psi}{\partial t}=\psi$$

Solving gives $\psi=ce^t$, where c is a constant. Therefore, assuming $\xi\psi\neq\psi$ in Theorem 6.8, implies $\boldsymbol{M}$ is η-Einstein.

6.2.2 Almost Kenmotsu manifolds satisfying Einstein-type equation

Making use of Lemma 3.1 and the result by Dileo and Pastore (Theorem 4.2, 2009), we can prove subsequent:

Theorem 6.9. *Let* $\mathbf{M}^{2n+1}(\varphi, \eta, \xi, g)$ *be a* $(\kappa, \mu)'$*-akm with the condition* $h' \neq 0$*. If* (g, ψ) *is a non-trivial solution of the equation (6.44) then* $\mathbf{M}^3$ *is locally isometric to the Riemannian product* $\mathbb{H}^2(-4) \times \mathbb{R}$ *and* $\mathbf{M}$ *is locally isometric to the WP*

$$\mathbb{H}^{n+1}(\alpha) \times_{\bar{\psi}} \mathbb{R}^n, \qquad \mathbb{B}^{n+1}(\alpha') \times_{\bar{\psi}'} \mathbb{R}^n,$$

for $n > 1$*. Here,* $\mathbb{H}^{n+1}(\alpha)$ *and* $\mathbb{B}^{n+1}(\alpha')$ *are the hyperbolic space of constant curvature* $\alpha = -\frac{2}{n} - \frac{1}{n^2} - 1$ *and space of constant curvature* $\alpha' = -\frac{1}{n^2} + \frac{2}{n} - 1$*, respectively. Also,* $\bar{\psi} = c_1 e^{(1-\frac{1}{n})t}$ *and* $\bar{\psi}' = c_1' e^{(1-\frac{1}{n})t}$ *where* c_1, c_1' *are positive constants.*

Proof. We first replace X by ξ in (6.47), then take its an inner product with ξ and utilizing Lemma 3.1, infer that

$$g(R(\xi, Y)D\psi, \xi) = 2n\kappa\{(\xi\psi)\eta(Y) - (Y\psi)\} + (Y\sigma) - (\xi\sigma)\eta(Y). \tag{6.56}$$

Also, we replace X by ξ in (1.33) and after taking inner product with $D\psi$ gives

$$g(R(\xi, Y)\xi, D\psi) = \kappa\{(\xi\psi)\eta(Y) - (Y\psi)\} - \mu h'(Y\psi). \tag{6.57}$$

Since scalar curvature $r = 2n(\kappa - 2n)$ is constant, in view of this (6.51) becomes $4nD\sigma - 2rD\psi = 0$. Combining (6.56), (6.57) and the last expression, we get

$$2n(\kappa + 1)\{(\xi\psi)\xi - D\psi\} = \mu h' D\psi. \tag{6.58}$$

Operating (6.58) by h' and using (1.34) yields $-2n(\kappa+1)h'D\psi = \mu(\kappa+1)\{-D\psi+$

$(\xi\psi)\xi\}$, then combining the obtained equation with (6.58), we obtain

$$\{\mu^2(\kappa+1)+4n^2(\kappa+1)^2\}\varphi^2 D\psi = 0. \tag{6.59}$$

Thus we get the following two cases, $\varphi^2 D\psi = 0$ or $\varphi^2 D\psi \neq 0$.

Case-I: $\varphi^2 D\psi \neq 0$, then (6.59) gives $\kappa = -1 - \frac{\mu^2}{4n^2}$. Since $\mu = -2$, in view of this in last expression yields $\kappa = -1 - \frac{1}{n^2}$. By using Dileo and Pastore (Theorem 4.2, 2009) we can conclude that $\mathbf{M}^3$ is locally isometric to the Riemannian product $\mathbb{H}^2(-4)\times\mathbb{R}$ and $\mathbf{M}$ is locally isometric to the WP

$$\mathbb{H}^{n+1}(\alpha)\times_{\bar{\psi}}\mathbb{R}^n, \qquad \mathbb{B}^{n+1}(\alpha')\times_{\bar{\psi}'}\mathbb{R}^n,$$

for $n > 1$.

Case-II: $\varphi^2 D\psi = 0$ which implies $D\psi = (\xi\psi)\xi$. Taking the covariant derivative and using (6.44) and (1.31), we get

$$\psi QX - \sigma X = X(\xi\psi)\xi + (\xi\psi)(X - \eta(X)\xi - \varphi hX). \tag{6.60}$$

Replacing X by ξ in (6.60) gives $\xi(\xi\psi) = 2n\kappa\psi - \sigma$. In view of this in the contraction of (6.60), we obtain $\xi\psi = -2n\psi - \sigma$.

Comparing (6.60) with Lemma 3.1, then operating the obtained result by φ gives $(2n\psi + (\xi\psi))hX = 0$. Making use of $\xi\psi = -2n\psi - \sigma$ and the fact that $h \neq 0$, we see that $\sigma = 0$. In consequence, (6.51) becomes $(\kappa - 2n)D\psi = 0$. As $\kappa < -1$, we get $D\psi = 0$, that is, ψ is constant, a contradiction. This completes the proof. □

Consider a generalized $(\kappa,\mu)'$-*akm* of dimension three with $\kappa < -1$. If we assume that κ is invariant along ξ, then from (Proposition 3.2 (Pastore and Saltarelli, 2011)) we have $\xi(\kappa) = -2(1+\kappa)(\mu+2)$ implies $\mu = -2$. Moreover, from (Lemma 3.3 (Saltarelli, 2015)), we have $h'(grad\mu) = grad\kappa - \xi(\kappa)\xi$ which implies κ is constant under our assumption. Therefore $\mathbf{M}^3$ becomes a $(\kappa,-2)'$-*akm*. By applying Theorem 6.9, we can conclude the following:

Corollary 6.2. *Let $\mathbf{M}^3(\varphi, \eta, \xi, g)$ be a generalized $(\kappa, \mu)'$-akm with $\kappa < -1$ invariant along ξ. If (g, ψ) is a non-trivial solution of the equation (6.44) then $\mathbf{M}^3$ is locally isometric to the Riemannian product $\mathbb{H}^2(4) \times \mathbb{R}$.*

Next, we investigate 3-dimensional *akm* admitting a non-trivial solution to the equation (6.44).

Theorem 6.10. *Let $\mathbf{M}^3(\varphi, \xi, \eta, g)$ be an almost Kenmotsu 3-H-manifold equipped with $h' \neq 0$. If (ψ, g) is a non-trivial solution of the equation (6.44) with smooth function ψ constant along the Reeb vector field, then it is locally isometric to a non-unimodular Lie group with a left invariant almost Kenmotsu structure.*

Proof. Under our hypothesis, from the first argument of Lemma 3.3, we obtain

$$e(\vartheta) = -2\vartheta c, \qquad \varphi e(\vartheta) = -\vartheta b. \tag{6.61}$$

Taking the inner product of (6.44) with vector filed Y, the equation (6.44) can be rewritten as the following:

$$g(\nabla_X D\psi, Y) = \psi Ric(X, Y) - \sigma g(X, Y), \tag{6.62}$$

for all X, Y on $\mathbf{M}$. Since the smooth function ψ is constant along the Reeb vector field ξ, we can write

$$D\psi = \psi_1 e + \psi_2 \varphi e,$$

for smooth functions ψ_1, ψ_2 on $\mathbf{M}$.

Replacing X and Y by ξ in (6.62), then making use of Lemma 3.2 and Lemma 3.3 we get

$$\sigma = -2\psi(\vartheta^2 + 1). \tag{6.63}$$

Substituting $X = e$ and $Y = \xi$ in (6.62) and using Lemma 3.3, Lemma 3.2 yield

$$\vartheta\psi_2 - \psi_1 = 0. \tag{6.64}$$

Similarly, taking $X = \varphi e$ and $Y = \xi$ in (6.62) gives

$$\vartheta\psi_1 - \psi_2 = 0. \tag{6.65}$$

Combining (6.64) and (6.65), we get $(\vartheta^2 - 1)\psi_1 = 0$. If $\psi_1 = 0$ then from (6.65) we see that $\psi_2 = 0$ which implies $D\psi = 0$, that is, ψ is constant, a contradiction. Therefore, we must have $\vartheta^2 = 1$ which implies ϑ is constant. Since ϑ is a positive function, we get $\vartheta = 1$. Making use of the fact that $\vartheta = 1$ in (6.61) gives $b = c = 0$. Moreover, eq. (6.64) implies $\psi_1 = \psi_2$.

Now consider the following open set:

$$\mathcal{O} = \{p \in \mathcal{U}_1 : \psi_1 = \psi_2 \neq 0 \ \text{ in a neighborhood of } \ p\}$$

Since Poincare's lemma $d^2\psi = 0$, i.e. the relation

$$g(\nabla_X D\psi, Y) = g(\nabla_Y D\psi, X) \tag{6.66}$$

holds for any vector fields X, Y in $\mathbf{M}$, letting $X = \xi$ and $Y = e$ in (6.66) using Lemma 3.2, we obtain

$$\xi(\psi_1) = a\psi_2. \tag{6.67}$$

Also, taking $X = \xi$ and $Y = \varphi e$ in (6.66) gives $\xi(\psi_2) = -a\psi_1$ and combining this with (6.67), we get $2a\psi_1 = 0$, that is, $a = 0$ in $\mathcal{O}$.

Making use of the fact that $a = b = c = 0$ and $\vartheta = 1$ along with Lemma 3.2, we obtain

$$[e, \varphi e] = 0, \qquad [\varphi e, \xi] = \varphi e - e, \qquad [\xi, e] = \varphi e - e.$$

According to Milnor's theorem (Milnor, 1976), we can conclude that $\mathbf{M}^3$ is locally isometric to a non-unimodular Lie group with a left invariant almost Kenmotsu structure. This completes the proof. □

Applying Wang's Theorem (Wang, 2017) and Theorem 6.10, we can now establish the following:

Corollary 6.3. *Let* $\mathbf{M}^3(\varphi, \xi, \eta, g)$ *be an almost Kenmotsu 3-H-manifold. If* (ψ, g) *is a non-trivial solution of the equation (6.44) with smooth function* ψ *constant along the Reeb vector field, then it is locally isometric to either the hyperbolic space* $\mathbb{H}^3(-1)$ *or the Riemannian product* $\mathbb{H}^2(-4) \times \mathbb{R}$.

Proof. We shall establish the theorem via the subsequent cases:

Case i: Let $h = 0$, then $\mathbf{M}^3$ be a Kenmotsu manifold. The Ricci operator of $\mathbf{M}^3$ is written by (see Cho (2014))

$$Q = (\frac{r}{2} + 1)id - (\frac{r}{2} + 3)\eta \otimes \xi. \tag{6.68}$$

Replacing X by ξ in (6.44), then taking it inner product with ξ and using (1.30), we get $\xi(\xi\psi) = -2\psi + \sigma$. If $\xi\psi = 0$, last equation becomes $\sigma = 2\psi$ which further implies $\xi\sigma = 0$. In consequence, for $n = 1$ Eq. (6.52) becomes $r = -6$, i.e. scalar curvature is constant. Moreover, (6.68) infer $Q = -2id$. Clearly $\mathbf{M}^3$ is conformally flat.

Case ii: $h \neq 0$ on some open subset of $\mathbf{M}^3$. By the proof of Theorem 6.10, we see that $a = b = c = 0$ and $\vartheta = 1$. Using this in Lemma 3.3, we get

$$Q\xi = -4\xi,$$
$$Qe = 2\varphi e - 2e,$$
$$Q\varphi e = 2e - 2\varphi e.$$

Also, the scalar curvature is constant, i.e. $r = -8$. Since r is constant and by (6.69), it is easy to see that $\mathbf{M}^3$ is conformally flat.

By applying Wang's theorem (Theorem 1.6, 2017), we conclude that it is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$. This completes the proof. □

Under the assumptions of Theorem 6.10, for non-Kenmotsu almost Kenmotsu

3-H-manifold, $\nabla_\xi h = 0$. Also, it is known that a *akm* of dimension 3 is Kenmotsu if and only if h vanishes (see Dileo and Pastore (2009)). In regard of Corollary 3.3 (Wang, 2017) and Corollary 6.3, we can write:

Corollary 6.4. *Let $\mathbf{M}^3(\varphi,\xi,\eta,g)$ be an almost Kenmotsu 3-H-manifold. If (ψ,g) is a non-trivial solution of the equation (6.44) with smooth function ψ constant along the Reeb vector field, then it is locally isometric to either the WP $\mathbb{R}\times_{ce^t}N(\kappa)$ ($N(\kappa)$: space of constant curvature κ) or the Riemannian product $\mathbb{H}^2(-4)\times\mathbb{R}$.*

Example 6.4. *In a strictly almost Kähler Einstein manifold $(\mathbf{M},J,\bar{g})$, we set $\eta = dt$, $\xi = \frac{\partial}{\partial t}$ and the tensor field φ is defined on $\mathbb{R}\times_\psi N$ by $\varphi X = JX$ for vector field X on $\mathbf{M}$ and $\varphi X = 0$ if X is tangent to $\mathbb{R}$. Consider a metric $g = g_0 + \bar{\sigma}^2\bar{g}$, where $\bar{\sigma}^2 = ce^{2t}$, g_0 indicates the Euclidean metric on $\mathbb{R}$ and c denotes a positive constant. Then it is easy to verify that the WP $\mathbb{R}\times_{\bar{\sigma}}\mathbf{M}, \bar{\sigma}^2 = ce^{2t}$, with the structure (φ,ξ,η,g) is an akm (Dileo and Pastore, 2007). Since $\mathbf{M}$ is Einstein $S = -2ng$. If we set a smooth function $\psi(x,t) = t^2$, then ψ solves the equation (6.44) for $\sigma = -2nt^2 - 2$.*

Now, we recollect the subsequent definition:

Definition 6.2. *A 3-dimensional akm is named a (κ,μ,ν)-akm if the Reeb vector field obeys the (κ,μ,ν)-nullity condition, that is,*

$$\begin{aligned} R(X,Y)\xi &= \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX \\ &\quad -\eta(X)hY) + \nu(\eta(Y)h'X - \eta(X)h'Y), \end{aligned}$$

for any X, Y and μ, κ and ν indicate smooth functions.

Example 6.5. *Let G^3 be a non-unimodular Lie group admitting a left invariant local orthonormal frame fields $\{\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3\}$ obeying (see Milnor (1976)):*

$$[\mathfrak{v}_2,\mathfrak{v}_3] = 0, \quad [\mathfrak{v}_1,\mathfrak{v}_2] = \alpha\mathfrak{v}_2 + \beta\mathfrak{v}_3, \quad [\mathfrak{v}_1,\mathfrak{v}_3] = \gamma\mathfrak{v}_2 + (2-\alpha)\mathfrak{v}_3, \tag{6.69}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$. *We define* g *on* G *by* $g(\mathfrak{v}_i, \mathfrak{v}_j) = \delta_{ij}$ *for* $1 \leq i, j \leq 3$. *Take* $\xi = -\mathfrak{v}_1$ *and its dual 1-form by* η. *Also, we define a* $(1,1)$ *tensor field* φ *by* $\varphi(\xi) = 0, \varphi(\mathfrak{v}_2) = \mathfrak{v}_3$ *and* $\varphi(\mathfrak{v}_3) = -\mathfrak{v}_2$. *We can easily verify that* $(G, \varphi, \xi, \eta, g)$ *admits a left invariant almost Kenmotsu structure. From (Theorem 3.2 (Wang, 2016)), we get that G has a* (κ, μ, ν)*-almost Kenmotsu structure where*

$$\kappa = -\alpha^2 + 2\alpha - \frac{1}{4}(\beta+\gamma)^2 - 2, \mu = \beta - \gamma, \nu = -2.$$

Moreover, from Wang (2016), we have

$$h\mathfrak{v}_2 = (\alpha-1)\mathfrak{v}_3 - \frac{1}{2}(\beta+\gamma)\mathfrak{v}_2, h\mathfrak{v}_3 = \frac{1}{2}(\beta+\gamma)\mathfrak{v}_3 + (\alpha-1)\mathfrak{v}_2. \tag{6.70}$$

The Ricci operator is determined as follows (see Wang, 2016):

$$Q\xi = (\frac{1}{2}(\beta-\gamma)^2 - \alpha^2 - \beta^2 - (\alpha-2)^2 - \gamma^2)\xi.$$

Clearly, taking $\alpha = \beta = \gamma = 1$ *in the above expressions shows that* G *is a non-Kenmotsu* $(\kappa, -2)'$*-akm with* $\kappa = -2$. *In view of this, we get* $Q\xi = -4\xi$ *and the scalar curvature as* $r = -8$ *(from Lemma 3.1). We define a function* $\psi = e^{-2t}, t \geq 0$. *Then by Laplace transformation, we get* $\Delta\psi = \frac{1}{s+2}$, *where* s *is a complex number. In view of this in (6.45) gives* $\sigma = -8e^{-2t}$. *Then it is easy to verify that* ψ *is a non-trivial solution of Einstein-type metric (6.44). Moreover, by Dileo and Pastore result (Theorem 4.2, 2009), we state that* G *is locally isometric to* $\mathbb{H}^2(-4) \times \mathbb{R}$, *the Riemannian product. Hence, Theorem 6.9 is verified.*

Next, we produce an example of almost Kenmotsu 3-H-manifold of dimension three (for detail see Wang (2017)).

Example 6.6. *Consider a cylindrical coordinates* (r, θ, z) *of* $\mathbb{R}^3$. *On* $\mathbf{M}^3$ *which is a simply connected domain of* $\mathbb{R}^3$ *excluding the origin, we define an almost*

Kenmotsu structure as (see Blair and Yildrim (2016)):

$$\xi = \frac{2}{\gamma}\frac{\partial}{\partial r}, \eta = \frac{\gamma}{2}dr, g = \frac{\gamma^2}{4}(dr^2 + r^2 d\theta^2 + dz^2),$$
$$\varphi(\frac{\partial}{\partial z}) = \frac{1}{r}\frac{\partial}{\partial \theta}, \varphi(\frac{\partial}{\partial r}) = 0, \varphi(\frac{\partial}{\partial \theta}) = -r\frac{\partial}{\partial z},$$

where $\gamma = \frac{1}{c_1\sqrt{r}-r}$, $\sqrt{r} > c_1 > 0$ *or* $\sqrt{r} < c_1$, c_1 *being a constant. If we set* $e_1 = \frac{2}{\gamma r}\frac{\partial}{\partial \theta}$ *and* $e_2 = \varphi e_1 = -\frac{2}{\gamma}\frac{\partial}{\partial z}$, *then in Wang (2017) it is shown that* ξ *is an eigenvector field of the Ricci operator. Therefore* $\mathbf{M}^3$ *is an almost Kenmotsu 3-H-manifold.*

Chapter 7

Summary and Conclusion

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Summary and Conclusion

In the present thesis, we give classification of almost contact metric manifolds admitting some geometrical structures and also studied their submanifold. The following objectives are taken up in the study:

1. To study semiconformal curvature tensor.
2. To study geometrical properties of (κ, μ)-contact metric manifolds.
3. To study the properties of certain Ricci solitons.
4. To characterize the invariant submanifolds of certain almost contact manifolds.

In Chapter 1, we give the general introduction of the study which includes the basic definitions and formulas of differential geometry such as topological manifolds, smooth manifolds, Riemannian manifolds, almost contact metric manifolds, Kenmotsu manifolds, f-Kenmotsu manifold, almost Kenmotsu manifolds, space forms, Lorentzian manifolds, generalized m-quasi-Einstein structure, almost Ricci-Yamabe soliton and Submanifolds, along with the review of the literature.

Chapter 2 is divided into three main sections. In first section, we introduce two types of generalized ϕ-recurrent (κ, μ)-contact metric manifolds known as hyper generalized ϕ-recurrent (k, μ)-contact metric manifolds and quasi generalized ϕ-recurrent (k, μ)-contact metric manifolds, and investigate their properties. We prove their existence by constructing non-trivial examples. In the second section, the geometric structures of generalized (k, μ)-space forms and their quasi-umbilical hypersurface are analyzed. First ξ-Q and conformally flat generalized (k, μ)-space form are investigated and shown that a conformally flat generalized (k, μ)-space form is Sasakian. Next, we prove that a generalized (k, μ)-space form satisfying Ricci pseudosymmetry is η-Einstein. We obtain the condition under which a quasi-umbilical hypersurface of a generalized (k, μ)-space form is a generalized quasi Einstein hypersurface. Also ξ-sectional curvature of a quasi-umbilical hypersurface of generalized (k, μ)-space form is obtained. Finally, the results obtained are verified by constructing an example of a 3-dimensional generalized (k, μ)-space form. In the third section, we introduce a type of Riemannian manifold, namely, an almost pseudo semiconformally symmetric manifold which is denoted by $A(PSCS)_n$. Several geometric properties of such a manifold are studied under certain curvature conditions. Some results on Ricci symmetric $A(PSCS)_n$ and Ricci-recurrent $A(PSCS)_n$ are obtained. Next, we consider the decomposability of $A(PSCS)_n$. Finally, two non-trivial examples of $A(PSCS)_n$ are constructed.

In Chapter 3, an extension of the m-Bakery-Emery Ricci tensor known as generalized m-quasi-Einstein metric is investigated. This chapter include two main sections. In the first section, we studied the generalized m-quasi-Einstein metric in the context of contact geometry. First, we prove if an H-contact manifold admits a generalized m-quasi-Einstein metric with non-zero potential vector field V collinear with ξ, then M is K-contact and η-Einstein. Moreover, it is also true when H-contactness is replaced by completeness under certain conditions. Next, we prove that if a complete K-contact manifold admits a closed generalized m-quasi-Einstein metric whose potential vector field is contact then M is compact, Einstein and Sasakian. Finally, we obtain some results on a 3-dimensional normal almost contact manifold admitting generalized m-quasi-Einstein metric. In the second section, we analyze the generalized m-quasi-Einstein structure in the context of almost Kenmotsu manifolds. Firstly we showed that a complete Kenmotsu manifold admitting a generalized m-quasi-Einstein structure (g, f, m, λ) is locally isometric to a hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or a warped product $\widetilde{M} \times_\gamma \mathbb{R}$ under certain conditions. Next, we proved that a $(\kappa, \mu)'$-almost Kenmotsu manifold with $h' \neq 0$ admitting a closed generalized m-quasi-Einstein metric is locally isometric to some warped product spaces. Finally, generalized m-quasi-Einstein metric (g, f, m, λ) in almost Kenmotsu 3-H-manifold is considered and proved that either it is locally isometric to the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$.

Chapter 4 is devoted to the characterization of almost Ricci-Yamabe solitons (shortly, ARYS). In the first section, we consider ARYS in certain contact metric manifolds such as K-contact and (κ, μ)-contact metric manifolds. Firstly, we prove that if the metric g admits an almost (α, β)-Ricci-Yamabe soliton with $\alpha \neq 0$ and potential vector field collinear with the Reeb vector field ξ on a complete contact metric manifold with the Reeb vector field ξ as an eigenvector of the Ricci operator, then the manifold is compact Einstein Sasakian and the potential vector field is a constant multiple of the Reeb vector field ξ. Next, if the complete K-contact manifold admits gradient ARYS with $\alpha \neq 0$, then it is compact Sasakian and isometric to unit sphere S^{2n+1}. Finally, gradient ARYS with $\alpha \neq 0$ in non-Sasakian (k, μ)-contact metric manifold is assumed and found that M^3 is flat and for $n > 1$, M is locally isometric to $E^{n+1} \times S^n(4)$ and the soliton vector field is tangential to the Euclidean factor E^{n+1}. An illustrative example is given to support the obtained result. In the second section, we examine ARYS within the framework of certain classes of almost Kenmotsu manifolds. Firstly, we prove that a complete Kenmotsu manifold, admitting ARYS with $\alpha \neq 0$ is locally isometric to hyperbolic space $\mathbb{H}^{2n+1}(-1)$ when Reeb vector field leaves the scalar curvature invariant. Secondly, we show that ARYS on the Kenmotsu manifold reduces to Ricci-Yamabe soliton under the certain conditions on the soliton function. Next, it is proved that if a $(\kappa, \mu)'$-almost Kenmotsu manifold with $h' \neq 0$ admits gradient ARYS then either it is locally isometric to $\mathbb{H}^{2n+1}(-4) \times \mathbb{R}^n$ or potential vector field is pointwise collinear with the Reeb vector field. Moreover, 3-dimensional non-Kenmotsu almost Kenmotsu manifolds admitting gradient ARYS are considered. Several examples have been constructed of ARYS on different classes of warped product spaces.

Chapter 5 is divided into two sections. The Invariant submanifolds of f-Kenmotsu manifolds are studied in the first section. Firstly, we show that any invariant submanifold of f-Kenmotsu manifold is again f-Kenmotsu manifold and minimal. Then, we give some characterizations of totally geodesic submanifolds of the f-Kenmotsu manifolds. Moreover, we study a 3-dimensional submanifolds and prove that a 3-dimensional submanifold of the f-Kenmotsu manifold is totally geodesic if and only if it is invariant. Also, η-Ricci soliton is considered on an invariant submanifold of f-Kenmotsu manifolds. Lastly, some non-trivial examples are constructed to verify the obtained results. In the second section, we derive Chen's inequalities involving Chen's δ-invariant δ_M, Riemannian invariant $\delta(m_1, \ldots, m_k)$, Ricci curvature, Riemannian invariant $\Theta_k(2 \leq k \leq m)$, the scalar curvature and the squared of the mean curvature for submanifolds of generalized Sasakian-space-forms endowed with a quarter-symmetric connection. As an application of the obtain inequality, we derived first Chen inequality for bi-slant submanifold of generalized Sasakian-space-forms.

In Chapter 6, we obtained some results on spacetime. This chapter include two sections. In the first section, we studied the geometrical aspects of a perfect fluid spacetime with torse-forming vector field ξ under certain curvature restrictions, and Ricci-Yamabe soliton and η-Ricci-Yamabe soliton in a perfect fluid spacetime. Conditions for the Ricci-Yamabe soliton to be steady, expanding or shrinking are also given. Moreover, when the potential vector field ξ of η-Ricci-Yamabe soliton is of gradient type, we derive a Poisson equation and also looked at its particular cases. Lastly, a non-trivial example of perfect fluid spacetime admitting η-Ricci-Yamabe soliton is constructed. Then in the second section, we classify the Einstein-type metric on Kenmotsu and almost Kenmotsu manifolds. In Kenmotsu case, we find that it is η-Einstein and if it is complete with the scalar curvature invariant along the Reeb vector field, then it is isometric either to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or the warped product $\widetilde{M} \times_\gamma \mathbb{R}$, provided $\xi\psi \neq \psi$. Next, we investigate non-Kenmotsu $(\kappa, \mu)'$-almost Kenmotsu manifolds obeying the Einstein-type metric and give some classification. Finally, we establish that if (ψ, g) is a non-trivial solution of Einstein-type metric with smooth function ψ constant along the Reeb vector field on almost Kenmotsu 3-H-manifold, then it is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$. Finally, we construct several non-trivial examples to verify our main results.

Lastly, we conclude that most of our works are an extension of previous works done by many geometers, some of which is A. Ghosh, U. C. De, D. M. Naik, A. De, V. Venkatesha and D. S. Patra. Some important geometrical structures such as generalized m-quasi-Einstein, almost Ricci-Yamabe soliton and Einstein-type metric were studied in the context of almost contact metric manifolds and several isometric classifications are obtained. Also, Chen's inequalities involving the intrinsic and extrinsic invariants are obtained for generalized Sasakian-space-forms by considering special connection. We also obtained some results which might be useful in theoretical physics, especially in the study of general relativity and spacetime. Moreover, we constructed several examples supporting our obtained results. Due to the abstract nature of the topic, most of our results are completely theoretical and do not have an immediate application at present but we are hopeful that it will help in our understanding and future research in the field of differential geometry.

Appendices

(A) LIST OF RESEARCH PUBLICATIONS/ ACCEPTED/ COMMUNICATIONS

(1) **M. Khatri** and J. P. Singh (2020). On a class of generalized recurrent (κ, μ)-contact metric manifolds, *Communications of the Korean Mathematical Society*, **35(4)**, 1283-1297.

(2) J. P. Singh and **M. Khatri** (2020). On almost pseudo semiconformally symmetric manifold, *Differential Geometry-Dynamical Systems*, **22**, 233-253.

(3) J. P. Singh and **M. Khatri** (2021). On the Geometric Structure of Generalized (κ, μ)-space forms, *Facta Universitatis, Series Mathematics and Informatics*, **36(5)**, 1129-1142.

(4) J. P. Singh and **M. Khatri** (2021). On Ricci-Yamabe soliton and geometrical structure in a perfect fluid spacetime, *Afrika Matematika*, **32(7)**, 1645-1656.

(5) Jay Prakash Singh and **Mohan Khatri** (2021). On semiconformal curvature tensor in (κ, μ)-contact metric manifold, *Conference Proceeding of Science and Technology*, **4(2)**, 215-225.

(6) Y. Li, **M. Khatri**, J. P. Singh and S. K. Chaubey (2022). Improved Chen's Inequalities for Submanifolds of Generalized Sasakian-space-forms, *Axioms*, **11(7)**, 324.

(7) J. P. Singh and **M. Khatri** (2022). On weakly cyclic B symmetric spacetime, *Balkan Journal of Geometry and Application*, **27(2)**, 122-138.

(8) J. P. Singh and **M. Khatri** (2022). Generalized m-quasi-Einstein metric on certain almost contact manifolds, Accepted to Filomat.

(9) **M. Khatri** and J. P. Singh (2022). Generalized m-quasi-Einstein structure in almost Kenmotsu manifolds, Accepted to Bulletin of Korean Mathematical Society.

(10) **M. Khatri** and J. P. Singh (2022). Almost Ricci-Yamabe soliton on contact metric manifolds, (Communicated).

(11) **M. Khatri** and J. P. Singh (2022). Almost Ricci-Yamabe soliton on almost Kenmotsu manifolds, (Communicated).

(12) **M. Khatri**, S. K. Chaubey and J. P. Singh (2022). Invariant submanifolds of f-Kenmotsu manifolds, International Journal of Geometric Methods in Modern Physics, https://doi.org/10.1142/S0219887822502255.

(13) U. C. De, **M. Khatri** and J. P. Singh (2022). Einstein-type metric on Almost Kenmotsu manifolds, (Communicated).

(14) **Mohan Khatri** and Jay Prakash Singh (2022). On a type of Static Equation on Certain Contact Metric Manifolds, (Communicated).

(15) **Mohan Khatri**, C. Zosangzuala and Jay Prakash Singh (2022). Isometries on Almost Ricci-Yamabe Solitons, Arabian Journal of Mathematics, https://doi.org/10.1007/s40065-022-00404-x.

(B) CONFERENCES/ SEMINARS/ WORKSHOPS

(1) Participated in the Instructional School for Teachers on "Mathematical Modelling in Continuum Mechanics and Ecology" held at Mizoram UNiversity, Aizawl-796004 from June 03-15, 2019.

(2) Attended "National Workshop on 'Ethics in Research and Preventing Plagiarism (ERPP 2019)'" organised by Department of Physics, School of Physical Sciences, Mizoram University held on 3^{rd} October 2019.

(3) Presented a paper "On Almost Pseudo Semiconformally Symmetric Manifolds" in the "2nd Annual Convention of North East(India) Academy of Science and Technology (NEAST) and International Seminar on Recent Advances in Science and Technology (IRSRAST)" during 16th–18th November 2020 (Virtual) organized by NEAST, Mizoram University, Aizawl-796004, Mizoram (India) .

(4) Presented a paper "On a class of generalized recurrent (κ, μ)-contact metric manifolds" at the "Mizoram Science Congress 2020 (online)" during December 3-4, 2020 organized by Mizoram Science, Technology and Innovation Council (MISTIC) and Directorate of Science and Technology, Planning Department, Govt. of Mizoram.

(5) Attended "NASI TMC Summer School on Differential Geometry" organised online by the Department of Mathematics and Statistics, Central University of Punjab, Bathinda during July 05-24, 2021.

(6) Presented a paper "On Almost Pseudo Semiconformally Symmetric Manifolds" during "18^{th} International Geometry Symposium" held on July 12-13, 2021 at the Malatya/TURKEY.

(7) Attended "Six day Faculty Development Programme on Mathematics and Statistics in Emerging Field" conducted by Department of Mathematics KPRIET held from 05.07.2021 to 10.07.2021.

(8) Participated as a trainee in the offline "One Week Training Program on Mathematical Modelling and Computing" organised by Department of Mathematics and Computer Science, Mizoram University, Aizawl-796004, Mizoram (India) held during 26^{th} April, 2022 to 2^{nd} May, 2022.

Bibliography

Agashe, N. S. and Chafle, M. R. (1992). A semi-symmetric non-metric connection on a Riemannian manifold, *Indian J. Pure Appl. Math.* **23**, 399-409.

Akbar, A. and Sarkar, A. (2015). Some curvature properties of (κ, μ)-contact space forms, *Malaya J. Mat.* **3(1)**(1), 45-50.

Alegre, P., Blair, D. E. and Carriazo, A. (2004). Generalized Sasakian-space-forms, *Israel J. Math.* **141**, 157-183.

Alegre, P. and Carriazo, A. (2008). Structures on generalized Sasakian-space-forms, *Diff. Geom. Appl.* **26(6)**, 656-666.

Alegre, P., Carriazo, A., Kim, Y. H. and Yoon, D. W. (2007). B.Y. Chen's inequality for submanifolds of generalized space forms, *Indian J. Pure Appl. Math.* **38(3)**, 185-201.

Alqahtani, L. S., Stankovic, M. S. and Uddin, S. (2017). Warped product bi-slant submanifolds of cosympletic manifolds, *Filomat* **(31)16**, 5065-5071.

Alkhaldi, A. H., Laurian-Ioan, P. and Abolarinwa A. (2021). Characterization of Almost Yamabe solitons and Gradient Almost Yamabe solitons with conformal vector fields, *Symmetry* **13(12)**, 2362.

Alkhaldi, A. H., Aquib, M., Aslam, M. and and Khan, M. A. (2021). Chen-Ricci Inequalities with a Quarter Symmetric Connection in Generalized Space Forms, *Adv. Math. Phys.* **2021**, 3221643.

Ambrozio, L. (2017). On static three-manifolds with positive scalar curvature, *J. Diff. Geom.* **107(1)**, 1–45.

Anitha, B. S. and Bagewadi, C. S. (2003). Invariant submanifolds of Sasakian manifolds, *Differ. Integral Equ.* **16(10)**, 1249-1280.

Arslan, K., Lumiste, U., Murathan, C. and Özgür, C. (1990). 2-semiparallel surfaces in space forms, I: two particular cases, *Proc. Estonian Acad. Sci.* **39**, 1-8.

Atceken, M. (2021). Certain results on invariant submanifolds of an almost Kenmotsu (κ, μ, ν)-space, *Arab. J. Math.* **10**, 543-554.

Atceken, M., Yildirim, $\ddot{U}$ and Dirik, S. (2020). Pseudoparallel invariant submanifolds of $(LCS)_n$-manifolds, *Korean J. Math.* **28(2)**, 275-284.

Atceken, M. and Uygun, P. (2021). Characterizations for totally geodesic submanifolds of (κ, μ)-paracontact metric manifolds, *Korean J. Math.* **28(3)**, 555-571.

Aytimur, H., Kon, M., Mihai, A., Özgür, C. and Takano, K. (2019). Chen Inequalities for Statistical Submanifolds of Kähler-Like Statistical Manifolds, *Mathematics* **7(12)**, 1202.

Baltazar, H. (2017). On critical point equation of compact manifolds with zero radial Weyl curvature, *Geom. Dedicata* **202(1)**, 337–355.

Barros, A. and Ribeiro Jr., E. (2012). Integral formulae on quasi-Einstein manifolds and applications, *Glasgow Math. J.* **54**, 213-223.

Barros, A. and Ribeiro Jr., E. (2014). Characterization and integral formulae for generalized m-quasi-Einstein metrics, *Bull. Braz. Math. Soc. (N.S.)* **45(2)**, 325-341.

Barros, A. and Gomes, J. N. (2017). Triviality of compact m-quasi-Einstein manifolds, *Results Math.* **71**, 241-250.

Barros, A., Batista, R. and Ribeiro E. (2021). Rigidity of gradient almost Ricci solitons, *Illinois J. Math.* **56(4)**, 1267-1279.

Barros, A., Gomes, J. N. and Ribeiro E. (2013). A note on rigidity of almost Ricci soliton, *Arch. Math.* **100**, 481-490.

Barros, A. and Gomes, J. N. (2013). A compact gradient generalized quasi-Einstein metric with constant scalar curvature, *J. Math. Anal. Appl.* **410(2)**, 702-705.

Barbosa, E. and Ribeiro, E. (2013). On conformal solutions of the Yamabe flow, *Archiv de Math.* **101(1)**, 79-89.

Barros, A., Dióǵenes, R. and Ribeiro Jr., E. (2015). Bach-Flat Critical Metrics of the Volume Functional on 4-Dimensional Manifolds with Boundary, *J. Geom. Anal.* **25(4)**, 2698–2715.

Bishop, R. L. and O'Neill, B. (1969). Manifolds of negative curvature, *Trans. Amer. Math. Soc.* **145**, 1-50.

Blaga, A. M. (2018). Almost η-Ricci solitons in $(LCS)_n$-manifolds, *Bull. Belgian. Math. Soc.-Simon Stevin* **25(5)**, 641-653.

Blaga, A. M. (2020). Solitons and geometrical structure in a perfect fluid spacetime, *Rocky Mountain J. Math.* **50(1)**, 41-53.

Blair, D. E. and Yildrim, H. (2016). On conformally flat almost contact metric manifolds, *Mediterr. J. Math.* **13**, 2759-2770.

Blair, D. E., Koufogiorgos, T. and Papantoniou, B. J. (1995). Contact metric manifolds satisfying a nullity condition, *Israel J. Math.* **91(1-3)**, 189-214.

Blair, D. E. (1976). Contact manifolds in Riemannian geometry, Lecture Notes in Math **509**, Springer-verlag, Berlin Heidelberg.

Blair, D. E. (2010). Riemannian geometry of contact and sympletic manifolds, Progress Math. **203**, Birkhauser.

Blair, D. E. (1977). Two remarks on contact metric structures, *Tohoku Math. J.* **29(3)**, 319-324.

Boeckx, E. (2000). A full classification of contact metric (k, μ)-spaces, *Illinois Math. J.* **44(1)**, 212-219.

Boucher, W., Gibbons, G. and Horowitz, G. (1984). Uniqueness theorem for anti-de Sitter spacetime, *Phys. Rev. D* **30(12)**, 2447-2451.

Boyer, C. P. and Galicki, K. (2001). Einstein manifolds and contact geometry, *Proc. Am. Math. Soc.* **129**, 2419-2430.

Caminha, A., Souza, P. and Camargo, F. (2010). Complete foliations of space forms by hypersurfaces, *Bull. Braz. Math. Soc. (N.S.)* **41(3)**, 339-353.

Cao, H. D. (2009). Recent progress on Ricci soliton, *Adv. Lect. Math.* **11**, 1-38.

Cao, H. D. (2006). Geometry of Ricci solitons, *Chin. Ann. Math.* bf 27(B), 121-142.

Cao, H. D. and Zhou, D. (2010). On complete gradient shrinking Ricci solitons, *Diff. Geom.* **85**, 175-185.

Cao, X., Wang, B. and Zhang, Z. (2011). On locally conformally flat gradient shrinking Ricci solitons, *Comm. Contemp. Math.* **13(2)**, 269-282.

Cartan, E. (1926). Sur une classe remarquable d'espaces de Riemann, *Bull. Soc. Math. France* **54**, 214-264.

Carriazo, A., Cho, J. T. and Molina, V. M. (2020). Generalized Sasakian Space Forms which are realized as Real Hypersurfaces in Complex Space forms, *Mathematics* **8**, 873.

Carriazo, A., Molina, V. M. and Tripathi, M. M. (2013). Generalized (κ, μ) space forms, *Mediterr. J. Math.* **10**, 175-196.

Carriazo, A. and Molina, V. M. (2011). Generalized (κ, μ)-space forms and D_α-homothetic deformations, *Balkan J. Geo. Appl.* **16(1)**, 37-47.

Case, J. (2010). On the non existence of quasi-Einstein metrics, *Pac. J. Math.* **248**, 227-284.

Case, J., Shu, Y. and Wei, G. (2010). Rigidity of quasi-Einstein metrics, *Diff. Geom. Appl.* **29**, 93-100.

Case, J., Shu, Y. J. and Wei, G. (2011). Rigidity of quasi-Einstein metrics, *Diff. Geom. Appl.* **29(1)**, 93-100.

Catino, G. (2012). Generalized quasi-Einstein manifold with harmonic Weyl tensor, *Math. Z.* **271**, 751-756.

Catino, G., Mastrolia, P., Monticellia, D. D. and Rigoli, M. (2017). On the geometry of gradient Einstein-type manifolds, *Pacific J. Math.* **286**, 39-67.

Chen, B. Y. (1973). Geometry of submanifolds, Marcel Dekker, Inc. New York.

Chaki, M. C. (1987). On pseudo symmetric manifolds, *An. Stiint. Univ. Al. I. Cuza Iasi Sect. Ia Mat.* **33**, 53-58.

Chaubey, S. K., Siddiqi, M. D. and Prakasha, D. G. (2022). Invariant Submanifolds of Hyperbolic Sasakian Manifolds and η-Ricci-Bourguignon Solitons, *Filomat* **36(2)**, 409-421.

Chaubey, S. K., De, U. C. and Suh, Y. J. (2021). Kenmotsu manifolds satisfying the Fischer-Marsden equation, *J. Korean Math. Soc.* **58(3)**, 597-607.

Chaubey, S. K., Lee, H. and Suh. Y. J. (2022). Yamabe and gradient Yamabe Solitons on real hypersurfaces in the complex quadric, *Int. J. Geom. Methods Mod. Phys.* **19(2)**, 2250026.

Chen, X. (2020). Quasi-Einstein structures and almost cosympletic manifolds, *RACSAM* **114(2)**, 1-14.

Chen, B. Y. (1993). Some pinching and classification theorems for minimal submanifolds, *Arch. Math.* **60**, 568-578.

Chen, B. Y. (1995). A Riemannian invariant and its applications to submanifold theorey, *Results Math.* **27**, 17-26.

Chen, B. Y. (1996). A general inequality for submanifolds in complex-space-forms and its applications, *Archiv. der. Mathematik.* **67(6)**, 519-528.

Chern, S. S. (1968). Minimal Submanifolds in a Riemannian Manifold, *University of Kansas Press*, Lawrence, Kansas.

Chinea, D. and Gonzalez, C. (1990). A classification of almost contact metric manifolds, *Annali. di. Matematica. pura. ed. applicata.* **156**, 15-36.

Cho, J. T. (2014). Reeb flow symmetry on almost contact three-manifolds, *Differ. Geom. Appl.* **35**, 266-273.

Cho, J. T. and Kimura, M. (2009). Ricci solitons and real hypersurfaces in a complex space form, *Tohoku Math. J.* **61(2)**, 205–212.

Cho, J. T. and Sharma, R. (2010). Contact geometry and Ricci solitons, *Int. J. Geom. Methods Mod. Phys.* **7(6)**, 951-960.

Chow, B., Lu, P. and Ni, L. (2006). Hamilton's Ricci Flow, Graduate Studies in Mathematics **77**, AMS, Providence, RI, USA.

Chu, Y. and Wang. X. (2013). On the scalar curvature estimates for gradient Yamabe Solitons, *Kodai Math. J.* **36(2)**, 246-257.

Cioroboiu, D. (2003). B. Y. Chen inequalities for semislant submanifolds in Sasakian space forms, *Int. J. Math. and Mathematical Sci.* **2003**, 354806.

Coutinho, F., Dióǵenes, R., Leandro, B. and Ribeiro Jr., E. (2019). Static perfect fluid space-time on compact manifolds, *Class. Quantum Grav.* **37(1)**, 015003.

De, U. C. and Gazi, A. K. (2008). On almost pseudo symmetric manifolds, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **51**, 53-68.

De, U. C., Shaikh, A. A. and Biswa, S. (2003). On ϕ-recurrent Sasakian manifolds, *Novi Sad J. Math.* **33(2)**, 43-48.

De, U. C. and Majhi, P. (2019). On the Q curvature tensor of a generalized sasakian-space-form, *Krag. J. Math.* **13(3)**, 333-319.

De, U. C. and Samui, S. (2016). The structure of some classes of (κ, μ)-contact space forms, *Diff. Geom.-Dynamical System* **18**, 1-13.

De, U. C. and Majhi, P. (2015a). φ-semisymmetric generalized Sasakian space-form, *Arab J. Math. Sci.* **21**, 170-178.

De, U. C. and Ghosh, G. C. (2001). On generalized quasi Einstein manifolds, *Kyungpook Math. J.* **44**, 607-615.

De, U. C. and Gazi, A. K. (2009). On almost pseudo conformally symmetric manifolds, *Demonstratio Math.* **42(4)**, 507-520.

De, U. C., Chaubey, S. K. and Suh, Y. J. (2021). Gradient Yamabe and gradient m-quasi einstein metrics on three-dimensional Cosympletic manifolds, *Mediterr. J. Math.* **18(3)**, 1-14.

De, U. C., Sardar, A. and De, K. (2022). Ricci-Yamabe solitons and 3-dimensional Riemannian manifolds, *Turkish J. Math.* **46**, 1078-1088.

De, U. C. and Majhi, P. (2015b). On invariant submanifolds of Kenmotsu manifolds, *J. Geom.* **106**, 109-122.

Decu, S. and Haesen, S. (2022). Chen Inequalities for Spacelike Submanifolds in Statistical Manifolds of type Para-Kähler Space Forms, *Mathematics* **10(3)**, 330.

Deshmukh. S., Turki, N. B. and Alsodais, H. (2020). Characterizations of Trivial Ricci Solitons, *Adv. Math. Phys.* **2020**, 9826570.

Deshmukh, S. (2012). Jacobi-type vector fields on Ricci solitons, *Bull. Math. SOc. Sci. Math. Roum.* **55(103)**, 1.

Deszcz, R. (1992). On pseudosymmetric spaces, *Bull. Belg. Math. Soc., Ser. A* **44**, 1-34.

Dey, D. (2020). Almost Kenmotsu metric as Ricci-Yamabe soliton, arXiv:2005.02322.

Dileo, G. and Pastore, A. M. (2007). Almost Kenmotsu manifolds and local symmetry, *Bull. Belg. Math. Soc. Simen Stewin* **14**, 343-354.

Dileo, G. and Pastore, A. M. (2009). Almost Kenmotsu manifolds and nullity distributions, *J. Geom.* **93**, 46-61.

Dubey, R. D. (1979). Generalized recurrent spaces, *Indian J. Pure Appl. Math.* **10(12)**, 1508-1513.

Endo, H. (1986). Invariant submaniffolds in contact metric manifolds, *Tensor N. S.* **43**, 83-87.

Faghfouri, M. and Ghaffarzadeh, N. (2015). On doubly warped product submanifolds of generalized (κ, μ)-space forms, *Afrika Mat.* **26(7-8)**, 1443-1455.

Ghosh, A. (2020a). m-Quasi-Einstein Metrics Satisfying Certain Conditions on the Potential Vector Field, *Mediterr. J. Math.* **17**, 115.

Ghosh, A. (2019a). m-quasi-Einstein metric and contact geometry, *RACSAM* **113(3)**, 2587–2600.

Ghosh, A. (2015a). Quasi-Einstein contact metric manifolds, *Glasgow Math. J.* **57**, 569-577.

Ghosh, A. (2015b). Generalized m-quasi-Einstein metric within the framework of Sasakian and K-contact manifolds, *Ann. Pol. Math.* **115(1)**, 33-40.

Ghosh, A. (2019b). Ricci soliton and Ricci almost soliton within the framework of Kenmotsu manifold, *Carpathian Math. Publ.* **11(1)**, 59-69.

Ghosh, A. (2014). Certain Contact Metrics as Ricci Almost Solitons, *Results Math* **65**, 81-94.

Ghosh, A. (2013). An η-Einstein Kenmotsu metric as a Ricci soliton, *Publ. Math. Debrecen* **82(3-4)**, 591-598.

Ghosh, A. (2011). Kenmotsu 3-metric as Ricci soliton, *Chaos Solitons Fractals* **14(8)**, 647-650.

Ghosh, A. and Patra, D. S. (2017). The critical point equation and contact geometry, *J. Geom.* **108(1)**, 185–194.

Ghosh, A. (2020b). Ricci almost soliton and almost Yamabe soliton on Kenmotsu manifold, *Asian-Eur. J. Math.* **14(8)**, 2150130.

Goldberg, S. I. and Yano, K. (1969). Integrability of almost cosympletic structures, *Pacific J. Math.* **31**, 373-382.

Graf, W. (2007). Ricci flow gravity, *PMC Phys. A* **1**, 3.

Guler, S. and Crasmareanu, M. (2019). Ricci-Yamabe maps for Riemannian flow and their volume variation and volume entropy, *Turk. J. Math.* **43**, 2631-2641.

Gupta, R. S. (2013). B. Y. Chen's inequalities for Bi-slant submanifolds in cosympletic space forms, *Sarajevo J. Math.* **9(21)**, 117-128.

Hamilton, R. S. (1998). The Ricci flow on surfaces, *Math. Gen. Relativ. (Santa Cruz, C.A., 1986) Contemp. Math.* **71**, 237-262.

Hamilton, R. S. (1982). Three-manifolds with positive Ricci curvature, *J. Diff. Geom.* **17(2)**, 255-306.

Hayden, H. A. (1932). Subspaces of a space with torsion, *Proc. London Math. Soc.* **34**, 27-50.

Hawking, S. W. and Ellis, G. F. R. (1973). The Large-Scale Structure of Space-time, Cambridge Univ. Press.

Hu, Z., Li, D. and Zhai, S. (2017). On generalized m-quasi-Einstein manifold with constant Ricci curvature, *J. Math. Anal. Appl.* **446**, 843-853.

Hu, Z., Li, D. and Xu, J. (2015). On generalized m-quasi-Einstein manifold with constant scalar curvature, *J. Math. Anal. Appl.* **432**, 733-743.

Hui, S. K. (2017). On generalized ϕ-recurrent generalized (k, μ)-contact metric manifolds, arXiv: 1707.00830v1[math.DG].

Hui, S. K., Uddin, S. and Mandal, P. (2018). Submanifolds of generalized (κ, μ)-space-forms, *Per. Math. Hung.* **77**, 329-339.

Hwang, S., Chang, J. and Yun, G. (2016). Nonexistence of multiple black holes in static space-times and weakly harmonic curvature, *Gen. Rel. Grav.* **48(9)**, 120.

Ishi, Y. (1957). On conharmonic transformation, *Tensor N. S.* **7**, 73-80.

Ivancevic, V. G. and Ivancevic, T. T. (2011). Ricci flow and nonlinear reaction-diffusion systems in biology, chemistry and physics, *Nonlinear Dyn.* **65**, 35-54.

Janssens, D. and Vanhecke, L. (1981). Almost contact structures and curvature tensors, *Kodai Math. J.* **4**, 1-27.

Kanai, M. (1983). On a Differential Equation Characterizing a Riemannian Structure of a Manifold, *Tokyo J. Math.* **6(1)**, 143-151.

Karchar, H. (1992). Infinitesimal characterization of Friedmann Universe, *Arch. Math. Basel.* **38**, 58-64.

Kenmotsu, K. (1972). A class of almost contact Riemannian manifolds, *Tohoku Math. J. II Ser.* **24**, 93-103.

Kim, T. W. and Pak, H. K. (2005). Canonical foliations of certain classes of almost contact metric structures, *Acta Math. Sin. (Engl. Ser.)* **21(4)**, 841-846.

Kim, U. K. (2006). Conformally flat generalized Sasakian space-forms and locally symmetric generalized Sasakian space forms, *Note Mat.* **25**, 55-67.

Kim, J. (2017). On pseudo semiconformally symmetric manifolds, *Bull. Korean Math. Soc.* **54**, 177-186.

Kim, J. (2016). A type of conformal curvature tensor, *Far East J. Math. Soc.* **99(1)**, 61-74.

Kim, J. S., Prasad, R. and Tripathi, M. M. (2002). On generalized Ricci-recurrent trans-Sasakian manifolds, *J. Korean Math. Soc.* **39(6)**, 953-961.

Kon, M. (1973). Invariant submanifolds of normal contact metric manifolds, *Kodai Math. Sem. rep.* **25**, 330-336.

Koufogiorgos, T. (1993). Contact metric manifolds, *Ann. Glob. Anal. Geom.* **11**, 25-34.

Koufogiorgos, T. (1997). Contact Riemannian manifolds with constant φ-sectional curvature, *Tokyo J. Math.* **20**, 13-22.

Koufogiorgos, T. and Tsichlias, C. (2000). On the existence of a new class of contact metric manifolds, *Canad. Math. Bull.* **20(1)**, 400-447.

Kumar, D. L. and Nagaraja, H. G. (2019). Second Order Parallel Tensor and Ricci solitons on Generalized (κ, μ)-Space forms, *Math. Adv. Pure Appl. Sci.* **2(1)**, 1-7.

Kumar, P. K., Gupta, R. S. and Sharfuddin, A. (2010). B. Y. Chen's inequalities for bi-slant submanifolds in Kenmotsu space forms, *Demonstration Mathematica* **XLIII(4)**, 887-898.

Kumara, H. A., Venkatesha, V. and Naik, D. M. (2021). Static Perfect Fluid Space-Time on Almost Kenmotsu Manifolds, *J. Geom. Symmetry Phys.* **61**, 41-51.

Leandro, B. (2021). Vanishing conditions on Weyl tensor for Einstein-type manifolds, *Pacific J. Math.* **314**, 99-113.

Leandro, B. and Solórzano, N. (2019). Static perfect fluid spacetime with half conformally flat spatial factor, *Manuscripta Math.* **160**, 51-63.

Limoncu, M. (2010). Modification of the Ricci tensor and applications, *Arch. Math.* **95**, 191-199.

Lotta, A. (1996). Slant submanifolds in contact geometry, *Bull. Math. Soc. Roum.* **39**, 183-198.

Lone, M. S., Bahadir, O., Park, C. and Hwang, I. (2022). Basic inequalities for statistical submanifolds in Golden-like statistical manifolds, *Open Mathematics* **20(1)**, 153-166.

Mangione, V. (2008). Harmonic maps and stability on f-Kenmotsu manifolds, *Internat. J. Math. Math. Sci.* **2008** 798317.

Mantica, C. A. and Suh, Y. J. (2013). Pseudo Q-symmetric Riemannian manifolds, *Int. J. Geom. Methods Mod. Phys.* **10(5)**, 1-25.

Masood-ul-Alam, A. K. M. (1987). On spherical symmetry of static perfect fluid space-times and the positive mass Theorem, *Class. Quantum Grav.* **4(3)**, 625-633.

Miao, P. and Tam, L. F. (2011). Einstein and conformally flat critical metrics of the volume functional, *Trans. Amer. Math. Soc.* **363(6)**, 2907-2937.

Miao, P. and Tam, L. F. (2009). On the volume functional of compact manifolds with boundary with constant scalar curvature, *Calc. Var. Partial Differential Equations* **36(2)**, 141–171.

Mihai, A. and Özgür, C. (2011). Chen inequalities for submanifolds of complex space forms and Sasakian space forms endowed with semi-symmetric metric connections, *Rocky Mountain J. Math.* **41(5)**, 1653-1673.

Mihai, A. (2004). B.Y. Chen inequalities for slant submanifolds in generalized complex space forms, *Radovi Mathematicki* **12**, 215-231.

Milnor, J. (1976). Curvature of left invariant metrics on Lie groups, *Adv. Math.* **21**, 293-329.

Munteanu, O. and Wang, J. (2017). Positively curved shrinking Ricci solitons are compact, *J. Diff. Geom.* **106**, 499-505.

Myers, S. B. (1935). Connections between differential geometry and topology, *Duke Math. J.* **1**, 376-391.

Obata, M. (1970). Conformal transformations of Riemannian manifolds, *J. Diff. Geom.* **4**, 311-333.

Obata. M. (1962). Certain conditions for a Riemannian manifold to be isometric with a sphere, *J. Math. Soc. Japan* **14**, 333-340.

Olszak, Z. (1989). Locally conformal almost cosympletic manifolds, *Colloq. Math.* **57**, 73-87.

Olszak, Z. (1986). Normal almost contact manifolds of dimension three, *Ann. Pol. Math.* **47**, 42-50.

Olszak, Z. (1989). Locally conformal almost cosympletic manifolds, *Colloq. Math.* **57(1)**, 73-87.

Olszak, Z. and Rosca, R. (1991). Normal locally conformal almost cosymplectic manifolds, *Publ. Math. Debrecen* **39(3-4)**, 315-323.

O'Neill, B. (1983). Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York.

Özgür, C. (2011). B. Y. Chen inequalities for submanifolds of a Riemannian manifold of a quasi-constant curvature, *Turk. J. Math.* **35**, 501-509.

Papantoniou, B. J. (1993). Contact Riemannian manifolds satisfying $R(\xi, X) \cdot R = 0$ and $\xi \in (k, \mu)$-nullity distribution, *Yokohama Math. J.* **40(2)**, 149-161.

Pastore, A. M. and Saltarelli, V. (2011). Generalized nullity distributions on almost Kenmotsu manifolds, *Intr. Elec. J. Geom.* **4(2)**, 168-183.

Patra, D. S., Ghosh, A. and Bhattacharyya, A. (2020). The critical point equation on Kenmotsu and almost Kenmotsu manifolds, *Pub. Math.* **97(1-2)**, 85-99.

Patra, D. S. (2021). K-contact metrics as Ricci almost solitons, *Beitr. Algebra Geom.* **62**, 737-744.

Patra, D. S. and Ghosh, A. (2021). On Einstein-type contact metric manifolds, *J. Geom. Phys.* **169**, 104342.

Patra, D. S. and Ghosh, A. (2018). Certain Almost Kenmotsu Metrics Satisfying the Miao-Tam Equation, *Publ. Math.* **93(1-2)**, 107-123.

D. Perrone, D. (2004). Contact metric manifolds whose characteristic vector field is a harmonic vector field, *Diff. Geom. Appl.* **20**, 367-378.

Petersen, P. and Wylie, W. (2009). Rigidity of gradient Ricci soliton, *Pacific J. Math.* **241(2)**, 329-345.

Pigola, S., Rigoli, M., Rimoldi, M. and Setti, A. (2011). Ricci almost solitons, *Ann. Sc. Norm. Sup. Pisa CL. Sci.* **10(5)**, 757-799.

Poyraz, N. Ö. (2022). Chen inequalities on lightlike hypersurfaces of GRW spacetime, *Int. J. Geom. Methods Mod. Phys.* **19(1)**, 2250011.

Praveena, M. M., Bagewadi, C. S. and Krishnamurthy, M. R. (2021). Solitons of Kählerian space-time manifolds, *Int. j. Geom. Methods Mod. Phys.* **18(2)**, 2150021.

Qing, J. and Yuan, W. (2013). A note on static spaces and related problems, *J. Geom. Phys.* **74**, 18-27.

Qu, Q. and Wang, Y. (2015). Multiply warped products with a quarter-symmetric connection, *J. Math. Anal. Appl.* **431**, 955-987.

Rehman, N. A. (2015). Stability on Generalized Sasakian space forms, *Math. Reports* **17(67)**, 57-64.

Rovenski, V. and Patra, D. S. (2021). On non-gradient (m, ρ)-quasi-Einstein contact metric manifolds, *J. Geom.* **112(14)**, https://doi.org/10.1007/s00022-021-00576-5.

Saltarelli, V. (2015). Three-dimensional almost Kenmotsu manifolds satisfying certain nullity conditions, *Bull. Malays. Math. Sci. Soc.* **38**, 437-459.

Sandhu, R. S., Geojou, T. T. and Tamenbaun, A. R. (2016). Ricci curvature: An economic indicator for market fragility and system risk, *Sci. Adv.* **2**, 1501495.

Sarkar, A., Sen, M. and Akbar, A. (2015). Generalized Sasakian-space-forms with conharmonic curvature tensor, *Palestine J. Math.* **4(1)**, 84-90.

Sardar, A. and Sarkar, A. (2022). Ricci-Yamabe solitons on a class of generalized Sasakian space forms, *Asian-European J. Math.* https://doi.org/10.1142/S1793557122501741.

Sasaki, S. (1960). On differentiable manifolds with certain structures which are closely related to almost contact structure I, *Tohoku Math. J.* **12(3)**, 459-476.

Sasaki, S. and Hatakeyama, Y. (1961). On differentiable manifolds with certain structures which are closely related to almost contact structure II, *Tohoku Math. J.* **13(1)**, 281-294.

Schmidt, H. J. (1993). On the de Sitter Space-Time - the Geometric Foundation of Inflationary Cosmology, *Fortschr. Phys.* **41(3)**, 179-199.

Schouten, J. A. (1954). Ricci-Calculus. An introduction to tensor Analysis and its Geometrical Applications, Springer-Virlay, Berlin-Gottingen-Hindelberg.

Seko, T. and Maeta, S. (2019). Classification of almost Yamabe solitons in Euclidean spaces, *J. Geom. Phys.* **136**, 97-103.

Sen, R. N. and Chaki, M. C. (1967). On curvature restrictions of a certain kind of conformally flat Riemannian space of class one, *Proc. Nat. Inst. Sci. India* **33**, 100-102.

Shaikh, A. A. and Ahmed, H. (2011). On generalized ϕ-recurrent Sasakian manifolds, *Appl. Math.* **2**, 1317-1322.

Shaikh, A. A. and Patra, A. (2010). On a generalized class of recurrent manifolds, *Arch. Math. (BRNO)* **46**, 71-78.

Shaikh, A. A. and Roy, I. (2010). On quasi-generalized recurrent manifolds, *Math. Pannonica.* **21(2)**, 251-263.

Shaikh, A. A., Deszcz, R., Hotlos, M. and Jelwicki, J. (2015). On pseudosymmetric manifolds, *Publicationes Mathematicae* **86(3-4)**, 433-456.

Shaikh, A. A., Mandal, P. and Mondal, C. K. (2021). Some characterizations of gradient Yamabe solitons, *J. Geom. Phys.* **167**, 104293.

Shaikh, A. A., Matsuyama, Y. and Hui, S. K. (2016). On invariant submanifolds of $(LCS)_n$-manifolds, *J. Egyp. Math. Soc.* **24**, 263-269.

Shammukha, B. and Venkatesha, V. (2019). Projective curvature tensor on generalized (κ, μ)-space forms, *Italian J. Pure Appl. Math.* **12**, 810-850.

Sharma, R. (2008). Certain results on K-contact and (k, μ)-contact manifolds, *J. Geom.* **89**, 138-147.

Shashidhar, S. and Nagaraja, H. G. (2015). η-Einstein (κ, μ)-space forms, *J. Info. Math. Sci.* **7(2)**, 109-120.

Shivaprasanna, G. S. (2016). Some results on generalized (κ, μ) space-forms, *Int. J. sci. Engg. Appl. Sci.* **2(7)**, 184-191.

Shivaprasanna, G. S., Maralabhavi, Y. B. and Somashekhara, G. (2014). On Semi-Symmetric Metric Connection in a Generalized (κ, μ) Space Forms, *Int. J. Math. Trends Tech.* **9(3)**, 173-188.

Siddiqi, M. D. and De, U. C. (2022). Relativistic perfect fluid spacetimes and Ricci-Yamabe solitons, *Lett. Math. Phys.* **112(1)**, https://doi.org/10.1007/s11005-021-01493-z.

Siddiqi, M. D., De, U. C. and Deshmukh S. (2022). Estimation of almost Ricci-Yamabe solitons on static spacetimes, *Filomat* **36(2)**, 397-407.

Siddiqi, M. D. and Akyol, M. A. (2020). η-Ricci-Yamabe solitons on Riemannian submersions from Riemannian manifolds, arXiv:2004.14124.

Siddiqi, M. D. and Siddiqui, S. A. (2020). Conformal Ricci soliton and geometrical structure in a perfect fluid spacetime, *Int. J. Geom. Methods Mod. Phys.* **17(6)**, 2050083.

Singh, J. P. and Khatri, M. (2021). On Ricci-Yamabe soliton and geometrical structure in a perfect fluid spacetime, *Afr. Math.* **32(7)**, 1645-1656.

Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C. and Herlt, E. (2003). Exact Solutions of Einstein's Field Equations, 2nd edn. Cambridge Monographs on Mathematical Physics, Cambridge Univ. Press, Cambridge.

Suh, Y. J. and De, U. C. (2020). Yamabe solitons and gradient Yamabe solitons on three-dimensional $N(\kappa)$-contact manifolds, *Int. J. Geom. Method Mod. Phys.* **17(12)**, 2050177.

Sular, S. (2016). Chen inequalities for submanifolds of generalized space forms with a semi-symmetric metric connection, *Hacettepe J. Math. Stat.* **45(3)**, 811-825.

Takahashi, T. (1977). Sasakian ϕ-symmetric spaces, *Tohoku Math. J.*, **29**, 99-113.

Tamássy, L. and Binh, T. Q. (1989). On weakly symmetric and weakly projective symmetric Riemannian manifolds, *Coll. Math. Soc., J. Bolyai* **56**, 663-670.

Tanno, S. (1969). The automorphism groups of almost contact Riemannian manifolds, *Tohoku Math. J.* **21**, 21-38.

Tashiro, Y. (1965). Complete Riemannian manifolds and some vector fields, *Trans. Amer. Math. Soc.* **117**, 251-275.

Venkatesha, V., Kumara, H. A. and Naik, D. M. (1019). On a class of generalized φ-recurrent Sasakian manifold, *J. Eqyptian Math. Soc.* **27(19)**, 1-14.

Venkatesha, V. and Kumara, H. A. (2019). Ricci soliton and geometrical structure in a perfect fluid spacetime with torse-forming vector field, *Afr. Math.* **30**, 725-736.

Vilcu, G. E. (2013). On Chen invariants and inequalities in quaternionic geometry, *J. Inequal. Appl.* **66**, https://doi.org/10.1186/1029-242X-2013-66.

Walker, A. G. (1950). On Ruses spaces of recurrent curvature, *Proc. London Math. Soc.* **52**, 36-64.

Wang, Y. and Liu, X. (2016). On almost Kenmotsu manifolds satisfying some nullity distributions, *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.* **86(3)**, 347-353.

Wang, Y. (2016). Gradient Ricci almost solitons on two classes of almost Kenmotsu manifolds, *J. Korean Math. Soc.* **53**, 1101-1114.

Wang, Y. (2017). Conformally Flat Almost Kenmotsu 3-Manifolds, *Mediterr. J. Math.* **14(5)**, 1-16.

Wang, L. F. (2011). On noncompact τ-quasi-Einstein metrics, *Pac. J. Math.* **254(2)**, 449-464.

Wang, Y. and Zhang, P. (2010). Chen inequalities for submanifolds of real space forms with a quarter-symmetric connection, *Taiwanese J. Math.* **14(4)**, 1465-1477.

Wang, Y. (2019). Chen inequalities for submanifolds of complex space forms and Sasakian space forms with quarter-symmetric connections, *Int. J. Geom. Methods Modern Phys.* **16(8)**, 1950118.

Wang, Y. and Wang, W. (2017). An Einstein-like metric on almost Kenmotsu manifolds, *Filomat* **31(15)**, 4695-4705.

Wang, Y. (2016). A class of 3-dimensional almost Kenmotsu manifolds with harmonic curvature tensors, *Open Math.* **14**, 977-985.

Wang, Y. (2016). Yamabe solitons on three-dimensional Kenmotsu manifolds, *Bull. Belg. Math. Soc.* **23(3)**, 345-355.

Wang, Y., Gu. X., Chan. T. F., Thompson, P. W. and Yau, S. T. (2012). Brain surface conformal parametrization with the Ricci flow, *IEEE Trans. Med. Imaging* **10**, 251-264.

Wylie, W. (2008). Complete shrinking Ricci solitons have finite fundamental group,]em Proc. Am. Math. Soc. **136**, 1803-1806.

Yang, F. and Zhang, L. (2017). Rigidity of gradient shrinking and expanding Ricci solitons, *Bull. Korean Math. Soc.* **54(3)**, 817-824.

Yano, K. (1970). Integral formulas in Riemannian geometry, Marcel Dekker, New York.

Yano, K. and Kon, M. (1984). Structures on manifolds,Series in Pure Math. Vol 3, World Scientific Publ. Co., Singapore.

Yano, K. (1910). Concircular geometry I. Concircular transformation, *Proc. Imp. Acad. Tokyo* **16**, 195-200.

Yano, K. and Ishihara, S. (1969). Invariant submanifolds of almost contact manifolds, *Kodai Math. Sem. Rep.* **21**, 350-364.

Yildiz, A. and Murathan, C. (2009). Invariant submanifolds of Sasakian space forms, *J. Geom.* **95**, 135-150.

Zhang, P., Zhang, L. and Song, W. (2014). Chen's inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature with a semi-symmetric metric connection, *Taiwanese J. Math.* **18(6)**, 1841-1862.

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ON A CLASS OF GENERALIZED RECURRENT (k,μ)-CONTACT METRIC MANIFOLDS

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Abstract. The goal of this paper is the introduction of hyper generalized ϕ-recurrent (k,μ)-contact metric manifolds and of quasi generalized ϕ-recurrent (k,μ)-contact metric manifolds, and the investigation of their properties. Their existence is guaranteed by examples.

1. Introduction

The concept of a (k,μ)-contact metric manifold was introduced by Blair et al. [4], and there are several reasons for studying it. One of its key features is that it contains both Sasakian and non-Sasakian manifolds. Sasakian manifolds were first studied by Sasaki [20]. A full classification of (k,μ)-spaces was given by Boeckx [5]. Recently, the properties of (k,μ)-spaces under certain conditions has been studied by many geometers; see [1, 2, 23] and references therein.

Cartan [6] introduced the concept of locally symmetric space, which has been weakened and studied by many geometers throughout the years to a great extent. The notion of locally ϕ-symmetric Sasakian manifolds was introduced by Takashi [24]. The generalization of ϕ-symmetric Sasakian manifolds was made by De et al. [9] and called it ϕ-recurrent Sasakian manifolds. Jun et al. [16] studied ϕ-recurrent (k,μ)-contact metric manifolds. De et al. [14] studied ϕ-Ricci symmetric (k,μ)-contact metric manifolds. Dubey [11] introduced the notion of generalized recurrent manifold. A non-flat Riemannian manifold is said to be a generalized recurrent manifold if its curvature tensor R satisfies

$$\nabla R = A\otimes R + B\otimes G, \tag{1}$$

where A and B are non-vanishing 1-forms defined by $A(X)=g(X,\gamma_1)$ and $B(X)=g(X,\gamma_2)$ and the tensor G is defined by

$$G(X,Y)Z = g(Y,Z)X - g(X,Z)Y \tag{2}$$

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for any vector fields X, Y, Z. Here, ∇ denotes the covariant differentiation with respect to the metric g. If the 1-form B vanishes, then (1) reduces to recurrent manifold [27].

A non-flat Riemannian manifold is said to be generalized Ricci-recurrent manifold [10] if the Ricci tensor S satisfies

$$\nabla S = A \otimes S + B \otimes g, \tag{3}$$

where A and B are 1-forms defined in (1). If 1-form B vanishes, then it reduces to the notion of Ricci-recurrent manifold [19].

Shaikh et al. [21] extended this concept to generalized ϕ-recurrent Sasakian manifold. Hui [15] studied generalized ϕ-recurrent generalized (k, μ)-contact metric manifold and obtained interesting results. A non-flat Riemannain manifold is said to be generalized ϕ-recurrent manifold if the curvature tensor R satisfies the condition

$$\phi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z + B(W)G(X, Y)Z \tag{4}$$

for all vector fields X, Y and Z. Here, tensor G is defined as in (2).

A Riemannain manifold is said to be hyper generalized recurrent manifold if its curvature tensor R satisfies the condition

$$\nabla R = A \otimes R + B \otimes (g \wedge S), \tag{5}$$

where A and B are 1-forms defined in (1).

Recently, Venkatesha et al. [25] extended the notion of hyper generalized recurrent manifolds (resp. quasi generalized recurrent manifolds) to hyper generalized ϕ-recurrent Sasakian manifolds (resp. quasi generalized ϕ-recurrent Sasakian manifolds) and obtained interesting results. Continuing this, we studied hyper generalized ϕ-recurrent (k, μ)-contact metric manifolds and prove its existence by giving a proper example. Similarly, quasi generalized ϕ-recurrent (k, μ)-contact metric manifolds was investigated. This paper has the following organization. After preliminaries, in Section 3, we study hyper generalized ϕ-recurrent (k, μ)-contact metric manifolds. And in Section 4, we construct an example to prove the existence of hyper generalized ϕ-recurrent (k, μ)-contact metric manifolds. Next, in Section 4, we study quasi generalized ϕ-recurrent (k, μ)-contact metric manifolds. Its existence is proved in Section 5 by constructing an example.

2. Preliminaries

In this section, we listed some of the basic formulae and definitions on (k, μ)-contact metric manifolds which will be used throughout the paper. It is well known that, the concept of (k, μ)-contact metric manifold contains both Sasakian and non-Sasakian manifolds. Recently, geometry of contact metric manifolds under various conditions has been studied by [10, 12, 13, 18, 19, 26]. A detailed study on (k, μ)-contact metric manifolds are available in [3–5, 8] and references therein.

Let M be a smooth connected manifold of dimension $(2n+1)$. Then, M is called an almost contact metric manifold if it is equipped with an almost contact structure (ϕ, ξ, η, g) which satisfies the following relations:

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ g(X, \xi) = \eta(X), \tag{6}$$

$$\phi\xi = 0,\ \eta\xi = 0,\ g(X, \phi Y) = -g(\phi X, Y),\ g(X, \phi X) = 0, \tag{7}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{8}$$

where η is a 1-form, ξ is a vector field, ϕ is a tensor field of type (1,1) and g is a Riemannian metric on M. An almost contact metric manifold satisfying $g(X, \phi Y) = d\eta(X, Y)$, is called a contact metric manifold. We consider on $M(\phi, \xi, \eta, g)$, a symmetric (1,1) tensor field h defined by $h = \frac{1}{2} L_\xi \phi$, where L denotes Lie differentiation, and satisfies $h\xi = 0$, $h\phi = -\phi h$, $trh = tr\phi h = 0$.

The (k, μ)-nullity distribution on the manifold $M(\phi, \xi, \eta, g)$ is a distribution [4]

$$\begin{aligned} N(k, \mu) : p \to N_p(k, \mu) = \{Z \in T_p(M) : R(X, Y)Z &= k(g(Y, Z)X - g(X, Z)Y) \\ &+ \mu(g(Y, Z)hX - g(X, Z)hY)\} \end{aligned} \tag{9}$$

for any $X, Y \in T_pM$ and $k, \mu \in \mathbb{R}^2$. A contact metric manifold with ξ belongings to (k, μ)-nullity distribution is called a (k, μ)-contact metric manifold. A (k, μ)-contact metric manifold becomes Sasakian manifold for $k = 1$, $\mu = 0$; and the notion of (k, μ)-nullity distribution reduces to k-nullity distribution for $\mu = 0$.

In a (k, μ)-contact metric manifold the following properties are true [4]:

$$h^2 = (k-1)\phi^2,\ k \leq 1, \tag{10}$$

$$\nabla_X \xi = -\phi X - \phi h X,\ (\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{11}$$

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{12}$$

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX], \tag{13}$$

$$\begin{aligned} S(X, Y) = [2(n-1) - n\mu]g(X, Y) + [2(n-1) + \mu]g(hX, Y) \\ + [2(1-n) + n(2k+\mu)]\eta(X)\eta(Y), \end{aligned} \tag{14}$$

$$S(X, \xi) = 2nk\eta(X), \tag{15}$$

$$r = 2n(2n - 2 + k - n\mu), \tag{16}$$

$$S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 2(2n - 2 + \mu)g(hX, Y), \tag{17}$$

where S is the Ricci tensor of type $(0, 2)$ and r is the scalar curvature of the manifold M. So

$$(\nabla_X \eta(Y)) = g(X + hX, \phi Y), \tag{18}$$

$$(\nabla_X hY) = \big[(1-k)g(X,\phi Y) + g(X,h\phi Y)\big]\xi$$
$$+ \eta(Y)\big[h(\phi X + \phi hX)\big] - \mu\eta(X)\phi hY \tag{19}$$

for all $X, Y \in \chi(M)$.

Definition. A $(2n+1)$-dimensional (k,μ)-contact metric manifold is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y),$$

for any vector fields X and Y, where α and β are constants. If $\beta = 0$, then the manifold M is an Einstein manifold.

3. Hyper generalized ϕ-recurrent (k,μ)-contact metric manifold

In the paper [22], the author studied hyper generalized recurrent manifolds. Recently, the author [25] studied hyper generalized ϕ-recurrent Sasakian manifold and obtained important results. By observing this, we extended it to (k,μ)-contact metric manifold. In this section, we study hyper generalized ϕ-recurrent (k,μ)-contact metric manifold.

Definition. A $(2n+1)$-dimensional (k,μ)-contact metric manifold is said to be a hyper generalized ϕ-recurrent if its curvature tensor R satisfies

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z + B(W)H(X,Y)Z \tag{20}$$

for all vector fields X, Y and Z. Here, A and B are two non-vanishing 1-forms such that $A(X) = g(X,\rho_1)$, $B(X) = g(X,\rho_2)$ and the tensor H is defined by

$$H(X,Y)Z = S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY \tag{21}$$

for all vector fields X, Y and Z. Here, Q is the Ricci operator, ρ_1 and ρ_2 are vector fields associated with 1-forms A and B respectively. If the 1-form B vanishes, then (20) reduces to the notion of ϕ-recurrent manifolds.

Theorem 3.1. *In a hyper generalized ϕ-recurrent (k,μ)-contact metric manifold, the 1-forms A and B satisfy the relation*

$$kA(W) + [n(2k-\mu+2) - 2]B(W) = 0.$$

Proof. Let us consider hyper generalized ϕ-recurrent (k,μ)-contact metric manifold. In view of (20) and (6) we obtain

$$-(\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi$$
$$= A(W)R(X,Y)Z + B(W)H(X,Y)Z. \tag{22}$$

Taking an inner product with U in (22), we get

$$-g((\nabla_W R)(X,Y)Z) + \eta((\nabla_W R)(X,Y)Z)\eta(U)$$
$$= A(W)g(R(X,Y)Z,U) + B(W)g(H(X,Y)Z,U). \tag{23}$$

Contracting over X and U in (22) gives

$$-(\nabla_W S)(Y,Z) + \eta((\nabla_W R)(\xi,Y)Z)$$

$$= [A(W) + (2n-1)B(W)]S(Y,Z) + rB(W)g(Y,Z). \tag{24}$$

Taking $Z = \xi$ in (24) and using the fact that $\eta((\nabla_W R)(\xi, Y)\xi) = 0$ we obtain

$$-(\nabla_W S)(Y,\xi) = [2nk(A(W) + (2n-1)B(W)) + rB(W)]\eta(Y). \tag{25}$$

Putting $Y = \xi$ in above equation gives

$$2nk[A(W) + (2n-1)B(W)] + rB(W) = 0. \tag{26}$$

Using (16) in (26), we obtain

$$kA(W) + [n(2k-\mu+2) - 2]B(W) = 0 \tag{27}$$

for any vector field W. This completes the proof. □

Taking $r = 0$ in (26), we are in a position to state the following corollary.

Corollary 3.2. *In a hyper generalized ϕ-recurrent (k,μ)-contact metric manifold, if the scalar curvature of the manifold vanishes then, either*

1. *1-forms A and B are co-directional, or*
2. *it is $\left(0, \frac{2(n-1)}{n}\right)$-contact metric manifold.*

Let $\{e_i\}_{i=1}^{2n+1}$ be an orthonormal basis of the manifold. Putting $Y = Z = e_i$ in (24) and taking summation over $i, 1 \le i \le 2n+1$, and using (6), (11) and (15) we obtain

$$-dr(W) = r[A(W) + 4nB(W)]. \tag{28}$$

This led us to the following theorem.

Theorem 3.3. *In a hyper generalized ϕ-recurrent (k,μ)-contact metric manifold, if the scalar curvature of the manifold is a non-zero constant, then $A(W) + 4nB(W) = 0$ for any vector field W.*

Theorem 3.4. *In a hyper generalized ϕ-recurrent (k,μ)-contact metric manifold, the associated vector fields ρ_1 and ρ_2 corresponding to 1-forms A and B satisfy the relation*

$$r\eta(\rho_1) + 2(2n-1)(r-2nk)\eta(\rho_2) = 0.$$

Proof. Changing X, Y, Z cyclically in (23) and using Bianchi's identity we get

$$\begin{aligned} &A(W)g(R(X,Y)Z,U) + A(X)g(R(Y,W)Z,U) \\ &+ A(Y)g(R(W,X)Z,U) + B(W)g(H(X,Y)Z,U) \\ &+ B(X)g(H(Y,W)Z,U) + B(Y)g(H(W,X)Z,U) = 0. \end{aligned} \tag{29}$$

Contracting over Y and Z and using (9), we obtain

$$\begin{aligned} &A(W)S(X,U) - A(X)S(W,U) - kg(X,U)A(W) + kg(W,U)A(X) \\ &- \mu g(hW,U)A(X) + B(W)[rg(X,U) + (2n-1)S(X,U)] \\ &+ B(X)[-rg(W,U) - (2n-1)S(W,U)] + B(QX)g(W,U) \\ &- B(QW)g(X,U) + B(X)S(W,U) - B(W)S(X,U) = 0. \end{aligned} \tag{30}$$

Again contracting (30) over X and U yields

$$\begin{aligned}&(r+2nk)A(W)-A(QW)+\mu A(hW)\\&+(4nr-2r)B(W)-(4n-2)B(QW)=0.\end{aligned}\tag{31}$$

Replacing W by ξ in (31) results in

$$r\eta(\rho_1)+2(2n-1)(r-2nk)\eta(\rho_2)=0.\tag{32}$$

This completes the proof. □

Making use of relation $g((\nabla_W R)(X,Y)Z,U)=-g((\nabla_W R)(X,Y)U,Z)$ we obtain the following relation

$$\begin{aligned}g((\nabla_W R)(\xi,Y)Z,\xi)=\mu[\{&(1-k)g(W,\phi Y)+g(W,h\phi Y)\\&-g(hY,\phi(W+hW))\}\eta(Z)-\mu\eta(W)g(\phi hY,Z)].\end{aligned}\tag{33}$$

Considering (33) and (23) we can state the following theorem.

Theorem 3.5. *A hyper generalized ϕ-recurrent (k,μ)-contact metric manifold is generalized Ricci recurrent if and only if the following relation holds*:

$$\begin{aligned}g((\nabla_W R)(\xi,Y)Z,\xi)=\mu[\{&(1-k)g(W,\phi Y)+g(W,h\phi Y)\\&-g(hY,\phi(W+hW))\}\eta(Z)-\mu\eta(W)g(\phi hY,Z)]=0.\end{aligned}$$

Theorem 3.6. *A hyper generalized ϕ-recurrent (k,μ)-contact metric manifold is an η-Einstein manifold.*

Proof. Since we have

$$(\nabla_W S)(Y,\xi)=\nabla_W S(Y,\xi)-S(\nabla_W Y,\xi)-S(Y,\nabla_W\xi).\tag{34}$$

Using (11) and (15) in (34) we get

$$(\nabla_W S)(Y,\xi)=-2nkg(\phi W+\phi hW,Y)+S(Y,\phi W+\phi hW).\tag{35}$$

From (27) and (35) we obtain

$$\begin{aligned}&2nkg(\phi W+\phi hW,Y)-S(Y,\phi W+\phi hW)\\&=\Big[2nk\{A(W)+(2n-1)B(W)\}+rB(W)\Big]\eta(Y).\end{aligned}\tag{36}$$

Taking $Y=\phi Y$ in (36) gives

$$\begin{aligned}S(Y,W)+S(Y,hW)=\;&2nkg(Y,W)+[2nk+2(2n-2+\mu)]g(Y,hW)\\&+2(2n-2+\mu)(k-1)g(Y,-W+\eta(W)\xi).\end{aligned}\tag{37}$$

Using

$$\begin{aligned}S(Y,hW)=\;&(2n-2-n\mu)g(Y,hW)-(2n-2+\mu)(k-1)g(Y,W)\\&+(2n-2+\mu)(k-1)\eta(W)\eta(Y),\end{aligned}$$

and (14) in (37) led us to the following relation

$$S(Y,W)=\alpha g(Y,W)+\beta\eta(Y)\eta(W),\tag{38}$$

where

$$\alpha = \frac{[2(nk+n-1)+\mu(n+2)][2(n-1)-n\mu]-[2(n-1)+\mu][\mu(1-k)+2(n-1)+2k]}{2nk+\mu(n+1)},$$

$$\beta = \frac{[2(nk+n-1)+\mu(n+2)][2(1-n)+n(2k+\mu)]-(k-1)[2(n-1)+\mu]^2}{2nk+\mu(n+1)}.$$

This completes the proof. □

Theorem 3.7. *In a hyper generalized ϕ-recurrent (k,μ)-contact metric manifold, the 1-forms A and B satisfy the relation*

$$2nkA(\phi W) + [r + 2nk(2n-1)]B(\phi W) = 0.$$

Proof. In view of (9), (11) and (12) we get

$$\begin{aligned}(\nabla_W R)(X,Y)\xi = {} & k[g(W+hW,\phi Y)X - g(W+hW,\phi X)Y] \\ & + \mu[g(W+hW,\phi Y)hX - g(W+hW,\phi X)hY \\ & + \{(1-k)g(W,\phi X) + g(W,h\phi X)\}\eta(Y)\xi \\ & - \{(1-k)g(W,\phi Y) + g(W,h\phi Y)\}\eta(X)\xi \\ & + \mu\eta(W)\{\eta(X)\phi hY - \eta(Y)\phi hX\}] \\ & + R(X,Y)\phi W + R(X,Y)\phi hW. \end{aligned} \tag{39}$$

Using (39) in (22) results in the following relation

$$\begin{aligned} & k[g(W+hW,\phi Y)\eta(X) - g(W+hW,\phi Y)\eta(Y)]\xi \\ & + \mu[(1-k)g(W,\phi X)\eta(Y) + g(W,h\phi X)\eta(Y) \\ & - (1-k)g(W,\phi Y)\eta(X) - g(W,h\phi Y)\eta(X)]\xi \\ & + k[g(Y,\phi W)\eta(X) - g(X,\phi W)\eta(Y) \\ & + g(Y,\phi hW)\eta(X) - g(X,\phi hW)\eta(Y)]\xi \\ & - k[g(W+hW,\phi Y)X - g(W+hW,\phi X)Y] \\ & - \mu[g(W+hW,\phi Y)hX - g(W+hW,\phi X)hY \\ & + \{(1-k)g(W,\phi X) + g(W,h\phi X)\}\eta(Y)\xi \\ & - \{(1-k)g(W,\phi Y) + g(W,h\phi Y)\}\eta(X)\xi \\ & + \mu\eta(W)\{\eta(X)\phi hY - \eta(Y)\phi hX\}] \\ & + R(X,Y)\phi W + R(X,Y)\phi hW \\ = {} & A(W)\{k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]\} \\ & + B(W)\{2nk[\eta(Y)X - \eta(X)Y] + \eta(Y)QX - \eta(X)QY\}. \end{aligned} \tag{40}$$

Putting $Y = \xi$ in (40) we get

$$\begin{aligned} & A(W)[k(X - \eta(X)\xi) + \mu hX] + B(W)[2nkX - 4nk\eta(X)\xi + QX] \\ & + \mu^2\eta(W)\phi hX = 0. \end{aligned} \tag{41}$$

Taking $W = \phi W$ and contracting over X in (41) gives

$$2nkA(\phi W) + [r + 2nk(2n-1)]B(\phi W) = 0. \tag{42}$$

This completes the proof. □

4. Example of hyper generalized ϕ-recurrent (k,μ)-contact metric manifold

In this section, we construct an example of hyper generalized ϕ-recurrent (k,μ)-contact metric manifold. We consider a 3-dimensional manifold $M^3 = \{(x,y,z) \in \mathbb{R}^3 : x \neq 0\}$ where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be linearly independent vector fields in M^3 which satisfy

$$[E_1, E_2] = 2xE_1,\ [E_2, E_3] = 0,\ [E_1, E_3] = 0.$$

Let g be Riemannian metric defined by

$$g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1,$$
$$g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0.$$

Let η be the 1-form defined by

$$\eta(X) = g(X, E_3)$$

for any vector field X. Let ϕ be (1,1)-tensor field defined by

$$\phi E_1 = E_2,\ \phi E_2 = -E_1,\ \phi E_3 = 0.$$

Then we have

$$\eta(E_3) = 1,\ \phi^2(X) = -X + \phi(X)E_3$$

and

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y).$$

Moreover

$$hE_3 = 0,\ hE_1 = -E_1,\ hE_2 = E_2.$$

Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines a contact metric structure on M^3. Let ∇ be the Riemannian connection of g. Using Koszul formula we obtain

$$\nabla_{E_1} E_1 = -2xE_2,\ \nabla_{E_1} E_2 = 2xE_1,\ \nabla_{E_1} E_3 = 0,$$
$$\nabla_{E_2} E_1 = 0,\ \nabla_{E_2} E_2 = 0,\ \nabla_{E_2} E_3 = 0,$$
$$\nabla_{E_3} E_1 = 0,\ \nabla_{E_3} E_2 = 0,\ \nabla_{E_3} E_3 = 0.$$

Thus the metric $M^3(\phi, \xi, \eta, g)$ under consideration is a (k,μ)-contact metric manifold. Now, we will show that it is a 3-dimensional hyper generalized ϕ-recurrent (k,μ)-contact metric manifold. The non-vanishing components of curvature tensor and Ricci tensor are

$$R(E_1, E_2)E_1 = 4x^2 E_2,\ R(E_1, E_2)E_2 = -4x^2 E_1,$$
$$S(E_1, E_1) = S(E_2, E_2) = -4x^2.$$

Since $\{E_1, E_2, E_3\}$ forms the orthonormal basis of the 3-dimensional (k,μ)-contact metric manifold any vector fields can be expressed as

$$X = a_1 E_1 + b_1 E_2 + c_1 E_3,$$
$$Y = a_2 E_1 + b_2 E_2 + c_2 E_3,$$

$$Z = a_3E_1 + b_3E_2 + c_3E_3.$$

Then,

$$R(X, Y)Z = u_1E_1 + u_2E_2, \tag{43}$$

where $u_1 = 4x^2b_3(a_2b_1 - a_1b_2)$ and $u_2 = -4x^2a_3(a_2b_1 - a_1b_2)$,
and

$$F(X, Y)Z = v_1E_1 + v_2E_2 + v_3E_3, \tag{44}$$

where

$$\begin{aligned} v_1 &= 4x^2[a_1(a_1a_2 + b_1b_2)(a_1a_3 + b_1b_3 + c_1c_3) \\ &\quad + b_3(a_2b_1 - a_1b_2) - a_2(a_1a_2 + b_1b_2)(a_2a_3 + b_2b_3 + c_2c_3)], \\ v_2 &= 4x^2[b_1(a_1a_3 + b_1b_3 + c_1c_3)(a_1a_2 + b_1b_2) \\ &\quad - a_3(a_2b_1 - a_1b_2) - b_2(a_1a_2 + b_1b_2)(a_2a_3 + b_2b_3 + c_2c_3)] \end{aligned}$$

and

$$\begin{aligned} v_3 &= 4x^2[c_1(a_1a_3 + b_1b_3 + c_1c_3)(a_1a_2 + b_1b_2) - c_1(a_2a_3 + b_2b_3) \\ &\quad + c_2(a_1a_3 + b_1b_3) - c_2(a_1a_2 + b_1b_2)(a_2a_3 + b_2b_3 + c_2c_3)]. \end{aligned}$$

By virtue of (43), we have the following

$$\begin{aligned} &(\nabla_{E_1}R)(X, Y)Z = 8x^3(a_1b_2 - a_2b_1)(b_3E_1 - a_3E_2), \\ &(\nabla_{E_2}R)(X, Y)Z = 0, \\ &(\nabla_{E_3}R)(X, Y)Z = 0. \end{aligned} \tag{45}$$

Form (43) one can easily obtain the following

$$\phi^2(\nabla_{E_i}R)(X, Y)Z = p_iE_1 + q_iE_2, \;\; i = 1, 2, 3, \tag{46}$$

where $p_1 = -8x^3b_3(a_1b_2 - a_2b_1),\; q_1 = 8x^3a_3(a_1b_2 - a_2b_1),\; p_2 = 0,\; q_2 = 0,\; p_3 = 0,\; q_3 = 0$.

Let the 1-forms be defined as

$$\begin{aligned} &A(E_1) = \frac{p_1v_2 - v_1q_1}{u_1v_2 - v_1u_2}, \;\; B(E_1) = \frac{u_1q_1 - p_1u_2}{u_1v_2 - v_1u_2}, \\ &A(E_2) = 0, \;\; B(E_2) = 0, \\ &A(E_3) = 0, \;\; B(E_3) = 0, \end{aligned} \tag{47}$$

satisfying, $p_1v_2 - v_1q_1 \neq 0$, $u_1v_2 - v_1u_2 \neq 0$, $u_1q_1 - p_1u_2 \neq 0$ and $v_3 = 0$.
In view of (43), (44) and (46) it is easy to show the following relation:

$$\phi^2(\nabla_{E_i}R)(X, Y)Z) = A(E_i)R(X, Y)Z + B(E_i)F(X, Y)Z, \;\; i = 1, 2, 3. \tag{48}$$

Hence, the metric M^3 under consideration is a 3-dimensional hyper generalized ϕ-recurrent (k, μ)-contact metric manifold which is neither ϕ-symmetric nor ϕ-recurrent.

We can state the following.

Theorem 4.1. *There exists a 3-dimensional hyper generalized ϕ-recurrent (k,μ)-contact metric manifold which is neither ϕ-symmetric nor ϕ-recurrent.*

5. Quasi generalized ϕ-recurrent (k,μ)-contact metric manifold

Recently, the author [25] studied quasi generalized ϕ-recurrent Sasakian manifolds. A brief study on quasi generalized recurrent manifolds was done by Shaikh [23] and obtained interesting results. In this section, we will study quasi generalized ϕ-recurrent (k,μ)-contact metric manifolds.

Definition. A $(2n+1)$-dimensional (k,μ)-contact metric manifold is said to be a quasi generalized ϕ-recurrent if its curvature tensor R satisfies

$$\phi^2((\nabla_W R)(X,Y)Z) = D(W)R(X,Y)Z + E(W)F(X,Y)Z \tag{49}$$

for all vector fields X, Y and Z. Here, D and E are two non-vanishing 1-forms such that $D(X) = g(X,\mu_1)$, $E(X) = g(X,\mu_2)$ and the tensor F is define by

$$\begin{aligned} F(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ & + g(Y,Z)\eta(Y)\xi - g(X,Z)\eta(Y)\xi \end{aligned} \tag{50}$$

for all vector fields X, Y and Z. Here, μ_1 and μ_2 are vector fields associated with 1-forms D and E respectively.

Theorem 5.1. *In a quasi generalized ϕ-recurrent (k,μ)-contact metric manifold, the associated 1-forms D and E are related by $kD(W) + 2E(W) = 0$.*

Proof. Consider a quasi generalized ϕ-recurrent (k,μ)-contact metric manifold. From (49) we get

$$\begin{aligned} & -((\nabla_W R)(X,Y)Z) + \eta((\nabla_W R)(X,Y)Z)\xi \\ & = D(W)R(X,Y)Z + E(W)F(X,Y)Z. \end{aligned} \tag{51}$$

Taking the same steps as in Theorem 3.1, we obtain the relation:

$$kD(W) + 2E(W) = 0. \tag{52}$$

This completes the proof. □

Contracting over X in (51) gives

$$\begin{aligned} & -(\nabla_W S)(Y,Z) + \eta((\nabla_W R)(\xi,Y)Z) \\ & = D(W)S(Y,Z) + [(2n+1)g(Y,Z) + (2n-1)\eta(Y)\eta(Z)]E(W). \end{aligned} \tag{53}$$

Putting $Y = Z = e_i$, (53) reduce to

$$-dr(W) = rD(W) + 2n(2n+3)E(W). \tag{54}$$

We are in a position to state the following.

Theorem 5.2. *In a quasi generalized ϕ-recurrent (k,μ)-contact metric manifold, if the scalar curvature is a non-zero constant, then*

$$rD(W) + 2n(2n+3)E(W) = 0.$$

From (53), we can state the following.

Theorem 5.3. *A quasi generalized ϕ-recurrent (k,μ)-contact metric manifold is a super generalized Ricci recurrent manifold if and only if*

$$g((\nabla_W R)(\xi,Y)Z,\xi) = \mu[\{(1-k)g(W,\phi Y) + g(W,h\phi Y) - g(hY,\phi(W+hW))\}\eta(Z) - \mu\eta(W)g(\phi hY,Z)] = 0.$$

Theorem 5.4. *In a quasi generalized ϕ-recurrent (k,μ)-contact metric manifold, the scalar curvature of the manifold satisfy the relation $r = k[n(5+2n^2)] + 2(2n-1)]$.*

Proof. Changing X, Y, Z cyclically in (51) and making use of Bianchi's identity we get

$$D(W)R(X,Y)Z + D(X)R(Y,W)Z + D(Y)R(W,X)Z + E(W)F(X,Y)Z + E(X)F(Y,W)Z + E(Y)F(W,X)Z = 0. \tag{55}$$

Contracting over X in (55) we get

$$\begin{aligned} &D(W)S(Y,Z) + D(R(Y,W)Z) - D(Y)S(W,Z) \\ &+ E(W)[(2n+1)g(Y,Z) + (2n-1)\eta(Y)\eta(Z)] + E(Y)g(W,Z) \\ &- g(Y,Z)E(W) + \eta(W)\eta(Z)E(X) - \eta(Y)\eta(Z)E(W) \\ &+ g(W,Z)\eta(Y)\eta(\mu_2) - g(Y,Z)\eta(W)\eta(\mu_2) \\ &- E(Y)[(2n+1)g(W,Z) + (2n+1)\eta(Z)\eta(W)] = 0. \end{aligned} \tag{56}$$

Putting $Y = Z = e_i, 1 \le i \le 2n+1$ in (56) we obtain

$$rD(W) - 2nkD(W) + \mu D(hW) - D(QW) + 2(2n^2+n-1)E(W) + 2(1-2n)\eta(W)\eta(\mu_2) = 0. \tag{57}$$

Replacing W with ξ in (57) gives

$$r = k[n(5+2n^2)] + 2(2n-1)]. \tag{58}$$

This completes the proof. □

Corollary 5.5. *In a quasi generalized ϕ-recurrent (k,μ)-contact metric manifold, if $k = 0$, then the scalar curvature is constant.*

Proceeding like in Theorem 3.6, one can easily show that the manifold is an η-Einstein manifold. Hence, we get the following statement.

Theorem 5.6. *A quasi generalized ϕ-recurrent (k,μ)-contact metric manifold is an η-Einstein manifold i.e.,*

$$S(Y,W) = \alpha g(Y,W) + \beta\eta(Y)\eta(W),$$

where

$$\alpha = \frac{[2(nk+n-1)+\mu(n+2)][2(n-1)-n\mu]-[2(n-1)+\mu][\mu(1-k)+2(n-1)+2k]}{2nk+\mu(n+1)},$$

$$\beta = \frac{[2(nk+n-1)+\mu(n+2)][2(1-n)+n(2k+\mu)]-(k-1)[2(n-1)+\mu]^2}{2nk+\mu(n+1)}.$$

6. Example of a quasi generalized ϕ-recurrent (k, μ)-contact metric manifold

In this section we give an example of a quasi generalized ϕ-recurrent (k,μ)-contact metric manifold. We consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : x \neq 0, y \neq 0\}$, where $\{x, y, z\}$ are the standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be the global coordinate frame on M given by

$$E_1 = \frac{\partial}{\partial y},\ E_2 = 2xy\frac{\partial}{\partial z},\ E_3 = \frac{\partial}{\partial z}.$$

Hui [15] has shown that M is a 3-dimensional (k,μ)-contact metric manifold with $k = -\frac{1}{y}$ and $\mu = -\frac{1}{y}$. We will show that the manifold M is a 3-dimensional quasi generalized ϕ-recurrent (k,μ)-contact metric manifold. Any vector fields X, Y, Z on M can be expressed as

$$\begin{aligned} X &= a_1E_1 + b_1E_2 + c_1E_3, \\ Y &= a_2E_1 + b_2E_2 + c_2E_3, \\ Z &= a_3E_1 + b_3E_2 + c_3E_3, \end{aligned}$$

where $a_i, b_i, c_i \in \mathbb{R}^+$ (set of positive numbers). Then the Riemannian curvature R becomes

$$R(X, Y)Z = v_1E_1 + v_2E_2, \tag{59}$$

where $v_1 = -\frac{2b_3}{y^2}(a_1b_2 - a_2b_1)$ and $v_2 = \frac{2a_3}{y^2}(a_1b_2 - a_2b_1)$.
Also

$$\begin{aligned} F(X, Y)Z = {} & (b_3u_1 + 2c_3u_2)E_1 + (2c_3u_3 - a_3u_1)E_2 \\ & - 2(a_3u_2 - b_3u_3)E_3, \end{aligned} \tag{60}$$

where $u_1 = (a_1b_2 - b_1a_2)$, $u_2 = (a_1c_2 - a_2c_1)$, $u_3 = (b_1c_2 - b_2c_1)$.

From (59) we obtained

$$(\nabla_{E_1}R)(X, Y)Z = \frac{4}{y^3}(a_1b_2 - a_2b_1)(b_3E_1 - a_3E_2), \tag{61}$$

$$(\nabla_{E_2}R)(X, Y)Z = 0, \tag{62}$$

$$(\nabla_{E_3}R)(X, Y)Z = 0. \tag{63}$$

Making use of (61), (62) and (63) we get the following

$$\phi^2((\nabla_{E_i}R)(X, Y)Z) = p_iE_1 + q_iE_2,\ i = 1, 2, 3, \tag{64}$$

where

$$\begin{aligned} & p_1 = -\frac{4b_3}{y^3}(a_1b_2 - a_2b_1),\ q_1 = \frac{4a_3}{y^3}(a_1b_2 - a_2b_1), \\ & p_2 = 0,\ q_2 = 0,\ p_3 = 0,\ q_3 = 0. \end{aligned}$$

Let us define 1-forms A and B by

$$
\begin{aligned}
&A(E_1)=\frac{a_3p_1(2c_3u_2-b_3u_1)-q_1b_3(b_3u_1+2c_3u_2)}{v_1a_3(2c_3u_2-b_3u_1)-b_3v_3u_2(a_3+2c_3)},\\
&B(E_1)=\frac{b_3(q_1v_1-p_1v_2)}{v_1a_3(2c_3u_2-b_3u_1)-b_3v_3u_2(a_3+2c_3)},\\
&A(E_2)=0,\ B(E_2)=0,\\
&A(E_3)=0,\ B(E_3)=0,
\end{aligned}
\tag{65}
$$

where $a_3p_1(2c_3u_2-b_3u_1)-q_1b_3(b_3u_1+2c_3u_2)\neq 0$, $b_3(q_1v_1-p_1v_2)\neq 0$ and $v_1a_3(2c_3u_2-b_3u_1)-b_3v_3u_2(a_3+2c_3)\neq 0$.

Using (61), (64) and (65) one can easily show that

$$
\phi^2((\nabla_{E_i}R)(X,Y)Z)=A(E_i)R(X,Y)Z+B(E_i)F(X,Y)Z,\ i=1,2,3. \tag{66}
$$

Hence, the manifold under consideration is a 3-dimensional quasi generalized ϕ-recurrent (k,μ)-contact metric manifold. Thus we can state the following.

Theorem 6.1. *There exists a 3-dimensional quasi generalized ϕ-recurrent (k,μ)-contact metric manifold which is neither ϕ-symmetric nor ϕ-recurrent.*

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References

[1] K. K. Baishya, S. Eyasmin, and A. A. Shaikh, *On ϕ-recurrent generalized (k,μ)-contact metric manifolds*, Lobachevskii J. Math. **27** (2007), 3–13.

[2] K. K. Baishya and P. Roy Chowdhury, *On generalized quasi-conformal $N(k,\mu)$-manifolds*, Commun. Korean Math. Soc. **31** (2016), no. 1, 163–176. https://doi.org/10.4134/CKMS.2016.31.1.163

[3] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics, Vol. 509, Springer-Verlag, Berlin, 1976.

[4] D. E. Blair, T. Koufogiorgos, and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math. **91** (1995), no. 1-3, 189–214. https://doi.org/10.1007/BF02761646

[5] E. Boeckx, *A full classification of contact metric (k,μ)-spaces*, Illinois J. Math. **44** (2000), no. 1, 212–219. http://projecteuclid.org/euclid.ijm/1255984960

[6] E. Cartan, *Sur une classe remarquable d'espaces de Riemann*, Bull. Soc. Math. France **54** (1926), 214–264.

[7] U. C. De, N. Guha, and D. Kamilya, *On generalized Ricci-recurrent manifolds*, Tensor (N.S.) **56** (1995), no. 3, 312–317.

[8] U. C. De, J.-B. Jun, and S. Samui, *Certain semisymmetry properties of (κ,μ)-contact metric manifolds*, Bull. Korean Math. Soc. **53** (2016), no. 4, 1237–1247. https://doi.org/10.4134/BKMS.b150638

[9] U. C. De, A. A. Shaikh, and S. Biswas, *On Φ-recurrent Sasakian manifolds*, Novi Sad J. Math. **33** (2003), no. 2, 43–48.

[10] M. N. Devaraja, H. A. Kumara, and V. Venkatesha, *Riemann soliton within the framework of contact geometry*, Quaestions Math. (2019). https://doi.org/10.2989/16073606.2020.1732495

[11] R. S. D. Dubey, *Generalized recurrent spaces*, Indian J. Pure Appl. Math. **10** (1979), no. 12, 1508–1513.

[12] A. Ghosh, *Ricci almost solitons and contact geometry*, Adv. Geome., 2019.

[13] ———, *m-quasi-Einstein metric and contact geometry*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113** (2019), no. 3, 2587–2600. https://doi.org/10.1007/s13398-019-00642-3

[14] S. Ghosh and U. C. De, *On ϕ-Ricci symmetric (k, μ)-contact metric manifolds*, Acta Math. Univ. Comenian. (N.S.) **86** (2017), no. 2, 205–213. https://doi.org/10.1111/1365-2656.12610

[15] S. K. Hui, *On generalized ϕ-recurrent generalized (k, μ)-contact metric manifolds*, arXiv: 1707.00830v1[math.DG], 2017.

[16] J.-B. Jun, A. Yildiz, and U. C. De, *On ϕ-recurrent (k, μ)-contact metric manifolds*, Bull. Korean Math. Soc. **45** (2008), no. 4, 689–700. https://doi.org/10.4134/BKMS.2008.45.4.689

[17] J. H. Kim, J. H. Park, and K. Sekigawa, *A generalization of contact metric manifolds*, Balkan J. Geom. Appl. **19** (2014), no. 2, 94–105.

[18] H. A. Kumara and V. Venkatesha, *Gradient Einstein-type contact metric manifolds*, Commun. Korean Math. Soc. **35** (2020), no. 2, 639–651. https://doi.org/10.4134/CKMS.c190247

[19] E. M. Patterson, *Some theorems on Ricci-recurrent spaces*, J. London Math. Soc. **27** (1952), 287–295. https://doi.org/10.1112/jlms/s1-27.3.287

[20] S. Sasaki, *On almost contact manifolds*, in Proc. U.S.-Japan Seminar in Differential Geometry (Kyoto, 1965), 128–136, Nippon Hyoronsha, Tokyo.

[21] A. A. Shaikh and H. Ahmed, *On generalized ϕ-recurrent Sasakian manifolds*, Appl. Math. **2**(2011), 1317–1322.

[22] A. A. Shaikh and A. Patra, *On a generalized class of recurrent manifolds*, Arch. Math. (Brno) **46** (2010), no. 1, 71–78.

[23] A. A. Shaikh and I. Roy, *On quasi-generalized recurrent manifolds*, Math. Pannon. **21** (2010), no. 2, 251–263.

[24] T. Takahashi, *Sasakian ϕ-symmetric spaces*, Tohoku Math. J. (2) **29** (1977), no. 1, 91–113. https://doi.org/10.2748/tmj/1178240699

[25] V. Venkatesha, H. A. Kumara, and D. M. Naik, *On a class of generalized φ-recurrent Sasakian manifold*, J. Egyptian Math. Soc. **27** (2019), no. 1, Paper No. 19, 14 pp. https://doi.org/10.1186/s42787-019-0022-0

[26] V. Venkatesha and D. M. Naik, *Yamabe solitons on 3-dimensional contact metric manifolds with $Q_\varphi = \varphi Q$*, Int. J. Geom. Methods Mod. Phys. **16** (2019), no. 3, 1950039, 9 pp. https://doi.org/10.1142/S0219887819500397

[27] A. G. Walker, *On Ruse's spaces of recurrent curvature*, Proc. London Math. Soc. (2) **52** (1950), 36–64. https://doi.org/10.1112/plms/s2-52.1.36

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ON THE GEOMETRIC STRUCTURES OF GENERALIZED (k, μ)-SPACE FORMS

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Abstract. In this paper, the geometric structures of generalized (k, μ)-space forms and their quasi-umbilical hypersurface are analyzed. First ξ-Q and conformally flat generalized (k, μ)-space form are investigated and shown that a conformally flat generalized (k, μ)-space form is Sasakian. Next, we prove that a generalized (k, μ)-space form satisfying Ricci pseudosymmetry and Q-Ricci pseudosymmetry conditions is η-Einstein. We obtain the condition under which a quasi-umbilical hypersurface of a generalized (k, μ)-space form is a generalized quasi Einstein hypersurface. Also ξ-sectional curvature of a quasi-umbilical hypersurface of generalized (k, μ)-space form is obtained. Finally, the results obtained are verified by constructing an example of 3-dimensional generalized (k, μ)-space form.
Keywords:(k, μ)-space form, Q curvature, Hypersurface, Sasakian, η-Einstein.

1. Introduction

The curvature tensor R of the Riemannian manifold mostly determines the nature of the manifold and the sectional curvature of the manifold completely determines the curvature tensor R. A Riemannian manifold having a constant sectional curvature c is known as real space-form. The sectional curvature $K(X, \phi X)$ of a plane section spanned by a unit vector X orthogonal to ξ is called a ϕ-sectional curvature. If the ϕ-sectional curvature of a Sasakian manifold is constant, then it is called Sasakian space form. Alegre et al. [2] introduced the notion of generalized Sasakian

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space forms and gave many examples of it. Throughout the years, many geometers [3, 4, 13, 15, 16, 17] focused on generalized Sasakian space forms under different geometric conditions.

Blair et al. [5] introduced the notion of (k, μ)-contact metric manifolds. Following this, Koufogiorgos [23] introduced and studied (k, μ) space forms. The (k, μ) space forms are studied by [1, 14, 23, 30]. Carriazo et al. [8] introduced generalized (k, μ) space form which generalizes the notion of (k, μ) space forms. An almost contact metric manifold $(M^{2n+1}, \phi, \xi, g, \eta)$ is said to be a generalized (k, μ) space form if there exists differentiable functions $f_1, f_2, f_3, f_4, f_5, f_6$ on the manifold whose curvature tensor R is given by

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_5R_5 + f_6R_6, \tag{1.1}$$

where $R_1, R_2, R_3, R_4, R_5, R_6$ are the following tensors:

$$\begin{aligned}
R_1(X,Y)Z &= g(Y,Z)X - g(X,Z)Y, \\
R_2(X,Y)Z &= g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z, \\
R_3(X,Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi, \\
R_4(X,Y)Z &= g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y, \\
R_5(X,Y)Z &= g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX, \\
R_6(X,Y)Z &= \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi,
\end{aligned}$$

for any $X, Y, Z \in \chi(M)$. Here, h is a symmetric tensor given by $2h = \mathcal{L}_\xi \phi$, where $\mathcal{L}$ is Lie derivative. In particular, for $f_4 = f_5 = f_6 = 0$ it reduces to the generalized Sasakian space form [2]. It is obvious that (k, μ) space form is an example of generalized (k, μ) space form when

$$f_1 = \frac{c+3}{4}, f_2 = \frac{c-1}{4}, f_3 = \frac{c+3}{4} - k, f_4 = 1, f_5 = \frac{1}{2}, f_6 = 1 - \mu$$

are constants. In [8], the author studied generalized (k, μ) space forms in contact metric and Trans-Sasakian manifolds. Carriazo and Molina [9] studied D_α-homothetic deformations of generalized (k, μ)-space forms and found that deformed spaces are again generalized (k, μ)-space forms in dimension 3, but not in general. In recent years, many geometers studied generalized (k, μ)-space forms under several conditions [21, 28, 22, 20, 27, 29].

In [26], Mantica and Suh introduced and studied Q curvature tensor. In a $(2n+1)$-dimensional Riemannian manifold (M, g), the Q curvature tensor is given by

$$Q(X,Y)Z = R(X,Y)Z - \frac{v}{2n}\big[g(Y,Z)X - g(X,Z)Y\big], \tag{1.2}$$

for any $X, Y, Z \in \chi(M)$ and v is an arbitrary scalar function on M. If $v = \frac{r}{2n+1}$, then Q curvature tensor reduces to concircular curvature tensor [32]. In [13], De and Majhi studied Q curvature tensor in a generalized Sasakian space form.

One of the most important curvature tensors for analyzing the intrinsic properties of Riemannian manifold is the conformal curvature tensor introduced by Yano and Kon [33]. This curvature is invariant under conformal transformation. The conformal curvature C of type (1,3) on a $(2n+1)$-dimensional Riemannian manifold $(M, g), n > 1$, is defined by

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}\big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)PX - g(X,Z)PY\big] + \frac{r}{2n(2n-1)}\big[g(Y,Z)X - g(X,Z)Y\big], \tag{1.3}$$

where R, S, P, r denote the Riemannian curvature tensor, the Ricci tensor, Ricci-operator and the scalar curvature of the manifold respectively. Kim [25] studied conformally flat generalized Sasakian space forms. De and Majhi [15] studied ϕ-conformal semisymmetric generalized Sasakian space forms.

Cartan [10] first initiated and completely classified complete simply connected locally symmetric spaces. A Riemannian manifold is said to be locally symmetric if the curvature tensor satisfies $\nabla R = 0$. The notion of local symmetry is weakened by many authors throughout the years. One such notion is pseudosymmetric spaces introduced by Deszcz [19]. It should be noted that pseudosymmetric spaces introduced by Deszcz is different from those introduced by Chaki [11]. In [31], authors obtained the necessary and sufficient condition for a Chaki pseudosymmetric manifold to be Deszcz pseudosymmetric. De and Samui [14] studied Ricci pseudosymmetric (k, μ)-contact space forms and show that it is an η-Einstein manifold.

The authors in [14], studied quasi-umbilical hypersurface on (k, μ)-space forms. A hypersurface $(\widetilde{M}^{2n+1}, \tilde{g})$ of a Riemannian manifold M^{2n+1} is called quasi-umbilical [12] if its second fundamental tensor has the form

$$H_\rho(X,Y) = \alpha g(X,Y) + \beta\omega(X)\omega(Y), \tag{1.4}$$

where ω is the 1-form, α, β are scalars and the vector field corresponding to the 1-form ω is a unit vector field. Here, the second fundamental tensor H_ρ is defined by $H_\rho(X,Y) = \widetilde{g}(A_\rho, Y)$, where A is (1,1) tensor and ρ is the unit normal vector field and X, Y are tangent vector fields.

A Riemannian manifold is called a generalized quasi-Einstein manifold [18] if its Ricci tensor S satisfies

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + c\lambda(X)\lambda(Y),$$

where a, b and c are non-zero scalars and η, λ are 1-forms. If $c = 0$, then the manifold reduces to a quasi-Einstein manifold.

The paper is organized as follows: After preliminaries, ξ-Q and conformally flat generalized (k,μ)-space forms are investigated in section 3. Next in section 4, it is shown that Q-Ricci pseudosymmetric and Ricci pseudosymmetric generalized (k,μ)-space forms are η-Einstein under certain conditions. Moreover, conformal Ricci pseudosymmetric generalized (k,μ)-space forms are studied. In section 5, quasi-umbilical hypersurface of generalized (k,μ)-space form are investigated and shown that it is a generalized quasi Einstein hypersurface. Also ξ-sectional curvature of a quasi-umbilical hypersurface of generalized (k,μ)-space form is obtained. Finally, the obtained results are verified by using an example of a 3-dimensional generalized (k,μ)-space form.

2. Preliminaries

In this section, we highlight some of the formulae and statements which will be used later in our studies.

A $(2n+1)$-dimensional smooth manifold M is said to be a contact metric manifold if there exists a global 1-form η, known as the contact form, such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M and there exists a unit vector field ξ, called the Reeb vector field, corresponding to 1-form η such that $d\eta(\xi,\cdot)=0$, a $(1,1)$ tensor field ϕ and Riemannian metric g such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(X) = g(X,\xi), \quad d\eta(X,Y) = g(X,\phi Y), \tag{2.1}$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie-algebra of all vector fields on M. The metric g is called the associate metric and the structure (ϕ,ξ,η,g) is called contact metric structure. A Riemannian manifold M together with contact structure (ϕ,ξ,η,g) is called contact metric manifold. It follows from (2.1) that

$$\begin{aligned} \phi(\xi) = 0, \quad \eta\cdot\phi = 0, \quad g(X,\phi Y) = -g(\phi X, Y), \\ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \end{aligned} \tag{2.2}$$

for any $X, Y \in \chi(M)$. Further we define two self-adjoint operators h and l by $h = \frac{1}{2}(\mathcal{L}_\xi \phi)$ and $l = R(\cdot,\xi)\xi$ respectively, where R is the Riemannian curvature of M. These operators satisfy

$$h\xi = l\xi = 0, \quad h\phi + \phi h = 0, \quad Tr.h = Tr.h\phi = 0. \tag{2.3}$$

Here, "Tr." denotes trace. When unit vector ξ is Killing (i.e. $h=0$ or $Tr.l = 2n$) then contact metric manifold is called K-contact. A contact structure is said to be normal if the almost complex structure J on $M \times \mathbb{R}$ defined by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$, where t is the coordinate of $\mathbb{R}$ and f is a real function on $M \times \mathbb{R}$, is integrable. A normal contact metric manifold is called Sasakian. A Sasakian manifold is K-contact but the converse is true only in dimension 3. The (k,μ)-nullity distribution of a contact metric manifold $M(\phi,\xi,\eta,g)$ is a distribution

$$\begin{aligned} N(k,\mu) : p \to N_p(k,\mu) = \{Z \in \chi(M) : R(X,Y)Z = k\{g(Y,Z)X \\ -g(X,Z)Y\} + \mu\{g(Y,Z)hX - g(X,Z)hY\}\}, \end{aligned}$$

for any $X, Y, Z \in \chi(M)$ and real numbers k and μ. A contact metric manifold M with $\xi \in N(k,\mu)$ is called a (k,μ)-contact metric manifold.
In a generalized (k,μ)-space form (M^{2n+1}, g) the following relations hold [2]:

$$\begin{aligned} R(X,Y)\xi &= (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\} \\ &+ (f_4 - f_6)\{\eta(Y)hX - \eta(X)hY\}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} PX &= (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi \\ &+ ((2n-1)f_4 - f_6)hX, \end{aligned} \tag{2.5}$$

$$r = 2n\{(2n+1)f_1 + 3f_2 - 2f_3\}, \tag{2.6}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2n(f_1 - f_3)\eta(X)\eta(Y). \tag{2.7}$$

where, R, S, P, r are respectively the curvature tensor of type (1,3), the Ricci tensor, the Ricci operator i.e. $g(PX, Y) = S(X, Y)$, for any $X, Y \in \chi(M)$ and the scalar curvature of the manifold respectively.

3. Flatness of generalized (k,μ)-space form

De and Samui [14] studied conformally flat (k,μ) space form and De and Majhi [13] analyzed ξ-Q flatness of generalized Sasakian space form. Generalizing the results obtained, in this section we studied ξ-Q flat and conformally flat generalized (k,μ)-space form.

3.1. ξ-Q flat generalized (k,μ)-space form

Definition 3.1. A generalized (k,μ)-space form (M^{2n+1}, g), is said to be ξ-Q flat if $Q(X,Y)\xi = 0$, for any $X, Y \in \chi(M)$ on M.

We have, from (1.2)

$$Q(X,Y)\xi = R(X,Y)\xi - \frac{v}{2n}\big[\eta(Y)X - \eta(X)Y\big], \tag{3.1}$$

for any $X, Y \in \chi(M)$. Using (2.4) in (3.1) we get

$$\begin{aligned} Q(X,Y)\xi &= \big(f_1 - f_3 - \frac{v}{2n}\big)\big[\eta(Y)X - \eta(X)Y\big] \\ &+ (f_4 - f_6)[\eta(Y)hX - \eta(X)hY]. \end{aligned} \tag{3.2}$$

Suppose non-Sasakian generalized (k, μ)-space form is $\xi - Q$ flat. Then from (3.2) we get

$$(3.3)\quad \big(f_1 - f_3 - \frac{v}{2n}\big)\big[\eta(Y)X - \eta(X)Y\big] + (f_4 - f_6)[\eta(Y)hX - \eta(X)hY] = 0.$$

Taking $X = \phi X$ in (3.3), we obtain

$$\Big\{\big(f_1 - f_3 - \frac{v}{2n}\big)\phi X + (f_4 - f_6)h\phi X\Big\}\eta(Y) = 0. \tag{3.4}$$

Since $\eta(Y) \neq 0$ and taking inner product with U in (3.4) gives

$$\big(f_1 - f_3 - \frac{v}{2n}\big)g(\phi X, U) + (f_4 - f_6)g(\phi X, hU) = 0. \tag{3.5}$$

Since $g(\phi X, U) \neq 0$ and $g(\phi X, hU) \neq 0$, we see that $f_1 - f_3 = \frac{v}{2n}$ and $f_4 = f_6$. Conversely, taking $f_1 - f_3 = \frac{v}{2n}$ and $f_4 = f_6$, and putting these values in (3.2) gives $Q(X, Y)\xi = 0$ and hence M is $\xi - Q$ flat. Therefore, we can state the following:

Theorem 3.1. *A non-Sasakian generalized (k, μ)-space form (M^{2n+1}, g), is ξ-Q flat if and only if $f_1 - f_3 = \frac{v}{2n}$ and $f_4 = f_6$.*

In particular, if $v = \frac{r}{2n+1}$ then Q tensor reduces to concircular curvature tensor. Making use of (2.6) in the forgoing equation gives $v = \frac{2n\{(2n+1)f_1+3f_2-2f_3\}}{2n+1}$. In regard of Theorem 3.1, for ξ-concircularly flat we obtain $f_3 = \frac{3f_2}{1-2n}$ and hence we can state the following corollary:

Corollary 3.1. *A non-Sasakian generalized (k, μ)-space form (M^{2n+1}, g), is ξ-concircularly flat if and only if $f_3 = \frac{3f_2}{1-2n}$ and $f_4 = f_6$.*

We can easily see that Theorem 3.1 and Corollary 3.1 obtained by the geometers in [13], are particular cases of Theorem 3.1 and Corollary 3.1 respectively for $f_4 = f_5 = f_6 = 0$.

Substituting the values, $f_4 - f_6 = \mu$ and $f_1 - f_3 = k$ in Theorem 3.1, we obtained the following corollary:

Corollary 3.2. *A (k, μ)-space form (M^{2n+1}, g), is ξ-Q flat if and only if $k = \frac{v}{2n}$ and $\mu = 0$.*

3.2. Conformally flat generalized (k, μ)-space form

Definition 3.2. A generalized (k, μ)-space form $(M^{2n+1}, g), n > 1$, is said to be conformally flat if $C(X, Y)Z = 0$, for any $X, Y, Z \in \chi(M)$ on M.

Suppose generalized (k,μ)-space form is conformally flat. Then from (1.3), we get

$$
\begin{aligned}
&R(X,Y)Z-\frac{1}{2n-1}\{S(Y,Z)X-S(X,Z)Y+g(Y,Z)PX-g(X,Z)PY\}\\
&\qquad+\frac{r}{2n(2n-1)}\{g(Y,Z)X-g(X,Z)Y\}=0.
\end{aligned}
\tag{3.6}
$$

In consequence of taking $X=\xi$ in (3.6) and using (2.1), (2.4) and (2.5). Eq.(3.6) becomes

$$
\begin{aligned}
&(f_1-f_3)\{g(Y,Z)\xi-\eta(Z)Y\}+(f_4-f_6)\{g(hY,Z)\xi-\eta(Z)hY\}\\
&-\frac{1}{2n-1}\{S(Y,Z)\xi-2n(f_1-f_3)\eta(Z)Y+2n(f_1-f_3)g(Y,Z)\xi\\
&\qquad-\eta(Z)PY\}+\frac{r}{2n(2n-1)}\{g(Y,Z)\xi-\eta(Z)Y\}=0.
\end{aligned}
\tag{3.7}
$$

Putting $Z=\phi Z$ in (3.7) and making use of (2.4), (2.5) and (2.6) results in the following

$$
2(n+1)f_6 g(hY,\phi Z)=0. \tag{3.8}
$$

This shows that either $f_6=0$ or $\phi h=0$. In the second case, from (2.1) we have $h=0$. Therefore, we can state the following:

Theorem 3.2. *A generalized (k,μ)-space form $(M^{2n+1},g), n>1$, is conformally flat, then either $f_6=0$ or M is Sasakian.*

Corollary 3.3. *A (k,μ)-space form $(M^{2n+1},g), n>1$, is conformally flat, then $\mu=1$ or M is Sasakian.*

4. Pseudosymmetric generalized (k,μ)-space form

In this section certain pseudo symmetry such as Ricci pseudo symmetry, Q-Ricci pseudo symmetry and conformal Ricci pseudo symmetry in the context of generalized (k,μ)-space form are studied. First, we review an important definition

Definition 4.1. [19, 31] A Riemannian manifold $(M,g), n\geq 1$, admitting a $(0,k)$-tensor field T is said to be T-pseudosymmetric if $R\cdot T$ and $D(g,T)$ are linearly dependent, i.e., $R\cdot T=L_T D(g,T)$ holds on the set $U_T=\{x\in M: D(g,T)\neq 0$ at $x\}$, where L_T is some function on U_T.

In particular, if $R\cdot R=L_R D(g,R)$ and $R\cdot S=L_S D(g,S)$ then the manifold is called pseudosymmetric and Ricci pseudosymmetric respectively. Moreover, if $L_R=0$ (resp., $L_S=0$) then pseudosymmetric (resp., Ricci pseudosymmetric) reduces to semisymmetric (resp., Ricci semisymmetric) introduced by Cartan in 1946.

4.1. Ricci pseudosymmetric generalized (k, μ)-space form

Definition 4.2. A generalized (k, μ)-space form (M^{2n+1}, g), is said to be Ricci pseudosymmetric if its Ricci curvature satisfies the following relation,

$$R \cdot S = f_{S_2} D(g, S),$$

holds on the set $U_{S_2} = \{x \in M : D(g, S) \neq 0 \text{ at } x\}$, where f_{S_2} is some function on U_{S_2}.

Suppose a generalized (k, μ)-space form (M^{2n+1}, g), is Ricci pseudosymmetric i.e.,

$$R \cdot S = f_{S_2} D(g, S),$$

which can be written as

$$\begin{aligned} S(R(X,Y)U,V) + S(U,R(X,Y)V) = -f_s\big[S(Y,V)g(X,U) \\ -S(X,V)g(Y,U) + S(U,Y)g(X,V) - S(U,X)g(Y,V)\big] \end{aligned} \tag{4.1}$$

Taking $X = U = \xi$ in (4.1) and using (2.4), (2.5) and (2.7), we get

$$\begin{aligned} \big(f_3 - f_1 + f_{S_2}\big)S(Y,V) + \big[2n(f_1 - f_3)(f_1 - f_3 - f_{S_2}) - (k-1)(f_4 \\ -f_6)((2n-1)f_4 - f_6)\big]g(Y,V) - (k-1)(f_4 - f_6)\big((2n-1)f_4 \\ -f_6\big)\eta(Y)\eta(V) + (f_4 - f_6)\big((1-2n)f_3 - 3f_2\big)g(hY,V) = 0. \end{aligned} \tag{4.2}$$

Considering $f_{S_2} \neq f_1 - f_3$ and further taking $(1-2n)f_3 - 3f_2 = 0$ in (4.2), the manifold is η-Einstein. Hence we can state the following:

Theorem 4.1. *A Ricci pseudosymmetric generalized (k, μ)-space form (M^{2n+1}, g), with $f_{S_2} \neq f_1 - f_3$, is η-Einstein manifold if $f_3 = \frac{3f_2}{1-2n}$.*

If $f_{S_2} = 0$, then Ricci pseudosymmetric generalized (k, μ)-space form reduces to Ricci semisymmetric generalized (k, μ)-space form. In view of Theorem (4.1) we obtain the following:

Corollary 4.1. *A Ricci semisymmetric generalized (k, μ)-space form (M^{2n+1}, g), with $f_1 - f_3 \neq 0$ is η-Einstein manifold if $f_3 = \frac{3f_2}{1-2n}$.*

4.2. Q-Ricci pseudosymmetric generalized (k, μ)-space form

Definition 4.3. A generalized (k, μ)-space form (M^{2n+1}, g), is said to be Q-Ricci pseudosymmetric if

$$Q \cdot S = f_{S_3} D(g, S),$$

holds on the set $U_{S_3} = \{x \in M : D(g, S) \neq 0 \text{ at } x\}$, where f_{S_3} is any function on U_{S_3}.

Proceeding similarly as in Theorem 4.1, one can easily obtain the following relation:

Theorem 4.2. *A Q-Ricci pseudosymmetric generalized (k,μ)-space form (M^{2n+1}, g), with $f_{S_3} \neq f_3 - f_1 - \frac{v}{2n}$ is η-Einstein manifold if $f_3 = \frac{3f_2}{1-2n}$.*

Taking $f_{S_3} = 0$ in Theorem 4.2, we easily obtain the following:

Corollary 4.2. *A Q-Ricci semisymmetric generalized (k,μ)-space form (M^{2n+1}, g), with $f_3 - f_1 \neq \frac{v}{2n}$ is η-Einstein manifold if $f_3 = \frac{3f_2}{1-2n}$.*

4.3. Conformal Ricci pseudosymmetric generalized (k,μ)-space form

Definition 4.4. A generalized (k,μ)-space form $(M^{2n+1}, g), n > 1$, is said to be conformal Ricci pseudosymmetric if

$$C \cdot S = f_{S_4} D(g, S),$$

holds on the set $U_{S_4} = \{x \in M : D(g,S) \neq 0 \text{ at } x\}$, where f_{S_4} is any function on U_{S_4}.

Suppose a generalized (k,μ)-space form is conformal Ricci pseudosymmetric. Then, we have

$$\begin{aligned} S(C(X,Y)U,V) + S(U,C(X,Y)V) = -f_{S_4}\big[S(Y,V)g(X,U) \\ -S(X,V)g(Y,U) + S(U,Y)g(X,V) - S(U,X)g(Y,V)\big]. \end{aligned} \tag{4.3}$$

Taking $X = U = \xi$ and $f_4 = f_6$ in (4.3) and making use of (1.3),(2.1) and (2.5), we obtain

$$\begin{aligned} S^2(Y,V) = \big(4nf_1 + 3f_2 - (2n+1)f_3 + 2n(2n-1)f_{S_4}\big)S(Y,V) \\ -(2n-1)f_{S_4}\eta(Y)\eta(V) - \big(2nf_1 + 3f_2 - f_3\big)g(Y,V). \end{aligned} \tag{4.4}$$

Thus, we can state the following:

Theorem 4.3. *If a generalized (k,μ)-space form $(M^{2n+1}, g), n > 1$, is conformal Ricci pseudosymmetric with $f_4 = f_6$, then the relation(4.4) holds.*

5. Quasi-umbilical hypersurface of generalized (k,μ)-space form

Let us consider a quasi-umbilical hypersurface $\widetilde{M}$ of a generalized (k,μ)-space form. From Gauss [12], for any vector fields X, Y, Z, W tangent to the hypersurface we have

$$\begin{aligned} R(X,Y,Z,W) &= \widetilde{R}(X,Y,Z,W) - g(H(X,W),H(X,Z)) \\ &+ g(H(X,Z),H(Y,W)), \end{aligned} \tag{5.1}$$

where, $R(X,Y,Z,W) = g(R(X,Y)Z,W)$ and $\widetilde{R}(X,Y,Z,W) = g(\widetilde{R}(X,Y)Z,W)$. Here, H is the second fundamental tensor of $\widetilde{M}$ given by

$$H(X,Y) = \alpha g(X,Y)\rho + \beta\omega(X)\omega(Y)\rho, \tag{5.2}$$

where, ρ is the only unit normal vector field. Here, ω is the 1-form, the vector field corresponding to the 1-form ω is a unit vector field and α, β are scalars.
Using (5.2) in (5.1), we obtain the following result

$$\begin{aligned} f_1\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big] + f_2\big[g(X,\phi Z)g(\phi Y,W) \\ -g(Y,\phi Z)g(\phi X,W) + 2g(X,\phi Y)g(\phi Z,W)\big] + f_3\big[\eta(X)\eta(Z)g(Y,W) \\ -\eta(Y)\eta(Z)g(X,W) + g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W)\big] \\ +f_4\big[g(Y,Z)g(hX,W) - g(Y,Z)g(hY,W) + g(hY,Z)g(X,W) \\ -g(hX,Z)g(Y,W)\big] + f_5\big[g(hY,Z)g(hX,W) - g(hX,Z)g(hY,W) \\ +g(\phi hX,Z)g(\phi hY,W) - g(\phi hY,Z)g(\phi hX,W)\big] + f_6\big[\eta(X)\eta(Z)g(hY,W) \\ -\eta(Y)\eta(Z)g(hX,W) + g(hX,Z)\eta(Y)\eta(W) - g(hY,Z)\eta(X)\eta(W)\big] \\ = \widetilde{R}(X,Y,Z,W) - \alpha^2 g(X,W)g(Y,Z) - \alpha\beta g(X,W)\omega(Y)\omega(Z) \\ -\alpha\beta g(Y,Z)\omega(X)\omega(W) + \alpha^2 g(Y,W)g(X,Z) + \alpha\beta g(Y,W)\omega(X)\omega(Z) \\ +\alpha\beta g(X,Z)\omega(Y)\omega(W). \end{aligned} \tag{5.3}$$

Contracting over X and W in (5.3), we obtain

$$\begin{aligned} \widetilde{S}(Y,Z) = \big(2nf_1 + 3f_2 - f_3 + 2n\alpha^2 + \alpha\beta\big)g(Y,Z) \\ -\big(3f_2 + (2n+1)f_3\big)\eta(Y)\eta(Z) + \big((2n-1)f_4 - f_6\big)g(hY,Z) \\ +\alpha\beta(2n-1)\omega(Y)\omega(Z). \end{aligned} \tag{5.4}$$

Hence, we can state the following:

Theorem 5.1. *A quasi-umbilical hypersurface of a generalized (k,μ)-space form is a generalized quasi Einstein hypersurface, provided $f_4 = \frac{f_6}{2n-1}$*

In particular, for a (k,μ)-space form, the above Theorem 5.1 reduces to the following:

Theorem 5.2. [14] *A quasi-umbilical hypersurface of a (k,μ)-contact space form is a generalized quasi-Einstein hypersurface, provided $\mu = 2 - 2n$.*

Corollary 5.1. *A quasi-umbilical hypersurface of a generalized Sasakian space form is a generalized quasi-Einstein hypersurface.*

For any vector fields X, Y, the tensor field $K(X,Y) = \widetilde{R}(X,Y,Y,X)$ is called the sectional curvature of $\widetilde{M}$ given by the sectional plane $\{X,Y\}$. The sectional curvature $K(X,\xi)$ of a sectional plane spanned by ξ and vector field X orthogonal to ξ is called the ξ-sectional curvature of $\widetilde{M}$.

Theorem 5.3. *A ξ-sectional curvature of a quasi-umbilical hypersurface of generalized (k,μ)-space form is given by*

$$K(X,\xi) = \big(f_1 - f_3 + \alpha^2\big)g(\phi X,\phi X) + (f_4 - f_6)g(hX,X) \\ +\alpha\beta\big[(\omega(\xi))^2 + (\omega(X))^2\big] - 2\alpha\beta\eta(X)\omega(X)\omega(\xi).$$

Proof. Taking $W = X$ and $Z = Y$ in (5.3) results in following

$$\begin{aligned} & f_1\big[g(Y,Y)g(X,X) - g(X,Y)g(Y,X)\big] + f_2\big[g(X,\phi Y)g(\phi Y,X) \\ & -g(Y,\phi Y)g(\phi X,X) + 2g(X,\phi Y)g(\phi Y,X)\big] + f_3\big[\eta(X)\eta(Y)g(X,Y) \\ & -\eta(Y)\eta(Y)g(X,X) - g(X,Y)\eta(X)\eta(Y) - g(Y,Y)\eta(X)\eta(X)\big] \\ & +f_4\big[g(Y,Y)g(hX,X) - g(X,Y)g(hY,X) + g(hY,Y)g(X,X) \\ & -g(hX,Y)g(Y,X)\big] + f_5\big[g(hY,Y)g(hX,X) - g(hX,Y)g(hY,X) \\ & +g(\phi hX,Y)g(\phi hY,X) - g(\phi hY,Y)g(\phi hX,X)\big] + f_6\big[\eta(x)\eta(Y)g(hY,X) \\ & -\eta(Y)\eta(Y)g(hX,X) + g(hX,Y)\eta(Y)\eta(X) - g(hY,Y)\eta(X)\eta(X)\big] \\ & = K(X,Y) - \alpha^2 g(X,X)g(Y,Y) - \alpha\beta g(X,X)\omega(Y)\omega(Y) \\ & -\alpha\beta g(Y,Y)\omega(X)\omega(X) + \alpha^2 g(X,Y)g(X,Y) + \alpha\beta g(X,Y)\omega(X)\omega(Y) \\ & +\alpha\beta g(X,Y)\omega(Y)\omega(X). \end{aligned} \tag{5.5}$$

Putting $Y = \xi$ in (5.5) gives

$$K(X,\xi) = \big(f_1 - f_3 + \alpha^2\big)g(\phi X,\phi X) + (f_4 - f_6)g(hX,X) \\ +\alpha\beta\big[(\omega(\xi))^2 + (\omega(X))^2\big] - 2\alpha\beta\eta(X)\omega(X)\omega(\xi).$$

This completes the proof. □

6. Examples of generalized (k,μ)-space forms

Now we will show the validity of obtained result by considering an example of a generalized (k,μ)-space form of dimension 3. Koufogiorgos and Tsichlias [24] constructed an example of generalized (k,μ)-space of dimension 3 which was later shown by Carriazo et al. [8] to be a contact metric generalized (k,μ)-space form $M^3(f_1,0,f_3,f_4,0,0)$ with non-constant f_1, f_3, f_4.

Example 6.1: Let M^3 be the manifold $M = \{(x_1,x_2,x_3) \in \mathbb{R}^3 | x_3 \neq 0\}$ where (x_1,x_2,x_3) are standard coordinates on $\mathbb{R}^3$. Consider the vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = -2x_2x_3\frac{\partial}{\partial x_1} + \frac{2x_1}{x_3^2}\frac{\partial}{\partial x_2} - \frac{1}{x_3^2}\frac{\partial}{\partial x_3}, \quad e_3 = \frac{1}{x_3}\frac{\partial}{\partial x_2},$$

are linearly independent at each point of M and are related by

$$[e_1,e_2] = \frac{2}{x_3^2}e_3, \quad [e_2,e_3] = 2e_1 + \frac{1}{x_3^3}e_3, \quad [e_3,e_1] = 0.$$

Let g be the Riemannian metric defined by $g(e_i, e_j) = \delta_{ij}, i, j = 1, 2, 3$ and η be the 1-form defined by $\eta(X) = g(X, e_1)$ for any X on M. Also, let ϕ be the $(1, 1)$-tensor field defined by $\phi e_1 = 0, \quad \phi e_2 = e_3 \quad \phi e_3 = -e_2$. Therefore, (ϕ, e_1, η, g) defines a contact metric structure on M. Put $\lambda = \frac{1}{x_3^2}, k = 1 - \frac{1}{x_3^4}$ and $\mu = 2(1 - \frac{1}{x_3^2})$, then symmetric tensor h satisfies $he_1 = 0, he_2 = \lambda e_2, he_3 = -\lambda e_3$. The non-vanishing components of the Riemannian curvature are as follows:

$$\begin{aligned}
R(e_1, e_2)e_1 &= -(k + \lambda\mu)e_2, \quad R(e_1, e_2)e_2 = (k + \lambda\mu)e_1, \\
R(e_1, e_3)e_1 &= (-k + \lambda\mu)e_3, \quad R(e - 1, e_3)e_3 = (k - \lambda\mu)e_1, \\
R(e_2, e_3)e_2 &= (k + \mu - 2\lambda^3)e_3, \quad R(e_2, e_3)e_3 = -(k + \mu - 2\lambda^3)e_2.
\end{aligned}$$

Therefore, M is a generalized (k, μ)-space with k, μ not constant. As a contact metric generalized (k, μ)-space is a generalized (k, μ)-space form with $k = f_1 - f_3$ and $\mu = f_4 - f_6$ (Theorem 4.1, [8]), the manifold under consideration is a generalized (k, μ)-space form $M^3(f_1, 0, f_3, f_4, 0, 0)$ where

$$\begin{aligned}
f_1 &= -3 + \frac{2}{x_3^2} + \frac{1}{x_3^4} + \frac{2}{x_3^6}, \\
f_3 &= -4 + \frac{2}{x_3^2} + \frac{2}{x_3^4} + \frac{2}{x_3^6} \\
f_4 &= 2(1 - \frac{1}{x_3^2}).
\end{aligned}$$

Next we obtain the non-vanishing components of Q-curvature tensor for arbitrary function v as follows:

$$\begin{aligned}
Q(e_1, e_2)e_1 = -(k + \lambda\mu - \frac{v}{2})e_2, \quad Q(e_1, e_2)e_2 = (k + \lambda\mu - \frac{v}{2})e_1, \\
Q(e_1, e_3)e_1 = (-k + \lambda\mu + \frac{v}{2})e_3, \quad Q(e_1, e_3)e_3 = (k - \lambda\mu - \frac{v}{2})e_1, \\
Q(e_2, e_3)e_2 = (k + \mu - 2\lambda^3 + \frac{v}{2})e_3, \quad Q(e_2, e_3)e_3 = -(k + \mu - 2\lambda^3 + \frac{v}{2})e_2.
\end{aligned}$$

From the above equations we see that $Q(X, Y)e_1 = 0$ for all X, Y on M if and only if $v = 2(1 - \frac{1}{x_3^4})$ and $x_3^2 = 1$. Hence, Theorem 3.1 is verified.

Example 6.2: In [2], it was shown that the warped product $\mathbb{R} \times_f \mathbb{C}^m$ with

$$f_1 = -\frac{(f')^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f},$$

is a generalized Sasakian space form. Since every generalized Sasakian space form is a particular case of generalized (k, μ)-space form, $\mathbb{R} \times_f \mathbb{C}^m$ with f_1, f_2, f_3 define as above and $f_4 = f_5 = f_6 = 0$ is a generalized (k, μ)-space form.

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REFERENCES

1. A. Akbar and A. Sarkar: *Some curvature properties of (k, μ)-contact space forms.* Malaya J. Mat. **3**(1) (2015), 45–50.
2. P. Alegre, D. E. Blair and A. Carriazo: *Generalized Sasakian space forms.* Israel J. Math. **141** (2004), 157–183.
3. P. Alegre and A. Carriazo: *Submanifolds of generalized Sasakian space forms.* Taiwanese J. Math. **13** (2009), 923–911.
4. P. Alegre and A. Carriazo: *Structures on generalized Sasakian space forms.* Diff. Geom. Appl. **26** (2008), 656–666.
5. D. E. Blair, T. Koufogiorgos and B. J. Papantoniou: *Contact Metric Manifolds satisfying a nullity condition.* Israel J. Math. **19**(1-3) (1995), 189–214.
6. D. E. Blair: *Contact manifolds in Riemannian geometry.* Lecture note in Mathematics, 509. Springer-Verlag, Berlin, New-York, 1976.
7. D. E. Blair: *Riemannian geometry of contact and sympletic manifolds.* Birkhauser Boston, 2002.
8. A. Carriazo, V. M. Molina and M. M. Tripathi: *Generalized (k, μ) space forms.* Mediterr. J. Math. **10** (2013), 175–196.
9. A. Carriazo and V. M. Molina: *Generalized (k, μ)-space forms and D_α-homothetic deformations.* Balkan J. Geo. Appl. **16**(1) (2011), 37–47.
10. E. Cartan: *Sur une classe remarquable d'espaces de Riemann.* Bull. Soc. Math. France **54** (1926), 214–264.
11. M. C. Chaki: *On pseudosymmetric manifolds.* An. Stiint. Univ. AL.I. Cuza din Iasi Sect. I-a Math. N. S. **33**(1) (1987), 53–58.
12. B. Y. Chen: *Geometry of submanifolds.* Marcel Dekker, Inc. New York, 1973.
13. U. C. De and P. Majhi: *On the Q curvature tensor of a generalized sasakian-space-form.* Krag. J. Math. **13**(3) (2019), 333–319.
14. U. C. De and S. Samui: *The structure of some classes of (k, μ)-contact space forms.* Diff. Geom.-Dynamical System, **18** (2016), 1–13.
15. U. C. De and P. Majhi: *ϕ-semisymmetric generalized Sasakian space-form.* Arab J. Math. Sci. **21** (2015), 170–178.
16. U. C. De and A. Sarkar: *Some results on generalized Sasakian space forms.* Thai J. Math. **8** (2010), 1–10.
17. U. C. De and A. Yildiz: *Certain curvature conditions on generalized Sasakian space forms.* Quaest. Math. **38**(1) (2015), 195–504.

18. U. C. De and G. C. Ghosh: *On generalized quasi Einstein manifolds.* Kyungpook Math. J. **44** (2001), 607–615.
19. R. Deszcz: *On pseudosymmetric spaces.* Bull. Belg. Math. Soc., Ser. A, **44** (1992), 1–34.
20. M. Faghfouri and N. Ghaffarzadeh: *On doubly warped product submanifolds of generalized (k, μ)-space forms.* Afrika Mat. **26**(7-8) (2015), 1443–1455.
21. S. K. Hui, S. Uddin and P. Mandal: *Submanifolds of generalized (k, μ)-space-forms.* Per. Math. Hung. **77** (2018), 329–339.
22. D. L. Kumar and H. G. Nagaraja: *Second Order Parallel Tensor and Ricci solitons on Generalized (k, μ)-Space forms.* Math. Adv. Pure Appl. Sci. **2**(1) (2019), 1–7.
23. T. Koufogiorgos: *Contact Riemannian manifolds with constant ϕ-sectional curvature.* Tokyo J. Math. **20** (1997), 13–22.
24. T. Koufogiorgos and C. Tsichlias: *On the existence of a new class of contact metric manifolds.* Canad. Math. Bull. **20**(1) (2000), 400–447.
25. U. K. Kim: *Conformally flat generalized Sasakian space-forms and locally symmetric generalized Sasakian space forms.* Note Mat. **25** (2006), 55–67.
26. C. A. Mantica and Y. J. Suh: *Pseudo Q-symmetric Riemannian manifolds.* Int. J. Geom. Methods Mod. Phys. **10**(5) (2013), 1–25.
27. B. Shammukha and V. Venkatesha: *Projective curvature tensor on generalized (k, μ)-space forms.* Italian J. Pure Appl. Math. **12** (2019), 810–850.
28. G. S. Shivaprasanna: *Some results on generalized (k, μ) space-forms.* Int. J. sci. Engg. Appl. Sci. **2**(7) (2016), 184–191.
29. G. S. Shivaprasanna, Y. B. Maralabhavi and G. Somashekhara: *On Semi-Symmetric Metric Connection in a Generalized (k, μ) Space Forms.* Int. J. Math. Trends Tech. **9**(3) (2014), 173–188.
30. S. Shashidhar and H. G. Nagaraja: *η-Einstein (k, μ)-space forms.* J. Info. Math. Sci. **7**(2) (2015), 109–120.
31. A. A. Shaikh, R. Deszcz, M. Hotlos and J. Jelwicki: *On pseudosymmetric manifolds.* Publicationes Mathematicae, **86**(3-4) (2015), 433–456.
32. K. Yano: *Concircular geometry I. Concircular transformation.* Proc. Imp. Acad. Tokyo, **16** (1910), 195–200.
33. K. Yano and M. Kon: *Structures on manifolds.* Series in Pure Math. Vol 3, World Scientific Publ. Co., Singapore, 1984.

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On Ricci–Yamabe soliton and geometrical structure in a perfect fluid spacetime

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Abstract

In this paper, we studied the geometrical aspects of a perfect fluid spacetime with torse-forming vector field ξ under certain curvature restrictions, and Ricci–Yamabe soliton and η-Ricci–Yamabe soliton in a perfect fluid spacetime. Conditions for the Ricci–Yamabe soliton to be steady, expanding or shrinking are also given. Moreover, when the potential vector field ξ of η-Ricci–Yamabe soliton is of gradient type, we derive a Poisson equation and also looked at its particular cases. Lastly, a non-trivial example of perfect fluid spacetime admitting η-Ricci–Yamabe soliton is constructed.

Keywords Ricci–Yamabe soliton · Perfect fluid · Poisson equation · Semiconformal curvature · Einstein's field equation

Mathematics Subject Classification 53B50 · 53C44 · 53C50 · 83C02

1 Introduction

Geometric flows plays a significant role in analyzing the geometric structures in Riemannian geometry. In 1982, Hamilton [12] introduced the concept of Ricci flow, defined as follows:

$$\frac{\partial}{\partial t}g(t) = -2S(t), \quad t \geq 0, \quad g(0) = g, \tag{1}$$

where g is the Riemannian metric and S denotes the (0, 2)-symmetric Ricci tensor. Solitons are physically the waves that propagate with little loss of energy and retains its shape and speed after colliding with another such wave. Solitons are important in the analytic treatment of initial-value problems for nonlinear partial differential equations describing wave propagation. It also explained the recurrence in the Fermi–Pasta–Ulam system. A Ricci soliton emerges as the limit of the solution of Ricci flow if it moves only by a one-parameter group

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of diffeomorphism and scaling. A Riemannian manifold (M^n, g) is said to be a Ricci soliton if there exists a vector field V and a constant μ such that

$$\mathcal{L}_V g + 2S = 2\mu g, \tag{2}$$

where $\mathcal{L}_V$ denotes the Lie derivative along V.

To tackle the Yamabe problem of finding a metric on a given compact Riemannian manifold (M^n, g) which is conformal to g such that it has a constant scalar curvature, Hamilton [11] introduced the concept of Yamabe flow, defined as follows:

$$\frac{\partial}{\partial t} g(t) = -r g(t), \quad t \geq 0, \quad g(0) = g. \tag{3}$$

Like Ricci soliton, Yamabe soliton is a self-similar solution to Yamabe flow and is defined as follows:

$$\frac{1}{2}\mathcal{L}_V g = (\mu - r) g, \tag{4}$$

The Ricci soliton and Yamabe soliton are the same in the two-dimensional study, but in a higher dimension, Yamabe soliton preserves the conformal class of the metric but the Ricci soliton does not in general. In the last two decades, the theory of geometric flows such as Ricci flow and Yamabe flow and their soliton has been the focus of attraction of many geometers.

Recently, Guler and Crasmareanu [10] introduced a new geometric flow which is a scalar combination of Ricci flow and Yamabe flow, and called it as Ricci–Yamabe map. The Ricci–Yamabe flow of type (α, β) is defined as follows:

Definition 1 [10] The map $RY^{(\alpha,\beta,g)} : I \to T_2^s(M)$ given by:

$$RY^{(\alpha,\beta,g)} = \frac{\partial g}{\partial t}(t) + 2\alpha S(t) + \beta r(t) g(t),$$

is called the (α, β)-Ricci–Yamabe map of the Riemannian manifold (M, g). If

$$RY^{(\alpha,\beta,g)} \equiv 0,$$

then $g(.)$ will be called an (α, β)-Ricci–Yamabe flow.

The Ricci–Yamabe flow can also be a Riemannian or semi-Riemannian or singular Riemannian flow due to the sign of the two scalars α and β. This flexibility of multiple choices can be useful in analyzing geometry or when dealing with the physical model of relativistic theories. The notion of (α, β)-Ricci–Yamabe soliton or simply Ricci–Yamabe soliton [9] is defined as follows:

Definition 2 A Riemannian or pseudo-Riemannian manifold (M^n, g) is said to be a Ricci–Yamabe soliton $(g, V, \mu, \alpha, \beta)$ if

$$\mathcal{L}_V g + 2\alpha S = (2\mu - \beta r) g. \tag{5}$$

If $\mu > 0$, $\mu < 0$ or $\mu = 0$, then the Ricci–Yamabe soliton is expanding, shrinking or steady respectively. This is said to be a gradient Ricci–Yamabe soliton if there exists a smooth function $f : M \to \mathbb{R}$ such that $V = Df$, where D denotes the gradient operator of g. The Ricci–Yamabe soliton is a generalization of Ricci and Yamabe soliton. Also, $(1, -1)$-type of Ricci–Yamabe soliton is a well-known Einstein soliton (for details see [5,26]). Therefore, it is worthwhile to study Ricci–Yamabe soliton as it generalizes a large group of solitons. Recently, in [9], the author studied Ricci–Yamabe soliton on almost Kenmotsu manifolds. He showed that a $(k, \mu)'$-almost Kenmotsu manifolds admitting a Ricci–Yamabe soliton or gradient Ricci–Yamabe soliton is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Extending the notion of Ricci soliton, Cho and Kimura [1] introduced η-Ricci soliton which is obtained by perturbing the equation (2) by a multiple of a certain (0, 2)-tensor field $\eta \otimes \eta$. A more general extension is obtained by Siddiqi and Akyol [22] and called such soliton as η-Ricci–Yamabe soliton of type (α, β) which is defined as:

$$\mathcal{L}_V g + 2\alpha S + (2\mu - \beta r)g + 2\omega\eta \otimes \eta = 0. \tag{6}$$

It is worth remarking that η-Ricci soliton [1] and η-Yamabe soliton [6] are η-Ricci–Yamabe soliton of type (1, 0) and (0, 2) respectively. If $\omega = 0$ in equation (6) then it reduces to Ricci–Yamabe soliton. For more details on η-Ricci soliton and η-Yamabe soliton see [3,4,7,8,18,19] and references therein.

In the last decade, a great deal of work had been done on η-Ricci soliton and η-Yamabe soliton in the framework of Riemannian geometry. Recently, geometric flows are initiated in the investigation of the cosmological model such as perfect fluid spacetime. In [2] , Blaga studied η-Ricci and η-Einstein soliton in perfect fluid spacetime and obtained the Poisson equation from the soliton equation when the potential vector field ξ is of gradient type. Kumara and Venkatesha [25] analyzed Ricci soliton in perfect fluid spacetime with torse-forming vector field. Also, Conformal Ricci soliton in perfect fluid spacetime [23] is studied. Praveena et al. [20] studied solitons in Kählerian space-time manifolds. As Ricci–Yamabe soliton is a scalar combination of Ricci and Yamabe soliton, it is fruitful to study it in the context of perfect fluid spacetime and obtain results that generalize the previously known results in perfect fluid spacetime.

The paper is organized as follows: Sect. 2 is devoted to the investigation of the geometrical structure of perfect fluid spacetime with torse-forming vector field ξ under certain curvature restrictions. Next in Sect. 3, the conditions under which it is expanding, steady and shrinking are obtained for Ricci–Yamabe soliton in perfect fluid spacetime. Generalizing the results obtained by Blaga [2], in Sect. 4, we analyzed η-Ricci–Yamabe soliton in perfect fluid spacetime and obtained the Poisson equation satisfied by function f where $\xi = grad\ f$. Section 5 is about the application of Poisson equation in Physics. Lastly, in Sect. 6, we constructed an example of perfect fluid spacetime admitting η-Ricci–Yamabe soliton.

2 Geometrical structure of perfect fluid spacetime with torse-forming vector field

According to Einstein's field equation, the energy-momentum tensor describes the curvature of the spacetime and hence plays a crucial role in the theory of relativity. Spacetime of general relativity is regarded as a connected four dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g with signature $(-, +, +, +)$. A spacetime is said to be a perfect fluid spacetime if the Ricci tensor is of the form:

$$S = ag + b\eta \otimes \eta, \tag{7}$$

where a, b are scalars and η is non-zero 1-form.

The general form of energy-momentum tensor T for a perfect fluid is [17]

$$T(X, Y) = \rho g(X, Y) + (\sigma + \rho)\eta(X)\eta(Y), \tag{8}$$

for any $X, Y \in \chi(M)$, where σ is the energy density, ρ is the isotropic pressure, g is the metric tensor of Minkowski spacetime, $\eta(X) = -g(X, \xi)$ is 1-form, equivalent to unit vector

ξ and $g(\xi,\xi) = -1$. If $\rho = \rho(\sigma)$ then perfect fluid spacetime is called isentropic [13] and if $\sigma = 3\rho$ then it is a radiation fluid.

The Einstein's field equation [17] governing the perfect fluid motion is defined as:

$$S(X,Y) + \left(\lambda - \frac{r}{2}\right) g(X,Y) = kT(X,Y), \tag{9}$$

for any $X, Y \in \chi(M)$, where λ is the cosmological constant, $k(\approx 8\pi G$, where G is universal Gravitational constant) is the gravitational constant.

Combining (8) and (9) we obtain

$$S(X,Y) = -\left(\lambda - \frac{r}{2} + k\rho\right) g(X,Y) + k(\sigma + \rho)\eta(X)\eta(Y). \tag{10}$$

Taking trace of (10), the scalar curvature becomes $r = 4\lambda + k(\sigma - 3\rho)$, using in (10) we infer

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{11}$$

where $a = \lambda + \frac{k(\sigma-\rho)}{2}$ and $b = k(\sigma + \rho)$.

Definition 3 A vector field ξ is called torse-forming [1] if it satisfies

$$\nabla_X \xi = X + \eta(X)\xi, \tag{12}$$

for any $X \in \chi(M)$ and 1-form η.

Lemma 1 [1,23,25] *In perfect fluid spacetime with torse-forming vector field ξ, the following relations hold:*

$$\begin{aligned}
\eta(\nabla_\xi \xi) &= 0, \quad \nabla_\xi \xi = 0,\\
(\nabla_X \eta)(Y) &= g(X,Y) + \eta(X)\eta(Y),\\
R(X,Y)\xi &= \eta(Y)X - \eta(X)Y,\\
(\mathcal{L}_\xi g)(X,Y) &= 2[g(X,Y) + \eta(X)\eta(Y)],\\
R(X,\xi)\xi &= -X - \eta(X)\xi.
\end{aligned}$$

In [15,16], Kim introduced curvature like tensor which is a scalar combination of conformal and conharmonic curvature tensor which is defined as follows:

$$\begin{aligned}
P(X,Y)Z = \alpha R(X,Y)Z - \frac{\alpha}{2}[&S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX\\
&-g(X,Z)QY] - \frac{\beta r}{3}[g(Y,Z)X - g(X,Z)Y],
\end{aligned} \tag{13}$$

for scalar α and β. Here, Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is, $S(X,Y) = g(QX,Y)$. If P vanishes then the spacetime is said to be semiconformally flat.

Let (M^4, g) be a semiconformally flat perfect fluid spacetime with torse-forming vector field ξ. As $P = 0$, we have $divP = 0$ where "div" is the divergent. Since r is constant, implies $X(r) = 0$ for any $X \in \chi(M)$. From (13) for $divP = 0$ we obtain

$$k(\sigma + \rho)[\eta(Y)X - \eta(X)Y] = 0. \tag{14}$$

As $k \neq 0$, in this case the equation of state $\rho + \sigma = 0$ emerges. This is the characteristic equation of state for dark energy in the universe and corresponds to the cosmological constant [24]. Essentially, as density cannot be negative, the pressure ρ must be negative which is useful in explaining the observed accelerated expansion of the universe problem.

Making use of $\rho = -\sigma$ in (11) and (13) gives

$$R(X, Y)Z = \frac{1}{3\alpha}(3\alpha - 4\beta)(\lambda + k\sigma)[g(Y, Z)X - g(X, Z)Y]. \tag{15}$$

Therefore, the spacetime has constant curvature. As de-Sitter space is a Lorentzian manifold of constant curvature with implied negative pressure driving cosmic inflation (see [21]) we can state the following:

Theorem 1 *If perfect fluid spacetime with torse-forming vector field ξ is semiconformally flat, then the spacetime represents de-Sitter space, provided $\alpha \neq 0$.*

We know that manifold of constant curvature is Einstein. Also from (15) we easily see that $R \cdot R = 0$. A perfect fluid spacetime satisfying $R \cdot R = 0$ and $R \cdot S = 0$ are called semi-symmetric and Ricci semi-symmetric respectively. A semi-symmetric implies Ricci semi-symmetric but conversely not true.

Proposition 1 *A semiconformally flat perfect fluid spacetime with torse-forming vector field ξ is*

(i) *Einstein.*
(ii) *semi-symmetric and Ricci semi-symmetric.*

According to Karchar [14], a Lorentzian manifold is called infinitesimal spatially isotropic relative to timelike unit vector field ρ if its curvature tensor R satisfies relations

$$R(X, Y)Z = l[g(Y, Z)X - g(X, Z)Y],$$

for all $X, Y, Z \in \rho^{\perp}$ and

$$R(X, \rho)\rho = mX,$$

for all $X \in \rho^{\perp}$, where l, m are real-valued functions on the manifold.

Let $\xi^{\perp}$ denote the 3-dimensional distribution in a semiconformally flat perfect fluid spacetime orthogonal to torse-forming vector field ξ, then from (15) we get

$$R(X, Y)Z = \frac{1}{3\alpha}(3\alpha - 4\beta)(\lambda + k\sigma)[g(Y, Z)X - g(X, Z)Y], \tag{16}$$

for all $X, Y, Z \in \xi^{\perp}$. Also taking $Y = Z = \xi$ in (16) gives

$$R(X, \rho)\rho = -\frac{1}{3\alpha}(3\alpha - 4\beta)(\lambda + k\sigma)X, \tag{17}$$

for every $X \in \xi^{\perp}$. Hence we can state the following:

Theorem 2 *A semiconformally flat perfect fluid spacetime with $a \neq 0$ and torse-forming vector field ξ is infinitesimally spatially isotropic relative to unit vector field ξ.*

Theorem 3 *Let (M^4, g) be a general relativistic perfect fluid spacetime with torse-forming vector field ξ.*

1. *If $P(\xi, \cdot) \cdot S = 0$ then $\rho = -\sigma$ or $\rho = \frac{3\alpha(\lambda-1)+\beta(4\lambda+k\sigma)}{3k(\alpha-\beta)}$.*
2. *If $S(\xi, \cdot) \cdot P = 0$ then $\rho = \frac{\lambda}{k}$ or $\rho = \frac{3\alpha(\lambda-1)+\beta(4\lambda+k\sigma)}{3k(\alpha-\beta)}$.*

Proof 1. Suppose perfect fluid spacetime with torse-forming vector field ξ satisfies $P(\xi, X)\cdot S(U, V) = 0$, implies

$$S(P(\xi, X)U, V) + S(U, P(\xi, X)V) = 0, \tag{18}$$

for all $X, U, V \in \chi(M)$. Inserting (11) and (13) in (18) results in

$$\begin{aligned}&-2\alpha k(\sigma+\rho)\left(\lambda+\frac{k}{2}(\sigma-\rho)\right)\eta(X)\eta(U)\eta(V)+k(\sigma+\rho)\left(\alpha-\frac{\beta r}{3}\right.\\&\left.-\alpha(\lambda-k\rho)\right)[-g(X,U)\eta(V)-2\eta(X)\eta(U)\eta(V)-g(X,V)\eta(U)]\\&+2\alpha k^2(\sigma+\rho)^2\eta(X)\eta(U)\eta(V)=0.\end{aligned} \tag{19}$$

Replacing U by ξ in (19) we obtain that either $\rho = -\sigma$ or $\rho = \frac{3\alpha(\lambda-1)+\beta(4\lambda+k\sigma)}{3k(\alpha-\beta)}$.

2. Suppose perfect fluid spacetime satisfies $S(\xi, X)\cdot P(U, V)W = 0$, implies

$$\begin{aligned}&S(X,P(U,V)W)\xi-S(\xi,P(U,V)W)X+S(X,U)P(\xi,V)W\\&-S(\xi,U)P(X,V)W+S(X,V)P(U,\xi)W-S(\xi,V)P(U,X)W\\&+S(X,W)P(U,V)\xi-S(\xi,W)P(U,V)X=0,\end{aligned} \tag{20}$$

for all $X, U, V, W \in \chi(M)$.
Taking $V = W = \xi$ in (20) and using (11) and (13), we obtain the following relation

$$(\lambda-k\rho)\left(\alpha-\frac{\beta r}{3}-\alpha(\lambda-k\rho)\right)[g(X,U)+\eta(X)\eta(U)]=0.$$

Thus either $\rho = \frac{\lambda}{k}$ or $\rho = \frac{3\alpha(\lambda-1)+\beta(4\lambda+k\sigma)}{3k(\alpha-\beta)}$.
This completes the proof. □

3 Ricci–Yamabe soliton in a perfect fluid spacetime

In this section, we study Ricci–Yamabe soliton in the framework of perfect fluid spacetime admitting a torse-forming vector field ξ.

Taking potential vector field, $V = \xi$ in (5) and using Lemma 1 we obtain

$$\alpha S(X,Y)=\left[\mu-\frac{\beta r}{2}-1\right]g(X,Y)-\eta(X)\eta(Y). \tag{21}$$

Inserting $X = Y = \xi$ in (21) yields

$$\mu=\lambda(\alpha+2\beta)+\frac{\beta k}{2}(\sigma-3\rho)-\frac{\alpha k}{2}(\sigma+3\rho). \tag{22}$$

Hence we can state the following:

Theorem 4 *If a perfect fluid spacetime with torse-forming vector field ξ admits Ricci–Yamabe soliton $(g, \xi, \mu, \alpha, \beta)$, then the Ricci–Yamabe soliton is expanding, steady or shrinking according to as $\lambda > \frac{k}{2(\alpha+2\beta)}\{\alpha(\sigma+3\rho)-\beta(\sigma-3\rho)\}$, $\lambda = \frac{k}{2(\alpha+2\beta)}\{\alpha(\sigma+3\rho)-\beta(\sigma-3\rho)\}$ or $\lambda < \frac{k}{2(\alpha+2\beta)}\{\alpha(\sigma+3\rho)-\beta(\sigma-3\rho)\}$ respectively, provided $\alpha+2\beta \neq 0$.*

Remark 1 Now we will look at some of the particular cases of Theorem 4. If a perfect fluid spacetime with torse-forming vector field ξ admits:

1. Ricci soliton ($\alpha = 1, \beta = 0$), then the Ricci soliton is expanding, steady or shrinking according as $\lambda > \frac{k}{2}(\sigma + 3\rho)$, $\lambda = \frac{k}{2}(\sigma + 3\rho)$ or $\lambda < \frac{k}{2}(\sigma + 3\rho)$ respectively. This was shown by Venkatesha [25].
2. Yamabe soliton ($\alpha = 0, \beta = 2$), then the Yamabe soliton is expanding, steady or shrinking according as $\lambda > \frac{k}{4}(3\rho - \sigma)$, $\lambda = \frac{k}{4}(3\rho - \sigma)$ or $\lambda < \frac{k}{4}(3\rho - \sigma)$ respectively.
3. Einstein soliton ($\alpha = 1, \beta = -1$), then $\mu = -\lambda - k\sigma$ implies Einstein soliton is expanding if $\lambda < -k\sigma$, steady if $\lambda = -k\sigma$ and shrinking if $\lambda > -k\sigma$.

Theorem 5 *If a perfect fluid spacetime with torse-forming vector field ξ admits Ricci–Yamabe soliton $(g, V, \mu, \alpha, \beta)$, then either every perfect fluid spacetime with torse-forming vector field ξ is a spacetime with the equal associated scalar or the Ricci–Yamabe soliton is expanding, steady or shrinking according to as Theorem 4.*

Proof Inserting (11) in (5) we get

$$(\mathcal{L}_V g)(X, Y) = 2\left(\mu - \frac{\beta r}{2} - a\alpha\right) g(X, Y) - 2\alpha b\eta(X)\eta(Y). \tag{23}$$

Taking Lie-differentiation of (11) and using it in (23) yields

$$\begin{aligned}(\mathcal{L}_V S)(X, Y) = 2a\left(\mu - \frac{\beta r}{2} - a\alpha\right) g(X, Y) - 2a\alpha b\eta(X)\eta(Y)\\ +b[(\mathcal{L}_V \eta)(X)\eta(Y) + (\mathcal{L}_V \eta)(Y)\eta(X)].\end{aligned} \tag{24}$$

Differentiating covariantly (11) along vector field Z and using Lemma 1 infer

$$(\nabla_Z S)(X, Y) = b[g(Z, X)\eta(Y) + g(Z, Y)\eta(X) + 2\eta(X)\eta(Y)\eta(Z)]. \tag{25}$$

According to Yano [27], we have the following commutative formula:

$$\begin{aligned}&(\mathcal{L}_V \nabla_Z g - \nabla_Z \mathcal{L}_V g - \nabla_{[V,Z]}(X, Y)\\ &= -g((\mathcal{L}_V \nabla)(Z, X), Y) - g((\mathcal{L}_V \nabla)(Z, Y), X).\end{aligned} \tag{26}$$

Combining (5) and (26) we obtain

$$g((\mathcal{L}_V \nabla)(X, Y), Z) = (\nabla_Z S)(X, Y) - (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z). \tag{27}$$

Inserting (25) in (27), we get the form

$$(\mathcal{L}_V \nabla)(X, Y) = -2b[g(X, Y)\xi + \eta(X)\eta(Y)\xi]. \tag{28}$$

Again considering the commutative formula given by Yano [27]:

$$(\mathcal{L}_V R)(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z). \tag{29}$$

Taking covariant differentiation of (28) and using it in (29), yields

$$\begin{aligned}(\mathcal{L}_V R)(X, Y)Z = 2b[g(X, Z)Y - g(Y, Z)X\\ +\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X].\end{aligned} \tag{30}$$

Contracting (30) with respect to X gives

$$(\mathcal{L}_V S)(Y, Z) = -6b[g(Y, Z) + \eta(Y)\eta(Z)]. \tag{31}$$

Putting $Y = Z = \xi$ in (31), we have

$$(\mathcal{L}_V S)(\xi, \xi) = 0. \tag{32}$$

Inserting $X = Y = \xi$ in (24) we obtain

$$-2a\left(\mu - \frac{\beta r}{2} - a\alpha\right) - 2a\alpha b + 2b(\mathcal{L}_V \eta)(\xi) = 0. \tag{33}$$

Also, taking $X = \xi$ in (23) infer

$$(\mathcal{L}_V g)(X, \xi) = \left[2\left(\mu - \frac{\beta r}{2} - a\alpha\right) + 2\alpha b\right]\eta(X). \tag{34}$$

Taking Lie-differentiation of $\eta(X) = g(X, \xi)$ and using it in (34) give us the relation:

$$(\mathcal{L}_V \eta)(X) - g(\mathcal{L}_V \xi, X) - \left[2\left(\mu - \frac{\beta r}{2} - a\alpha\right) + 2b\alpha\right]\eta(X) = 0. \tag{35}$$

Again, taking Lie-differentiation of $g(\xi, \xi) = -1$ along V and using (24) gives

$$\eta(\mathcal{L}_V \xi) = \mu - \frac{\beta r}{2} - a\alpha + \alpha b. \tag{36}$$

Making use of (36), (33) and substituting the values of a and b we obtain the following relation:

$$[2\lambda - k(\sigma + 3\rho)]\left[\mu - \lambda(\alpha + 2\beta) + \frac{\beta k}{2}(\sigma - 3\rho) - \frac{\alpha k}{2}(\sigma + 3\rho)\right] = 0. \tag{37}$$

Thus we see that either $\lambda = \frac{k}{2}(\sigma + 3\rho)$ or $\mu = \lambda(\alpha + 2\beta) + \frac{\beta k}{2}(\sigma - 3\rho) - \frac{\alpha k}{2}(\sigma + 3\rho)$. We obtain the following two cases:
Case I If $\lambda \neq \frac{k}{2}(\sigma + 3\rho)$, then $\mu = \lambda(\alpha + 2\beta) + \frac{\beta k}{2}(\sigma - 3\rho) - \frac{\alpha k}{2}(\sigma + 3\rho)$. In this case Ricci–Yamabe soliton is expanding, steady or shrinking accordingly as Theorem 4.
Case II If $\lambda = \frac{k}{2}(\sigma + 3\rho)$ and $\mu \neq \lambda(\alpha + 2\beta) + \frac{\beta k}{2}(\sigma - 3\rho) - \frac{\alpha k}{2}(\sigma + 3\rho)$, implies $\mu \neq 3\beta k(\sigma + 3\rho)$. Then we get

$$S(X, Y) = k(\sigma + \rho)[g(X, Y) + \eta(X)\eta(Y)], \tag{38}$$

i.e. perfect fluid spacetime is a spacetime with equal associated scalar constant. This completes the proof. □

Taking $X = Y = \xi$ in (28) yields

$$(\mathcal{L}_V \nabla)(\xi, \xi) = 0. \tag{39}$$

Using the commutative formula:

$$(\mathcal{L}_V \nabla)(X, Y) = \nabla_X \nabla_Y V - \nabla_{\nabla_X Y} V + R(V, X)Y. \tag{40}$$

Replacing X, Y by ξ in (40) and using (39) gives

$$\nabla_\xi \nabla_\xi V - \nabla_{\nabla_\xi \xi} V + R(V, \xi)\xi = 0. \tag{41}$$

Since ξ is torse-forming vector field, $\nabla_\xi \xi = 0$ then (41) becomes

$$\nabla_\xi \nabla_\xi V + R(V, \xi)\xi = 0. \tag{42}$$

This implies that potential vector field V is a Jacobi vector field along the geodesic of ξ. Hence we can state the following:

Theorem 6 *If a perfect fluid spacetime with torse-forming vector field ξ admits a Ricci–Yamabe soliton $(V, g, \mu, \alpha, \beta)$, then the potential vector field V is a Jacobi vector field along the geodesics of ξ.*

4 η-Ricci–Yamabe soliton in a perfect fluid spacetime

In this section we consider η-Ricci–Yamabe soliton in the context of perfect fluid spacetime admitting torse-forming vector field ξ and obtain the Poisson equation.

Writing explicitly the Lie derivative $\mathcal{L}_\xi g$ and taking potential vector $V = \xi$ in (6) we get

$$[g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)] + 2\alpha S(X, Y) + (2\mu - \beta r)g(X, Y) + 2\omega\eta(X)\eta(Y) = 0, \tag{43}$$

for any $X, Y \in \chi(M)$. Contracting (43) yields

$$div(\xi) + \alpha r + \left(\mu - \frac{\beta r}{2}\right) dim(M) = \omega. \tag{44}$$

Let (M^4, g) be a general relativistic perfect fluid spacetime and $(g, \xi, \mu, \omega, \alpha, \beta)$ be η-Ricci–Yamabe soliton in M. From (6) and (11) we get

$$\frac{1}{2}[g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)] + \left(a\alpha + \mu - \frac{\beta r}{2}\right) g(X, Y) + (\alpha b + \omega)\eta(X)\eta(Y) = 0. \tag{45}$$

Consider $\{e_i\}_{1\le i\le 4}$ an orthonormal frame field and let $\xi = \Sigma_{i=1}^4 \xi^i e_i$, we have $\Sigma_{i=1}^4 \epsilon_{ii}(\xi^i)^2 = -1$ and $\eta(e_i) = \epsilon_{ii}\xi^i$.

Multiplying (45) by ϵ_{ii} and summing over i for $X = Y = e_i$ we obtain

$$4\mu - \omega = (2\beta - \alpha)r - div(\xi). \tag{46}$$

Taking $X = Y = \xi$ in (45) gives

$$\omega - \mu = \alpha(a - b) - \frac{\beta r}{2}. \tag{47}$$

Therefore,

$$\mu = (2\beta - \alpha)\lambda + \frac{\beta k}{2}(\sigma - 3\rho) - \frac{\alpha k}{2}(\sigma - \rho) - \frac{div(\xi)}{3} \tag{48}$$

$$\omega = -\alpha k(\sigma + \rho) - \frac{div(\xi)}{3} \tag{49}$$

Hence we can state the following:

Theorem 7 *Let (M, g) be a 4-dimensional pseudo-Riemannian manifold and let η be the g-dual 1-form of the gradient vector field $\xi = grad(f)$ with $g(\xi, \xi) = -1$. If (6) defines an η-Ricci–Yamabe soliton in M, then the Poisson equation satisfies by f is*

$$\Delta(f) = -3[\omega + \alpha k(\sigma + \rho)].$$

In view of (6), taking $\alpha = 0$ and $\beta = 1$ it gives η-Yamabe soliton. Thus we can state the following:

Corollary 1 *Let (M, g) be a 4-dimensional pseudo-Riemannian manifold and let η be the g-dual 1-form of the gradient vector field $\xi = grad(f)$ with $g(\xi, \xi) = -1$. If (6) defines an η-Yamabe soliton in M, then the Poisson equation satisfies by f is*

$$\Delta(f) = -3\omega.$$

Remark 2 Now we look at some of the particular cases of Theorem 7. Under similar hypothesis as in Theorem 7, if g admits:

1. η-Ricci soliton ($\alpha = 1, \beta = 0$), then the Poisson equation satisfies by f is $\Delta(f) = -3[\omega + k(\sigma + \rho)]$.
2. η-Einstein soliton ($\alpha = 1, \beta = -1$), then the Poisson equation becomes $\Delta(f) = -3[\omega + k(\sigma + \rho)]$. These results were obtained by Blaga [1].

Example 1 An η-Ricci–Yamabe soliton $(g, \xi, \mu, \omega, \alpha, \beta)$ in a radiation fluid is given by

$$\mu = (4\beta - \alpha)\lambda - \alpha kp - \frac{div(\xi)}{3}$$
$$\omega = -4\alpha kp - \frac{div(\xi)}{3}$$

From this example, we deduce that Ricci–Yamabe soliton in radiation fluid is steady if $p = \frac{(\alpha-4\beta)\lambda}{3\alpha k}$, expanding if $p > \frac{(\alpha-4\beta)\lambda}{3\alpha k}$ and shrinking if $p < \frac{(\alpha-4\beta)\lambda}{3\alpha k}$ for $\alpha \neq 0$.

5 Applications of Poisson equation in physics

The fundamental forces of nature such as gravity and electrostatic forces could be modeled using functions called gravitational potential and electrostatic potential both of which satisfy the Poisson equation. For example, Gauss's law of gravitational in differential form is

$$\nabla\psi = -4\pi G\rho, \tag{50}$$

where ψ is the gravitational field, ρ the mass density and G the gravitational constant. Since ψ is conservative and can be expressed as the negative gradient of gravitational potential i.e. $\psi = -grad\ f$ then the Poisson equation of gravitation is

$$\nabla^2 = 4\pi G\rho. \tag{51}$$

Similarly, Poisson's equation for electrostatics is

$$\nabla^2\varphi = -\frac{\rho}{\varepsilon}, \tag{52}$$

where ρ is charge distribution, ε permittivity of the medium and φ is gradient scalar function such that $E = -\nabla\varphi$ for electric field E. Solving the Poisson equation amounts to finding the electric potential φ for a given charge distribution.

These physical phenomenon's are directly identical to above Theorem 7 which is a Poisson equation with potential vector field of gradient type.

6 Example of η-Ricci–Yamabe soliton in a perfect fluid spacetime

In this section, we constructed a non-trivial example of a perfect fluid spacetime admitting η-Ricci–Yamabe soliton in a 4-dimensional pseudo-Riemannian manifold. Let $M = \{(x, y, z, t) \in \mathbb{R}^4; t \neq 0\}$, where (x, y, z, t) are the standard coordinates of $\mathbb{R}^4$. Consider a Lorentzian metric g on M is given by

$$ds^2 = e^{2t}[dx^2 + dy^2 + dz^2] - dt^2. \tag{53}$$

The non-vanishing components of the Christoffel symbol, the curvature tensor and Ricci tensor are

$$\Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = e^{2t}, \Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = 1,$$
$$R_{1441} = R_{2442} = R_{3443} = e^{2t}, R_{1221} = R_{1331} = R_{2332} = -e^{4t},$$
$$S_{11} = S_{22} = S_{33} = -3e^{2t}, S_{44} = 3.$$

Therefore, the scalar curvature of the manifold is $r = -12$. Thus, (M^4, g) is a perfect fluid spacetime whose isotropic pressure and energy density are $\rho = \frac{1}{k}(\lambda+3)$ and $\sigma = -\frac{1}{k}(\lambda+3)$ respectively.

Let η be the 1-form defined by $\eta(Z) = -g(Z, t)$ for any $Z \in \chi(M)$. Take $\xi = t$. Replacing $V = \xi$ in (6) and using $(\mathcal{L}_\xi g)(X, Y) = 2[g(X, Y) + \eta(X)\eta(Y)]$ we see that the soliton equation becomes

$$2[g_{ii} + \eta_i \otimes \eta_i] + 2\alpha S_{ii} + (2\mu - \beta r)g_{ii} + 2\omega\eta_i \otimes \eta_i = 0, \tag{54}$$

for all $i \in \{1, 2, 3, 4\}$. Thus the data $(\xi, g, \mu, \omega, \alpha, \beta)$ is η-Ricci–Yamabe soliton on (M^4, g) where $\mu = 3\alpha - 4\beta - 1$ and $\omega = -1$, which is expanding if $3\alpha - 4\beta > 1$, shrinking if $3\alpha - 4\beta < 1$ and steady if $3\alpha - 4\beta = 1$.

7 Conclusions

This work is an extension of previous work done on perfect fluid spacetime by Blaga [2] and Venkatesha and Kumara [25]. Blaga [2] obtained the Poisson equations in perfect fluid spacetime admitting η-Ricci soliton and η-Einstein soliton. We generalized the result of Blaga in Sect. 4 and obtained a more general expression of the Poisson equation in perfect fluid spacetime admitting η-Ricci–Yamabe soliton. The significance of this result is that it holds for a large group of solitons. Next, the conditions under which a perfect fluid spacetime admitting torse-forming vector field ξ is expanding, shrinking and steady Ricci–Yamabe soliton is obtained. Further, it is shown that the results in [2,25] are particular cases. Ricci–Yamabe soliton in the context of Riemannian and semi-Riemannian manifolds need further research.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Blaga, A.M.: Almost eta-Ricci solitons in $(LCS)_n$-manifolds. Bull. Belgian Math. Soc.-Simon Stevin **25**(5), 641–653 (2018)
2. Blaga, A.M.: Solitons and geometrical structure in a perfect fluid spacetime. Rocky Mt. J. Math. (2020). https://projecteuclid.org/euclid.rmim
3. Blaga, A.M.: η-Ricci solitons on Lorentzian para-Sasakian manifolds. Filomat **30**(2), 489–496 (2016)

4. Blaga, A.M., Crasmareanu, M.: Torse-forming η-Ricci solitons in almost paracontact η-Einstein geometry. Filomat **31**(2), 499–504 (2017)
5. Catino, G., Mazzieri, L.: Gradient Einstein solitons. Nonlinear Anal. **132**, 66–94 (2016)
6. Chen, B.Y., Deshmukh, S.: Yamabe and quasi-Yamabe solitons on Euclidean submanifolds. Mediterr. J. Math. **15**, 194 (2018)
7. Chen, B.Y., Deshmukh, S.: Yamabe and quasi-Yamabe solitons on Euclidean submanifolds. Mediterr. J. Math. **15**(5), 194 (2018)
8. De, U.C., De, K.: Almost quasi-Yamabe solitons and gradient almost quasi-Yamabe solitons in paracontact geometry. Quaest. Math. (2020). https://doi.org/10.2989/16073606.2020.1799882
9. Dey, D.: Almost Kenmotsu metric as Ricci–Yamabe soliton. arXiv:2005.02322
10. Guler, S., Crasmareanu, M.: Ricci–Yamabe maps for Riemannian flow and their volume variation and volume entropy. Turk. J. Math. **43**, 2631–2641 (2019)
11. Hamilton, R.S.: Lectures on geometric flows. Unpublished manuscript (1989)
12. Hamilton, R.S.: The Ricci flow on surfaces. Contemp. Math. **71**, 237–262 (1998)
13. Hawking, S.W., Ellis, G.F.R.: The Large-Scale Structure of Spacetime. Cambridge University Press, Cambridge (1973)
14. Karchar, H.: Infinitesimal characterization of Friedmann Universe. Arch. Math. Basel **38**, 58–64 (1992)
15. Kim, J.: A type of conformal curvature tensor. Far East J. Math. Soc. **99**(1), 61–74 (2016)
16. Kim, J.: On pseudo semiconformally symmetric manifolds. Bull. Korean Math. Soc. **54**, 177–186 (2017)
17. O'Neill, B.: Semi-Riemannian Geometry with Applications to Relativity. Academic Press, New York (1983)
18. Pirhadi, V., Razavi, A.: On the almost quasi-Yamabe solitons. Int. J. Geom. Methods Mod. Phys. **14**(11), 1750161 (2017)
19. Prakasha, D., Hadimani, B.: η-Ricci solitons on para-Sasakian manifolds. J. Geom. **108**(2), 383–392 (2017)
20. Praveena, M.M., Bagewadi, C.S., Krishnamurthy, M.R.: Solitons of Kählerian space-time manifolds. Int. J. Geom. Methods Mod. Phys., 2150021 (2021)
21. Schmidt, H.J.: On the de Sitter space-time—the geometric foundation of inflationary cosmology. Fortschr. Phys. **41**(3), 179–199 (1993)
22. Siddiqi, M.D., Akyol, M.A.: η-Ricci–Yamabe solitons on Riemannian submersions from Riemannian manifolds. arXiv:2004.14124
23. Siddiqi, M.D., Siddiqui, S.A.: Conformal Ricci soliton and geometrical structure in a perfect fluid spacetime. Int. J. Geom. Methods Mod. Phys. **17**, 2050083 (2020)
24. Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C., Herlt, E.: Exact Solutions of Einstein's Field Equations, 2nd edn. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge (2003)
25. Venkatesha, V., Kumara, H.A.: Ricci soliton and geometrical structure in a perfect fluid spacetime with torse-forming vector field. Afr. Mat. **30**, 725–736 (2019)
26. Venkatesha, V., Kumara, H.A.: Gradient ρ-Einstein soliton on almost Kenmotsu manifolds. Annali Dell'universita'di Ferrara **65**(2), 375–388 (2019)
27. Yano, K.: Integral Formulas in Riemannian Geometry. Marcel Dekker, New York (1970)

On weakly cyclic B symmetric spacetime

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Abstract. The object of the present paper is to investigate some geometric and physical properties of weakly cyclic B symmetric $(WCBS)_4$ spacetime under certain conditions. At first, the existence of $(WCBS)_4$ spacetime is showed by constructing a non-trivial example. Then it is shown that a $(WCBS)_4$ spacetime with harmonic Weyl tensor is a Yang Pure space or the integral curve of vector field ρ are geodesic and vector field ρ is irrotational, provided $r = \frac{b}{a}$. Moreover some geometric properties of $(WCBS)_4$ spacetime satisfying certain curvature restrictions are investigated and shown that conformally flat $(WCBS)_4$ spacetime is infinitesimally spatially isotropic relative to the unit timelike vector field ρ. Next we characterize viscous fluid, dust and perfect fluid $(WCBS)_4$ spacetimes and obtained interesting results. Finally, we showed that in a $(WCBS)_4$ spacetime with non-constant scalar curvature satisfying $divC = 0$ and fulfilling the condition $r = \frac{b}{a}$, if ρ is Killing vector then it is Weyl compatible, purely electric spacetime and its possible Petrov types are I or D.

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1 Introduction

A Lorentzian manifold is a special case of pseudo-Riemannian manifold. A pseudo-Riemannian manifold of dimension n is a smooth n-dimensional differentiable manifold equipped with a pseudo-Riemannian metric of signature (p, q) where $n = p+q$. Due to non-degeneracy of Lorentzian metric, the tangent vector can be classified into timelike, null or spacetime vector. A Lorentzian manifold has many applications especially in the field of relativity and cosmology. The casuality of the vector fields plays an important role and hence it becomes a convenient choice for researchers for the study of General Relativity. If a Lorentzian manifold admits a globally timelike vector field, it is called time oriented Lorentzian manifold, physically known as spacetime. In general, a Lorentzian manifold may not have a globally timelike vector field. For more details see [1, 8, 23, 4, 17] and references therein.

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In [6], it is showed that a generalistic spacetime with covariant constant energy momentum tensor is Ricci symmetric, that is, $\nabla S = 0$, where S is the Ricci tensor of the spacetime and ∇ denotes the covariant differentiation with respect to the metric tensor g. If however, $\nabla S \neq 0$, then such a spacetime may be called weakly Ricci symmetric [26]. De and Ghosh [9] studied weakly Ricci symmetric spacetimes and proved that if in a weakly Ricci symmetric spacetime of non-zero scalar curvature the matter distribution is perfect fluid, then the acceleration vector and the expansion scalar are zero and such a spacetime can not admit heat flux. A non-flat Riemannian or pseudo-Riemannian manifold $(M^n, g)(n > 2)$ is called weakly Ricci symmetric if the Ricci tensor S is of the form

$$\begin{aligned} (\nabla_X S)(U, V) &= A(X)S(U, V) + D(U)S(V, X) \\ &+ E(V)S(X, U), \end{aligned} \tag{1.1}$$

where A, D and E are 1-forms which are non-zero simultaneously. Such an n-dimensional Riemannian manifold is denoted by $(WRS)_n$. If $A = B = D = 0$, then the manifold reduces to a Ricci symmetric manifold.

A (0,2) symmetric tensor is a generalized Z tensor if

$$Z_{ij} = S_{ij} + \phi g_{ij}, \tag{1.2}$$

where ϕ is an arbitrary scalar function. Recently, Mantica and Molinari [19] introduced weakly Z symmetric manifolds. It is further weaken by De et al. [10] into weakly cyclic Z symmetric manifolds and it is denoted by $(WCZS)_n$. A non-flat Riemannian or pseudo-Riemannian manifold $(M^n, g)(n > 2)$ is called weakly cyclic Z symmetric if the Z tensor is non-zero and satisfies the following condition

$$\begin{aligned} &(\nabla_X Z)(U, V) + (\nabla_U Z)(V, X) + (\nabla_V Z)(X, U) \\ &= A(X)Z(U, V) + D(U)Z(V, X) + E(V)Z(X, U), \end{aligned} \tag{1.3}$$

for all vector fields X, U and V. Here, Z is the generalized Z tensor. De et al. [11] studied weakly cyclic Z symmetric spacetimes and showed that if a $(WCZS)_4$ spacetime satisfies $divC = 0$ and fulfills the condition $r = a$, then the spacetime is the Robertson-Walker spacetime. De et al. [20] introduced a new symmetric (0,2) tensor B_{ij} as

$$B_{ij} = aS_{ij} + brg_{ij}, \tag{1.4}$$

where a and b are non-zero arbitrary scalar functions and r is the scalar curvature. For $a = 1$ and $b = \frac{\phi}{r}$ the tensor reduces to generalized Z tensor. Thus generalized Z tensor is a particular case of B tensor and hence it give us a reason to study B tensor. Contracting (1.4) we get, scalar B as $B = (a + nb)r$. In [20], the authors introduced pseudo B symmetric manifold which is a generalization of pseudo Z symmetric manifold [21]. Motivated by this we introduced weakly cyclic B symmetric manifold. A non-flat Riemannain or pseudo-Riemannain manifold $(M^n, g)(n > 1)$ is called a weakly cyclic B symmetric manifold of dimension n if the B tensor is non-zero and satisfies the condition

$$\begin{aligned} &(\nabla_X B)(Y, Z) + (\nabla_Y B)(Z, X) + (\nabla_Z B)(X, Y) \\ &= A(X)B(Y, Z) + D(Y)B(Z, X) + E(Z)B(X, Y), \end{aligned} \tag{1.5}$$

where A, D and E are non-zero 1-forms. It will be denoted by $(WCBS)_n$ manifold. In [11], the authors investigated weakly cyclic Z symmetric spacetime and obtained interesting results. This inspired us to study weakly cyclic B symmetric spacetime.

The notion of quasi Einstein manifolds arose during the study of exact solutions of the Einstein's field equation as well as during the considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. Chaki and Maity [5] introduced the notion of quasi Einstein manifolds as a generalization of the Einstein manifolds. A pseudo-Riemannain manifold $(M^n, g)(n > 2)$ is said to be a quasi Einstein manifold if its Ricci curvature is non-zero and satisfies the condition

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y), \tag{1.6}$$

where α and β are real valued non-zero scalar functions on M. In [12], it is proved that a quasi Einstein manifolds can be taken as a model of perfect fluid spacetime in General Relativity. Also, the Robertson-Walker spacetimes are quasi Einstein manifolds. Thus quasi Einstein manifolds are important in theoretical physics, especially in General Relativity and cosmology.

The Weyl (or conformal curvature) tensor plays an important role in differential geometry and also in General Relativity providing curvature to the spacetime when the Ricci tensor is zero. The Weyl conformal tensor C in a Lorentzian manifold $(M^n, g)(n > 3)$ is defined by [29]

$$\begin{aligned} C(X,Y)U &= R(X,Y)U - \frac{1}{n-2}[g(Y,U)QX - g(X,U)QY \\ &+ S(Y,U)X - S(X,U)Y] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,U)X - g(X,U)Y], \end{aligned} \tag{1.7}$$

where Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is, $g(QX, Y) = S(X, Y)$. The Lorenzian manifold of dimension $n(n > 3)$ is said to be conformally flat if the conformal curvature tensor C is identically zero. In [16], Endean studied cosmology in conformally flat spacetime.

Ahsan and Siddiqui [1] proved that a concircularly flat perfect fluid spacetime admits a conformal Killing vector field if and only if the energy-momentum tensor has a symmetry inheritance property. The concircular curvature tensor in a Lorentzian manifold $(M^n, g)(n > 3)$ is defined by

$$\begin{aligned} \widetilde{C}(X,Y)U &= R(X,Y)U \\ &+ \frac{r}{n(n-1)}[g(U,X)Y - g(Y,Z)X], \end{aligned} \tag{1.8}$$

for all vector fields X, Y, Z in M. For $n = 3$, the Weyl tensor as well as concircular curvature tensor vanishes identically. The Lorenzian manifold of dimension $n(n > 3)$ is said to be concircularly flat if the concircular curvature tensor $\widetilde{C}$ is identically zero.

The paper is organized as follows:

In Section 2, the existence of $(WCBS)_4$ spacetime is established by constructing a non-trivial example. Next in Section 3 it is shown that a $(WCBS)_4$ spacetime is quasi Einstein spacetime. Moreover conformally flat $(WCBS)_4$ spacetime and $(WCBS)_4$ spacetime with $divC = 0$ are studied and prove that a $(WCBS)_4$ spacetime satisfying

$divC = 0$ with assumption $r = \frac{b}{a}$, the integral curve of vector field ρ are geodesic and vector field ρ is irrotational or the spacetime is Yang Pure space. In the next section, we investigate some geometric and physical properties of this spacetime under certain curvature conditions. The last section deal with the application of $(WCBS)_4$ spacetime in General Relativity. We prove that if a perfect fluid $(WCBS)_4$ spacetime with vanishing scalar B obeys Einstein's field equation without cosmological constant then the spacetime is characterized by the following cases:
(i) The spacetime represents inflation and the fluid behaves as a cosmological constant. This is also termed as a phantom barrier.
(ii) The spacetime represents quintessence barrier and the fluid behaves as exotic matter.
The energy density and isotropic pressure for viscous fluid $(WCBS)_4$ spacetime are obtained and also we prove that a relativistic fluid $(WCBS)_4$ spacetime obeying Einstein's field equation with the cosmological constant admit heat flux, provided $\lambda + k\sigma \neq \frac{3B-2br}{2a}$. Finally, it is shown that in a $(WCBS)_4$ spacetime with non-constant scalar curvature satisfying $divC = 0$ and fulfilling the condition $r = \frac{b}{a}$, if ρ is Killing vector then it is Weyl compatible, purely electric spacetime and its possible Petrov types are I or D.

2 Existence of $(WCBS)_4$ spacetime

In this section, we prove the existence of the $(WCBS)_4$ spacetime by constructing a non-trivial example (see [24]). Now, we shall consider a Lorentzian metric g on the 4-dimensional real number space $\mathbb{R}^4$ by

$$ds^2 = e^{2z}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2, \tag{2.1}$$

where $z = x^4 \neq 0$ and x^1, x^2, x^3, x^4 are the standard coordinates of $\mathbb{R}^4$. Then the non-vanishing components of the Christoffel symbol, the curvature tensor and the Ricci tensor are

$$\Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = e^{2z}, \qquad \Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = 1, \tag{2.2}$$

$$R_{1441} = R_{2442} = R_{3443} = e^{2z}, R_{1221} = R_{1331} = R_{2332} = -e^{4z}, \tag{2.3}$$

$$S_{11} = S_{22} = S_{33} = -3e^{2z}, \qquad S_{44} = 3, \tag{2.4}$$

and the components which can be obtained from this by symmetric properties. One can easily show that the scalar curvature r of the manifold is $r = -12$.
Let us choose an arbitrary scalar function as $a = e^{-z}$ and $b = e^{-2z}$. Making use of (1.4) the non-vanishing components of symmetric B tensor and their covariant derivatives are as follows

$$B_{11} = B_{22} = B_{33} = -3(e^z + 4), \qquad B_{44} = 3(e^{-z} + 4e^{-2z}, \tag{2.5}$$

$$B_{11,4} = B_{22,4} = B_{33,4} = -3, \qquad B_{44,4} = -3(e^{-z} + 8e^{-2z}). \tag{2.6}$$

Let us choose the associate 1-forms as follows:

$$A_i(x) = \begin{cases} \dfrac{1}{e^z + 4} & \text{for } i = 4 \\ 0 \,, & \text{otherwise,} \end{cases} \tag{2.7}$$

$$(2.8) \qquad D_i(x) = \begin{cases} \dfrac{-37e^z}{(e^z+4)^2} & \text{for } i = 4 \\ 0\ , & \text{otherwise} \end{cases}$$

and

$$(2.9) \qquad E_i(x) = \begin{cases} \dfrac{-3e^{2z}-13}{(e^z+4)^2} & \text{for } i = 4 \\ 0\ , & \text{otherwise,} \end{cases}$$

at any point $x \in \mathbb{R}^4$. In consequence of (2.5), (2.6), (2.7), (2.8) and (2.9) we obtain

$$(2.10) \qquad B_{11,4} + B_{14,1} + B_{14,1} = A_4B_{11} + D_1B_{41} + E_1B_{14},$$

$$(2.11) \qquad B_{22,4} + B_{24,2} + B_{24,2} = A_4B_{22} + D_2B_{42} + E_2B_{24},$$

$$(2.12) \qquad B_{33,4} + B_{34,3} + B_{34,3} = A_4B_{33} + D_3B_{43} + E_3B_{34},$$

$$(2.13) \qquad B_{44,4} + B_{44,4} + B_{44,4} = A_4B_{44} + D_4B_{44} + E_4B_{44},$$

for all other cases (1.5) holds trivially. Therefore, this proves that the manifold $(\mathbb{R}^4, g)$ under consideration is a $(WCBS)_4$ spacetime with non-zero scalar curvature. Hence we can state that:

Theorem 2.1. *Let $(\mathbb{R}^4, g)$ be a Lorentzian manifold endowed with the metric given by*

$$ds^2 = g_{ij}dx^i dx^j = e^{2z}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2$$

where $z = x^4 \neq 0$ and x^1, x^2, x^3, x^4 are the standard coordinates of $\mathbb{R}^4$. Then $(\mathbb{R}^4, g)$ is an $(WCBS)_4$ spacetime with non-zero scalar curvature $r = -12$.

3 $(WCBS)_4$ spacetime

A Lorentzian manifold (M^4, g) is said to be weakly cyclic B symmetric $(WCBS)_4$ spacetime if the B tensor is non-zero and satisfies

$$(3.1) \qquad \begin{aligned} &(\nabla_X B)(Y,Z) + (\nabla_Y B)(Z,X) + (\nabla_Z B)(X,Y) \\ &= A(X)B(Y,Z) + D(Y)B(Z,X) + E(Z)B(X,Y), \end{aligned}$$

for all vector fields X, Y, Z in M^4. Here, 1-forms A, D and E are given by

$$A(X) = g(X, \rho_1), D(X) = g(X, \rho_2), E(X) = g(X, \rho_3),$$

where ρ_1, ρ_2, ρ_3 are timelike vector fields, that is, $g(\rho_i, \rho_i) = -1, i = 1, 2, 3$ corresponding to 1-forms A, D and E respectively.
Interchanging Y and Z in (3.1) we obtain

$$(3.2) \qquad \begin{aligned} &(\nabla_X B)(Z,Y) + (\nabla_Z B)(Y,X) + (\nabla_Y B)(X,Z) = \\ &A(X)B(Z,Y) + D(Z)B(Y,X) + E(Y)B(X,Z). \end{aligned}$$

Combining (3.1) and (3.2) yields

$$[D(Y) - E(Y)]B(X,Z) = [D(Z) - E(Z)]B(X,Z). \tag{3.3}$$

Define a 1-form as $H(X) = D(X) - E(X) = g(X, \rho)$ for all vector fields X. Using this in (3.3) gives

$$H(Y)B(X,Z) = H(Z)B(X,Y). \tag{3.4}$$

Taking a frame field and contracting $X = Z = e_i$ where $\{e_i\}$ is the orthonormal basis of the tangent space at each point in spacetime we get

$$BH(Y) = B(\rho, Y). \tag{3.5}$$

Taking $Z = \rho$ in (3.4) gives

$$H(Y)[aS(X,\rho) + brH(X)] = -B(X,Y). \tag{3.6}$$

Replacing X by ρ in (1.4) in using it in (3.5) we obtain

$$aS(\rho, Y) = (a + 3b)rH(Y). \tag{3.7}$$

In regard of (3.6) and (3.7), we see that

$$S(X,Y) = \alpha g(X,Y) + \beta H(X)H(Y), \tag{3.8}$$

where $\alpha = -\frac{br}{a}$ and $\beta = -\frac{B}{a}$. Hence we can state the following:

Theorem 3.1. *A $(WCBS)_4$ spacetime is a quasi Einstein spacetime.*

Theorem 3.2. *A conformally flat $(WCBS)_4$ spacetime is infinitesimally spatially isotropic relative to the unit timelike vector field ρ.*

Proof. Suppose $(WCBS)_4$ spacetime is conformally flat. Making use of (3.8) and (1.7) in conformally flat $(WCBS)_4$ spacetime we obtain

$$\begin{aligned} R(X,Y)Z &= \frac{1}{2}\Big[-\frac{2br}{a}g(Y,Z)X - \frac{B}{a}H(Y)H(Z)X \\ &+ \frac{2br}{a}g(X,Z)Y + \frac{B}{a}H(X)H(Z)Y \\ &- \frac{B}{a}H(X)g(Y,Z)\rho + \frac{B}{a}H(Y)g(X,Z)\rho\Big] \\ &- \frac{r}{6}\Big[g(Y,Z)X - g(X,Z)Y\Big]. \end{aligned} \tag{3.9}$$

Let $\rho^{\perp}$ denote the 3-dimensional distribution in a conformally flat $(WCBS)_4$ spacetime orthogonal to ρ, then from (3.9) we get

$$R(X,Y)Z = \Big(\frac{br}{a} - \frac{r}{6}\Big)[g(Y,Z)X - g(X,Z)Y], \tag{3.10}$$

for all $X, Y, Z \in \rho^{\perp}$. Also taking $Y = Z = \rho$ in (3.9) gives

$$R(X,\rho)\rho = \frac{r}{6a}(6b + a)X, \tag{3.11}$$

for every $X \in \rho^{\perp}$.
According to Karchar [18], a Lorentzian manifold is called infinitesimal spatially isotropic relative to timelike unit vector field ρ if its curvature tensor R satisfies relations

$$R(X,Y)Z = l[g(Y,Z)X - g(X,Z)Y],$$

for all $X, Y, Z \in \rho^{\perp}$ and

$$R(X,\rho)\rho = mX,$$

for all $X \in \rho^{\perp}$, where l, m are real valued functions on the manifold. Thus in view of (3.10) and (3.11) we see that a conformally flat $(WCBS)_4$ spacetime is infinitesimal spatially isotropic relative to timelike unit vector field ρ.
This completes the proof. □

Theorem 3.3. *In a $(WCBS)_4$ spacetime satisfying $divC = 0$ with assumption $r = \frac{b}{a}$, the integral curve of vector field ρ are geodesic and vector field ρ is irrotational or the spacetime is Yang Pure space.*

Proof. Suppose $(WCBS)_4$ spacetime has harmonic conformal curvature, that is, $divC = 0$. Then (1.7) gives

$$\begin{aligned}(\nabla_X S)(Y,U) \quad &- \quad (\nabla_U S)(Y,X)\\ &= \frac{1}{6}[g(Y,U)dr(X) - g(X,Y)dr(U)]. \end{aligned} \tag{3.12}$$

Making use of (3.8) in (3.12) we obtain

$$\begin{aligned}&\Big\{\frac{adr(U) - rda(U)}{a^2}\Big\}[bg(X,Y) + (a+4b)H(X)H(Y)]\\ &+\frac{r}{a}\Big[db(U)g(X,Y) + \{da(U) + 4db(U)\}H(X)H(Y)\\ &+(a+4b)\{(\nabla_U H)(X)H(Y) + (\nabla_U H)(Y)H(X)\}\Big]\\ &-\Big\{\frac{adr(X) - rda(X)}{a^2}\Big\}[bg(Y,U) + (a+4b)H(Y)H(U)]\\ &-\frac{r}{a}\Big[db(X)g(Y,U) + \{da(X) + 4db(X)\}H(Y)H(U)\\ &+(a+4b)\{(\nabla_X H)(Y)H(U) + (\nabla_X H)(U)H(Y)\}\Big]\\ &= \frac{1}{6}[g(Y,U)dr(X) - g(X,Y)dr(U)]. \end{aligned} \tag{3.13}$$

Taking a frame field and contracting (3.13) over X and Y gives

$$\begin{aligned}&-\Big(1+\frac{b}{a}\Big)dr(U) + \frac{br}{a^2}da(U) - \frac{(a+4b)}{a^2}\big[adr(\rho)\\ &-rda(\rho)\big]H(U) - \frac{r}{a}db(U) - \frac{r}{a}\big[da(\rho)H(U)\\ &+4db(\rho)H(U)\big] - \frac{B}{a}\big[(\delta H)H(U)\\ &+(\nabla_\rho H)(U)\big] = -\frac{1}{2}dr(U), \end{aligned} \tag{3.14}$$

where $(\delta H)=\Sigma_{i=1}^{n}\epsilon_i(\nabla_{e_i}H)(e_i)$. Putting $X=Y=\rho$ in (3.13) we get

$$\begin{aligned}\Big\{\frac{adr(U)-rda(U)}{a^2}\Big\}(a+3b)+\frac{r}{a}\big[3db(U)+da(U)\big]\\ -\frac{b}{a^2}\big\{adr(\rho)-rda(\rho)\big\}H(U)+\frac{(a+4b)}{a^2}\big\{adr(\rho)\\ -rda(\rho)\big\}H(U)+\frac{r}{a}\big\{adb(\rho)+da(\rho)\big\}H(U)\\ +\frac{B}{a}(\nabla_\rho H)(U)=\frac{1}{6}\big[dr(\rho)H(U)+dr(U)\big].\end{aligned}\tag{3.15}$$

Combining (3.14) and (3.15) yields

$$\begin{aligned}\frac{-2br}{a^2}da(U)+\frac{2r}{a}db(U)-\frac{r}{a}db(\rho)H(U)\\ -\frac{B}{a}(\delta H)H(U)+\Big(\frac{4}{3}-\frac{b}{a}\Big)dr(U)\\ -\Big(\frac{1}{6}+\frac{b}{a}\Big)dr(\rho)H(U)+\frac{br}{a^2}da(\rho)H(U)=0.\end{aligned}\tag{3.16}$$

Replacing U by ρ in (3.16) and using it in (3.16) results in the following

$$\begin{aligned}\frac{2r}{a}\Big[\frac{adb(U)-bda(U)}{a}\Big]+\frac{2r}{a}\Big[\frac{adb(\rho)-bda(\rho)}{a}\Big]H(U)\\ +\Big(\frac{4}{3}-\frac{b}{a}\Big)\big[dr(U)+dr(\rho)H(U)\big]=0.\end{aligned}\tag{3.17}$$

If possible, suppose $r=\frac{b}{a}$, then

$$dr(U)=\frac{adb(U)-bda(U)}{a^2},\tag{3.18}$$

for any vector field U. In consequence of (3.17) and (3.18) we see that either $4a+3b=0$ or $dr(U)=-dr(\rho)H(U)$. Considering the case when $4a+3b=0$, we see that $r=\frac{-4}{3}$ is a constant, and hence $dr=0$. Using this facts in (3.12) gives

$$(\nabla_X S)(Y,U)-(\nabla_U S)(Y,X)=0.$$

This means that $(WCBS)_4$ spacetime is a Yang Pure space [30].
Suppose $4a+3b\neq 0$. Replacing Y by ρ in (3.13) and using $dr(U)=-dr(\rho)H(U)$ yields

$$(\nabla_X H)(U)-(\nabla_U H)(X)=0.\tag{3.19}$$

This means that the 1-form H is closed. Thus we get

$$g(\nabla_X\rho,U)=g(\nabla_U\rho,X)$$

for all X,U. Taking $U=\rho$ gives

$$g(\nabla_\rho\rho,X)=g(\nabla_X\rho,\rho).$$

Since $g(\nabla_X\rho,\rho)=0$ implies $g(\nabla_\rho\rho,X)=0$ for all X. Hence $\nabla_\rho\rho=0$. This means that the integral curve of the vector field ρ are geodesic and vector field is irrotational. This completes the proof. □

A vector field ρ is a Killing vector if

$$g(Y,\nabla_\rho X)+g(\nabla_\rho Y,X)=0, \tag{3.20}$$

for any vector fields X, Y. Hence we can state the following:

Corollary 3.4. *If a $(WCBS)_4$ spacetime with non-constant scalar curvature satisfies $divC = 0$ and fulfills the condition $r = \frac{b}{a}$, then the vector field ρ is a Killing vector if and only if ρ is parallel vector.*

4 Some geometrical properties of $(WCBS)_4$ spacetime

The k-nullity distribution $N(k)$ of a pseudo-Riemannian manifold M^n is defined by [27]

$$\begin{aligned} N(k) &: p\to N_p(k) \\ &= \{Z\in T_p(M)|R(X,Y)Z=k[g(Y,Z)X-g(X,Z)Y]\} \end{aligned}$$

for all $X, Y \in TM$, where k is some smooth function. If the generator ρ of the quasi-Einstein manifold M^n belongs to the k-nullity distribution $N(k)$ for some smooth function k, then M^n is called $N(k)$-quasi Einstein manifold [28].

According to Deszcz [2, 13, 14], for (0,4)-tensor field T if $R\cdot T$ and $Q(g,T)$ are linearly dependent, that is, $R\cdot T = L_T Q(g,T)$ holds on the set $U_T = \{x \in M : Q(g,T) \neq 0 \text{ at } x\}$, where L_T is some function on U_T. In particular, if $T = R$(resp., $S, C, \widetilde{C}$) then the manifold is called pseudosymmetric (resp., Ricci-pseudosymmetric, conformally pseudosymmetric, concircularly pseudosymmetric). De and Velimirović [8] studied spacetimes with semisymmetric Energy-Momentum tensor and showed that such a spacetime is Ricci semisymmetric.

In this section, $(WCBS)_4$ spacetime under certain curvature conditions such as Ricci-pseudosymmetric, conformal Ricci semisymmetric and concircular Ricci-pseudosymmetric are studied.

Theorem 4.1. *Every Ricci-pseudosymmetric $(WCBS)_4$ spacetime with non-vanishing scalar B is an $N(\frac{B-br}{3a})$-quasi Einstein spacetime.*

Proof. Suppose $(WCBS)_4$ spacetime is Ricci-pseudosymmetric, that is, $R\cdot S = L_S Q(g,S)$ holds on U_s and L_S is a certain function on U_S. Thus we get

$$\begin{aligned} &S(R(X,Y)U,V)+S(U,R(X,Y)V)= \\ &L_S[g(Y,U)S(X,V)-g(X,U)S(Y,V) \\ &+g(Y,V)S(U,X)-g(X,V)S(Y,U)]. \end{aligned} \tag{4.1}$$

In consequence of (3.8) in (4.1) we obtain

$$\begin{aligned} &H(R(X,Y)U)H(V)+H(U)H(R(X,Y)V)= \\ &L_S[g(Y,U)H(X)H(V)-g(X,U)H(Y)H(V) \\ &+g(Y,V)H(X)H(U)-g(X,V)H(Y)H(U)]. \end{aligned} \tag{4.2}$$

Contracting (4.2) over X and V and using (3.8) yields

$$R(\rho, Y)U = L_S g(Y,U)\rho + \Big[\frac{B}{a} - \frac{br}{a} - 4L_S\Big]H(Y)U. \tag{4.3}$$

Substituting $Y = U = \rho$ in (4.3) we get the following relation

$$L_S = \frac{(B - br)}{3a}. \tag{4.4}$$

Taking $U = \rho$ in (4.2) gives

$$R(X,Y)\rho = L_S[H(Y)X - H(X)Y]. \tag{4.5}$$

Making use of (4.4) in (4.5) we obtain

$$R(X,Y)\rho = \frac{B - br}{3a}[H(Y)X - H(X)Y]. \tag{4.6}$$

This means that the vector field ρ belongs to the $(\frac{B-br}{3a})$-nullity distribution. This completes the proof. □

If we take $L_S = 0$, then the manifold satisfies the condition $R \cdot S = 0$ and so it is Ricci semisymmetric. In this case, we see that $B = br$ implies $a = -3b$. Hence we can state the following:

Corollary 4.2. *In a Ricci semisymmetric $(WCBS)_4$ spacetime with non-vanishing scalar B the relation $a + 3b = 0$ holds.*

Theorem 4.3. *In a $(WCBS)_4$ spacetime with non-vanishing scalar B satisfying $C(X,Y) \cdot S = 0$, $\Big(\frac{5B-8br}{12a}\Big)$ is an eigenvalue of the Ricci operator Q.*

Proof. Proceeding similarly as in Theorem 4.1, we obtain the following relation

$$\begin{aligned} g(R(X,Y)U,\rho) &= \Big(\frac{5B - 8br}{12a}\Big)[g(Y,U)H(X) \\ &- g(X,U)H(Y)]. \end{aligned} \tag{4.7}$$

Contracting (4.7) over X and U yields

$$S(Y,\rho) = \Big(\frac{5B - 8br}{12a}\Big)g(Y,\rho), \tag{4.8}$$

i.e., $QY = \Big(\frac{5B-8br}{12a}\Big)Y$ for all vector field Y. Thus $\Big(\frac{5B-8br}{12a}\Big)$ is an eigenvalue of the Ricci operator Q. This completes the proof. □

Suppose $(WCBS)_4$ spacetime is concircularly pseudosymmetric, that is, $R \cdot \widetilde{C} = L_S Q(g,S)$. Then proceeding similarly as in Theorem 4.1 and Theorem 4.3 one can easily obtained the following:

Theorem 4.4. *In a $(WCBS)_4$ spacetime with non-vanishing scalar B satisfying $R \cdot \widetilde{C} = L_S Q(g,S)$, $\frac{3(B-br)}{5a}$ is an eigenvalue corresponding to Ricci operator Q and the timelike vector field ρ belongs to the $N(\frac{B-br}{5a})$-quasi Einstein spacetime.*

5 Application of $(WCBS)_4$ spacetime in General Relativity

The general theory of relativity postulate that the spacetime should be described as a curved manifold. The Einstein's field equation [23] relate the geometry of spacetime with the distribution of matter within it. Einstein's field equation is conferred by

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = kT(X,Y), \tag{5.1}$$

for all vector fields X, Y where S is the Ricci tensor of type (0,2), r is the scalar curvature, λ is the cosmological constant and k is the gravitational constant. Eq. (5.1) imply that the matter detrmines the geometry of spacetime and conversely that the motion of matter is determined by the metric tensor of the space which is not flat. Here, T is the energy momentum tensor which is a symmetric $(0,2)$-tensor with divergence zero.

The energy momentum tensor is said to describe a perfect fluid [23] if

$$T(X,Y) = (\sigma + p)H(X)H(Y) + pg(X,Y), \tag{5.2}$$

where σ is the energy density and p is the isotropic pressure of the fluid, H is a non-zero 1-form such that

$$g(X,\rho) = H(X),$$

for all X, ρ being the velocity vector field of the fluid which is a timelike vector, that is, $g(\rho,\rho) = H(\rho) = -1$.
Combining (3.8) and (5.1), the energy momentum tensor can be written as

$$T(X,Y) = \frac{r[a(2\lambda - 1) - 2b]}{2ak}g(X,Y) - \frac{B}{ak}H(X)H(Y), \tag{5.3}$$

Thus we can state the following:

Proposition 5.1. *A $(WCBS)_4$ spacetime satisfying Einstein's field equation with cosmological constant can be considered as a model of perfect fluid spacetime, in General Relativity.*

Inserting (5.2) in (5.1) without cosmological constant, we obtain

$$S(X,Y) = k(\sigma + p)H(X)H(Y) + (kp + \frac{r}{2})g(X,Y). \tag{5.4}$$

Comparing (3.8) and (5.4), we see that in a perfect fluid $(WCBS_4)$ spacetime the following relations hold

$$\alpha = -\frac{br}{a} = (kp + \frac{r}{2}) \text{ and } \beta = -\frac{B}{a} = k(\sigma + p). \tag{5.5}$$

Replacing X by QX in (5.4) and using (3.8) gives

$$\begin{aligned} S^2(X,Y) &= k(\sigma + p)(\alpha - \beta)H(X)H(Y) \\ &+ \frac{k}{2}(\sigma - p)[\alpha g(X,Y) + \beta H(X)H(Y)], \end{aligned} \tag{5.6}$$

where $S^2(X,Y) = S(QX,Y)$. Taking a frame field and contracting (5.6) over X and Y yields

$$||Q||^2 = k^2(\sigma^2 + 2p^2 - \sigma p). \tag{5.7}$$

Hence we can state the following:

Theorem 5.2. *If a perfect fluid $(WCBS)_4$ spacetime obeying Einstein's field equation without cosmological constant, then the square of the length of the Ricci operator is $k^2(\sigma^2 + 2p^2 - \sigma p)$.*

In view of (5.4), if perfect fluid $(WCBS)_4$ spacetime satisfies the timelike convergence condition, that is, $S(\rho,\rho) \geq 0$ then $\sigma + 3p \geq 0$, thus the spacetime obeys cosmic strong energy condition. Thus we can state

Proposition 5.3. *If a perfect fluid $(WCBS)_4$ spacetime obeying Einstein's field equation without the cosmological constant satisfies timelike convergence condition, then the spacetime obeys strong energy condition.*

In cosmology we know that when $\sigma = -p$ this lead to rapid expansion of the spacetime which is termed as inflation. Also $\sigma + p = 0$ is known as Phantom Barrier [7]. Here the fluid behaves as a cosmological constant [25]. And if $\sigma + 3p = 0$ then strong energy condition begins to violate and fluid behaves as exotic matter. This is termed as a Quintessence Barrier. Recent observations have indicated that our universe is in quintessence era [3].

In consequence of (5.5), we get $\sigma + p = -\frac{B}{ak}$ and $\sigma + 3p = -\frac{r}{k}$.
Suppose that the scalar B vanishes, it follows that either (i) $a + 4b = 0$ or (ii) $r = 0$. Now (i) $a + 4b = 0$ implies $\sigma + p = 0$. Thus the spacetime represents phantom barrier. Again (ii) $r = 0$ implies $\sigma + 3p = 0$. Thus the spacetime represents quintessence barrier. Hence we can state the following:

Theorem 5.4. *If a perfect fluid $(WCBS)_4$ spacetime with vanishing scalar B obeys Einstein's field equation without cosmological constant then the spacetime is characterized by the following cases:*
(i) The spacetime represents inflation and the fluid behaves as a cosmological constant. This is also termed as a phantom barrier.
(ii) The spacetime represents quintessence barrier and the fluid behaves as exotic matter.

Next we state and proof the following:

Theorem 5.5. *A relativistic fluid $(WCBS)_4$ spacetime obeying Einstein's field equation with the cosmological constant admit heat flux, provided $\lambda + k\sigma \neq \frac{3B-2br}{2a}$.*

Proof. For a relativistic fluid matter distribution, the energy momentum tensor is as follows [15]

$$\begin{aligned} T(X,Y) &= pg(X,Y) + (\sigma + p)A(X)A(Y) \\ &+ A(X)B(Y) + B(Y)A(Y), \end{aligned} \tag{5.8}$$

where $A(X) = g(X,\rho), A(\rho) = -1, B(X) = g(X,\mu), B(\mu) > 0, g(\rho,\mu) = 0$. Here ρ is the velocity vector field and μ is the heat conduction vector field.
Making use of (5.8), the Einstein's field equation becomes

$$\begin{aligned} S(X,Y) &= (kp - \lambda + \frac{r}{2})g(X,Y) + k(\sigma+p)A(X)A(Y) \\ &+ k[A(X)B(Y) + B(X)A(Y)]. \end{aligned} \tag{5.9}$$

Inserting (3.8) in (5.9) gives

$$\begin{aligned} \Big(\alpha &- kp + \lambda - \frac{r}{2}\Big)g(X,Y) + [\beta - k(\sigma+p)]A(X)A(Y) \\ &- k[A(X)B(Y) + B(X)A(Y)] = 0. \end{aligned} \tag{5.10}$$

Replacing X by ρ in (5.10) we obtain

$$B(Y) = \frac{1}{k}\Big(\frac{br}{a} - \frac{B}{a} - k\sigma + \frac{r}{2} - \lambda\Big)A(Y). \tag{5.11}$$

Thus the spacetime admit heat flux if $\lambda + k\sigma \neq \frac{3B-2br}{2a}$. This completes the proof. □

Next we consider viscous fluid matter, under which the energy momentum tensor is of form:

$$T(X,Y) = pg(X,Y) + (\sigma+p)H(X)H(Y) + P(X,Y), \tag{5.12}$$

where P denotes the anisotropic pressure tensor of the fluid.
Combining (5.12), (5.1) and (3.8) yields

$$\begin{aligned} (\alpha - \frac{r}{2} - kp)g(X,Y) &+ [\beta - k(\sigma+p)]H(X)H(Y) \\ &= kP(X,Y). \end{aligned} \tag{5.13}$$

Replacing X and Y by ρ in (5.13) we get

$$-(\alpha - \frac{r}{2} - kp) + \beta - k(\sigma+p) = kI, \tag{5.14}$$

where $I = P(\rho,\rho)$. Contracting (5.13) over X and Y gives

$$4(\alpha - \frac{r}{2} - kp) - \beta + k(\sigma+p) = kJ, \tag{5.15}$$

where $J =$ Trace of P. Adding (5.14) and (5.15) the expression for isotropic pressure is given by

$$p = \frac{1}{k}\Big\{\lambda - \frac{br}{a} - \frac{r}{2} - \frac{k(I+J)}{3}\Big\}. \tag{5.16}$$

In consequence of (5.16) in (5.14) the expression for energy density is given by

$$\sigma = \frac{1}{k}\Big(\frac{br}{a} + \frac{r}{2} - \lambda - \frac{B}{a}\Big). \tag{5.17}$$

Thus we can state the following:

Theorem 5.6. *In a viscous fluid $(WCBS)_4$ spacetime obeying Einstein's field equation with cosmological constant the energy density and isotropic pressure are given by (5.17) and (5.16) respectively.*

For a pressureless fluid spacetime (dust), the energy momentum tensor is of form $T(X,Y) = \sigma H(X)H(Y)$. Proceeding similarly as in Theorem 5.6 one can easily obtain the follow:

Proposition 5.7. *A dust $(WCBS)_4$ spacetime obeying Einstein's field equation with cosmological constant is vacuum if and only if scalar B vanishes.*

Definition 5.1. A symmetric tensor b_{ij} is Weyl compatible if

$$b_{im}C^m_{jkl} + b_{jm}C^m_{kil} + b_{km}C^m_{ijl} = 0. \tag{5.18}$$

Now we examine the Weyl compatibility of $(WCBS)_4$ spacetime. In accordance of Corollary 3.4, suppose ρ is Killing vector field then ρ is parallel vector and hence we get

$$R(X,Y)\rho = [\nabla_X, \nabla_Y]\rho - \nabla_{[X,Y]}\rho = 0. \tag{5.19}$$

Contracting (5.19) over X and using (3.8) we see that $(a + 3b)r = 0$. But, $r \neq 0$, hence $a + 3b = 0$. Making use of this in (3.8) yields

$$S(X,Y) = -\frac{1}{9}[g(X,Y) + H(X)H(Y)]. \tag{5.20}$$

In consequence of (5.20) the Weyl tensor is of the form

$$\begin{aligned} C_{ijkl} = R_{ijkl} + \frac{1}{12}[g_{jk}g_{il} - g_{ik}g_{jl}] + \frac{1}{18}[g_{il}H_jH_k \\ -g_{jl}H_iH_k + g_{jk}H_iH_l - g_{ik}H_jH_l]. \end{aligned} \tag{5.21}$$

Since the generator ρ is parallel so transvecting (5.21) by H^l we get

$$H^lC_{ijkl} = \frac{1}{36}[g_{jk}H_i - g_{ik}H_j]. \tag{5.22}$$

In view of (5.22) we can obtain the following relation

$$(H_iC_{jklm} + H_jC_{kilm} + H_kC_{ijlm})H^m = 0. \tag{5.23}$$

Thus we can state the following:

Theorem 5.8. *In a $(WCBS)_4$ spacetime with non-constant scalar curvature satisfying $divC = 0$ and fulfilling the condition $r = \frac{b}{a}$, if ρ is Killing vector then the spacetime is Weyl compatible.*

In General Relativity, given a timelike vector field u with $u^iu_i = -1$, then the electric and magnetic components of Weyl tensor are defined by

$$E_{kl} = u^ju^mC_{jklm}, \tag{5.24}$$

$$H_{kl} = \frac{1}{2}\varepsilon_{jkrs}u^j u^m C^{rs}_{lm}, \tag{5.25}$$

where the components C^{rs}_{lm} is of type (2,2) of the Weyl tensor and ε_{jkrs} denotes the completely skew-symmetric Levi-Civita symbol. In [22] it is shown that on a 4-dimensional spacetime a timelike vector field is Weyl compatible if and only if the magnetic part of the Weyl tensor vanishes i.e., $H_{kl} = 0$. In regard of Theorem 5.8 and above result we obtain the following:

Proposition 5.9. *In a $(WCBS)_4$ spacetime with non-constant scalar curvature satisfying $divC = 0$ and fulfilling the condition $r = \frac{b}{a}$, if ρ is Killing vector then it is a purely electric spacetime.*

If the electric and magnetic parts of the Weyl tensor are proportional i.e., $\gamma E = \mu H$ for some scalar fields γ and μ including the case when one of them is zero, then the space is of type I, D or O. But $E_{kl} = \frac{R_{kl}}{4} \neq 0$, the Weyl tensor is non-vanishing so the space cannot be of type O. Thus we can state

Proposition 5.10. *In a $(WCBS)_4$ spacetime with non-constant scalar curvature satisfying $divC = 0$ and fulfilling the condition $r = \frac{b}{a}$, if ρ is Killing vector then the possible Petrov types are I or D.*

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References

[1] Z. Ahsan, S. A. Siddiqui, *Concircular curvature tensor and fluid spacetimes*, Int. J. Theor. Phys., 48, 11 (2009), 3202-3212.

[2] K. Arslan, Y. Celik, R. Deszcz, R. Ezentas, *On the equivalence of the Ricci-semisymmetry and semisymmetry*, Colloquium Math., 76, 2 (1998), 279-294.

[3] L. Amendola, S. Tsujikawa, *Dark Energy: Theory and Observations*, Cambridge University Press, Cambridge, 2010.

[4] J. K. Beem, P. E. Ehrlich, *Global Lorentzian Geometry*, Marcel Dekker, New York, 1981.

[5] M. C. Chaki, R. K. Maity, *On quasi Einstein manifolds*, Publ. Math. Debrecen, 57, (2000), 297-306.

[6] M. C. Chaki, S. Roy, *Space-times with covariant constant energy-momentum tensor*, Int. J. Theor. Phys., 35, (1996), 1027-1032.

[7] S. Chakraborty, N. Mazumder, R. Biswas, *Cosmological evolution across phantom crossing and the nature of the horizon*, Astrophys. Space Sci., 334, (2011), 183-186.

[8] U. C. De, L. Velimirović, *Spacetimes with semisymmetric energy-momentum tensor*, Int. J. Theor. Phys., 54, (2015), 1779-1783.

[9] U. C. De, G. C. Ghosh, *On weakly Ricci symmetric spacetime manifolds*, Radovi Matematicki, 13, (2004), 93-101.

[10] U. C. De, C. A. Mantica, Y. J. Suh, *On weakly cyclic Z symmetric manifolds*, Acta Math. Hungar., 146, 1 (2015), 153-167.

[11] U. C. De, C. A. Mantica, L. G. Molinair, Y. J. Suh, *On Weakly cyclic Z symmetric spacetimes*, Acta Math. Hungar., 149, (2016), 462-477.

[12] U. C. De, G. C. Ghosh, *On quasi Einstein and special quasi Einstein manifolds*, In: Proc. of the Int. Conf. of Mathematics and its Applications, Kuwait university, 2004, 178-191.

[13] R. Deszcz, *On pseudosymmetric spaces*, Bull. Belg. Math. soc. Ser. A, 44, (1992), 1-34.

[14] R. Deszcz, M. Hotlos, Z. Senturk, *On the equivalence of the Ricci-pseudosymmetry and pseudosymmetry*, Colloquium Math., 79, 2 (1999), 211-227.

[15] G. F. R. Ellis, Sachs R K (ed.), *Relativistic Cosmology in General Relativity and Cosmology*, Academic Press, London, 1971.

[16] G. Endean, *Cosmology in conformally flat spacetime*, The Astrophysics Journal, 479, (1997), 40-45.

[17] B. P. Geroch, *Spacetime Structure from a Global View Point*, Academic Press, New York, 1971.

[18] H. Karchar, *Infinitesimal characterization of Friedmann Universe*, Arch. Math. Basel., 38, (1992), 58-64.

[19] C. A. Mantica, L. G. Molinari, *Weakly Z symmetric manifolds*, Acta Math. Hungar., 135, 1-2 (2012), 80-96.

[20] C. A. Mantica, U. C. Dey, Y. J. Suh, P. Pal, *On pseudo B-symmetric manifolds*, Int. J. Geom. Meth. Mod. Phys., 14, 9 (2017), 1750119.

[21] C. A. Mantica, Y. J. Suh, *Pseudo-Z symmetric space-times*, J. Math. Phys., 55, (2014), 042502.

[22] C. A. Mantica, L. G. Molinari, *Riemann compatible tensors*, Colloq. Math., 128, 2 (2012), 197-210.

[23] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.

[24] J. P. Singh, M. Khatri, *On Ricci-Yamabe soliton and geometrical structure in a perfect fluid spacetime*, Afr. Mat., 32, (2021), 1645-1656.

[25] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt,*Exact Solutions of Einstein's Field Equations*, 2nd edn. Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 2003.

[26] L. Tamássy, T. Q. Binh, *On weakly symmetries of Einstein and Sasakian manifolds*, Tensor N.S., 53, (1993), 140-148.

[27] S. Tanno, *Ricci curvatures of contact Riemannain manifolds*, Tohoku Math. J., 40, 3 (1988), 441-448.

[28] M. M. Triphati, J. S. Kim, *On $N(k)$-Einstein manifolds*, Commun. Korean Math. Soc., 22, 3 (2007), 411-417.

[29] K. Yano, M. Kon, *Structures of Manifolds*, World Scientific Publishing, Singapore, 1984.

[30] C. N. Yang, *Integral formalism for gauge fields*, Phys. Rev. Lett., 33 (1974), 445-447.

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Article

Improved Chen's Inequalities for Submanifolds of Generalized Sasakian-Space-Forms

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Abstract: In this article, we derive Chen's inequalities involving Chen's δ-invariant δ_M, Riemannian invariant $\delta(m_1, \ldots, m_k)$, Ricci curvature, Riemannian invariant $\Theta_k(2 \leq k \leq m)$, the scalar curvature and the squared of the mean curvature for submanifolds of generalized Sasakian-space-forms endowed with a quarter-symmetric connection. As an application of the obtain inequality, we first derived the Chen inequality for the bi-slant submanifold of generalized Sasakian-space-forms.

Keywords: Chen inequalities; quarter-symmetric connection; generalized Sasakian-space-form; bi-slant; Riemannian invariants

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1. Introduction

In submanifold theory, obtaining the relationship between an intrinsic invariant and an extrinsic invariant has been the primary goal of many geometers in recent decades. Chen invariants were introduced by B.Y. Chen [1] to tackle the question raised by Chen concerning the existence of minimal immersions into a Euclidean space of arbitrary dimension [2]. Chen's δ-invariant δ_M of a Riemannian manifold M introduced by Chen is

$$\delta_M(x) = \tau(x) - \inf\{K(\Pi) | \Pi \text{ is a plane section } \subset T_xM\}, \tag{1}$$

where τ is the scalar curvature of M.

In [1], Chen obtained an inequality for a Riemannian submanifold M^m of a real space form $\widetilde{M}$ with constant sectional curvature c as

$$\delta_M \leq \frac{m^2(m-2)}{2(m-1)} \parallel H \parallel^2 + \frac{1}{2}(m+1)(m-2)c, \tag{2}$$

where H is the mean curvature of the submanifold M^m. Equation (2) is known as the first Chen inequality.

Then in [3], Chen gave the inequality for a Riemannian submanifold M^m of complex-space-form $\widetilde{M}^n(4c)$ as follows:

$$\delta_M \leq \frac{m^2(m-2)}{2(m-1)} \parallel H \parallel^2 + \frac{1}{2}(m+1)(m-2)c + \frac{3}{2} \parallel P \parallel^2 c - 3\Theta(\pi)c. \tag{3}$$

Afterward, many authors obtained Chen's inequalities for different submanifolds in various ambient spaces, such as the Kenmotsu space form [4], the Sasakian-space-form [5], the Cosympletic space form [6], the Riemannian manifold of quasi-constant curvature [7], generalized space forms [8,9], Statistical manifolds [10–12], quaternionic space forms [13] and the GRW spacetime [14].

Qu and Wang [15] introduced the notion of a special type of quarter-symmetric connection as a generalization of a semi-symmetric metric connection [16] and a semi-symmetric non-metric connection [17]. They studied the Einstein warped product and multiple warped products with a quarter-symmetric connection [15]. In [18], the authors obtained Chen's inequalities for submanifolds of real space forms endowed with a quarter-symmetric connection. Mihai and Özgür [19] obtained the Chen inequalities for submanifolds of complex space forms and Sasakian-space-forms with a semi-symmetric metric connection. Wang [20] obtained Chen inequalities for submanifolds of complex space forms and Sasakian-space-forms with quarter-symmetric connections which improved the results of Mihai and Özgür [19]. Sular [21] obtained Chen inequalities for submanifolds of generalized space forms with a semi-symmetric metric connection. Al-Khaldi et al. [22] obtained the Chen–Ricci inequalities Lagrangian submanifold in generalized complex space form and a Legendrian submanifold in a generalized Sasakian-space-form endowed with the quarter-symmetric connection.

As a continuation of their studies, we obtained Chen inequalities for submanifolds of generalized Sasakian-space-form admitting a quarter-symmetric connection. The significance of this study is that it generalizes a large number of previously obtained results, some of which are [20,21]. The paper is organized as follows. In Section 2, we recall the properties of the quarter-symmetric connection. In Section 3, we establish the B.Y. Chen inequalities for submanifolds of a generalized Sasakian-space-form endowed with a quarter-symmetric connection. First, we prove the following inequality and also look at its equality case.

Theorem 1. *Let $M^m, m \geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a quarter-symmetric connection $\overline{\nabla}$, then*

$$\begin{aligned}\tau(x) - K(\Pi) \leq{}& (m-2)\Big(\frac{m^2}{2(m-1)} \parallel H \parallel^2 + (m+1)\frac{f_1}{2}\Big) \\ &+ \Big(3 \parallel \mathcal{T} \parallel^2 - 6\Theta^2(\Pi)\Big)\frac{f_2}{2} + \Big(\parallel \xi_\Pi \parallel^2 - (m-1) \parallel \xi^T \parallel^2 \Big) f_3 \\ &+ \frac{(\psi_1+\psi_2)}{2}\Big(tr(\beta_1 \mid_\Pi) - \lambda(m-1)\Big) + \frac{\psi_2(\psi_1-\psi_2)}{2}\Big(tr(\beta_2 \mid_\Pi) \\ &- \mu(m-1)\Big) + \frac{(\psi_1-\psi_2)}{2}\Big(\Lambda(tr(h \mid_\Pi)) - m(m-1)\Lambda(H)\Big),\end{aligned}$$

where Π is a two-plane section T_xM, $x \in M$.

Next, we obtain bounds for the Riemannian invariant $\delta(m_1, \ldots, m_k)$ and a Ricci curvature in terms of the scalar curvature of the r-plane section L, squared mean curvature and some special functions. Among others, we obtain the inequality involving the Riemannian invariant $\Theta_k, 2 \leq k \leq m$, as follows:

$$\begin{aligned}\parallel H \parallel^2 (x) \geq{}& \Theta_k(x) - f_1 - \frac{3f_2}{m(m-1)} \parallel \mathcal{T} \parallel^2 + \frac{2f_3}{m} \parallel \xi^T \parallel^2 \\ &+ \frac{\lambda}{m}(\psi_1+\psi_2) + \frac{\mu}{m}\psi_2(\psi_1-\psi_2) + (\psi_1-\psi_2)\Lambda(H).\end{aligned}$$

Using Theorem 1 in Section 4, we derive Chen inequalities for the bi-slant submanifold of generalized Sasakian-space-forms.

2. Preliminaries

Suppose that $\widetilde{M}^{m+p}$ is an $(m+p)$-dimensional Riemannian manifold with Riemannian metric g. A linear connection $\overline{\nabla}$ is known as a quarter-symmetric connection if its torsion tensor T is presented by

$$T(X_1, X_2) = \overline{\nabla}_{X_1} X_2 - \overline{\nabla}_{X_2} X_1 - [X_1, X_2]$$

satisfies

$$T(X_1, X_2) = \Lambda(X_2)\varphi X_1 - \Lambda(X_1)\varphi X_2,$$

where Λ is a 1-form, P is a vector field given by $\Lambda(X_1) = g(X_1, P)$, and φ is $(1,1)$-tensor. In [15], the authors introduced a special type of quarter-symmetric connection defined as:

$$\overline{\nabla}_{X_1} X_2 = \widehat{\widehat{\nabla}}_{X_1} X_2 + \psi_1 \Lambda(X_2) X_1 - \psi_2 g(X_1, X_2) P, \tag{4}$$

where $\widehat{\widehat{\nabla}}$ denote the Levi–Civita connection. It is easy to see that the quarter-symmetric connection $\overline{\nabla}$ includes the semi-symmetric metric connection ($\psi_1 = \psi_2 = 1$) and the semi-symmetric non-metric connection ($\psi_1 = 1, \psi_2 = 0$). Let the curvature tensor of $\overline{\nabla}$ be

$$\overline{R}(X_1, X_2)X_3 = \overline{\nabla}_{X_1}\overline{\nabla}_{X_2}X_3 - \overline{\nabla}_{X_2}\overline{\nabla}_{X_1}X_3 - \overline{\nabla}_{[X_1,X_2]}X_3.$$

Similarly, the curvature tensor $\widehat{\widehat{R}}$ of $\widehat{\widehat{\nabla}}$ can be defined as the same.

Let M^m be an m-dimensional submanifold of an $(m+p)$-dimensional Riemannian manifold $\widetilde{M}^{m+p}$ endowed with the quarter-symmetric connection $\overline{\nabla}$ and the Levi–Civita connection $\widehat{\widehat{\nabla}}$. Let ∇ and $\widehat{\nabla}$ denote the induced quarter-symmetric connection and the induced Levi–Civita connection on the submanifold M. The Gauss formula with respect to ∇ and $\widehat{\nabla}$ can be presented as

$$\overline{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + h(X_1, X_2), \quad X_1, X_2 \in \Gamma(TM)$$
$$\widehat{\widehat{\nabla}}_{X_1} X_2 = \widehat{\nabla}_{X_1} X_2 + \widehat{h}(X_1, X_2), \quad X_1, X_2 \in \Gamma(TM)$$

where h and $\widehat{h}$ are the second fundamental forms associated with the quarter-symmetric connection ∇ and the Levi-Civita connection $\widehat{\nabla}$, respectively, and are related as follows:

$$h(X_1, X_2) = \widehat{h}(X_1, X_2) - \psi_2 g(X_1, X_2) P^{\perp}, \tag{5}$$

where $P^{\perp}$ is the normal component of the vector field P on M. If P^T represents that tangent component of the vector field P on M, then $P = P^T + P^{\perp}$.

The curvature tensor $\overline{R}$ with respect to the quarter-symmetric connection $\overline{\nabla}$ on $\widetilde{M}^{m+p}$ can be expressed as [15]:

$$\begin{aligned}\overline{R}(X_1, X_2, X_3, X_4) &= \widehat{\widehat{R}}(X_1, X_2, X_3, X_4) + \psi_1 \beta_1(X_1, X_3) g(X_2, X_3) \\ -\psi_1 \beta_1(X_2, X_3) g(X_1, X_4) &+ \psi_2 g(X_1, X_3)\beta_1(X_2, X_4) - \psi_2 g(X_2, X_3)\beta_1(X_1, X_4) \\ +\psi_2(\psi_1 - \psi_2) g(X_1, X_3)&\beta_2(X_2, X_4) - \psi_2(\psi_1 - \psi_2) g(X_2, X_3)\beta_2(X_1, X_4),\end{aligned} \tag{6}$$

where β_1 and β_2 are symmetric $(0,2)$-tensor fields defined as

$$\beta_1(X_1, X_2) = (\widehat{\widehat{\nabla}}_{X_1}\Lambda)(X_2) - \psi_1 \Lambda(X_1)\Lambda(X_2) + \frac{\psi_2}{2} g(X_1, X_2)\Lambda(P),$$

and

$$\beta_2(X_1, X_2) = \frac{\Lambda(P)}{2} g(X_1, X_2) + \Lambda(X_1)\Lambda(X_2).$$

Moreover, we assume that $tr(\beta_1) = \lambda$ and $tr(\beta_2) = \mu$.

Suppose that R and $\widehat{R}$ are the curvature tensors of ∇ and $\widehat{\nabla}$, respectively. Then the Gauss equation with respect to the quarter-symmetric connection is as follows [15]:

$$\begin{aligned}\overline{R}(X_1, X_2, X_3, X_4) &= R(X_1, X_2, X_3, X_4) - g(h(X_1, X_4), h(X_2, X_3)) \\ &+ g(h(X_2, X_4), h(X_1, X_3)) + (\psi_1 - \psi_2) g(h(X_2, X_3), P) g(X_1, X_4) \\ &+ (\psi_2 - \psi_1) g(h(X_1, X_3), P) g(X_2, X_4).\end{aligned} \tag{7}$$

Let $\{e_1, \ldots, e_m\}$ and $\{e_{m+1}, \ldots, e_{m+p}\}$ be an orthonormal frame of T_xM and $T_x^{\perp}M$ at the point $x \in M$. Then the mean curvature vector of M associated with ∇ is $H = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_j)$. Similarly, the mean curvature vector of M associated to $\widehat{\nabla}$ is $\widehat{H} = \frac{1}{m}\sum_{i=1}^{m} \widehat{h}(e_i, e_j)$. In addition, the squared length of h is $\| h \|^2 = \sum_{i,j=1}^{m} g(h(e_i, e_j), h(e_i, e_j))$.

Now, we recall some of the Riemannian invariants introduced by Chen [23] in a Riemannian manifold. Let L be an r-dimensional subspace of T_xM, $x \in M$, $r \geq 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of L. The scalar curvature τ of the r-plane section L is given by

$$\tau(L) = \sum_{1 \leq i < j \leq r} K_{ij}, \tag{8}$$

where K_{ij} is the sectional curvature of the plane section spanned by e_i and e_j at $x \in M$. Suppose that $\Pi \subset T_xM$ is a two-plane section and $K(\Pi)$ is the sectional curvature of M for a plane section Π in T_xM, $x \in M$. Then

$$K(\Pi) = \frac{1}{2}[R(e_1, e_2, e_2, e_1) - R(e_1, e_2, e_1, e_2)]. \tag{9}$$

The scalar curvature $\tau(x)$ of M at the point x is presented by

$$\tau(x) = \sum_{i<j} K_{ij}, \tag{10}$$

where $\{e_1, \ldots, e_m\}$ is an orthonormal basis for T_xM.

3. B. Y. Chen Inequalities

First, we recall the well-known lemma obtained by Chen [1], which is as follows:

Lemma 1. *If $a_1, \ldots, a_m, a_{m+1}$ are $m+1$ $(m \geq 2)$ real numbers such that*

$$\Big(\sum_{i=1}^{m} a_i\Big)^2 = (m-1)\Big(\sum_{i=1}^{m} a_i^2 + a_{m+1}\Big),$$

then $2a_1a_2 \geq a_{m+1}$, with equality holding if and only if $a_1 + a_2 = a_3 = \ldots = a_m$.

Now, let $\widetilde{M}$ be a $(2n+1)$-dimensional almost contact metric manifold with the structure (φ, η, g, ξ) where φ is a $(1,1)$-tensor, η is a 1-form which is dual to the Reeb vector field ξ, and g is a Riemannian metric on $\widetilde{M}$ which satisfies the follows [24]:

$$\varphi^2 = -I + \eta \otimes \xi, \;\; \eta(\xi) = 1, \;\;\; g(\varphi X_1, \varphi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2).$$

Because of these conditions, we have

$$\varphi\xi = 0, \;\; \eta \cdot \varphi = 0, \quad \eta(X_1) = g(X_1, \xi),$$

for any vector fields $X_1, X_2 \in \Gamma(T\widetilde{M})$.

An almost contact metric manifold $(\widetilde{M}, \varphi, \eta, \xi, g)$ whose curvature tensor satisfies

$$\begin{aligned}\widehat{\widetilde{R}}(X_1, X_2)X_3 = {} & f_1\{g(X_2, X_3)X_1 - g(X_1, X_3)X_2\} + f_2\{g(X_1, \varphi X_3)\varphi X_2 \\ & -g(X_2, \varphi X_3)\varphi X_1 + 2g(X_1, \varphi X_2)\varphi X_3\} + f_3\{\eta(X_1)\eta(X_3)X_2 \\ & -\eta(X_2)\eta(X_3)X_1 + g(X_1, X_3)\eta(X_2)\xi - g(X_2, X_3)\eta(X_1)\xi\},\end{aligned} \tag{11}$$

for any vector field $X_1, X_2, X_3 \in \Gamma(T\widetilde{M})$ and f_1, f_2, f_3 being differentiable functions on $\widetilde{M}$ is said to be a generalized Sasakian-space-form denoted by $\widetilde{M}(f_1, f_2, f_3)$. The notion of a generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ was introduced by Alegre et al. [25], generalizing three important contact space forms, that is, the Sasakian-space-form $(f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4})$, the Kenmotsu space form $(f_1 = \frac{c-3}{4}, f_2 = f_3 = \frac{c+1}{4})$ and the Cosympletic space form $(f_1 = f_2 = f_3 = \frac{c}{4})$.

From (6) and (11), we obtain

$$\begin{aligned}\overline{R}(X_1, X_2, X_3, X_4) = {} & f_1\{g(X_2, X_3)g(X_1, X_4) - g(X_1, X_3)g(X_2, X_4)\} \\ & + f_2\{g(X_1, \varphi X_3)g(\varphi X_2, X_4) - g(X_2, \varphi X_3)g(\varphi X_1, X_4) \\ & + 2g(X_1, \varphi X_2)g(\varphi X_3, X_4)\} + f_3\{\eta(X_1)\eta(X_3)g(X_2, X_4) \\ & - \eta(X_2)\eta(X_3)g(X_1, X_4) + g(X_1, X_3)\eta(X_2)\eta(X_4) \\ & - g(X_2, X_3)\eta(X_1)\eta(X_4)\} + \psi_1\beta_1(X_1, X_3)g(X_2, X_3) \\ & - \psi_1\beta_1(X_2, X_3)g(X_1, X_4) + \psi_2 g(X_1, X_3)\beta_1(X_2, X_4) \\ & - \psi_2 g(X_2, X_3)\beta_1(X_1, X_4) + \psi_2(\psi_1 - \psi_2)g(X_1, X_3)\beta_2(X_2, X_4) \\ & - \psi_2(\psi_1 - \psi_2)g(X_2, X_3)\beta_2(X_1, X_4), \end{aligned} \tag{12}$$

Let M^m be a submanifold of a generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ of dimension $(2n+1)$. For any tangent vector field X_1 on M, we can write $\varphi X_1 = \mathcal{T}X_1 + \mathcal{F}X_1$, where $\mathcal{T}X_1$ is the tangential component, and $\mathcal{F}X_1$ is the normal component of φX_1. The squared norm of $\mathcal{T}$ at $x \in M$ is defined as

$$\| \mathcal{T} \|^2 = \sum_{i,j=1}^{m} g^2(\varphi e_i, e_j), \tag{13}$$

where $\{e_1, \dots, e_m\}$ is any orthonormal basis of the tangent space T_xM and decomposing the structural vector field $\xi = \xi^T + \xi^\perp$, where ξ^T and $\xi^\perp$ denotes the tangential and normal components of ξ. Moreover, we set $\Theta^2(\Pi) = g^2(\mathcal{T}e_1, e_2) = g^2(\varphi e_1, e_2)$, where $\{e_1, e_2\}$ is the orthonormal basis of two-plane section Π.

Theorem 2. *Let $M^m, m \geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a quarter-symmetric connection $\overline{\nabla}$, then*

$$\begin{aligned}\tau(x) - K(\Pi) \leq {} & (m-2)\Big(\frac{m^2}{2(m-1)} \| H \|^2 + (m+1)\frac{f_1}{2}\Big) \\ & + \Big(3 \| \mathcal{T} \|^2 - 6\Theta^2(\Pi)\Big)\frac{f_2}{2} + \Big(\| \xi_\Pi \|^2 - (m-1) \| \xi^T \|^2 \Big) f_3 \\ & + \frac{(\psi_1 + \psi_2)}{2}\Big(tr(\beta_1 |_\Pi) - \lambda(m-1)\Big) + \frac{\psi_2(\psi_1 - \psi_2)}{2}\Big(tr(\beta_2 |_\Pi) \\ & - \mu(m-1)\Big) + \frac{(\psi_1 - \psi_2)}{2}\Big(\Lambda(tr(h |_\Pi)) - m(m-1)\Lambda(H)\Big), \end{aligned}$$

where Π is a two-plane section T_xM, $x \in M$.

If in addition, P is a tangent vector field on M^m, then $H = \widehat{H}$ and the equality case holds at a point $x \in M$ if and only if there exists an orthonormal basis $\{e_1, \dots, e_m\}$ of T_xM and an orthonormal basis $\{e_{m+1}, \dots, e_{2n+1}\}$ of $T_x^\perp M$ such that the shape operators of M in $\widetilde{M}(f_1, f_2, f_3)$ at x have the following forms:

$$A_{e_{m+1}} = \begin{pmatrix} h_{11}^{m+1} & 0 & 0 & \dots & 0 \\ 0 & h_{22}^{m+1} & 0 & \dots & 0 \\ 0 & 0 & h_{11}^{m+1} + h_{22}^{m+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & h_{11}^{m+1} + h_{22}^{m+1} \end{pmatrix}$$

Theorem and

$$A_{e_r} = \begin{pmatrix} h^r_{11} & h^r_{12} & 0 & \dots & 0 \\ h^r_{12} & -h^r_{11} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, m+2 \leq r \leq 2n+1$$

Proof. Let $x \in M$ and $\{e_1, e_2, \dots, e_m\}$, $\{e_{m+1}, \dots, e_{2n+1}\}$ be an orthonormal basis of $T_x M$ and $T_x^\perp M$, respectively, then from (7), (10) and (12) we obtain

$$\begin{aligned} 2\tau(x) = {} & m^2 \parallel H \parallel^2 - \parallel h \parallel^2 + m(m-1)f_1 + 3f_2 \parallel \mathcal{T} \parallel^2 - 2(m-1)f_3 \parallel \xi^T \parallel^2 \\ & -(\psi_1+\psi_2)\lambda(m-1) - \psi_2(\psi_1-\psi_2)\mu(m-1) - m(m-1)(\psi_1-\psi_2)\Lambda(H). \end{aligned} \tag{14}$$

We set,

$$\begin{aligned} c = {} & 2\tau(x) - \frac{m^2(m-2)}{m-1} \parallel H \parallel^2 - m(m-1)f_1 - 3f_2 \parallel \mathcal{T} \parallel^2 + 2(m-1)f_3 \parallel \xi^T \parallel^2 \\ & +(\psi_1+\psi_2)\lambda(m-1) + \psi_2(\psi_1-\psi_2)\mu(m-1) + m(m-1)(\psi_1-\psi_2)\Lambda(H), \end{aligned} \tag{15}$$

then (14) becomes

$$m^2 \parallel H \parallel^2 = (m-1)(\parallel h \parallel^2 + c). \tag{16}$$

For a chosen orthonormal basis, (16) can be written as:

$$\Big(\sum_{i=1}^{m} h^{m+1}_{ii} \Big)^2 = (m-1)\Big[\sum_{i=1}^{m} (h^{m+1}_{ii})^2 + \sum_{i\neq j} (h^{m+1}_{ij})^2 + \sum_{r=m+2}^{2n+1} \sum_{i,j=1}^{m} (h^r_{ij})^2 + c \Big],$$

then using Lemma 1, we have

$$2h^{m+1}_{11} h^{m+1}_{22} \geq \sum_{i\neq j} (h^{m+1}_{ij})^2 + \sum_{r=m+2}^{2n+1} \sum_{i,j=1}^{m} (h^r_{ij})^2 + c. \tag{17}$$

Now, let $\Pi = span\{e_1, e_2\}$, then from (7) and (12) we obtain

$$\begin{aligned} R(e_1,e_2,e_2,e_1) = {} & \sum_{r=m+1}^{2n+1} [h^r_{11}h^r_{22} - (h^r_{12})^2] - (\psi_1-\psi_2)g(h(e_2,e_2),P) \\ & + f_1 + 3f_2 g^2(\varphi e_1, e_2) - f_3(\eta^2(e_1) + \eta^2(e_2)) \\ & -\psi_1\beta_1(e_2,e_2) - \psi_2\beta_1(e_1,e_1) - \psi_2(\psi_1-\psi_2)\beta_2(e_1,e_1). \end{aligned} \tag{18}$$

and

$$\begin{aligned} R(e_1,e_2,e_1,e_2) = {} & \sum_{r=m+1}^{2n+1} [(h^r_{12})^2 - h^r_{11}h^r_{22}] + (\psi_1-\psi_2)g(h(e_1,e_1),P) \\ & - f_1 - 3f_2 g^2(\varphi e_1, e_2) + f_3(\eta^2(e_1) + \eta^2(e_2)) \\ & +\psi_1\beta_1(e_1,e_1) + \psi_2\beta_1(e_2,e_2) + \psi_2(\psi_1-\psi_2)\beta_2(e_2,e_2). \end{aligned} \tag{19}$$

Making use of (18) and (19) in (9), we obtain

$$K(\Pi) = \sum_{r=m+1}^{2n+1} [h_{11}^r h_{22}^r - (h_{12}^r)^2] - \frac{(\psi_1 - \psi_2)}{2} \Lambda(tr(h\mid_\Pi))$$
$$+ f_1 + 3f_2\Theta^2(\Pi) - f_3(\| \xi_\Pi \|^2)$$
$$- \frac{\psi_1}{2} tr(\beta_1 \mid \Pi) - \frac{\psi_2}{2} tr(\beta_1 \mid_\Pi) - \frac{\psi_2}{2}(\psi_1 - \psi_2) tr(\beta_2 \mid_\Pi). \tag{20}$$

Combining (14) and (20) gives

$$\tau(x) - K(\Pi) = (m-2)\Big(\frac{m^2}{2(m-1)} \| H \|^2 + (m+1)\frac{f_1}{2}\Big)$$
$$+ \Big(3 \| \mathcal{T} \|^2 - 6\Theta^2(\Pi)\Big)\frac{f_2}{2} + \Big(\| \xi_\Pi \|^2 - (m-1) \| \xi^T \|^2 \Big) f_3$$
$$+ \frac{(\psi_1 + \psi_2)}{2}\Big(tr(\beta_1 \mid_\Pi) - \lambda(m-1)\Big) + \frac{\psi_2(\psi_1 - \psi_2)}{2}\Big(tr(\beta_2 \mid_\Pi)$$
$$- \mu(m-1)\Big) + \frac{(\psi_1 - \psi_2)}{2}\Big(\Lambda(tr(h \mid_\Pi)) - m(m-1)\Lambda(H)\Big)$$
$$+ \sum_{r=m+1}^{2n+1}\Big[\sum_{1\le i<j\le m} h_{ii}^r h_{jj}^r - h_{11}^r h_{22}^r - \sum_{1\le i<j\le m} (h_{ij}^r)^2 + (h_{12}^r)^2\Big]. \tag{21}$$

Making use of Lemma 2.4 [26], we have

$$\sum_{r=m+1}^{2n+1}\Big[\sum_{1\le i<j\le m} h_{ii}^r h_{jj}^r - h_{11}^r h_{22}^r - \sum_{1\le i<j\le m} (h_{ij}^r)^2 + (h_{12}^r)^2\Big] \le \frac{m^2(m-2)}{2(m-1)} \| H \|^2 . \tag{22}$$

In view of the last expression in (21), we obtain

$$\tau(x) - K(\Pi) \le (m-2)\Big(\frac{m^2}{2(m-1)} \| H \|^2 + (m+1)\frac{f_1}{2}\Big)$$
$$+ \Big(3 \| \mathcal{T} \|^2 - 6\Theta^2(\Pi)\Big)\frac{f_2}{2} + \Big(\| \xi_\Pi \|^2 - (m-1) \| \xi^T \|^2 \Big) f_3$$
$$+ \frac{(\psi_1 + \psi_2)}{2}\Big(tr(\beta_1 \mid_\Pi) - \lambda(m-1)\Big) + \frac{\psi_2(\psi_1 - \psi_2)}{2}\Big(tr(\beta_2 \mid_\Pi)$$
$$- \mu(m-1)\Big) + \frac{(\psi_1 - \psi_2)}{2}\Big(\Lambda(tr(h \mid_\Pi)) - m(m-1)\Lambda(H)\Big). \tag{23}$$

Now, if P is a tangent vector field on M, then (5) implies $h = \widehat{h}$ and $H = \widehat{H}$. If the equality case (23) holds at a point $x \in M$, then the equality cases of (17) and (22) hold, which gives

$$h_{11}^{m+1} = h_{22}^{m+1} = h_{33}^{m+1} = \cdots = h_{mm}^{m+1}$$
$$h_{1j}^{m+1} = h_{2j}^{m+1} = 0, j > 2$$
$$h_{11}^r + h_{22}^r = 0, r = m+2, \ldots, 2n+1$$
$$h_{ij}^r = 0, i \ne j, r = m+1, \ldots, 2n+1$$
$$h_{ij}^{m+1} = 0, i \ne j, i, j > 2$$

Therefore, choosing a suitable orthonormal basis, the shape operators take the desired forms. □

Corollary 1. *Under the same arguments as in Theorem 2,*

1. *If the structure vector field ξ is tangent to M, we have*

$$\tau(x) - K(\Pi) \leq (m-2)\Big(\frac{m^2}{2(m-1)} \parallel H \parallel^2 + (m+1)\frac{f_1}{2}\Big) + \Big(3 \parallel \mathcal{T} \parallel^2 - 6\Theta^2(\Pi)\Big)\frac{f_2}{2} + \Big(\parallel \xi_\Pi \parallel^2 - (m-1)\Big) f_3 + \frac{(\psi_1+\psi_2)}{2}\Big(tr(\beta_1 |_\Pi) - \lambda(m-1)\Big) + \frac{\psi_2(\psi_1-\psi_2)}{2}\Big(tr(\beta_2 |_\Pi) - \mu(m-1)\Big) + \frac{(\psi_1-\psi_2)}{2}\Big(\Lambda(tr(h |_\Pi)) - m(m-1)\Lambda(H)\Big).$$

2. *If the structure vector field ξ is normal to M, we have*

$$\tau(x) - K(\Pi) \leq (m-2)\Big(\frac{m^2}{2(m-1)} \parallel H \parallel^2 + (m+1)\frac{f_1}{2}\Big) + \Big(3 \parallel \mathcal{T} \parallel^2 - 6\Theta^2(\Pi)\Big)\frac{f_2}{2} + + \frac{(\psi_1+\psi_2)}{2}\Big(tr(\beta_1 |_\Pi) - \lambda(m-1)\Big) + \frac{\psi_2(\psi_1-\psi_2)}{2}\Big(tr(\beta_2 |_\Pi) - \mu(m-1)\Big) + \frac{(\psi_1-\psi_2)}{2}\Big(\Lambda(tr(h |_\Pi)) - m(m-1)\Lambda(H)\Big).$$

Remark 1. *It should be noted that Theorem 2 generalizes the Theorem 6 obtained in [20]. Moreover, taking different values of $f_i, i = 1,2,3$, we can obtain similar inequalities as Theorem 1 for the Kenmotsu space form and the Cosympletic space form endowed with certain types of connections by restricting the values of $\psi_i, i = 1,2$.*

Remark 2. *If in Theorem 2, we take $\psi_1 = \psi_2 = 1$ then we obtain Theorem 5.1 [21].*

Corollary 2. *Let $M^m, m \geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a semi-symmetric non-metric connection, then*

$$\tau(x) - K(\Pi) \leq (m-2)\Big(\frac{m^2}{2(m-1)} \parallel H \parallel^2 + (m+1)\frac{f_1}{2}\Big) + \Big(3 \parallel \mathcal{T} \parallel^2 - 6\Theta^2(\Pi)\Big)\frac{f_2}{2} + \Big(\parallel \xi_\Pi \parallel^2 - (m-1) \parallel \xi^T \parallel^2 \Big) f_3 + \frac{1}{2}\Big(tr(\beta_1 |_\Pi) - \lambda(m-1)\Big) + \frac{1}{2}\Big(\Lambda(tr(h |_\Pi)) - m(m-1)\Lambda(H)\Big),$$

where Π is a two-plane section T_xM, $x \in M$.

For an integer $k \geq 0$, we denote by $S(m,k)$ the set of k-tuples $(m_1, \dots, m_k)$ of integers ≥ 2 satisfying $m_1 < m$ and $m_1, \dots, m_k \leq m$. In addition, let $S(m)$ be the set of unordered k-tuples with $k \geq 0$ for a fixed m. Then, for each k-tuple $(m_1, \dots, m_k) \in S(m)$, Chen introduced a Riemannian invariant $\delta(m_1, \dots, m_k)$ as follows [23]

$$\delta(m_1, \dots, m_k)(x) = \tau(x) - inf\{\tau(L_1) + \dots + \tau(L_k)\}, \quad (24)$$

where $L_1, \dots, L_k$ run over all k mutually orthogonal subspaces of T_xM such that $\dim L_j = m_j, j \in \{1, \dots, k\}$. For simplicity, we set

$$\Psi_1(L_j) = \sum_{1\le i<j\le r} g^2(\mathcal{T}e_i, e_j),\;\; \Psi_2(L_j) = \sum_{1\le i<j\le r} [g(\xi^T, e_i)^2 + g(\xi^T, e_j)^2]$$
$$\Psi_3(L_j) = \sum_{1\le i<j\le r} [\beta_1(e_i, e_i) + \beta_1(e_j, e_j)],\;\; \Psi_4(L_j) = \sum_{1\le i<j\le r} [\beta_2(e_i, e_i) + \beta_2(e_j, e_j)]$$
$$\Psi_5(L_j) = \sum_{1\le i<j\le r} \Lambda(h(e_i, e_i) + h(e_j, e_j))$$

As the generalization of Theorem 2, we state and prove the following results using the methods used in [26].

Theorem 3. *Let $M^m, m \ge 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a quarter-symmetric connection $\overline{\nabla}$, then*

$$\begin{aligned}\delta(m_1, \dots, m_k) &\le b(m_1, \dots, m_k) \parallel H \parallel^2 + a(m_1, \dots, m_k) f_1\\
&+3f_2\Big(\frac{\parallel \mathcal{T} \parallel^2}{2} - \sum_{j=1}^{k} \Psi_1(L_j)\Big) - f_3\Big((m-1) \parallel \xi^T \parallel^2 - \sum_{j=1}^{k} \Psi_2(L_j)\Big)\\
&-\frac{(\psi_1+\psi_2)}{2}\Big((m-1)\lambda - \sum_{j=1}^{k} \Psi_3(L_j)\Big) - \frac{\psi_2}{2}(\psi_1 - \psi_2)\Big((m-1)\mu\\
&- \sum_{j=1}^{k} \Psi_4(L_j)\Big) + \frac{(\psi_1-\psi_2)}{2}\Big(m(m-1)\Lambda(H) - \sum_{j=1}^{k} \Psi_5(L_j)\Big),\end{aligned}$$

for any k-tuples $(m_1, \dots, m_k) \in S(m)$. If P is a tangent vector field on M, the equality case holds at $x \in M^m$ if and only if there exists an orthonormal basis $\{e_1, \dots, e_m\}$ of T_xM and an orthonormal basis $\{e_{m+1}, \dots, e_{2n+1}\}$ of $T_x^\perp M$ such that the shape operators of M in $\widetilde{M}(f_1, f_2, f_3)$ at x have the following forms:

$$A_{e_{m+1}} = \begin{pmatrix} a_1 & 0 & \dots & 0\\ 0 & a_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & a_m \end{pmatrix}, A_{e_r} = \begin{pmatrix} A_1^r & \dots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & A_k^r & 0\\ 0 & \dots & 0 & \varsigma_r I \end{pmatrix}, r = m+2, \dots, 2n+1,$$

where $a_1, \dots, a_m$ satisfy

$$a_1 + \dots + a_{m_1} = \dots = a_{m_1+\dots+m_{k-1}+1} + \dots + a_{m_1+\dots+m_k+1} = \dots = a_m$$

and each A_j^r is a symmetric $m_j \times m_j$ submatrix satisfying $tr(A_1^r) = \dots = tr(A_k^r) = \varsigma_r$, I is an identity matrix.

Proof. Choose an orthonormal basis $\{e_1, \dots, e_m\}$ of T_xM and an orthonormal basis $\{e_{m+1}, \dots, e_{2n+1}\}$ of $T_x^\perp M$ such that mean curvature vector H is in the direction of the normal vector to e_{m+1}. We set

$$\begin{aligned} a_i &= h_{ii}^{m+1}, \quad i = 1, \dots, m\\
b_1 &= a_1, b_2 = a_2 + \dots + a_{m_1}, b_3 = a_{m_1+1} + \dots + a_{m_1+m_2}, \dots,\\
b_{k+1} &= a_{m_1+\dots+m_{k-1}+1} + \dots + a_{m_1+\dots+m_{k-1}+m_k}, \dots, b_{\gamma+1} = a_m,\end{aligned}$$

and consider the following sets

$$\begin{aligned} D_1 &= \{1, \dots, m_1\}, \quad D_2 = \{m_1+1, \dots, m_1+m_2\}, \dots,\\
D_k &= \{(m_1 + \dots + m_{k-1}) + 1, \dots, (m_1 + \dots + m_{k-1}) + m_k\}.\end{aligned}$$

Let $L_1, \ldots, L_k$ be a mutually orthogonal subspace of T_xM with $\dim L_j = m_j$, defined by

$$L_j = Span\{e_{m_1+\cdots+m_{j-1}+1}, \ldots, e_{m_1+\cdots+m_j}\}, \quad j = 1, \ldots, k.$$

From (7), (8) and (12), we obtain

$$\begin{aligned}\tau(L_j) = \frac{m_j(m_j-1)}{2} f_1 + 3f_2\Psi_1(L_j) - f_3\Psi_2(L_j) \\ -\frac{(\psi_1+\psi_2)}{2}\Psi_3(L_j) - \frac{\psi_2}{2}(\psi_1-\psi_2)\Psi_4(L_j) - \frac{(\psi_1-\psi_2)}{2}\Psi_5(L_j) \\ + \sum_{r=m+1}^{2n+1}\sum_{\alpha_j<\beta_j}[h^r_{\alpha_j\alpha_j}h^r_{\beta_j\beta_j} - (h_{\alpha_j\beta_j})^2].\end{aligned} \tag{25}$$

We set

$$\begin{aligned}\varepsilon = 2\tau - 2b(m_1, \ldots, m_k) \parallel H \parallel^2 - m(m-1)f_1 - 3f_2 \parallel \mathcal{T} \parallel^2 \\ +2(m-1)f_3 \parallel \xi^T \parallel^2 + (\psi_1+\psi_2)\lambda(m-1) \\ +\psi_2(\psi_1-\psi_2)\mu(m-1) + m(m-1)(\psi_1-\psi_2)\Lambda(H),\end{aligned} \tag{26}$$

where

$$b(m_1, \ldots, m_k) = \frac{m^2\Big(m+k-1-\sum_{j=1}^{k} m_j\Big)}{2\Big(m+k-\sum_{j=1}^{k} m_j\Big)},$$

for each $(m_1, \ldots, m_k) \in S(m)$.

In addition, let $\gamma = m + k - \sum_{j=1}^{k} m_j$. Then in view of this and (26), Equation (14) becomes

$$m^2 \parallel H \parallel^2 = (\parallel h \parallel^2 + \varepsilon)\gamma,$$

which can be written as

$$\begin{aligned}\Big(\sum_{i=1}^{\gamma+1} b_i\Big)^2 = \gamma\Big[\varepsilon + \sum_{i=1}^{\gamma+1} b_i^2 + \sum_{i\neq j}(h^{m+1}_{ij})^2 + \sum_{r=m+2}^{2n+1}\sum_{i,j=1}^{m}(h^r_{ij})^2 \\ -2\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1} - \cdots - 2\sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k}\Big],\end{aligned} \tag{27}$$

where $\alpha_j, \beta_j \in D_j$ for all $j = 1, \ldots, k$.

Now applying Lemma 2.3 [26] in (27), we obtain

$$\begin{aligned}\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1} + \cdots + \sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k} \geq \\ \frac{1}{2}\Big[\varepsilon + \sum_{i\neq j}(h^{m+1}_{ij})^2 + \sum_{r=m+2}^{2n+1}\sum_{i,j=1}^{m}(h^r_{ij})^2\Big],\end{aligned}$$

which further implies

$$\sum_{j=1}^{k}\sum_{r=m+1}^{2n+1}\sum_{\alpha_j<\beta_j}\left[h^r_{\alpha_j\alpha_j}h^r_{\beta_j\beta_j}-(h^r_{\alpha_j\beta_j})^2\right]\geq\frac{\varepsilon}{2}$$

$$+\frac{1}{2}\sum_{r=m+1}^{2n+1}\sum_{(\alpha,\beta)\notin D^2}(h^r_{\alpha\beta})^2+\sum_{r=m+2}^{2n+1}\sum_{\alpha_j\in D_j}(h^r_{\alpha_j\alpha_j})^2\leq\frac{\varepsilon}{2}, \tag{28}$$

where $D^2=(D_1\times D_1)\cup\cdots\cup(D_k\times D_k)$. Combining (14), (25) and (28) gives

$$\tau-\sum_{j=1}^{k}\tau(L_j)\leq b(m_1,\ldots,m_k)\parallel H\parallel^2+a(m_1,\ldots,m_k)f_1$$
$$+3f_2\Big(\frac{\parallel\mathcal{T}\parallel^2}{2}-\sum_{j=1}^{k}\Psi_1(L_j)\Big)-f_3\Big((m-1)\parallel\xi^T\parallel^2-\sum_{j=1}^{k}\Psi_2(L_j)\Big)$$
$$-\frac{(\psi_1+\psi_2)}{2}\Big((m-1)\lambda-\sum_{j=1}^{k}\Psi_3(L_j)\Big)-\frac{\psi_2}{2}(\psi_1-\psi_2)\Big((m-1)\mu$$
$$-\sum_{j=1}^{k}\Psi_4(L_j)\Big)+\frac{(\psi_1-\psi_2)}{2}\Big(m(m-1)\Lambda(H)-\sum_{j=1}^{k}\Psi_5(L_j)\Big), \tag{29}$$

where, $a(m_1,\ldots,m_k)=\frac{1}{2}\Big[m(m-1)-\sum_{j=1}^{k}m_j(m_j-1)\Big]$.

The equality case (29) at a point $x\in M$ holds if and only if all the previous inequalities hold; thus, the shape operators take the desired forms. □

Remark 3. *Restricting the values of $f_i, i=1,2,3$ and ψ_i for $i=1,2$, we can obtain similar bounds as Theorem 3 for certain contact space forms endowed with certain connections.*

Theorem 4. *Let $M^m, m\geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian-space-form $\widetilde{M}(f_1,f_2,f_3)$ endowed with a quarter-symmetric connection $\overline{\nabla}$, then*

(i) For each unit vector X_1 in T_xM, we have

$$Ric(X_1)\leq(m-1)f_1+3f_2\sum_{j=2}^{m}g^2(\varphi X_1,e_j)+f_3((2-m)\eta^2(X_1)-\parallel\xi^T\parallel^2)$$
$$+[\psi_1+(1-m)\psi_2]\beta_1(X_1,X_1)-\psi_1\lambda+\psi_2(\psi_1-\psi_2)(1-m)\beta_2(X_1,X_1)$$
$$-(\psi_1-\psi_2)[m\Lambda(H)-\Lambda(h(X_1,X_1))]+\frac{m^2}{4}\parallel H\parallel^2. \tag{30}$$

(ii) If $H(x)=0$, then a unit tangent vector X_1 at x satisfies the equality case of (30) if and only if $X_1\in\mathcal{M}(x)=\{X_1\in T_xM\mid h(X_1,X_2)=0,\forall X_2\in T_xM\}$.

(iii) The equality of (30) holds for all unit tangent vectors at x if and only if either

1. *$m\neq 2, h^r_{ij}=0, i,j=1,2\ldots,m.r=m+1,\ldots,2n+1$, or*
2. *$m=2, h^r_{11}=h^r_{22}, h^r_{12}=0, r=3,\ldots,2n+1$.*

Proof. Choosing the orthonormal basis $\{e_1,\ldots,e_m\}$ such that $e_1=X_1$, where $X_1\in T_xM$ is a unit tangent vector at the point x on M. In view of (7) and (12), then proceeding similarly as the proof of Theorem 4 in [20], one can easily obtain the desire results. □

By choosing an orthonormal frame $\{e_1, \ldots, e_k\}$ of L such that $e_1 = X_1$, a unit tangent vector, Chen [23] defined the k-Ricci curvature of L at X_1 by

$$Ric_L(X_1) = K_{12} + K_{13} + \cdots + K_{1k}. \tag{31}$$

For an integer $k, 2 \leq k \leq m$, the Riemannian invariant Θ_k on M is defined by

$$\Theta_k(x) = \frac{1}{k-1} inf\{Ric_L(X_1) \mid L, X_1\}, x \in M$$

where L runs over all k-plane sections in T_xM and X_1 runs over all unit vectors in L. From [26], we have

$$\tau(x) \geq \frac{m(m-1)}{2}\Theta_k(x). \tag{32}$$

Let us choose $\{e_1, \ldots, e_m\}$ and $\{e_{m+1}, \ldots, e_{2n+1}\}$ as an orthonormal basis of T_xM and $T_x^{\perp}M, x \in M$, respectively, where e_{m+1} is parallel to the mean curvature vector H. In addition, let $\{e_1, \ldots, e_m\}$ diagonalize the shape operator $A_{e_{m+1}}$. Then,

$$A_{e_{m+1}} = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_m \end{pmatrix}$$

and

$$A_{e_r} = h^r_{ij}, \quad i,j = 1, \ldots, m, \quad r = m+2, \ldots, 2n+1, \quad trA_{e_r} = 0. \tag{33}$$

In consequence of the above assumptions, Equation (14) can be written as follows:

$$\begin{aligned} m^2 \parallel H \parallel^2 = 2\tau + \sum_{i=1}^{m} a_i^2 + \sum_{r=m+2}^{2n+1}\sum_{i,j=1}^{m}(h^r_{ij})^2 - m(m-1)f_1 \\ -3f_2 \parallel \mathcal{T} \parallel^2 + 2(m-1)f_3 \parallel \xi^T \parallel^2 + (\psi_1+\psi_2)\lambda(m-1) \\ +\psi_2(\psi_1-\psi_2)\mu(m-1) + m(m-1)(\psi_1-\psi_2)\Lambda(H). \end{aligned} \tag{34}$$

Using the Cauchy–Schwartz inequality, we have

$$\sum_{i=1}^{m} a_i^2 \geq m \parallel H \parallel^2. \tag{35}$$

Combining (32) and (34), we can state the following:

Theorem 5. *Let $M^m, m \geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a quarter-symmetric connection $\overline{\nabla}$. Then for any integer $k, 2 \leq k \leq m$ and any point $x \in M$, we have*

$$\begin{aligned} \parallel H \parallel^2 (x) \geq \Theta_k(x) - f_1 - \frac{3f_2}{m(m-1)} \parallel \mathcal{T} \parallel^2 + \frac{2f_3}{m} \parallel \xi^T \parallel^2 \\ + \frac{\lambda}{m}(\psi_1+\psi_2) + \frac{\mu}{m}\psi_2(\psi_1-\psi_2) + (\psi_1-\psi_2)\Lambda(H). \end{aligned}$$

As a particular case of Theorem 5, we obtained Theorem 6.2 [21] which is as follows:

Corollary 3 ([21]). *Let $M^m, m \geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a semi-symmetric metric connection. Then for any integer $k, 2 \leq k \leq m$ and any point $x \in M$, we have*

$$\| H \|^2 (x) \geq \Theta_k(x) - f_1 - \frac{3f_2}{m(m-1)} \| \mathcal{T} \|^2 + \frac{2f_3}{m} \| \xi^T \|^2 + \frac{2\lambda}{m}.$$

Corollary 4. *Let $M^m, m \geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a semi-symmetric non-metric connection. Then for any integer $k, 2 \leq k \leq m$ and any point $x \in M$, we have*

$$\| H \|^2 (x) \geq \Theta_k(x) - f_1 - \frac{3f_2}{m(m-1)} \| \mathcal{T} \|^2 + \frac{2f_3}{m} \| \xi^T \|^2 + \frac{\lambda}{m} + \Lambda(H).$$

Remark 4. *Restricting function $f_i, i = 1, 2, 3$, we can easily obtain similar inequality in the case of the Sasakian, Kenmotsu and Cosympletic space forms.*

4. Some Applications

The notion of slant submanifolds in almost contact geometry was introduced by Lotta [27]. A submanifold M of an almost contact metric manifold $(\widetilde{M}, \varphi, \xi, \eta, g)$ tangent to the structure vector field ξ is said to be a contact slant submanifold if, for any point $x \in M$ and any vector $X_1 \in T_x M$ linearly independent on ξ_x, the angle between the vector φX_1 and the tangent space $T_x M$ is constant. This angle is known as the slant angle of M. The concept of slant submanifold is further generalized as follows:

Definition 1 ([28]). *A submanifold M of an almost contact metric manifold M is called a bi-slant submanifold, whenever we have*

1. $TM = \mathcal{D}_{\theta_1} \oplus \mathcal{D}_{\theta_2} \oplus \xi$
2. $\varphi\mathcal{D}_{\theta_1} \perp \mathcal{D}_{\theta_2}$ *and* $\varphi\mathcal{D}_{\theta_2} \perp \mathcal{D}_{\theta_1}$.
3. *For $i = 1, 2$, the distribution D_i is slant with slant angle θ_i.*

Now, as a consequence of Theorem 2, we can state the following:

Theorem 6. *Let M be a $(m = 2d_1 + 2d_2 + 1)$-dimensional bi-slant submanifold of a $(2n+1)$-dimensional generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a quarter-symmetric connection $\overline{\nabla}$, then we have*

$$\begin{aligned}\tau(x) - K(\Pi) \leq{}& (m-2)\Big(\frac{m^2}{2(m-1)} \| H \|^2 + (m+1)\frac{f_1}{2}\Big) \\ &+3\Big((d_1-1)cos^2\theta_1 + d_2 cos^2\theta_2\Big)\frac{f_2}{2} - (m-1)f_3 \\ &+\frac{(\psi_1+\psi_2)}{2}\Big(tr(\beta_1 |_\Pi) - \lambda(m-1)\Big) + \frac{\psi_2(\psi_1-\psi_2)}{2}\Big(tr(\beta_2 |_\Pi) \\ &-\mu(m-1)\Big) + \frac{(\psi_1-\psi_2)}{2}\Big(\Lambda(tr(h|_\Pi)) - m(m-1)\Lambda(H)\Big),\end{aligned}$$

for any plane Π invariant by $\mathcal{T}$ and tangent to slant distribution $\mathcal{D}_{\theta_1}$ and

$$\begin{aligned}\tau(x) - K(\Pi) \leq{}& (m-2)\Big(\frac{m^2}{2(m-1)} \| H \|^2 + (m+1)\frac{f_1}{2}\Big) \\ &+3\Big(d_1 cos^2\theta_1 + (d_2-1)cos^2\theta_2\Big)\frac{f_2}{2} - (m-1)f_3 \\ &+\frac{(\psi_1+\psi_2)}{2}\Big(tr(\beta_1 |_\Pi) - \lambda(m-1)\Big) + \frac{\psi_2(\psi_1-\psi_2)}{2}\Big(tr(\beta_2 |_\Pi) \\ &-\mu(m-1)\Big) + \frac{(\psi_1-\psi_2)}{2}\Big(\Lambda(tr(h|_\Pi)) - m(m-1)\Lambda(H)\Big),\end{aligned}$$

for any plane Π invariant by $\mathcal{T}$ and tangent to slant distribution $\mathcal{D}_{\theta_2}$. Moreover, the ideal case is the same as Theorem 2.

Proof. Let M be a bi-slant submanifold of a generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ of dimension $(m = 2d_1 + 2d_2 + 1)$ and let $\{e_1, \ldots, e_m = \xi\}$ be an orthonormal frame of tangent space $T_x M$ at a point $x \in M$, such that

$$e_1, e_2 = sec\theta_1 \mathcal{T} e_1, \ldots, e_{2d_1-1}, e_{2d_1} = sec\theta_1 \mathcal{T} e_{2d_1-1}, e_{2d_1+1}, e_{2d_1+2}$$
$$= sec\theta_2 \mathcal{T} e_{2d_1+1}, \ldots, e_{2d_1+2d_2-1}, e_{2d_1+2d_2} = sec\theta_2 \mathcal{T} e_{2d_1+2d_2-1}, e_{2d_1+2d_2+1} = \xi,$$

which gives

$$g^2(\varphi e_{i+1}, e_i) = \begin{cases} cos^2\theta_1, & \text{for} \quad i = 1, 2, \ldots, 2d_1 - 1 \\ cos^2\theta_2, & \text{for} \quad i = 2d_1 + 1, \ldots, 2d_1 + 2d_2 - 1. \end{cases}$$

Thus we have

$$\| \mathcal{T} \|^2 = 2\{d_1 cos^2\theta_1 + d_2 cos^2\theta_2\}$$

Making use of the above facts in Theorem 2, the proof is straightforward. □

In a similar manner, Theorems 3, 4 and 5 can be stated for a bi-slant submanifold of a generalized Sasakian-space-form. Moreover, restricting the values of $\theta_i, i = 1, 2$, similar results can be obtained for a large class of submanifolds such as slant, semi-slant, hemi-slant, semi-invariant submanifolds. Moreover, by taking different values of $f_i, i = 1, 2, 3$, we can derive similar inequalities for the Sasakian, Kenmotsu and Cosympletic space forms.

5. Conclusions and Future Work

In this article, we established the general form of Chen's inequalities are obtained for generalized Sasakian-space-forms endowed with a special type of quarter-symmetric connection. This work is in continuation of the previous works by Wang [20], Mihai and Özgür [19], Sular [21] and Wang and Zhang [18]. By using the obtained inequality, we derived the Chen inequality for the bi-slant submanifold of generalized Sasakian-space-forms. Recently, Chen inequality for lightlike hypersurfaces of GRW spacetime was obtained by Poyraz [14]. For future research, we would try to combine the methods and results in [29–52] to obtain the Chen inequalities for submanifolds of indefinite space forms such as spacelike and lightlike.

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References

1. Chen, B.Y. Some pinching and classification theorems for minimal submanifolds. *Arch. Math.* **1993**, *60*, 568–578. [CrossRef]
2. Chern, S.S. *Minimal Submanifolds in a Riemannian Manifold*; University of Kansas Press: Lawrence, KS, USA, 1968.

3. Chen, B.Y. A general inequality for submanifolds in complex-space-forms and its applications. *Arch. Math.* **1996**, *67*, 519–528. [CrossRef]
4. Kumar, P.K.; Gupta, R.S.; Sharfuddin, A. B. Y. Chen's inequalities for bi-slant submanifolds in Kenmotsu space forms. *Demonstr. Math.* **2010**, *4*, 887–898.
5. Cioroboiu, D. B. Y. Chen inequalities for semislant submanifolds in Sasakian-space-forms. *Int. J. Math. Math. Sci.* **2003**, *2003*, 354806. [CrossRef]
6. Gupta, R.S. B. Y. Chen's inequalities for Bi-slant submanifolds in cosympletic space forms. *Sarajevo J. Math.* **2013**, *9*, 117–128. [CrossRef]
7. Özgür, C. B. Y. Chen inequalities for submanifolds of a Riemannian manifold of a quasi-constant curvature. *Turk. J. Math.* **2011**, *35*, 501–509.
8. Alegre, P.; Carriazo, A.; Kim, Y.H.; Yoon, D.W. B.Y. Chen's inequality for submanifolds of generalized space forms. *Indian J. Pure Appl. Math.* **2007**, *38*, 185–201.
9. Mihai, A. B.Y. Chen inequalities for slant submanifolds in generalized complex space forms. *Rad. Math.* **2004**, *12*, 215–231.
10. Decu, S.; Haesen, S. Chen Inequalities for Spacelike Submanifolds in Statistical Manifolds of type Para-Kähler Space Forms. *Mathematics* **2022**, *10*, 330. [CrossRef]
11. Aytimur, H.; Kon, M.; Mihai, A.; Özgür, C.; Takano, K. Chen Inequalities for Statistical Submanifolds of Kähler-Like Statistical Manifolds. *Mathematics* **2019**, *7*, 1202. [CrossRef]
12. Lone, M.S.; Bahadir, O.; Park, C.; Hwang, I. Basic inequalities for statistical submanifolds in Golden-like statistical manifolds. *Open Math.* **2022**, *20*, 153–166. [CrossRef]
13. Vilcu, G.E. On Chen invariants and inequalities in quaternionic geometry. *J. Inequal. Appl.* **2013**, *2013*, 66. [CrossRef]
14. Poyraz, N.Ö. Chen inequalities on lightlike hypersurfaces of GRW spacetime. *Int. J. Geom. Methods Mod. Phys.* **2022**, *19*, 2250011. [CrossRef]
15. Qu, Q.; Wang, Y. Multiply warped products with a quarter-symmetric connection. *J. Math. Anal. Appl.* **2015**, *431*, 955–987. [CrossRef]
16. Hayden, H.A. Subspaces of a space with torsion. *Proc. London Math. Soc.* **1932**, *34*, 27–50. [CrossRef]
17. Agashe, N.S.; Chafle, M.R. A semi-symmetric non-metric connection on a Riemannian manifold. *Indian J. Pure Appl. Math.* **1992**, *23*, 399–409.
18. Wang, Y.; Zhang, P. Chen inequalities for submanifolds of real space forms with a quarter-symmetric connection. *Taiwanese J. Math.* **2010**, *14*, 1465–1477.
19. Mihai, A.; Özgür, C. Chen inequalities for submanifolds of complex space forms and Sasakian-space-forms endowed with semi-symmetric metric connections. *Rocky Mountain J. Math.* **2011**, *41*, 1653–1673. [CrossRef]
20. Wang, Y. Chen inequalities for submanifolds of complex space forms and Sasakian-space-forms with quarter-symmetric connections. *Int. J. Geom. Methods Modern Phys.* **2019**, *16*, 1950118. [CrossRef]
21. Sular, S. Chen inequalities for submanifolds of generalized space forms with a semi-symmetric metric connection. *Hacettepe J. Math. Stat.* **2016**, *45*, 811–825. [CrossRef]
22. Al-Khaldi, A.H.; Aquib, M.; Aslam, M.; Khan, M.A. Chen-Ricci Inequalities with a Quarter Symmetric Connection in Generalized Space Forms. *Adv. Math. Phys.* **2021**, *2021*, 3221643. [CrossRef]
23. Chen, B.Y. A Riemannian invariant and its applications to submanifold theorey. *Results Math.* **1995**, *27*, 17–26. [CrossRef]
24. Blair, D.E. *Riemannian Geometry of Contact and Sympletic Manifolds*; Birkhauser: Boston, MA, USA, 2002.
25. Alegre, P.; Blair, D.E.; Carriazo, A. Generalized Sasakian-space-forms. *Israel J. Math.* **2004**, *141*, 157–183. [CrossRef]
26. Zhang, P.; Zhang, L.; Song, W. Chen's inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature with a semi-symmetric metric connection. *Taiwanese J. Math.* **2014**, *18*, 1841–1862. [CrossRef]
27. Lotta, A. Slant submanifolds in contact geometry. *Bull. Math. Soc. Roum.* **1996**, *39*, 183–198.
28. Alqahtani, L.S.; Stankovic, M.S.; Uddin, S. Warped product bi-slant submanifolds of cosympletic manifolds. *Filomat* **2017**, *17*, 5065–5071. [CrossRef]
29. Abedi, E.; Ziabari, R.B. Slant submanifolds of a conformal Sasakian manifold. *Acta Universitatis Apulensis* **2014**, *40*, 35–49.
30. Abedi, E.; Ziabari, R.B.; Tripathi, M.M. Ricci and scalar curvatures of submanifolds of a conformal Sasakian space form. *Arch. Math.* **2016** , *52*, 113–130. [CrossRef]
31. Akram, A.; Piscoran, L.I. Geometric classification of warped products isometrically immersed into Sasakian space form. *Math. Nachr.* **2019**, *292*, 234–251. [CrossRef]
32. Akram, A.; Piscoran, L.I. Geometry of warped product immersions of Kenmotsu space forms and its applications to slant immersions. *J. Geom. Phys.* **2017**, *114*, 276–290.
33. Akram, A.; Uddin, S.; Othman, W.A.M. Geometry of warped product pointwise semi-slant submanifolds of Kahler manifolds. *Filomat* **2017**, *12*, 3771–3778.
34. Al-Solamy, F.R.; Khan, V.A. Warped product semi-slant submanifolds of a Sasakian manifold. *Serdica Math. J.* **2008**, *34*, 597–606.
35. Li, Y.L.; Ali, A.; Ali, R. A general inequality for CR-warped products in generalized Sasakian-space-form and its applications. *Adv. Math. Phys.* **2021**, *2021*, 5777554. [CrossRef]
36. Li, Y.L.; Alkhaldi, A.H.; Ali, A.; Laurian-Ioan, P. On the Topology of Warped Product Pointwise Semi-Slant Submanifolds with Positive Curvature. *Mathematics* **2021**, *9*, 3156. [CrossRef]

37. Li, Y.L.; Abolarinwa, A.; Azami, S.; Ali, A. Yamabe constant evolution and monotonicity along the conformal Ricci flow. *AIMS Math.* **2022**, *7*, 12077–12090. [CrossRef]
38. Li, Y.L.; Ganguly, D.; Dey, S.; Bhattacharyya, A. Conformal η-Ricci solitons within the framework of indefinite Kenmotsu manifolds. *AIMS Math.* **2022**, *7*, 5408–5430. [CrossRef]
39. Li, Y.L.; Zhu, Y.S.; Sun, Q.Y. Singularities and dualities of pedal curves in pseudo-hyperbolic and de Sitter space. *Int. J. Geom. Methods Mod. Phys.* **2021**, *18*, 1–31. [CrossRef]
40. Li, Y.L.; Ali, A.; Mofarreh, F.; Alluhaibi, N. Homology groups in warped product submanifolds in hyperbolic spaces. *J. Math.* **2021**, *2021*, 8554738. [CrossRef]
41. Li, Y.L.; Lone, M.A.; Wani, U.A. Biharmonic submanifolds of Kähler product manifolds. *AIMS Math.* **2021**, *6*, 9309–9321. [CrossRef]
42. Li, Y.L.; Dey, S.; Pahan, S.; Ali, A. Geometry of conformal η-Ricci solitons and conformal η-Ricci almost solitons on Paracontact geometry. *Open Math.* **2022**, *20*, 1–20. [CrossRef]
43. Li, Y.L.; Ali, A.; Mofarreh, F.; Abolarinwa, A.; Ali, R. Some eigenvalues estimate for the ϕ-Laplace operator on slant submanifolds of Sasakian space forms. *J. Funct. Space* **2021**, *2021*, 6195939.
44. Yang, Z.C.; Li, Y.L.; Erdoğdub, M.; Zhu, Y.S. Evolving evolutoids and pedaloids from viewpoints of envelope and singularity theory in Minkowski plane. *J. Geom. Phys.* **2022**, *176*, 104513. [CrossRef]
45. Todorčević, V. Subharmonic behavior and quasiconformal mappings. *Anal. Math. Phys.* **2019**, *9*, 1211–1225. [CrossRef]
46. Todorčević, V. *Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics*; Springer International Publishing: Berlin/Heidelberg, Germany, 2019.
47. Kojić, V. Quasi-nearly subharmonic functions and conformal mappings. *Filomat* **2007**, *21*, 243–249. [CrossRef]
48. Kojić, V.; Pavlović, M. Subharmonicity of $|f|^p$ for quasiregular harmonic functions, with applications. *J. Math. Anal. Appl.* **2008**, *342*, 742–746. [CrossRef]
49. Manojlović, V.; Vuorinen, M. On quasiconformal maps with identity boundary values. *Trans. Am. Math. Soc.* **2011**, *363*, 2367–2479. [CrossRef]
50. Manojlović, V. Bilipschitz mappings between sectors in planes and quasi-conformality. *Funct. Anal. Approx. Comput.* **2009**, *1*, 1–6.
51. Manojlović, V. On bilipschicity of quasiconformal harmonic mappings. *Novi Sad J. Math.* **2015**, *45*, 105–109. [CrossRef]
52. Manojlović, V. On conformally invariant extremal problems. *Appl. Anal. Discret. Math.* **2009**, *3*, 97–119. [CrossRef]

On almost pseudo semiconformally symmetric manifolds

J. P. Singh and M. Khatri

Abstract. The object of the present paper is to study a type of Riemannian manifold, namely, an almost pseudo semiconformally symmetric manifold which is denoted by $A(PSCS)_n$. Several geometric properties of such a manifold are studied under certain curvature conditions. Some results on Ricci symmetric $A(PSCS)_n$ and Ricci-recurrent $A(PSCS)_n$ are obtained. Next, we consider the decomposability of $A(PSCS)_n$. Finally, two non-trivial examples of $A(PSCS)_n$ are constructed.

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1 Introduction

Riemannian symmetric spaces have an important role in differential geometry. They were first classified by Cartan [4] in the late twenties and he also gave a classification of Riemannian symmetric spaces. In 1926, Cartan [4] studied the certain class of Riemannian spaces and introduced the notation of a symmetric space. According to him, an n-dimensional Riemannian manifold M is said to be locally symmetric if its curvature tensor R satisfies $R_{hijk,l} = 0$, where "," represent the covariant differentiation with respect to the metric tensor and R_{hijk} are the components of the curvature tensor of the manifold M. This condition of locally symmetry is equivalent to the fact that the local geodesic symmetry $F(P)$ is an isometry [20] at every point $P \in M$.

After Cartan, the notation of locally symmetric manifolds has been reduced by many authors in several ways to a different extent such as pseudo symmetric manifolds introduced by Chaki [6], recurrent manifolds introduced by Walker [27], conformally symmetric manifolds introduced by Chaki and Gupta [5], conformally recurrent manifolds introduced by Adati and Miyazawa [2], weakly symmetric manifolds introduced by Tamássy and Binh [26], etc.

In 1967, Sen and Chaki [24] obtained an expression for the covariant derivative of the curvature tensor while studying conformally flat space of class one with certain

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curvature restrictions on the curvature tensor, which is as follows:

$$R^h_{ijk,l} = 2\lambda_l R^h_{ijk} + \lambda_i R^h_{ljk} + \lambda_j R^h_{ilk} + \lambda_k R^h_{ijl} + \lambda^h R^l_{ijk}\,, \tag{1.1}$$

where R^h_{ijk} are the components of the curvature tensor R, $R_{lijk} = g_{hl} R^h_{ijk}$, λ_i is a non-zero covariant vector. Later in 1987, Chaki [6] introduced a manifold whose curvature tensor satisfies (1.1) and called it a pseudo symmetric manifold. In the index-free notation this can be defined as:

$$\begin{aligned}(\nabla_E R)(X,Y)W &= 2A(E)R(X,Y)W + A(X)R(E,Y)W \\ &+ A(Y)R(X,E)W + A(W)R(X,Y)E \\ &+ g(R(X,Y)W,E)\rho,\end{aligned} \tag{1.2}$$

where A is a non-zero 1-form called the associate 1-form of the manifold. Here, ρ is a vector field corresponding to 1-form A and is defined by

$$g(E,\rho) = A(E), \tag{1.3}$$

for all vector field E, and ∇ represents the operator of covariant differentiation with respect to the metric tensor g. Taking $A = 0$ in (1.2) the manifold reduces to a symmetric manifold in the sense of Cartan. An n-dimensional pseudo symmetric manifold is denoted by $(PS)_n$. It should be taken into account that the notation of pseudo symmetric manifold studied in particular by Deszez([3],[8],[9],[10]) differ from that of Chaki [6].

In 2008, De and Gazi [11] introduced a type of Riemannian manifold which is a generalization of pseudo symmetric manifolds. Such manifold is called an almost pseudo symmetric manifold and is denoted by $(APS)_n$. A Riemannian manifold $(M_n, g), (n > 2)$ is said to be an almost pseudo symmetric [11] if its curvature tensor R of type $(0,4)$ satisfies the following relation:

$$\begin{aligned}(\nabla_E R)(X,Y,W,V) &= [A(E)+B(E)]R(X,Y,W,V) + A(X)R(E,Y,W,V) \\ &+ A(Y)R(X,E,W,V) + A(W)R(X,Y,E,V) \\ &+ A(V)R(X,Y,W,X),\end{aligned} \tag{1.4}$$

where A, B are non-zero 1-forms given by

$$g(E,\rho) = A(E), g(E,\sigma) = B(E), \tag{1.5}$$

for all vector fields E. In the paper ([12],[13]) it has been mentioned that $(PS)_n$ is a particular case of an $(AP)_n$.

Gray[16] introduced two groups of Riemannian manifolds based on the covariant differentiation of the Ricci tensor. The first group contains all Riemannian manifolds whose Ricci tensor S is a Codazzi tensor, that is,

$$(\nabla_E S)(X,Y) = (\nabla_X S)(E,Y). \tag{1.6}$$

The second group contains all Riemannian manifolds whose Ricci tensor S is cyclic parallel, that is,

$$(\nabla_E S)(X,Y) + (\nabla_X S)(E,Y) + (\nabla_Y S)(E,X) = 0. \tag{1.7}$$

In 1952, Patterson [22] introduced the notion of Ricci-recurrent manifolds. A non-flat Riemannian manifold $(M, g), (n > 2)$ is said to be a Ricci-recurrent manifold [22] if its non-zero Ricci tensor S of type (0,2) satisfies the following condition

$$(\nabla_E S)(X, Y) = \tilde{H}(E) S(X, Y), \tag{1.8}$$

where $\tilde{H}$ is non-zero 1-form called 1-form of recurrence, which is defined by

$$g(E, \mu) = \tilde{H}(E). \tag{1.9}$$

In 2016, Kim [18] introduced a type of curvature tensor which is a combination of conformal and conharmonic curvature tensor, called semiconformal curvature tensor. The semiconformal curvature tensor of type $(1, 3)$ remains invariant under conharmonic transformation [1]. More precisely, the semiconformal curvature tensor $\tilde{P}$ of type $(1, 3)$ on a Riemannian manifold (M_n, g) is defined as follows:

$$\tilde{P}(X, Y)W = -(n-2)b\, C(X, Y)W + [a + (n-2)b]\, H(X, Y)W, \tag{1.10}$$

where a, b are constants not simultaneously zero, $C(X, Y)W$ denotes the conformal curvature tensor of type $(1, 3)$, and $H(X, Y)W$ denotes the conharmonic curvature tensor of type $(1, 3)$. The conformal curvature tensor and the conharmonic curvature tensor[25] are given as follows:

$$\begin{aligned} C(X, Y)W &= R(X, Y)W - \frac{1}{(n-2)}\Big[S(Y, W)X - S(X, W)Y + g(Y, W)LX \\ &- g(X, W)LY\Big] + \frac{r}{(n-1)(n-2)}\Big[g(Y, W)X - g(X, W)Y\Big], \end{aligned} \tag{1.11}$$

and,

$$\begin{aligned} H(X, Y)W &= R(X, Y)W - \frac{1}{(n-2)}\Big[S(Y, W)X - S(X, W)Y + g(Y, W)LX \\ &- g(X, W)LY\Big], \end{aligned} \tag{1.12}$$

where L is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is, $g(LE, X) = S(E, X)$ and r is the scalar curvature of the manifold. From equations (1.10),(1.11) and (1.12) we obtain an expression for semiconformal curvature tensor $P(X, Y, W, V)$ of type (0,4) as follows:

$$\begin{aligned} P(X, Y, W, V) &= aR(X, Y, W, V) - \frac{a}{(n-2)}\Big[S(Y, W)g(X, V) \\ &- S(X, W)g(Y, V) + S(X, V)g(Y, W) - S(Y, V)g(X, W)\Big] \\ &- \frac{br}{(n-1)}\Big[g(Y, W)g(X, V) - g(X, W)g(Y, V)\Big], \end{aligned} \tag{1.13}$$

where $P(X, Y, W, V) = g(\tilde{P}(X, Y)W, V)$.

For $a = 1$ and $b = -\frac{1}{(n-2)}$, the semiconformal curvature becomes conformal curvature tensor and for $a = 1$ and $b = 0$, such a tensor reduces to conharmonic curvature tensor. A Riemannian manifold (M_n, g) of dimension $n \geq 4$ is said to be pseudo semiconformally symmetric [17] if its semiconformal curvature tensor P of type $(0, 4)$ satisfies the relation

$$\begin{aligned}(\nabla_E P)(X,Y,W,V) &= 2A(E)P(X,Y,W,V) + A(X)P(E,Y,W,V)\\ &+ A(Y)P(X,E,W,V) + A(W)P(X,Y,E,V)\\ &+ A(V)P(X,Y,W,E).\end{aligned} \tag{1.14}$$

The semiconformal curvature tensor is further studied in the recent paper by De and Suh [14]. An almost pseudo symmetric manifold introduced by De and Gazi [11] is an important generalization of symmetric space which is studied by several geometers ([15],[7],[21],[19]), and many others. Motivated by there studies in an almost pseudo symmetric manifold and semiconformal curvature tensor, in the present paper, we introduced a type of non-flat Riemannian manifold $(M_n, g), (n \geq 4)$ whose semiconformal curvature tensor P of type $(0, 4)$ satisfies the condition

$$\begin{aligned}(\nabla_E P)(X,Y,W,V) &= [A(E) + B(E)]P(X,Y,W,V) + A(X)P(E,Y,W,V)\\ &+ A(Y)P(X,E,W,V) + A(W)P(X,Y,E,V)\\ &+ A(V)R(X,Y,W,E),\end{aligned} \tag{1.15}$$

where A and B are non-zero 1-forms and are called the associated 1-forms, defined as in (1.5), and ∇ has the meaning previously introduced. The vector fields ρ and σ corresponding to the associated 1-forms A and B respectively shall be called the basic vector fields of the manifold. We shall be calling such a manifold as an almost pseudo semiconformally symmetric manifold and an n-dimensional manifold of this kind shall be denoted by $A(PSCS)_n$. If in (1.15) $A = B$, then the manifold becomes a pseudo semiconformally symmetric manifold defined by (1.14). The manifold $A(PSCS)_n$ includes an almost pseudo conformally symmetric manifold [13] and an almost pseudo conharmonically symmetric manifold [21].

The present paper is organized as follows: After preliminaries, in section 3 we investigated some geometric properties of $A(PSCS)_n$ with non-zero constant scalar curvature and Codazzi type of Ricci tensor. In section 4, Ricci symmetric $A(PSCS)_n$ and Ricci recurrent $A(PSCS)_n$ are studied. Section 5 deals with an Einstein $A(PSCS)_n$. In section 6, it is concerned with the decomposition of $A(PSCS)_n$ and exactly defined each product manifolds of an $A(PSCS)_n$. Finally, we constructed two non-trivial examples of $A(PSCS)_n$.

2 Preliminaries

Let r and S denote the scalar curvature and the Ricci tensor of type (0,2) respectively and L has the meaning already mentioned, that is,

$$g(LE, X) = S(E, X). \tag{2.1}$$

In this section, we will derive some formulas, which we will be using in the study of $A(PSCS)_n$ throughout this paper. Let $\{e_i\}$ be an orthonormal basis of the tangent

space at each point of the manifold where $1 \leq i \leq n$.
Now from equation (1.13), we have

$$\sum_{i=1}^{n} P(X, Y, e_i, e_i) = 0 = \sum_{i=1}^{n} P(e_i, e_i, X, Y), \tag{2.2}$$

and,

$$\sum_{i=1}^{n} P(e_i, Y, W, e_i) = \sum_{i=1}^{n} P(Y, e_i, e_i, W) = -\frac{\{a + (n-2)b\}r}{(n-2)} g(Y, W), \tag{2.3}$$

where, $r = \sum_{i=1}^{n} S(e_i, e_i)$ is the scalar curvature.

Making use of equation (1.13) we obtain the following relations:

$$\begin{aligned} &(i) \quad P(X, Y, W, V) = -P(Y, X, W, V), \\ &(ii) \quad P(X, Y, W, V) = -P(X, Y, V, W), \\ &(iii) \quad P(X, Y, W, V) = P(W, V, X, Y), \\ &(iv) \quad P(X, Y, W, V) + P(Y, W, X, V) + P(W, X, Y, V) = 0. \end{aligned} \tag{2.4}$$

3 An $A(PSCS)_n, (n \geq 4)$ with non-zero constant scalar curvature and Codazzi type of Ricci tensor.

Theorem 3.1. *In $A(PSCS)_n, (n \geq 4)$ the scalar curvature is a non-zero constant if and only if $(4+n)A(E) + nB(E) = 0$, provided $[a + (n-2)b] \neq 0$.*

Proof. Taking covariant derivative of equation (1.13) with respect to E we get,

$$\begin{aligned} a(\nabla_E R)(X, Y, W, V) &= (\nabla_E P)(X, Y, W, V) + \frac{a}{(n-2)}\Big\{(\nabla_E S)(Y, W)g(X, V) \\ &- (\nabla_E S)(X, W)g(Y, V) + (\nabla_E S)(X, V)g(Y, W) \\ &- (\nabla_E S)(Y, V)g(X, W)\Big\} + \frac{b\, dr(E)}{(n-1)}\Big\{g(Y, W)g(X, V) \\ &- g(X, W)g(Y, V)\Big\}. \end{aligned} \tag{3.1}$$

Inserting equation (1.15) in equation (3.1) we obtain,

$$\begin{aligned} a(\nabla_E R)(X, Y, W, V) &= [A(E) + B(E)]P(X, Y, W, V) + A(X)P(E, Y, W, V) \\ &+ A(Y)P(X, E, W, V) + A(W)P(X, Y, E, V) \\ &+ A(V)R(X, Y, W, E) + \frac{a}{(n-2)}\Big\{(\nabla_E S)(Y, W)g(X, V) \\ &- (\nabla_E S)(X, W)g(Y, V) + (\nabla_E S)(X, V)g(Y, W) \\ &- (\nabla_E S)(Y, V)g(X, W)\Big\} + \frac{b\, dr(E)}{(n-1)}\Big\{g(Y, W)g(X, V) \\ &- g(X, W)g(Y, V)\Big\}. \end{aligned} \tag{3.2}$$

Putting $X = V = e_i, (i = 1, 2, ..., n)$ and $\lambda = \dfrac{\{a + (n-2)b\}r}{(n-2)}$ in equation (3.2), we obtain

$$\begin{aligned} a(\nabla_E S)(Y,W) &= [A(E)+B(E)]\Big[-\lambda r g(Y,W)\Big] + A(\tilde{P}(E,Y)W) \\ &+ A(Y)\Big[-\lambda r g(E,W)\Big] + A(W)\Big[-\lambda r g(Y,E)\Big] - A(\tilde{P}(W,E)Y) \\ &+ \frac{a}{(n-2)}\Big[n(\nabla_E S)(Y,W) - (\nabla_E S)(W,Y) + dr(E)g(Y,W) \\ &- (\nabla_E S)(Y,W)\Big] + b\,dr(E)g(Y,W). \end{aligned} \tag{3.3}$$

Contracting over Y and W in equation (3.3), the above equation reduces to

$$n[a+(n-2)b]\,dr(E) = [a+(n-2)b]r[(4+n)A(E)+nB(E)]. \tag{3.4}$$

Assuming $[a+(n-2)b] \neq 0$, then equation (3.4) reduces to

$$n\,dr(E) = r[(4+n)A(E)+nB(E)]. \tag{3.5}$$

Clearly if $[(4+n)A(E)+nB(E)] = 0$ then r is a non-zero constant.
Conversely, if r is a non-zero constant then $[(4+n)A(E)+nB(E)] = 0$.
This completes the proof. □

Theorem 3.2. *If Ricci tensor in $A(PSCS)_n$ is of Codazzi type then the semiconformal curvature tensor P satisfies Bianchi's second identity.*

Proof. Making use of equation (1.13) we can obtain

$$\begin{aligned} (\nabla_E P)(X,Y,W,V) &+ (\nabla_X P)(Y,E,W,V) + (\nabla_Y P)(E,X,W,V) \\ &= a\Big[(\nabla_E R)(X,Y,W,V) + (\nabla_X R)(Y,E,W,V) \\ &+ (\nabla_Y R)(E,X,W,V)\Big] - \frac{a}{(n-2)}\Big[(\nabla_E S)(Y,W)g(X,V) \\ &- (\nabla_E S)(X,W)g(Y,V) + (\nabla_E S)(X,V)g(Y,W) \\ &- (\nabla_E S)(Y,V)g(X,W) + (\nabla_X S)(E,W)g(Y,V) \\ &- (\nabla_X S)(Y,W)g(E,V) + (\nabla_X S)(Y,V)g(E,W) \\ &- (\nabla_X S)(E,V)g(Y,W) + (\nabla_Y S)(X,W)g(E,V) \\ &- (\nabla_Y S)(E,W)g(X,V) - (\nabla_Y S)(X,V)g(E,W) \\ &+ (\nabla_Y S)(E,V)g(X,W)\Big] - \frac{b}{(n-1)}\Big[dr(E)\{g(Y,W)g(X,V) \\ &- g(X,W)g(Y,V)\} + dr(X)\{g(E,W)g(Y,V) \\ &- g(Y,W)g(E,V)\} + dr(Y)\{g(X,W)g(E,V) \\ &- g(E,W)g(X,V)\}\Big]. \end{aligned} \tag{3.6}$$

Since the Ricci tensor is of Codazzi type, S satisfies the relation:

$$(\nabla_E S)(X,Y) = (\nabla_X S)(E,Y), \tag{3.7}$$

implies $r =$ constant.
Moreover, inserting equation (3.7) in equation (3.6), we have

$$(\nabla_E P)(X,Y,W,V) + (\nabla_X P)(Y,E,W,V) + (\nabla_Y P)(E,X,W,V) = 0. \tag{3.8}$$

Hence, the theorem is proved. □

Theorem 3.3. *In $A(PSCS)_n$, if the semiconformal curvature tensor P satisfies Bianchi's second identity then $A(PSCS)_n$ reduces to a pseudo semiconformally symmetric manifold, provided $[a+(n-2)b] \neq 0$ and $r \neq 0$.*

Proof. Suppose that the semiconformal tensor P in $A(PSCS)_n$ satisfies Bianchi's second identity. Then making use equation (1.15), we get

$$\begin{aligned}[B(E)-A(E)]P(X,Y,W,V) + [B(X)-A(X)]P(Y,E,W,V)\\ +[B(Y)-A(Y)]P(E,X,W,V) = 0.\end{aligned} \tag{3.9}$$

Let $Q(E) = B(E) - A(E)$ and ρ_1 be a basic vector such that

$$g(E,\rho_1) = Q(E), \tag{3.10}$$

for all E. Equation (3.9) with the help of equation (3.10) may be written as

$$Q(E)P(X,Y,W,V) + Q(X)P(Y,E,W,V) + Q(Y)P(E,X,W,V) = 0. \tag{3.11}$$

Putting $X = V = e_i$ in equation (3.11), the above equation reduces to

$$\begin{aligned}Q(E)\left\{-\frac{[a+(n-2)b]r}{(n-2)}g(Y,W)\right\} + Q(\tilde{P}(Y,E)W)\\ -Q(Y)\left\{-\frac{[a+(n-2)b]r}{(n-2)}g(E,W)\right\} = 0,\end{aligned} \tag{3.12}$$

and contracting over Y and W, we infer

$$[a+(n-2)b]rQ(E) = 0. \tag{3.13}$$

Suppose $r \neq 0$ and $[a+(n-2)b] \neq 0$ in above equation implies $Q(E) = 0$.
This completes the proof. □

Theorem 3.4. *If $A(PSCS)_n$ satisfies Bianchi's second identity then the scalar curvature is constant provided $[a+(n-2)b] \neq 0$.*

Proof. Suppose $A(PSCS)_n$ satisfies Bianchi's second identity. Then, from equation (1.13), we obtain

$$\begin{aligned}&\frac{a}{(n-2)}\Big\{(\nabla_E S)(Y,W)g(X,V) - (\nabla_E S)(X,W)g(Y,V) + (\nabla_E S)(X,V)g(Y,W)\\ &\quad -(\nabla_E S)(Y,V)g(X,W) + (\nabla_X S)(E,W)g(Y,V) - (\nabla_X S)(Y,W)g(E,V)\\ &\quad +(\nabla_X S)(Y,V)g(E,W) - (\nabla_X S)(E,V)g(Y,W) + (\nabla_Y S)(X,W)g(E,V)\\ &\quad -(\nabla_Y S)(E,W)g(X,V) - (\nabla_Y S)(X,V)g(E,W) + (\nabla_Y S)(E,V)g(X,W)\Big\}\\ &+\frac{b}{(n-1)}\Big\{dr(E)\{g(Y,W)g(X,V) - g(X,W)g(Y,V)\} + dr(X)\{g(E,W)g(Y,V)\\ &\quad -g(Y,W)g(E,V)\} + dr(Y)\{g(X,W)g(E,V) - g(E,W)g(X,V)\}\Big\} = 0.\end{aligned} \tag{3.14}$$

Contracting equation (3.14) over Y and W, the equation reduces to

$$\frac{a}{(n-2)}\Bigg[\frac{1}{2}dr(E)g(X,V)+(n-2)(\nabla_E S)(X,V)+(2-n)(\nabla_X S)(E,V)$$
$$-\frac{1}{2}dr(X)g(E,V)-(\nabla_E S)(X,V)+(\nabla_X S)(E,V)\Bigg]+bg(X,V)dr(E)$$
$$-bg(E,V)dr(X)+\frac{b}{(n-1)}\Bigg[dr(X)g(E,V)-dr(E)g(X,V)\Bigg]=0. \tag{3.15}$$

Substituting $X=V=e_i$ in equation (3.15) yields

$$[a+(n-2)b]\,dr(E)=0. \tag{3.16}$$

This completes the proof. □

4 Ricci Symmetric $A(PSCS)_n, (n\geq 4)$ and Ricci-recurrent $A(PSCS)_n, (n\geq 4)$.

Theorem 4.1. *In a Ricci symmetric $A(PSCS)_n, (n\geq 4)$, the Bianchi's second identity holds for semiconformal curvature tensor.*

Proof. Since $A(PSCS)_n$ is Ricci symmetric, the Ricci tensor S satisfies the condition

$$\nabla S=0$$

and $dr=0$.
Using this, we have

$$(\nabla_E P)(X,Y,W,V)=a(\nabla_E R)(X,Y,W,V).$$

Hence,

$$(\nabla_E P)(X,Y,W,V)+(\nabla_X P)(Y,E,W,V)+(\nabla_Y P)(E,X,W,V)=$$
$$a[(\nabla_E R)(X,Y,W,V)+(\nabla_X R)(Y,E,W,V)+(\nabla_Y R)(E,X,W,V)], \tag{4.1}$$

implies,

$$(\nabla_E P)(X,Y,W,V)+(\nabla_X P)(Y,E,W,V)+(\nabla_Y P)(E,X,W,V)=0. \tag{4.2}$$

Hence, the theorem is proved. □

Theorem 4.2. *In a Ricci symmetric $A(PSCS)_n, (n\geq 4)$ the vector fields corresponding to the 1-forms A and B are in opposite direction, provided $r\neq 0$ and $[a+(n-2)b]\neq 0$.*

Proof. Contracting equation (1.15) over E, we get

$$(div\tilde{P})(X,Y)W=A(\tilde{P}(X,Y)W)+B(\tilde{P}(X,Y)W)-A(X)\left\{\frac{[a+(n-2)b]r}{(n-2)}\right\}$$
$$g(Y,W)+A(Y)\left\{\frac{[a+(n-2)b]r}{(n-2)}\right\}g(X,W)+A(\tilde{P}(X,Y)W). \tag{4.3}$$

Moreover we have,

$$(div\tilde{P})(X,Y)W = \frac{a(n-3)}{(n-2)}\Big\{(\nabla_X S)(Y,W) - (\nabla_Y S)(X,W)\Big\}$$
$$-\Big\{\frac{[a(n-1)+b(n-2)]}{2(n-1)(n-2)}\Big\}\Big\{dr(X)g(Y,W) - dr(Y)g(X,W)\Big\}. \tag{4.4}$$

Combining equations (4.3) and (4.4), the above equations reduces to

$$A(\tilde{P}(X,Y)W) + B(\tilde{P}(X,Y)W) - A(X)\Big\{\frac{[a+(n-2)b]r}{(n-2)}\Big\}$$
$$g(Y,W) + A(Y)\Big\{\frac{[a+(n-2)b]r}{(n-2)}\Big\}g(X,W) + A(\tilde{P}(X,Y)W)$$
$$= \frac{a(n-3)}{(n-2)}\Big\{(\nabla_X S)(Y,W) - (\nabla_Y S)(X,W)\Big\}$$
$$-\Big\{\frac{[a(n-1)+b(n-2)]}{2(n-1)(n-2)}\Big\}\Big\{dr(X)g(Y,W) - dr(Y)g(X,W)\Big\}. \tag{4.5}$$

Suppose the manifold is Ricci symmetric, then equation (4.5) becomes

$$2A(\tilde{P}(X,Y)W) + B(\tilde{P}(X,Y)W) - A(X)\Big\{\frac{[a+(n-2)b]r}{(n-2)}\Big\}g(Y,W)$$
$$+A(Y)\Big\{\frac{[a+(n-2)b]r}{(n-2)}\Big\}g(X,W) = 0. \tag{4.6}$$

Inserting $Y = W = e_i$ in equation (4.6) and taking summation over $1 \leq i \leq n$, we obtain

$$[a+(n-2)b]r[(n+1)A(X) + B(X)] = 0. \tag{4.7}$$

If $r \neq 0$ and $[a+(n-2)b] \neq 0$, then above equation gives $B(X) = -(n+1)A(X)$. Therefore, this led to the statement of the above theorem. □

Corollary 4.3. *In a Ricci symmetric* $A(PSCS)_n, (n \geq 4)$ *the scalar curvature vanishes if* $[(n+1)A(X) + B(X)] \neq 0$*, provided* $[a+(n-2)b] \neq 0$.

Theorem 4.4. *In a Ricci-recurrent* $A(PSCS)_n, (n \geq 4)$*, if the scalar curvature is non-zero and* $[a+(n-2)b] \neq 0$*, then* $\tilde{H}(E) = 3A(E) + B(E)$*, for all* E.

Proof. Equation (1.13) making use of (1.15) results in the following

$$[A(E)+B(E)]P(X,Y,W,V) + A(X)P(E,Y,W,V) + A(Y)P(X,E,W,V)$$
$$+A(W)P(X,Y,E,V) + A(V)R(X,Y,W,E) = a(\nabla_E R)(X,Y,W,V)$$
$$-\frac{a}{(n-2)}\Big\{(\nabla_E S)(Y,W)g(X,V) - (\nabla_E S)(X,W)g(Y,V)$$
$$+(\nabla_E S)(X,V)g(Y,W) - (\nabla_E S)(Y,V)g(X,W)\Big\}$$
$$-\frac{b\,dr(E)}{(n-1)}\Big\{g(Y,W)g(X,V) - g(X,W)g(Y,V)\Big\}. \tag{4.8}$$

Now, contracting above equation yields

$$dr(E) = r\tilde{H}(E). \tag{4.9}$$

The use of equations (1.8) and (4.9) in equation (4.8) gives

$$\begin{aligned}
[A(E) + B(E)]P(X,Y,W,V) + A(X)P(E,Y,W,V) + A(Y)P(X,E,W,V) \\
+A(W)P(X,Y,E,V) + A(V)R(X,Y,W,E) = a(\nabla_E R)(X,Y,W,V) \\
-\frac{a}{(n-2)}\Big\{S(Y,W)g(X,V) - S(X,W)g(Y,V) \\
+S(X,V)g(Y,W) - S(Y,V)g(X,W)\Big\}H(E) \\
-\frac{br\tilde{H}(E)}{(n-1)}\Big\{g(Y,W)g(X,V) - g(X,W)g(Y,V)\Big\}.
\end{aligned} \tag{4.10}$$

Putting $X = V = e_i$ in equation (4.10), we get

$$\begin{aligned}
[A(E) + B(E)]\Big\{-\frac{[a + (n-2)b]r}{(n-2)}\Big\}g(Y,W) + A(\tilde{P}(E,Y)W) \\
-A(Y)\Big\{\frac{[a + (n-2)b]r}{(n-2)}\Big\}g(E,W) - A(W)\Big\{\frac{[a + (n-2)b]r}{(n-2)}\Big\}g(Y,E) \\
-A(\tilde{P}(W,E)Y) = -r\Big\{\frac{[a + (n-2)b]}{(n-2)}\Big\}g(Y,W)\tilde{H}(E).
\end{aligned} \tag{4.11}$$

Moreover, inserting $Y = W = e_i$ in equation (4.11), the above equation becomes

$$[(n+4)A(E) + nB(E)] = n\tilde{H}(E). \tag{4.12}$$

Similarly, taking $E = Y = e_i$ in equation (4.11) gives,

$$(1+n)A(W) + B(W) = \tilde{H}(W), \tag{4.13}$$

and replacing $W = E$ in above equation, we get

$$(1+n)A(E) + B(E) = \tilde{H}(E). \tag{4.14}$$

Again, contracting the equation (4.11) over E and W, we infer

$$(n+1)A(Y) + B(Y) = \tilde{H}(Y). \tag{4.15}$$

Substituting $Y = E$ in equation (4.15) gives

$$(1+n)A(E) + B(E) = \tilde{H}(E). \tag{4.16}$$

Combining equations (4.12),(4.14) and (4.16), we obtain

$$\tilde{H}(E) = 3A(E) + B(E). \tag{4.17}$$

Hence, $\tilde{H}(E) = 3A(E) + B(E)$ provided $r \neq 0$ and $[a + (n-2)b] \neq 0$. □

5 Einstein $A(PSCS)_n, (n \geq 4)$

Theorem 5.1. *If an Einstein $A(PSCS)_n, (n \geq 4)$ is an $A(PS)_n$ and $2a(n-1) - bn(n-2) \neq 0$ and $3A(E) + B(E) \neq 0$, then its scalar curvature vanishes, provided $a \neq 0$.*

Proof. In Einstein manifold the Ricci tensor is given by

$$S(E, X) = \frac{r}{n} g(E, X), \tag{5.1}$$

implies,

$$dr(E) = 0 \text{ and } (\nabla_E S)(X, Y) = 0. \tag{5.2}$$

Using equations (1.13),(5.1) and (5.2), we obtain

$$\begin{aligned} P(X, Y, W, V) = aR(X, Y, W, V) \quad &- \quad r\left[\frac{2a(n-1) - bn(n-2)}{n(n-1)(n-2)}\right]\Big[g(Y, W)g(X, V) \\ &- \quad g(X, W)g(Y, V)\Big]. \end{aligned} \tag{5.3}$$

The covariant derivative of equation (5.3) gives

$$(\nabla_E P)(X, Y, W, V) = a(\nabla_E R)(X, Y, W, V). \tag{5.4}$$

Now, inserting equation (5.4) in equation (1.13), we obtain

$$\begin{aligned} a(\nabla_E R)(X, Y, W, V) &= [A(E) + B(E)]\Big\{aR(X, Y, W, V) \\ &- r\Big\{\frac{[2a(n-1) - bn(n-2)]}{n(n-1)(n-2)}\Big\}\Big[g(Y, W)g(X, V) \\ &- g(X, W)g(Y, V)\Big]\Big\} + A(X)\Big\{aR(E, Y, W, V) \\ &- r\Big\{\frac{[2a(n-1) - bn(n-2)]}{n(n-1)(n-2)}\Big\}\Big[g(Y, W)g(E, V) \\ &- g(E, W)g(Y, V)\Big]\Big\} + A(Y)\Big\{aR(X, E, W, V) \\ &- r\Big\{\frac{[2a(n-1) - bn(n-2)]}{n(n-1)(n-2)}\Big\}\Big[g(E, U)g(Y, V) \\ &- g(Y, U)g(E, V)\Big]\Big\} + A(W)\Big\{aR(Y, Z, E, V) \\ &- r\Big\{\frac{[2a(n-1) - bn(n-2)]}{n(n-1)(n-2)}\Big\}\Big[g(Y, E)g(X, V) \\ &- g(X, E)g(Y, V)\Big]\Big\} + A(V)\Big\{aR(X, Y, W, E) \\ &- r\Big\{\frac{[2a(n-1) - bn(n-2)]}{n(n-1)(n-2)}\Big\}\Big[g(Y, W)g(X, E) \\ &- g(X, W)g(Y, E)\Big]\Big\}. \end{aligned} \tag{5.5}$$

Assume $a \neq 0$. Suppose that an Einstein $A(PSCS)_n$ is an $A(PS)_n$. Then equation (5.5) becomes

$$\begin{aligned}&\Big[\frac{r\{2a(n-1)-bn(n-2)\}}{n(n-1)(n-2)}\Big]\Big[\{A(E)+B(E)\}\Big[g(Y,W)g(X,V)\\&-g(X,W)g(Y,V)\Big]+A(X)\Big[g(Y,W)g(E,V)-g(E,W)g(Y,V)\Big]\\&+A(Y)\Big[g(E,W)g(X,V)-g(X,W)g(E,V)\Big]+A(W)\Big[g(Y,E)g(X,V)\\&-g(X,E)g(Y,V)\Big]+A(V)\Big[g(Y,W)g(X,E)-g(X,W)g(Y,E)\Big]\Big]=0.\end{aligned} \tag{5.6}$$

Putting $X = V = e_i$ in equation (5.6), the above equation reduces to

$$\begin{aligned}&r\Big[2a(n-1)-bn(n-2)\Big]\Big[\{A(E)+B(E)\}(n-1)g(Y,W)+A(E)g(Y,W)\\&-A(Y)g(E,W)+A(Y)(n-1)g(E,W)+A(W)(n-1)g(Y,E)\\&+A(E)g(Y,W)-A(W)g(Y,E)\Big]=0.\end{aligned} \tag{5.7}$$

Moreover, taking $Y = W = e_i$ in equation (5.7) gives

$$r[2a(n-1)-bn(n-2)][(n+4)A(E)+nB(E)]=0. \tag{5.8}$$

Similarly, contracting equation (5.7) over Y and E we infer

$$r[2a(n-1)-bn(n-2)][(n+1)A(W)+B(W)]=0. \tag{5.9}$$

Substituting $W = E$ in equation (5.9) gives

$$r[2a(n-1)-bn(n-2)][(n+1)A(E)+B(E)]=0. \tag{5.10}$$

Again, putting $W = E = e_i$ in equation (5.7), we get

$$r[2a(n-1)-bn(n-2)][(n+1)A(Y)+B(Y)]=0, \tag{5.11}$$

and substituting $Y = E$ in equation (5.11) gives,

$$r[2a(n-1)-bn(n-2)][(n+1)A(E)+B(E)]=0. \tag{5.12}$$

Combining the equations (5.8),(5.10) and (5.12), we obtain the following result

$$r[2a(n-1)-bn(n-2)][3A(E)+B(E)]=0. \tag{5.13}$$

Hence, the theorem is proved. □

Suppose $r = 0$ in equation (5.5) then Einstein $A(PSCS)_n$ is an $A(PS)_n$, provided $a \neq 0$. Thus, we can state the following:

Theorem 5.2. *If $a \neq 0$ and scalar curvature vanishes in Einstein $A(PSCS)_n$, $(n \geq 4)$ then such a manifold is an $A(PS)_n$.*

Theorem 5.3. *If the vector field ρ_1 defined by $g(E, \rho_1) = B(E) - A(E)$, for all E, is a parallel vector field in an Einstein $A(PSCS)_n, (n \geq 4)$ with $a \neq 0$ and $||\rho_1||^2 \neq 0$, then it is an $A(PS)_n$.*

Proof. Let us consider that the vector field ρ_1 defined in equation (3.10) is parallel in an Einstein $A(PSCS)_n$. Then, we get

$$\nabla_E \rho_1 = 0, \tag{5.14}$$

for all E.
Which gives,

$$R(E, X, \rho_1, V) = 0.$$

Contracting the above equation we get

$$S(X, \rho_1) = 0.$$

Then, from equation (5.1), we have

$$rg(X, \rho_1) = 0. \tag{5.15}$$

If $||\rho_1||^2 \neq 0$, then above equation follows that $r = 0$.
Therefore, by equation (5.5), Einstein $A(PSCS)_n$ reduces to $A(PS)_n$, provided $a \neq 0$.
Hence, this completes the theorem. □

6 Decomposition of $A(PSCS)_n, (n \geq 4)$

A Riemannian manifold (M^n, g) is said to be decomposable or a product manifold[23] if it can be written as $M_1^p \times M_2^{n-p}$ for $2 \leq p \leq (n-2)$, that is, in some coordinate neighborhood of the Riemannian manifold (M^n, g) the metric can be expressed as

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ab}dx^a dx^b + g^*_{\alpha\beta}dx^\alpha dx^\beta, \tag{6.1}$$

where $\bar{g}_{ab}$ are functions of $x^1, x^2, ..., x^p$ denoted by $\bar{x}$ and $g^*_{\alpha\beta}$ are functions of $x^{p+1}, x^{p+2}, ..., x^n$ denoted by x^* : $a, b, c, ...$run from 1 to p and $\alpha, \beta, \gamma, ...$, run from $p+1$ to n. In (6.1), $\bar{g}_{ab}$ and $g^*_{\alpha\beta}$ are the matrices of $M_1^p (p \geq 2)$ and $M_2^{n-p}(n-p \geq 2)$ respectively, which are called the components of the decomposable manifold $M^n = M_1^p \times M_2^{n-p}(2 \leq p \leq n-2)$.

We will assume throughout this section that all objects indicated by a 'bar' belong to M_1 and all objects indicated by a 'star' belongs to M_2.

Let $\bar{E}, \bar{X}, \bar{Y}, \bar{W}, \bar{V} \in \chi(M_1)$ and $E^*, X^*, Y^*, W^*, V^* \in \chi(M_2)$. Then in a decomposable Riemannian manifold $M^n = M_1^p \times M_2^{n-p}(2 \leq p \leq n-2)$, the following relations hold

$$\begin{aligned} R(E^*, \bar{X}, \bar{Y}, \bar{W}) = 0 = R(\bar{E}, X^*, \bar{Y}, W^*) = R(\bar{E}, X^*, Y^*, W^*), \\ (\nabla_{E^*} R)(\bar{X}, \bar{Y}, \bar{W}, \bar{V}) = 0 = (\nabla_{\bar{E}} R)(\bar{X}, Y^*, \bar{W}, V^*) = (\nabla_{E^*} R)(\bar{X}, Y^*, \bar{W}, V^*), \\ R(\bar{E}, \bar{X}, \bar{Y}, \bar{W}) = \bar{R}(\bar{E}, \bar{X}, \bar{Y}, \bar{W}); R(E^*, X^*, Y^*, W^*) = R^*(E^*, X^*, Y^*, W^*), \\ S(\bar{E}, \bar{X}) = \bar{S}(\bar{E}, \bar{X}); S(E^*, X^*) = S^*(E^*, X^*), \\ (\nabla_{\bar{E}} S)(\bar{X}, \bar{Y}) = (\bar{\nabla}_{\bar{E}} S)(\bar{X}, \bar{Y}); (\nabla_{E^*} S)(X^*, Y^*) = (\nabla^*_{E^*} S)(X^*, Y^*), \end{aligned} \tag{6.2}$$

where $\bar{r}$,r^* and r are scalar curvature of M_1,M_2 and M respectively and are related as $r = \bar{r} + r^*$. Also $S(\bar{E}, X^*) = 0$ and $g(\bar{E}, X^*) = 0$.

Theorem 6.1. *Let an $A(PSCS)_n$ be a decomposable space such that $M^n = M_1^p \times M_2^{n-p}$ for $(2 \leq p \leq n-2)$, then the following holds:*
i) In the case of $A = B = 0$ on M_2, the manifold M_2 is Ricci symmetric and scalar curvature r^* is constant in M_2, provided $d\bar{r}(E^*) = 0$ and $\frac{a(n-p-2)}{(n-2)} \neq \frac{bp(n-p)}{(n-1)}$.
ii) when M_1 is semiconformally flat, then M_1 is an Einstein manifold.

Proof. Let us consider a Riemannian manifold (M^n, g) which is a decomposable $A(PSCS)_n$, then

$$M^n = M_1^p \times M_2^{n-p} (2 \leq p \leq n-2).$$

Now from equation (1.13), we obtain

$$\begin{aligned}
P(X^*, \bar{Y}, \bar{W}, \bar{V}) &= 0 = P(\bar{X}, Y^*, W^*, V^*) \\
&= P(\bar{X}, Y^*, \bar{W}, \bar{V}) = P(\bar{X}, \bar{Y}, W^*, \bar{V}); \\
P(X^*, \bar{Y}, \bar{W}, V^*) &= -\frac{a}{(n-2)}\Big[S(\bar{Y}, \bar{W})g(X^*, W^*) + S(X^*, V^*)g(\bar{Y}, \bar{W})\Big] \\
&- \frac{rb}{(n-1)}\Big[g(\bar{Y}, \bar{W})g(X^*, V^*)\Big]; \\
P(X^*, Y^*, \bar{W}, \bar{V}) &= 0 = P(\bar{X}, \bar{Y}, W^*, V^*); \\
P(X^*, \bar{Y}, W*, \bar{V}) &= \frac{a}{(n-2)}\Big[S(\bar{Y}, \bar{V})g(X^*, W^*) + S(X^*, W^*)g(\bar{Y}, \bar{V})\Big] \\
&+ \frac{rb}{(n-1)}\Big[g(\bar{Y}, \bar{V})g(X^*, W^*)\Big].
\end{aligned} \tag{6.3}$$

Further simplifying the above equation, we get

$$\begin{aligned}
&(\nabla_{\bar{E}} P)(\bar{X}, \bar{Y}, \bar{W}, \bar{V}) = [A(\bar{E}) + B(\bar{E})]P(\bar{X}, \bar{Y}, \bar{W}, \bar{V}) + A(\bar{X})P(\bar{E}, \bar{Y}, \bar{W}, \bar{V}) \\
&\quad + A(\bar{Y})P(\bar{X}, \bar{E}, \bar{W}, \bar{V}) + A(\bar{W})P(\bar{X}, \bar{Y}, \bar{E}, \bar{V}) + A(\bar{V})P(\bar{X}, \bar{Y}, \bar{W}, \bar{E})
\end{aligned} \tag{6.4}$$

Putting $\bar{X} = X^*$ in equation (6.4) gives

$$A(X^*)P(\bar{E}, \bar{Y}, \bar{W}, \bar{V}) = 0. \tag{6.5}$$

Also, inserting $\bar{E} = E^*$ in equation (6.4), we have

$$[A(E^*) + B(E^*)]P(\bar{X}, \bar{Y}, \bar{W}, \bar{V}) = 0. \tag{6.6}$$

Similarly inserting $\bar{E} = E^*$ and $\bar{X} = X^*$ in equation (6.4), we infer

$$A(\bar{W})P(X^*, \bar{Y}, E^*, \bar{V}) + A(\bar{V})P(X^*, \bar{Y}, \bar{W}, E^*) = 0. \tag{6.7}$$

Putting $\bar{E} = E^*$ and $\bar{W} = W^*$ in equation (6.4), we get

$$A(\bar{X})P(E^*, \bar{Y}, W^*, \bar{V}) + A(\bar{Y})P(\bar{X}, E^*, W^*, \bar{V}) = 0. \tag{6.8}$$

And, taking $\bar{X} = X^*, \bar{Y} = Y^*$ and $\bar{W} = W^*$ in equation (6.4) results in

$$A(X^*)P(\bar{E}, Y^*, W^*, \bar{V}) + A(Y^*)P(X^*, \bar{E}, W^*, \bar{V}) = 0. \tag{6.9}$$

Substituting $\bar{Y} = Y^*, \bar{W} = W^*$ and $\bar{V} = V^*$ in equation (6.4), we have

$$A(W^*)P(\bar{X}, Y^*, \bar{E}, V^*) + A(V^*)P(\bar{X}, Y^*, W^*, \bar{E}) = 0. \tag{6.10}$$

Moreover, using equation (1.13) gives

$$\begin{aligned}(\nabla_{E^*}P)(X^*, Y^*, W^*, V^*) = {} & [A(E^*) + B(E^*)]P(X^*, Y^*, W^*, V^*)\\ & +A(X^*)P(E^*, Y^*, W^*, V^*) + A(Y^*)P(X^*, E^*, W^*, V^*)\\ & +A(W^*)P(X^*, Y^*, E^*, V^*) + A(V^*)P(X^*, Y^*, W^*, E^*).\end{aligned} \tag{6.11}$$

From equation (6.11), we obtain

$$[A(\bar{E} + B(\bar{E})]P(X^*, Y^*, W^*, V^*) = 0, \tag{6.12}$$

and,

$$A(\bar{X})P(E^*, Y^*, W^*, V^*) = 0. \tag{6.13}$$

Putting $\bar{E} = E^*, \bar{X} = X^*$ and $\bar{V} = V^*$ in equation (6.4) gives

$$\begin{aligned}(\nabla_{E^*}P)(X^*, \bar{Y}, \bar{W}, V^*) = {} & [A(E^*) + B(E^*)]P(X^*, \bar{Y}, \bar{W}, V^*)\\ & +A(X^*)P(E^*, \bar{Y}, \bar{W}, V^*) + A(V^*)P(X^*, \bar{Y}, \bar{W}, E^*).\end{aligned} \tag{6.14}$$

Similarly, putting $E^* = \bar{E}, X^* = \bar{X}$ and $V^* = \bar{V}$ in equation (6.11) gives

$$\begin{aligned}(\nabla_{\bar{E}}P)(\bar{X}, Y^*, W^*, \bar{V}) = {} & [A(\bar{E}) + B(\bar{E})]P(\bar{X}, Y^*, W^*, \bar{V})\\ & +A(\bar{X})P(\bar{E}, Y^*, W^*, \bar{V}) + A(\bar{V})P(\bar{X}, Y^*, W^*, \bar{E}).\end{aligned} \tag{6.15}$$

In regard of equations (6.5) and (6.6), we have the following two cases:

i) $A = B = 0$ on M_2.
ii) M_1 is semiconformally flat.

First, we consider the case (i). Then, equation (6.14) becomes

$$(\nabla_{E^*}P)(X^*, \bar{Y}, \bar{W}, V^*) = 0, \tag{6.16}$$

implies,

$$\begin{aligned} a(\nabla_{E^*}R)(X^*, \bar{Y}, \bar{W}, V^*) - \frac{a}{(n-2)}(\nabla_{E^*}S)(X^*, V^*)g(\bar{Y}, \bar{W})\\ -\frac{b\,dr(E^*)}{(n-1)}g(\bar{Y}, \bar{W})g(X^*, V^*) = 0.\end{aligned} \tag{6.17}$$

Now, Putting $\bar{Y} = \bar{W} = \bar{e}_\alpha, 1 \leq \alpha \leq p$ in equation (6.17), we get

$$\frac{a(n-p-2)}{(n-2)}(\nabla_{E^*}S)(X^*, V^*) - \frac{b\,dr(E^*)}{(n-1)}pg(X^*, V^*) = 0. \tag{6.18}$$

Also, taking $X^* = V^* = e_i^*, p+1 \leq i \leq n$ in equation (6.18) gives

$$\frac{a(n-p-2)}{(n-2)}\,dr^*(E^*) - \frac{bp(n-p)}{(n-1)}\,dr(E^*) = 0. \tag{6.19}$$

If possible let $d\bar{r}(E^*) = 0$. The equation (6.19) becomes

$$\left[\frac{a(n-p-2)}{(n-2)} - \frac{bp(n-p)}{(n-1)}\right] dr^*(E^*) = 0. \tag{6.20}$$

Thus r^* is constant in M_2 provided, $\dfrac{a(n-p-2)}{(n-2)} \neq \dfrac{bp(n-p)}{(n-1)}$. Then from equation (6.18), we get

$$(\nabla_{E^*} S)(X^*, V^*) = 0.$$

Therefore, M_2 is Ricci symmetric.
Secondly, we will consider the case (ii). Since M_1 is semiconformally flat, we get

$$\begin{aligned} aR(\bar{X},\bar{Y},\bar{W},\bar{V}) - \frac{a}{(n-2)}\Big[&S(\bar{Y},\bar{W})g(\bar{X},\bar{V}) - S(\bar{X},\bar{W})g(\bar{Y},\bar{V}) \\ &+S(\bar{X},\bar{V})g(\bar{Y},\bar{W}) - S(\bar{Y},\bar{V})g(\bar{X},\bar{W})\Big] \\ -\frac{br}{(n-1)}\Big[&g(\bar{Y},\bar{W})g(\bar{X},\bar{V}) - g(\bar{X},\bar{W})g(\bar{Y},\bar{V})\Big] = 0. \end{aligned} \tag{6.21}$$

Putting $\bar{X} = \bar{V} = \bar{e}_\alpha$ in equation (6.21), the above equation becomes

$$S(\bar{Y},\bar{W}) = \left[\frac{a\bar{r}(n-1) + br(p-1)(n-2)}{a(n-p-2)}\right] g(\bar{Y},\bar{W}). \tag{6.22}$$

Therefore, M_1 is an Einstein manifold.
Hence, the theorem is proved. □

Theorem 6.2. *Let an $A(PSCS)_n$ be a decomposable space such that $M^n = M_1^p \times M_2^{n-p}$ for $(2 \leq p \leq n-2)$, then the following holds:*
i) In the case of $A = B = 0$ on M_1, the manifold M_1 is Ricci symmetric and scalar curvature $\bar{r}$ is constant in M_1, provided $dr^*(\bar{E}) = 0$ and $\dfrac{a(p-2)}{(n-2)} \neq \dfrac{bp(n-p)}{(n-1)}$.
ii) when M_2 is semiconformally flat, then M_2 is an Einstein manifold.

Proof. Making use of equations (6.12) and (6.13), we get the following two cases:

i) $A = B = 0$ on M_1.
ii) M_2 is semiconformally flat.

Proceeding in a similar manner as in Theorem 6.1,
Hence, we will obtain the required result. □

Corollary 6.3. *If $A(PSCS)_n$ is a decomposable space such that $M^n = M_1^p \times M_2^{n-p}$ for $(2 \leq p \leq n-2)$, then one of the decomposed manifold is semiconformally flat while on other manifold both the associate 1-form A and B vanishes.*

7 Examples of $A(PSCS)_4$

In this section, we have constructed two examples of an $A(PSCS)_4$ on coordinate space $\mathbb{R}^4$ (with coordinates(x^1, x^2, x^3, x^4)) and obtain all the non-vanishing components of the curvature tensor, the Ricci tensor, the scalar curvature and the semiconformal curvature tensor along with its covariant derivatives. Then we verified the relation (1.15).

Example 7.1. Let us consider a Riemannian metric g defined on 4-dimensional manifold $M^4 = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : x^1 \neq -1\}$ given by

$$ds^2 = (x^1+1)(x^4)^2(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (dx^4)^2. \tag{7.1}$$

A similar Riemannian metric g is given by De and Gazi[13].
Then the covariant and contravariant components of the metric are as follows

$$\begin{gathered} g_{11} = (x^1+1)(x^4)^2, g_{12} = g_{21} = 1, g_{33} = g_{44} = 1 \\ g^{11} = 0, g^{12} = g^{21} = 1, g^{33} = g^{44} = 1, g^{22} = -(x^1+1)(x^4)^2 \end{gathered} \tag{7.2}$$

All non-vanishing components of the Christoffel symbols and the curvature tensor in the considered metric are as follows:

$$\begin{aligned} \Gamma^4_{11} = -(x^1+1)(x^4), \Gamma^2_{11} &= \frac{1}{2}(x^4)^2, \Gamma^2_{14} = (x^1+1)(x^4) \\ R_{1441} &= (x^1+1) \end{aligned} \tag{7.3}$$

From equations (7.2) and (7.3), the non-vanishing components of Ricci tensor are

$$S_{11} = x^1 + 1. \tag{7.4}$$

The scalar curvature of metric considered is given by,

$$r = 0. \tag{7.5}$$

The only non-vanishing components of the semiconformal curvature tensor are

$$P_{1441} = \frac{a}{2}(x^1+1) \neq 0. \tag{7.6}$$

Clearly, the only non-vanishing term of $\nabla_l P_{hijk}$ are

$$\nabla_1 P_{1441} = \frac{a}{2} \neq 0. \tag{7.7}$$

In term of the local coordinate system, let us define the components of the 1-form A and B as

$$A_i = \begin{cases} \dfrac{1}{6(x^1+1)} \text{ for } i = 1 \\ 0, \text{ otherwise} \end{cases}$$

and,

$$B_i = \begin{cases} \dfrac{1}{2(x^1+1)} \text{ for } i = 1 \\ 0\ \text{, otherwise} \end{cases} \tag{7.8}$$

at any point in M^4.
In (M^4, g) the considered 1-form reduces the equation (1.15) in the following equations

$$\nabla_1 P_{1441} = (3A_1 + B_1)P_{1441} + A_4 P_{1141} + A_4 P_{1411}. \tag{7.9}$$

$$\nabla_4 P_{1141} = [A_4 + B_4]P_{1141} + A_1 P_{4141} + A_1 P_{1441} + A_4 P_{1141} + A_1 P_{1144}. \tag{7.10}$$

$$\nabla_4 P_{1411} = [A_4 + B_4]P_{1411} + A_1 P_{4411} + A_4 P_{1411} + A_1 P_{1441} + A_1 P_{1414}. \tag{7.11}$$

In all other cases excluding (7.9),(7.10), and (7.11), the relation (1.15) either holds trivially or the components of each term vanishes identically.
By (7.8), we get

$$\begin{aligned} \text{RHS of (7.9)} &= (3A_1 + B_1)P_{1441} + A_4 P_{1141} + A_4 P_{1411} \\ &= \left[\frac{3}{6(x^1+1)} + \frac{1}{2(x^1+1)}\right]\frac{a}{2}(x^1+1) \\ &= \frac{a}{4} + \frac{a}{4} \\ &= \frac{a}{2} \\ &= \nabla_1 P_{1441} \\ &= \text{LHS of (7.9)}. \end{aligned} \tag{7.12}$$

By proceeding in a similar manner, it can be shown that the equations (7.10) and (7.11) are also true.
Thus, (M^4, g) is an $A(PSCS)_4$.

Example 7.2. Let us consider a Riemannian metric g defined on 4-dimensional manifold $M^4 = (x^1, x^2, x^3, x^4) \in \mathbb{R}^4$ given by

$$ds^2 = (1 + 2q)[(dx^1)^2 + (dx^2)^2] + (dx^3)^2 + (dx^4)^2, \tag{7.13}$$

where $q = \frac{e^{x^1}}{k^2}$, where k is non-zero constant.
Then the covariant and contravariant components of the metric are as follows:

$$\begin{aligned} g_{11} = g_{22} = 1 + 2q,\ g_{33} = g_{44} = 1 \\ g^{11} = g^{22} = \frac{1}{1+2q},\ g^{33} = g^{44} = 1 \end{aligned} \tag{7.14}$$

All the non-vanishing components of the Christoffel symbols and the curvature tensor in the considered metric are

$$\begin{aligned} \Gamma^1_{11} = \Gamma^2_{12} = \frac{q}{1+2q},\ \Gamma^1_{22} = -\frac{q}{1+2q} \\ R_{1221} = \frac{q}{1+2q} \end{aligned} \tag{7.15}$$

By (7.14) and (7.15), the non-vanishing components of Ricci tensor are

$$S_{11} = \frac{q}{(1+2q)^2}. \tag{7.16}$$

The Scalar curvature is given by

$$r = g^{ij}S_{ij} = g^{11}S_{11} + g^{22}S_{22} + g^{33}S_{33} + g^{44}S_{44} = \frac{q}{(1+2q)^3}. \tag{7.17}$$

The only non-vanishing components of semiconformal curvature tensors are

$$P_{1221} = \frac{q}{1+2q}\left\{\frac{a}{2} - \frac{b}{3}\right\}. \tag{7.18}$$

From equation (7.18), it can be shown that only non-zero term of $\nabla_l P_{hijk}$ are

$$\nabla_1 P_{1221} = \frac{1}{(1+2q)^2}\left\{\frac{a}{2} - \frac{b}{3}\right\}, \tag{7.19}$$

and all other components of $\nabla_l P_{hijk}$ vanishes identically.
In term of the local coordinate system, let us consider the components of the 1-form A and B as

$$A_i = \begin{cases} \dfrac{1}{6q(1+2q)} \text{ for } i = 1 \\ 0, \text{ otherwise} \end{cases}$$

and,

$$B_i = \begin{cases} \dfrac{1}{2q(1+2q)} \text{ for } i = 1 \\ 0 \text{ , otherwise} \end{cases} \tag{7.20}$$

at any point in M^4.
In (M^4, g), the considered 1-form reduces equation (1.15) into the following equations

$$\nabla_1 P_{1221} = (3A_1 + B_1)P_{1221} + A_2 P_{1121} + A_2 P_{1211}. \tag{7.21}$$

$$\nabla_2 P_{1121} = (A_2 + B_2)P_{1121} + A_1 P_{2121} + A_1 P_{1221} + A_2 P_{1121} + A_1 P_{1122}. \tag{7.22}$$

$$\nabla_2 P_{1211} = [A_2 + B_2]P_{1211} + A_1 P_{2211} + A_2 P_{1211} + A_1 P_{1221} + A_1 P_{1212}. \tag{7.23}$$

The relation (1.15) either holds trivially or the components of each term vanishes identically excluding the above cases.
By (7.21) we get

$$\begin{aligned} RHS \text{ of } (7.21) &= (3A_1 + B_1)P_{1221} + A_2 P_{1121} + A_2 P_{1211}. \\ &= \left[\frac{3}{6q(1+2q)} + \frac{1}{2q(1+2q)}\right]\frac{q}{(1+2q)}\left\{\frac{a}{2} - \frac{b}{3}\right\} \\ &= \frac{1}{(1+2q)^2}\left\{\frac{a}{2} - \frac{b}{3}\right\} \\ &= \nabla_1 P_{1221} \\ &= LHS \text{ of } (7.21). \end{aligned} \tag{7.24}$$

By proceeding similarly it can be shown that the equations (7.22) and (7.23) also holds.
Thus, (M^4, g) is an $A(PSCS)_4$.

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References

[1] D. B. Abdussatar, *On Conharmonic transformation in general relativity*, Bull. Calcutta. Math. Soc. 88, 6 (1996), 467-470.

[2] T. Adati and T. Miyazawa, *On conformally symmetric spaces*, Tensor (N.S.) 18 (1967), 335-342.

[3] M. Belkhelfa, R. Deszcz, M. Glogowska, M. Hotloś, D. Kowalczyk, L. Verstraelen, *On some type of curvature conditions*, Banach Center Publ. 57 (2002), 179-194.

[4] E. Cartan, *Sur une classe remarquable d'espaces de Riemann*, Bull. Soc. Math. France 54 (1926), 214-264.

[5] M. C. Chaki and B. Gupta,*On conformally symmetric spaces*, Indian J. Math. 5 (1963), 113-122.

[6] M. C. Chaki, *On pseudo symmetric manifolds*, An. Stiint. Univ. Al. I. Cuza Iasi Sectia Mat. 33 (1987), 53-58.

[7] M. C. Chaki, *On generalized pseudo symmetric manifolds*, Publ. Math. Debrecen, 51 (1997), 35-42.

[8] R. Deszcz, *On pseudosymmetric spaces*, Bull. Belg. Math. Soc. Series A, 44 (1992), 1-34.

[9] R. Deszcz and W. Grycak, *On some classes of warped product manifolds*, Bull. Inst. Math. Acad. Sinica, 19 (1987), 311-322.

[10] R. Deszcz, S. Haesen and L. Verstraelen, *On natural symmetries*, Topics in Differential Geometry, Ed. Rom. Acad. Bucharest, (2008), 249-308.

[11] U. C. De and A. K. Gazi, *On almost pseudo symmetric manifolds*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 51 (2008), 53-68.

[12] U. C. De and A. K. Gazi, *On conformally flat almost pseudo Ricci symmetric manifolds*, Kyunpook Math. J. 19,2 (1971), 97-103.

[13] U. C. De and A. K. Gazi, *On almost pseudo conformally symmetric manifolds*, Demonstratio Math. 42,4 (2009), 507-520.

[14] U. C. De and Y. J. Suh, *On Weakly Semiconformally symmetric manifolds*, Acta. Math. Hungar. 157, 2 (2019), 503-521.

[15] U. C. De and S. Mallick, *On almost pseudo concircularly symmetric manifolds*, The Journal of Mathematics and Computer Science, 4, 3 (2012), 317-330.

[16] A. Gray, *Einstein-like manifolds which are not Einstein*, Geom. Dedicata, 7 (1978), 259-280.

[17] J. Kim, *On pseudo semiconformally symmetric manifolds*, Bull. Korean Math. Soc. 54 (2017), 177-186.

[18] J. Kim, *A type of conformal curvature tensor*, Far East J. Math. Soc. 99, 1 (2016),61-74.

[19] C. Lalmalsawma and J. P. Singh, *On almost pseudo m-projectively symmetric manifolds*, Novi Sad J. Math. 48, 2 (2018), 81-95.

[20] B. O'Neill, *Semi-Riemannian Geometry*, Pure and Applied Mathematics, 103, Academic Press, Inc., New York, 1983.

[21] P. Pal, *On almost pseudo conharmonically symmetric manifolds*, Kyungpook Math. J. 54 (2014), 699-714.

[22] E. M. Patterson, *Some theorems on Ricci-recurrent spaces.* J. London Math. Soc. 27 (1952), 287-295.

[23] J. A. Schouten, *Ricci-Calculus. An introduction to tensor Analysis and its Geometrical Applications*,Springer-Virlay (Berlin-Gottingen-Hindelberg, 1954.

[24] R. N. Sen and M. C. Chaki, *On curvature restrictions of a certain kind of conformally flat Riemannian space of class one*, Proc. Nat. Inst. Sci. India, 33 (1967), 100-102.

[25] S. A. Siddiqui and Z. Ahsan, *Conharmonic curvature tensor and the spacetime of general relativity*, Differ. Geom. Dyr. Syst. 12 (2010), 213-220.

[26] L. Tamaássy and T. Q. Binh, *On Weakly symmetric and Weakly projectively symmetric Riemannian manifolds*, Colloq. Math. Soc. Janos Bolyai, 56 (1989), 663-670.

[27] A. G. Walker, *On Rusis space of recurrent curvature*, Proc. London Math. Soc. 52 (1951), 36-64.

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