P-ADIC VALUATIONS OF CERTAIN CLASSES OF STIRLING NUMBERS OF THE SECOND KIND

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A. LALCHHUANGLIANA

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BY

A. LALCHHUANGLIANA
Department of Mathematics and Computer Science

Supervisor : Prof. S. Sarat Singh

Submitted
In partial fulfillment of the requirement of the Degree of Doctor of Philosophy in Mathematics of Mizoram University, Aizawl.

DEPARTMENT OF MATHEMATICS & COMPUTER SCIENCE
MIZORAM UNIVERSITY

Dr. S. Sarat Singh
Professor

Aizawl – 796 004, Mizoram: INDIA
Fax: 0389 - 2330644
Phone: +919862626776(m)
+919863264645(m)
e-mail: saratcha32@yahoo.co.uk

CERTIFICATE

This is to certify that the thesis entitled "p-adic Valuations of Certain Classes of Stirling Numbers of the Second Kind", submitted by Mr. A. Lalchhuangliana for the award of the degree of Doctor of Philosophy (Ph. D.) in Mathematics, is a bonafide record of the original research carried out by him under my supervision. He has been duly registered, and the thesis is worthy of being considered for the award of the Ph. D. degree.

I, the undersigned, have declared that this research work has been done under my supervision and has not been submitted for any degree of any other university.

(Prof. S. SARAT SINGH)

Supervisor

DECLARATION

Mizoram University

June, 2023

I, **A Lalchhuangliana**, hereby declare that the subject matter of this thesis is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to do the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in any other University/Institute.

This is being submitted to the Mizoram University for the degree of Doctor of Philosophy in Mathematics.

(A. LALCHHUANGLIANA)

Candidate

(Prof. M. SUNDARARAJAN)

Head

(Prof. S. SARAT SINGH)

Supervisor

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Dated: ....

Place: Aizawl (A. LALCHHUANGLIANA)

PREFACE

The present thesis entitled "**p-adic Valuations of Certain Classes of Stirling Numbers of the Second Kind**" is an outcome of the research carried out under the supervision of Prof. S. Sarat Singh, Department of Mathematics & Computer Science, Mizoram University, Aizawl - 796 004, Mizoram, INDIA.

The thesis consists of various approaches to determine the p-adic valuations of certain classes of Stirling numbers of the second kind for an odd prime p. The p-adic valuations are mainly obtained through congruence relations. Some cases are also tackled through an algebraic and combinatorial approach. It consists of six chapters. The first chapter is General Introduction which contains basic definitions, divisibility and congruence, p-adic Valuation, Stirling Numbers, Periodicity, applications of Stirling numbers and review of literature.

The second chapter deals with the problem of divisibility of certain classes of Stirling numbers of the second kind. It includes the derivation of a new identity of Stirling numbers of the second kind. A combinatorial approach helps to obtain the lower bounds of p-adic valuations of some classes of $S(n,k)$ for an odd prime p. We also extend an existing congruence relation in modulo of a power of an odd prime, which is useful in determining the lower bound of $v_p(S(p^n, kp))$ when k is odd and less than $p-1$. We obtain the lower bound of $v_p(S(p^2, kp))$ when k is even and its value is greater than the one when k is odd. We also discuss the congruence behaviour of $S(p^n, k)$ and the involvement of p-adic digits of k on the congruence when k is not divisible by p.

In Chapter 3, we study the p-adic valuations of $S(n,k)$ when n is a power of a prime. We find that the results when k is divisible by p (or p^m) are quite different from the ones where k is not divisible by p. We have proved that $v_p(S(p^2, kp)) \geq 5$ when k is even, which confirms the lower bound of the Conjecture 2.3.1 in Chapter 2. Furthermore, we find that the values of $v_p(S(n, kp^m))$ are affected by the parity of n and k. In fact, if n and k are opposite in parity, i.e., $n-k$ is odd, then $v_p(S(n, kp^m)) \geq 2m$ when $(p-1) \nmid (n-k)$ and $v_p(S(n, kp^m)) \geq m$ when $(p-1) \mid (n-k)$. However, if the parity of n and k are the same, i.e., $n-k$ is even, then $v_p(S(n, kp^m)) \geq m$ when $(p-1) \nmid (n-k)$. We further investigate the divisibility of $S(p^n, k)$ when p does not divide k and we have found that the divisibility depends on the sum of the p-adic digits of k.

The fourth chapter focuses on the congruence relation between Stirling numbers of the first and the second kind. Their generating function is the bridge between the two numbers. We present their congruence relations with sums involving binomial coefficients for the two numbers. We also express the first kind in terms of sums involving the second kind modulo a power of a prime and vice versa. The congruence obtained helps to acquire the p-adic valuations of some classes of the two numbers. We even establish a congruence relation between the two numbers in modulo p^n for any positive integer n.

In the fifth chapter, the relationship between minimum periods and p-adic valuations of Stirling numbers of the second kind has been studied. We discuss the periodicity, period, and minimum period of the sequence $\{S(n, k) \pmod{p^N}\}_{n \geq 0}$ for some fixed positive integers N and k. We find that the cycle of the sequence sometimes starts even when n is less than k. We present some results about the divisibility of a partial Stirling number, which is effective in evaluating some classes of $S(n, k)$. The periodicity and minimum periods help to determine a class of $S(n, k)$ holding the same p-adic valuation.

Chapter 6 is the summary and conclusions of the thesis.

A list of references is presented at the end.

Contents

Chapter 1

General Introduction

1.1 Introduction

Combinatorics is a branch of mathematics that can be interpreted as a study of counting and its technique. This subject is related to many other areas of mathematics and has many applications from logic to statistical physics and from evolutionary biology to computer science. Currently, combinatorics has tremendous growth due to its application and major impact on the computers. We know that computers can solve large-scale problems with the increase of their speed, which previously would not possible. But computers do not function independently and they need to be programmed to perform. The bases for these programs are often combinatorial algorithms. The analysis of these algorithms for efficiency and storage requirements, demand more concepts of combinatorics.

The study of combinatorics includes the concept of permutations, combinations and partitions. *An ordered set,* $\{a_1,\ a_2,\ \dots,a_r\}$ *of* r *distinct objects selected from a set of* n *objects is called a permutation of* n *things taken* r *at a time.* The number of permutations is given by

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1). \tag{1.1}$$

A set of r objects selected from a set of n objects without regard to order is called a combination of n things taken r at a time. The number of combination is given by

$$C(n,r) = \frac{n!}{r!(n-r)!}. \tag{1.2}$$

A partition of a positive integer n is a representation of n as a sum of positive integers

$$n = x_1 + x_2 + \cdots + x_k, \qquad x_i \geq 1, i = 1, 2, \cdots, k. \tag{1.3}$$

The numbers, x_i are called the parts of the partition. The number of ordered partitions, n into k parts is $\binom{n-1}{k-1}$, which is the number of ways of putting $k-1$ separating marks in the $n-1$ spaces between n dots in a row. A standard unordered partition is represented by listing all the parts in a non-increasing order, say

$$n = x_1 + x_2 + \cdots + x_k, \qquad x_1 \geq x_2 \geq \cdots \geq x_k \geq 1. \tag{1.4}$$

One of the basic problems of combinatorics is determining the number of possible configurations for graphs, designs or arrays. Enumeration may be difficult even when the rules specifying the configuration are relatively simple. It is the mathematician who may have to be content with finding an approximate answer or at least a good lower and upper bound. An important and interesting subject of pure mathematics is Number theory which is one of the oldest branches. The mystery of Number theory has captivated many mathematicians. A basic understanding of Number theory is a critical precursor to cutting-edge software engineering, specifically security-based software. Number theory is at the heart of cryptography, which is experiencing a fascinating period of rapid evolution, ranging from the famous RSA (Rivest-Shamir-Adleman) algorithm to the wildly-popular blockchain world (Kraft and Washington, 2018). It may be noted that one of the oldest and most interesting topics of number theory is the divisibility of sequences of integers and rational numbers.

1.2 Divisibility and Congruence

In this section, we present some definitions and well-known results which are used in the present work.

Definition 1.2.1. *An integer b is said to be divisible by another integer $a \neq 0$ if there exists an integer c such that $b = ac$ and denoted by $a \mid b$, otherwise $a \nmid b$.*

We have the following important properties (Niven *et al.*, 1999):

1. $a \mid b$ and $b \mid c$ imply $a \mid c$, i.e., divisibility is associative.

2. $a \mid b$ and $a \mid c$ imply $a \mid (bx + cy)$ for any integer x and y.

3. $a \mid b$ implies $a \mid bc$ for any integer c.

4. $a \mid b$ and $b \mid a$ for $a \neq 0$ and $b \neq 0$ if and only if $a = \pm b$,

5. If $m \neq 0$, then $a \mid b$ implies $ma \mid mb$.

Definition 1.2.2. *Given any integers a and b, with $a > 0$, there exist unique integers q and r such that $b = qa + r$, $0 \leq r < a$. The integers q and r in the expression of b are called quotient and remainder, respectively.*

Definition 1.2.3. *An integer a is called a common divisor of b and c in case $a \mid b$ and $a \mid c$. Since there is only a finite number of divisors of any non-zero integer, there is only a finite number of common divisors of b and c. If at least one of b and c is not 0, then the greatest among their common divisors is called greatest common divisor of b and c and is denoted by $gcd(b, c)$ or simply (b, c). Similarly, we denote the greatest common divisor g of the all non-zero integers, b_1, b_2, $\cdots$ and b_n as $(b_1,\ b_2,\ \cdots,\ b_n)$.*

Definition 1.2.4. *If a non-zero integer m divides the difference $a - b$, then a is said to be congruent to b modulo m and we write $a \equiv b \pmod{m}$. If $a - b$ is not divisible by m, we say that a is not congruent to b modulo m and is denoted by $a \not\equiv b \pmod{p}$.*

The following properties hold for congruences:

1. $a \equiv b \pmod{m}$, $b \equiv a \pmod{m}$ and $a - b \equiv 0 \pmod{m}$ are equivalent.
2. If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$.
3. $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ imply $a + c \equiv b + d \pmod{m}$.
4. $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ imply $ac \equiv bd \pmod{m}$.
5. If $a \equiv b \pmod{m}$ and $d \mid m$, $d > 0$, then $a \equiv b \pmod{d}$.
6. If $a \equiv b \pmod{m}$, then $ac \equiv bc \pmod{mc}$ for $c > 0$.
7. If $a \equiv b \pmod{m}$, then $f(a) \equiv f(b) \pmod{m}$ for a polynomial f over $\mathbb{Z}$.

Theorem 1.2.1. *If p is a prime, then for any non-zero integer a such that $(a, p) = 1$,*

$$a^{p-1} \equiv 1 \pmod{p}. \tag{1.5}$$

The above theorem is called Fermat's little theorem and an alternate version of the theorem can be written as

$$a^p \equiv a \pmod{p}, \quad a \in \mathbb{Z}. \tag{1.6}$$

Fermat's theorem is a special case of Euler's theorem:

Theorem 1.2.2. *If a and m are integers such that $(a, m) = 1$, then*

$$a^{\phi(m)} \equiv 1 \pmod{m},$$

where $\phi(m)$ *is the Euler's phi function which counts the number of positive integers less than* m *and relatively prime to* m.

One of the important classical result related with congruence is Wilson's theorem which states as

Theorem 1.2.3. *If* p *is a prime, then*

$$(p-1)! \equiv -1 \pmod{p}.$$

1.3 p-adic Valuation

Sequences of integers and their divisibility properties are interesting topic in number theory. There are many Mathematicians who have been introducing different results, particularly powers of primes dividing integers. Nowadays, the divisibility properties of integers and more general, rational numbers are expressed in terms of p-adic valuations.

Definition 1.3.1. *Let* p *be a prime, and* a *be any non-zero integer. The* p*-adic valuation of* a*, denoted by* $v_p(a)$*, is defined as the exponent of the highest power of* p *dividing* a*. Note that* $v_p(0) = \infty$*. Thus,* $v_p(a)$*, for a non-zero integer* a*, is a non-negative integer.*

Example 1.3.1. *Since* $3 \nmid 25$, $v_3(25) = 0$, *whereas* $v_5(25) = 2$ *since* $5^2 \mid 25$ *and* $5^3 \nmid 25$.

Note that for any prime p, $v_p(\pm 1) = 0$.

It is easy to see that for any two integers a and b, the following inequality and equality hold:

$$v_p(a+b) \geq min\{v_p(a), v_p(b)\} \tag{1.7}$$

and

$$v_p(ab) = v_p(a) + v_p(b). \tag{1.8}$$

The p-adic valuation v_p can further be extended to the field of rational numbers. Given any rational number r such that $r = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$, then

$$v_p(r) = v_p(a) - v_p(b). \tag{1.9}$$

It can be easily verified that $v_p(r)$ is independent of the representation of r as a ratio of integers. It can further be confirmed that inequality (1.7) and equality (1.8) still hold for rational numbers. Also, note that $v_p(r) = v_p(-r)$ for any rational number r. The inequality (1.7), for rational number, is known as the **non-archimedean property** or **the triangle inequality** of the p-adic valuation and this, however, can still be strengthened to the isosceles triangle property of the p-adic valuation:

$$v_p(r + s) = min\{v_p(r), v_p(s)\} \tag{1.10}$$

if $v_p(r) \neq v_p(s)$ for any $r, s \in \mathbb{Q}$.

The properties of p-adic valuations of rational numbers show that if

$$| \, r \, |_p = p^{-v_p(r)} \tag{1.11}$$

for any rational number r, then $| \quad |_p$ is a norm on the field $\mathbb{Q}$ of rational numbers. The p-adic norm on $\mathbb{Q}$, unlike the usual absolute norm, is a non-archimedean norm due to inequality (1.7). This p-adic norm on $\mathbb{Q}$ give rises to the p-adic metric d_p on $\mathbb{Q}$ as follows:

$$d_p(r, s) = | \, r - s \, |_p \tag{1.12}$$

and $(\mathbb{Q}, d_p)$ is a metric space. A complete metric space can be constructed, which results a p-adic field $\mathbb{Q}_p$ containing $\mathbb{Q}$ as a sub-field. Moreover, the p-adic norm on $\mathbb{Q}$ can be extended to a non-archimedean norm, denoted by $| \quad |_p$, on $\mathbb{Q}_p$. The resultant metric space structure on $\mathbb{Q}_p$ allows us to do analysis in $\mathbb{Q}_p$. This is known as **p-adic analysis**. For more details, one can refer to Koblitz (1977) and

Gouvea (1993).

It is a well-known fact that every element $\alpha \in \mathbb{Q}_p$ has a unique p-adic expansion in the following sense:

$$\alpha = \sum_{k=n}^{\infty} a_k p^k, \tag{1.13}$$

where $n = v_p(\alpha)$ and $0 \le a_k \le p-1$ for all k with $a_n \neq 0$. This series converges in $\mathbb{Q}_p$ with respect to the p-adic norm. Moreover, α determines the coefficients a_k uniquely(Koblitz, 1977). In particular, any integer a has a unique finite p-adic expansion.

Definition 1.3.2. *For a unique p-adic expansion*

$$a = a_0 + a_1 p + \cdots + a_n p^n$$

($0 \le a_i \le p-1$) of a positive integer a, the coefficients a_0, a_1, ... , a_n are called the p-adic digits of a. The sum of the p-adic digits of the integer a is denoted by $s_p(a)$. Thus,

$$s_p(a) = \sum_{i=0}^{n} a_i. \tag{1.14}$$

Example 1.3.2. *For $a = 20$ and $p = 3$, the 3-adic expansion of 20 is*

$$20 = 2 \cdot 3^2 + 0 \cdot 3^1 + 2 \cdot 3^0.$$

Thus, $a_0 = 2$, $a_1 = 0$ and $a_2 = 2$. Therefore, $s_3(20) = 2 + 0 + 2 = 4$.

In 1808, Legendre proved that the p-adic valuation of $n!$ (where $n! = n(n-1)\cdots 3 \cdot 2 \cdot 1$ if $n > 0$ and $0! = 1$), for a positive integer n, can be expressed neatly in terms of the sum $s_p(n)$ of the p-adic digits of n. This result has been referred to as **Legendre's Theorem** (Mihet, 2010). It is also known as **Legendre's formula** though some authors named it as **de Polignac's formula**:

$$v_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor, \tag{1.15}$$

where $\lfloor\ \rfloor$ is the greatest integer function, sometimes called the floor function. An alternate version of Legendre's formula, in terms of the p-adic digits of n, is given as

$$v_p(n!) = \frac{n - s_p(n)}{p-1}. \tag{1.16}$$

Definition 1.3.3. *Let $n \in \mathbb{Z}$ and x be an indeterminate. Then the expansion*

$$(1+x)^n = \sum_{k \geq 0} c_k x^k,$$

where c_k's are integers, is called the binomial expansion of $(1+x)^n$. The coefficients c_k's are called binomial coefficients and are denoted by $\binom{n}{k}$. The binomial coefficient has the following explicit formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

If n is a positive integer, then $\binom{n}{k}$ is equal to the number of ways one can choose k objects from n distinct objects.

Application of Legendre's formula to the binomial coefficient gives a handy tool for p-adic valuation:

$$v_p\left(\binom{n}{k}\right) = \frac{s_p(k) + s_p(n-k) - s_p(n)}{p-1}. \tag{1.17}$$

for any two integers n and k such that $n \geq k$. The above formula is sometimes referred to as **Kummer's formula** after the great German number theorist Ernst Kummer. The p-adic valuation of the binomial coefficient $\binom{n}{k}$ is simply the number of carry-overs when one adds the p-adic expansions of k and $n-k$, or, equivalently, the number of borrows required when subtracting the p-adic expansion of m from n(Kummer, 1852).

The following basic results give an equivalent statement of congruence relation in terms of p-adic valuation.

Proposition 1.3.1. *Let p be a prime and N be a positive integer. If any two integers a and b satisfy the congruence, $a \equiv b \pmod{p^N}$, then the following results hold:*

$$a)\ v_p(a) \geq N, \quad \text{if } b = 0, \tag{1.18}$$

$$b)\ v_p(a-b) \geq N, \tag{1.19}$$

$$c)\ v_p(a) = v_p(b), \quad \text{if } v_p(b) < N, \tag{1.20}$$

$$d)\ ac \equiv bc \pmod{p^{N+M}}, \quad \text{if } c \in \mathbb{Q} \text{ and } v_p(c) \geq M, \tag{1.21}$$

$$e)\ a^{p^M} \equiv b^{p^M} \pmod{p^{N+M}}, \quad \text{if } M \text{ is a non-negative integer.} \tag{1.22}$$

The following results about binomial coefficients are also easy to obtain:

Proposition 1.3.2. *If p is an odd prime, then*

$$a)\ p \mid \binom{p}{k}, \quad \text{if } 0 < k < p, \tag{1.23}$$

$$b)\ \binom{p^N - 1}{k} \equiv (-1)^k \pmod{p}, \quad \text{if } 0 \leq k < p^N, \tag{1.24}$$

$$c)\ v_p\left(\binom{n}{k}\right) = v_p(n) - v_p(k), \quad \text{if } v_p(k) < v_p(n) \text{ and } k \leq n. \tag{1.25}$$

Lucas (1878) introduced a congruence property for binomial coefficients known as **Lucas congruence**: if $a = \sum_{i\geq 0} a_i p^i$ and $b = \sum_{i \geq 0} b_i p^i$ are the p-adic expansion of non-negative integers a and b, respectively, then

$$\binom{a}{b} \equiv \prod_{i\geq 0} \binom{a_i}{b_i} \pmod{p}. \tag{1.26}$$

Sagan (1985) employed the concept of group action on abelian groups and obtained the following congruence: if a and b are integers, then

$$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^2}. \tag{1.27}$$

Later, using sums of binomial coefficients, Bailey (1990) obtained a stronger version of the above congruence in modulo p^3 as

$$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^3}. \tag{1.28}$$

Davis and Webb (1993) obtained another stronger result for $p > 3$, which is

$$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^e}, \tag{1.29}$$

where $e = 3 + v_p(n) + v_p(k) + v_p(n-k) + v_p(\binom{n}{k})$.

1.4 Stirling Numbers

Stirling numbers of the first and second kind were introduced by Scottish Mathematician, James Stirling (1692-1770) in his book *Methodus Differentialis* (Stirling, 1730). Since then, these numbers have been found to be of great utility in various branches of Mathematics, such as combinatorics, number theory, calculus of finite differences, theory of algorithms, etc. The name "Stirling numbers" was first used by a Danish Mathematician, Niels Nielsen (1865-–1931) (Nielsen, 1906). For details about Stirling numbers, we refer to Comtet (1974), Graham *et al.* (2007), and Quaintance and Gould (2015).

Definition 1.4.1. *For a positive integer n, the n^{th} rising factorial of x denoted by $x^{\bar{n}}$, is defined as*

$$x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1). \tag{1.30}$$

Similarly, the n^{th} falling factorial of x denoted by $x^{\underline{n}}$, is represented as

$$x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1). \tag{1.31}$$

We extend the notation to non-negative integers by setting, $x^{\overline{0}} = x^{\underline{0}} = 1$.

Definition 1.4.2. *Given a non-negative integer n and k, not both zero, Stirling* *numbers of the second kind $S(n,k)$ is defined as the number of ways one can partition a set with n elements into exactly k non-empty subsets. By convention, $S(0,0) = 1$ and $S(0,k) = 0$ for $k \geq 1$.*

Thus, $S(n,k)$ is the number of ways of distributing n distinct balls into k indistinguishable boxes (the order of the boxes does not count) such that no box is empty. From the definition, it is clear that $S(n,k) = 0$ if $1 \leq k \leq n$, $S(n,k) = 0$ if $0 \leq n < k$ and $S(n,n) = 1$ for all $n \geq 0$. It is easy to work out the exact value of $S(n,k)$ for small values of k. Since there is only one way of putting n elements in a single non-empty set, $S(n,1) = 1$ if $n > 0$.
Using a combinatorial approach, we can derive the following particular values:

$$S(n,2) = 2^{n-1} - 1, \quad S(n,n-1) = \binom{n}{2}, \quad S(n,n-2) = \binom{n}{3} + 3\binom{n}{4}$$

for any positive integer n.

For a fixed positive integer n, the sum $\sum_{k=0}^{n} S(n,k)$ of Stirling numbers of the second kind is called **Bell number** and is denoted by B_n. The number $S(n,k)$ rapidly increases as n and k increase. For example, while $S(4,2) = 7$ has only one digit in base 10, $S(400,200)$ has as many as 531 digits in base 10, which is almost impossible to compute with pen and paper. Therefore, it is difficult to work with these numbers without the help of modern computers. We use **PARI/GP**, a software specific for number theoretical computations, which is very helpful in cross-checking results and estimating the valuations. Stirling numbers of the second kind may be denoted as $S(n,k)$ (Stanley, 1986) or $\left\{ {n \atop k} \right\}$ (Marx, 1962; Salmeri, 1962). Stirling numbers of the second kind, $S(n,k)$ have an explicit formula known as **Euler's formula** for Stirling numbers (Quaintance and Gould, 2015, p. 118)

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} i^n = \sum_{i=0}^{k} \binom{k}{i} (-1)^i (k-i)^n. \tag{1.32}$$

It is also known that $S(n,k)$ satisfies the following recurrence:

$$S(n+1,k+1) = S(n,k) + (k+1)S(n,k+1). \tag{1.33}$$

There are several generating functions for $S(n,k)$:

1. Rational generating function which generates $S(n,k)$ vertically:

$$\frac{1}{(1-x)(1-2x)(1-3x)\cdots(1-kx)} = \sum_{n=0}^{\infty} S(n+k,k)x^n. \tag{1.34}$$

This generating function can be easily modified in the following form:

$$\frac{1}{(1+x)(1+2x)(1+3x)\cdots(1+kx)} = \sum_{n=0}^{\infty}(-1)^n S(n+k,k)x^n. \tag{1.35}$$

2. Exponential generating function which also generates vertically:

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S(n,k)\frac{t^n}{n!}. \tag{1.36}$$

3. There are two horizontal generating functions, namely

$$x^n = \sum_{k=0}^{n}(-1)^{n-k} S(n,k)x^{\overline{k}} \tag{1.37}$$

and

$$x^n = \sum_{k=0}^{n} S(n,k)x^{\underline{k}}. \tag{1.38}$$

The Stirling numbers of the second kind have the following important identities:

$$a)\quad \binom{a+b}{b} S(n,a+b) = \sum_{j=0}^{n}\binom{n}{j} S(j,a)S(n-j,b), \tag{1.39}$$

$$b)\quad S(n+m,k) = \sum_{i=0}^{k}\sum_{j=i}^{k}\binom{j}{i}\frac{(k-i)!}{(k-j)!} S(n,k-i)S(m,j), \tag{1.40}$$

$$c)\quad S(n+1,k+1) = \sum_{j=k}^{n}\binom{n}{j} S(j,k). \tag{1.41}$$

For more details about Stirling numbers of the second kind, we refer to Stirling (1730), Nielsen (1906), Gould (1972), Comtet (1974), Aigner and Axler (2007), Graham *et al.* (2007) and Quaintance and Gould (2015).

We define Stirling numbers of the first kind from the combinatorial approach.

Definition 1.4.3. *The unsigned Stirling numbers of the first kind denoted by $c(n,k)$ or $\left[{n \atop k}\right]$ are defined as the number of permutations of n symbols with exactly k cycles. By convention, $c(0,0)=1$.*

From the definition, $c(0,n)=c(n,0)=0$ for $n>0$.

The unsigned Stirling numbers of the first kind satisfy the following recurrence:

$$c(n+1,k+1)=c(n,k)+nc(n,k+1). \tag{1.42}$$

Comparing the above recurrence with the recurrence (1.33) for Stirling numbers of the second kind and then comparing with recurrence for binomial coefficients

$$\binom{n+1}{k+1}=\binom{n}{k}+\binom{n}{k+1}, \tag{1.43}$$

we can observe that the recurrences of the three sequences of integers differ only in the multiplier. Consequently, these three numbers have some similarities in various identities and properties. Thus, the Stirling numbers have a deep impact and importance in the heart of combinatorics.

Definition 1.4.4. *Stirling numbers of the first kind denoted by $s(n,k)$ are defined as*

$$s(n,k)=(-1)^{n-k}c(n,k). \tag{1.44}$$

The following particular values are easy to calculate:

$$s(n,n)=1,\quad s(n,1)=(-1)^{n-1}(n-1)!,\quad s(n,n-1)=-\binom{n}{2},$$
$$s(n,n-2)=\frac{1}{4}(3n-1)\binom{n}{3},\quad s(n,n-3)=-\binom{n}{2}\binom{n}{4}.$$

From recurrence (1.42), it is trivial that

$$s(n+1,k+1)=s(n,k)-ns(n,k+1). \tag{1.45}$$

Stirling numbers of the first kind have the following generating functions:

$$x^{\underline{n}} = \sum_{i=0}^{n} s(n,i)x^i, \tag{1.46}$$

$$x^{\overline{n}} = \sum_{i=0}^{n} (-1)^{n-i} s(n,i)x^i, \tag{1.47}$$

and

$$\prod_{i=1}^{n-1}(1-ix) = \sum_{i=0}^{n-1} s(n,n-i)x^i, \tag{1.48}$$

$$\prod_{i=1}^{n-1}(1+ix) = \sum_{i=0}^{n-1} (-1)^i s(n,n-i)x^i. \tag{1.49}$$

Observing the generating functions of $S(n,k)$ and $s(n,k)$, we can see the similarity and the difference. The generating functions are useful in constructing the congruence relation between the two numbers. We have the following identities between the two numbers:

$$\sum_{j=0}^{n} S(n,j)s(j,k) = \binom{0}{n-k}, \tag{1.50}$$

$$\sum_{j=0}^{n} s(n,j)S(j,k) = \binom{0}{n-k}. \tag{1.51}$$

The preceding two identities are known as **orthogonality relations for Stirling numbers**. These two equations led to the following two inversion formulas:

a) For any two sets of constants, a_j and b_j, both independent of n,

$$a_n = \sum_{j=0}^{n} S(n,j)b_j \text{ if and only if } b_n = \sum_{j=0}^{n} s(n,j)a_j, \tag{1.52}$$

b) For any two sets of constants, a_j and b_j, both independent of n and m is an integer such that $m \geq n$, then

$$a_n = \sum_{j=0}^{m} S(j,n)b_j \text{ if and only if } b_n = \sum_{j=0}^{m} s(j,n)a_j. \tag{1.53}$$

Interestingly, some sequences of numbers are similar to Stirling numbers in certain ways regarding definitions, generating functions and properties. Now, we introduce sequences of integers arising from Stirling numbers.

Definition 1.4.5. *For an integer n and variable q, let $[n]_q$ be defined as*

$$[n]_q = \frac{1-q^n}{1-q}. \tag{1.54}$$

It follows that $[n]_q = 1 + q + \cdots + q^{n-1}$. Note that $[n]_1 = n$. We call $[n]_q$ as the q-integer n. Next, we define factorials as

$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q \tag{1.55}$$

with $[0]_q! = 1$.

For integers n and k with $n \geq k \geq 0$, the Gaussian coefficient, $\begin{bmatrix} n \\ k \end{bmatrix}_q$ can be obtained as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}. \tag{1.56}$$

Definition 1.4.6. *For a non-negative integer n, the polynomial $g_n(x) = (x-1)(x-q)\cdots(x-q^{n-1})$ with $g_0(x) = 1$ is called a Gaussian polynomial. The coefficients a_k of $g_k(x)$ in the expression*

$$x^n = \sum_{k=0}^{n} a_k g_k(x) \tag{1.57}$$

are called Gaussian coefficients or q-binomial coefficients and are denoted by $\begin{bmatrix} n \\ k \end{bmatrix}_q$ or $\binom{n}{k}_q$ (Aigner and Axler, 2007).

Now, we define the **q-falling** and **q-rising factorial** polynomials respectively as

$$x_q^{\underline{n}} = x[x-[1]_q](x-[2]_q)\cdots(x-[n-1]_q),$$

$$x_q^{\overline{n}} = x[x+[1]_q](x+[2]_q)\cdots(x+[n-1]_q).$$

In the following expansions:

$$x^n = \sum_{k=0}^{n} S(n,k;q) x_{\underline{q}}^{\underline{k}},$$

$$x_{\underline{q}}^{\underline{n}} = \sum_{k=0}^{n} s(n,k;q) x^k,$$

$S(n,k;q)$ is called **q-Stirling numbers of the second kind** and $s(n,k;q)$ is called **q-Stirling numbers of the first kind**. It should be noted that $S(n,k;1) = S(n,k)$ and $s(n,k;1) = s(n,k)$. More details of q-Stirling numbers can be seen from Gould (1961), Leroux (1990), Wachs and White (1991), Park (1994), Bennett *et al.* (1994), Ehrenborg (2003), Balogh and Schlosser (2016), Cai and Readdy (2017), and Duran *et al.* (2017).

There is another sequence arising from the normal Stirling numbers called the r-Stirling numbers (Broder, 1984).

Definition 1.4.7. *The r-Stirling numbers of the first kind denoted by $s_r(n,k)$ are defined as the number of permutations of the set $\{1,\ 2, \cdots,\ n\}$ having k cycles, such that the numbers $1,\ 2,\ \cdots,\ r$ are in distinct cycles. Note that $s_r(0,0) = 1$.*

Definition 1.4.8. *The r-Stirling numbers of the second kind for positive integers n and k are the number of ways to partition the set $\{1,\ 2, \cdots,\ n\}$ into k non-empty disjoint subsets, such that the numbers $1,\ 2,\ \cdots,\ r$ are all in distinct subsets. It is denoted by $S_r(n,k)$. It may be noted that $S_r(0,0) = 1$.*

It is clear that $s_1(n,k) =\mid s(n,k) \mid= c(n,k)$ and $S_1(n,k) = S(n,k)$. For more details, one can explore from Corcino *et al.* (1990), Mező (2008a), Mező (2008b), Corcino and Fernandez (2014), Kim and Kim (2014), Bényi *et al.* (2018), Morrow (2020), and Ma and Wang (2023).

The next sequences of integers are called generalized Stirling numbers. The generalized Stirling numbers deviate from the normal ones by simply modifying Identity (1.32). Some of the generalized Stirling numbers and their notations are given below (Cakic and Milovanovic, 2004):

$$S^{(\alpha)}(n,k) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{i}(-1)^{k-j}(\alpha+j)^n,$$

$$S^{(\alpha)}(n,k,r) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{i}(-1)^{k-j}(\alpha+rj)^n,$$

$$S^{(\alpha,\lambda)}(n,k,r) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{i}(-1)^{k-j}(\alpha+j)^{(\lambda-1,n)},$$

where

$$a^{(\lambda-1,n)} = \left(\frac{a}{\lambda-1}\right)^{\overline{n}}(\lambda-1)^n.$$

More details about the generalized Stirling numbers can be found in d'Ocagne (1887), Chak (1956), Toscano (1949), Singh (1967), Sinha and Dhawan (1969), Wang (1969), Shrivastava (1970), Toscano (1970), Carlitz (1975), Singh Chandel and Dwiwedi (1979), Singh Chandel (1977), Cakic (1980), and Milovanovic and Cakic (1994).

1.5 Periodicity

Definition 1.5.1. *A sequence $\{x_n\}_{n\geq 0}$ is said to be periodic of period π if there exists a non-negative integer γ such that $x_n = x_{n+\pi}$ for every integer $n \geq \gamma$.*

The period of a sequence is not unique. It is easy to verify that any multiple of a period is also a period.

Example 1.5.1. *a) The sequence of integers*

$$0,\ -2,\ 8,\ 9,\ -3,\ 2,\ 8,\ 9,\ -3,\ 2,\ 8,\ 9,\ -3,\ 2,\ \cdots$$

is periodic with periods 4, 8, *and so on, and the periodicity begins from the third term.*

b) However, the sequence

$$0,\ 1,\ 0,\ 0,\ 1,\ 1, 0,\ 0,\ 0,\ 1,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0,\ \cdots$$

is not periodic.

Definition 1.5.2. *For a given periodic sequence* $\{x_n\}_{n\geq 0}$*, the smallest positive integer* π *such that* $x_n = x_{n+\pi}$ *for any integer* $n \geq \gamma$ *for some positive integer* γ *is called a minimum period of* $\{x_n\}_{n\geq 0}$*.*

If the sequence is periodic from the μ*-th term, then any ordered set*

$$(x_n,\ x_{n+1},\ x_{n+2}, \cdots,\ x_{n+\pi-1})$$

for any integer $n \geq \mu$ *is called a cycle of the sequence* $\{x_n\}_{n\geq 0}$*.*

The minimum period of a given sequence is unique and divides any other period of the given sequence.

Example 1.5.2. *The minimum period of the sequence*

$$a,\ a,\ a,\ b,\ a,\ b,\ a,\ b,\ a,\ b,\ a,\ b,\ a,\ b,\ \cdots$$

is 2*, which is unique and divides the other periods* 4*,* 6*, and so on. The cycle of the sequence is* $(a,\ b)$ *or* $(b,\ a)$*.*

Carlitz (1955) showed that if $k > p > 2$ and $p^{b-1} < k \leq p^b$, where $b \geq 2$, $(p-1)p^{N+b-2}$ is a period for $\{S(n,k) \pmod{p^N}\}_{n\geq 0}$.

1.6 Applications of Stirling Numbers

Stirling numbers of the second kind appear in various literature in many branches of Mathematics. The numbers $S(n,k)$ has several applications to the partition of numbers and sets (Merca, 2016). Another simple example of its application is to represent the total number of rhyme schemes for a poem of n

lines. $S(n,k)$ gives the number of possible rhyming schemes for n lines using k unique rhyming syllables. As an example, for a poem of three lines, there is one rhyme scheme using just one rhyme (aaa), three rhyme schemes using two rhymes (aab, aba, abb), and one rhyme scheme using three rhymes (abc).

In finite differences, the k^{th} forward difference of a function $f(x)$ is given by

$$\Delta^k f(x) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x+i). \tag{1.58}$$

There are similarities between Equations (1.32) and (1.58). Thus, $S(n,k)$ plays a vital role in finite differences. If X is a random variable of a Poisson distribution with expected value λ, then its n^{th} moment is

$$E(X^n) = \sum_{k=0}^{n} S(n,k)\lambda^k. \tag{1.59}$$

In particular, the n^{th} moment of the Poisson distribution with 1 as the expected value is the n^{th} Bell number, which is equal to $\sum_{k=0}^{n} S(n,k)$ and this fact is also called Dobiñski's formula (Dobiñski, 1877). More applications in the same area can be found in Singh (1975), Berg (1975), Koutras (1982), Sibuya (1988), Hennecart (1994), Quaintance and Gould (2015), and Adell (2022).
In Zeon Algebra, Neto and dos Anjos (2014) evaluated the integrals in term of Stirling number as

$$\frac{1}{k!}\int [log(1+\varphi_n)]^k dv_n = s(n,k),$$

$$\text{and} \quad \frac{1}{k!}\int (e^{\varphi_n}-1)^k dv_n = S(n,k).$$

More applications in calculus can be seen in Butzer *et al.* (2003), Boyadzhiev (2012), and Komatsu and Simsek (2017).

In Graph Theory, Stirling numbers of the second kind are used to determine explicitly the chromatic polynomial of certain graphs (Mohr and Porter, 2009). Other applications to graph theory were discussed in Duncan and Peele (2009), Duncan (2010), Galvin and Thanh (2013) and Balogh and Nyul (2014).

In Linear Algebra, the 2-adic valuations of certain ratios of factorials are used to prove the conjecture of Falikman-Friedland-Lowery (Falikman *et al.*, 2002) based on the parity of degrees of projective varieties of $n \times m$ complex symmetric matrices (Friedland and Krattenthaler, 2007).

In Algebraic Topology, Davis (2012) used p-adic valuations of Stirling numbers of the second kind to obtain significant results related to James numbers, periodic homotopy groups, and exponents of $SU(n)$. More details about applications to Algebraic topology can be found in Lundell (1974), Selick (1984), Crabb and Knapp (1988), Bendersky and Davis (1991), Davis and Potocka (2007), Davis and Sun (2007), and Davis (2008).

1.7 Review of Literature

The study of sequences of special types of integers and their divisibility properties have led to enormous advances in number theory. The work of German Mathematician, Johann Peter Gustav Lejeune Dirichlet in the presence of prime in arithmetic progression (Dirichlet, 1837) has opened up new areas in analytic and algebraic number theory. The p-adic numbers were first introduced by Hensel (1897). The study of p-adic valuations and p-adic analysis can be explored through the following books; Bachman (1964), Koblitz (1977), Borevich and Shafarevich (1986), Gouvea (1993), Escassut (1995) and Robert (2000).

Kummer's result about the p-adic valuation of binomial coefficients in Equation (1.17) was generalized by Knuth and Wilf (1989) using Fibonacci numbers. Lengyel (1995) characterized the p-adic valuations $v_p(F_n)$ and $v_p(L_n)$, where F_n and L_n are Fibonacci and Lucas numbers respectively. It was found that the method employed by Knuth and Wilf (1989) can be modified to include Lucas numbers too. Later, Sanna (2016a) generalized the result of Lengyel (1995). The results of periodic property and divisibility of Fibonacci and Lucas numbers can be found in Wall (1960), Robinson (1963), Wilcox (1986), and Ribenboim (1990).

Bell (2007) found that the sequence $\{v_p(f(n))\}$, where f is a polynomial over $\mathbb{Z}$, is periodic if f has no zeros in the ring of the p-adic integers and obtained a bound for the length of the minimal period. Medina *et al.* (2017) strengthened the results of Bell and confirmed that f is either periodic or unbounded; $\{v_p(f(n))\}$ is periodic if and only if f has no zeros in $\mathbb{Z}_p$, in which case, the minimal period is a power of p. Castro *et al.* (2015) constructed a tree whose nodes contain information about the p-adic valuation of Eulerian numbers. The tree constructed and some classical results for Bernoulli numbers are then used to compute the exact p divisibility of the Eulerian numbers for some specific cases. The p-adic valuations of sequences of integers and rational numbers were discussed in Somer (1980), Cohen (1999), Cohn (1999), Young (1999), Lengyel (2003), Postnikov and Sagan (2007), Amdeberhan *et al.* (2008b), Straub *et al.* (2009), Sun and Moll (2009), Sun and Moll (2010), Beyerstedt *et al.* (2011), Heuberger and Prodinger (2011), Marques (2012), Pan and Sun (2012), Amdeberhan *et al.* (2013), Lengyel (2013), Renault (2013), Lengyel (2014), Marques and Lengyel (2014), Medina and Rowland (2015), Sanna (2016b), Katz *et al.* (2017), Lengyel and Marques (2017), Sobolewski (2017), Murru (2018), Choi (2019), Boultinghouse *et al.* (2021), Bunder and Tonien (2020), Bayarmagnai *et al.* (2022), and Cao (2022).

Several sequences of integers, especially those involving factorials, can be linked to Stirling numbers of the second kind. Such relations have been studied over many years and frequently appear in literature (Riordan, 1979; Srivastava, 2000; Boyadzhiev, 2012). For a fixed positive integer n, determining the value of K_n which satisfies

$$S(n,1) < \cdots < S(n,K_n) \leq S(n,K_n+1) > S(n,K_n+2) > \cdots > S(n,n)$$

is one of the interesting old problem(Dobson, 1968; Kanold, 1968a). Wegner (1973) presented a long-standing conjecture, i.e., there is no integer $n > 2$ such that $S(n, K_n) = S(n, K_n + 1)$. Some problems related with sequences of integers and Stirling numbers are Kanold (1968b), Harborth (1968), Bach (1968), Kanold (1969), Menon (1973), Canfield (1978), Canfield and Pomerance (2002), Kemkes *et al.* (2008), and Adell and Cardenas-Morales (2021).

Katsuura (2009) extended the Identity (1.32): For any two complex numbers x and y, and any two positive integers k and m, the following result holds:

$$\sum_{i=1}^{k} \binom{k}{i} (-1)^i (ix + y)^m = \begin{cases} 0, & \text{if } m < k; \\ (-1)^k x^k k!, & \text{if } m = k. \end{cases} \tag{1.60}$$

This result is independently obtained by Ruiz (1996) using induction. However, Identity (1.60) is not new as a generalized version was already given by Gould (1972).

Guo and Qi (2014a) obtained the following two identities on Stirling numbers of the second kind for any positive integer k:

$$\frac{2k+1}{2k+2} \sum_{m=1}^{2k+1} \frac{S(2k+1, m) S(2k+2, 2k-m+2)}{\binom{2k+1}{m-1}} - \sum_{m=1}^{2k} \frac{S(2k+2, m+1) S(2k+1, 2k-m+1)}{\binom{2k+1}{m}} = 1,$$

$$\sum_{m=1}^{k+1} \frac{S(k+2, m) S(k+2, k-m+2)}{\binom{k+1}{m-1}} - \sum_{m=1}^{k} \frac{S(k+3, m+1) S(k+1, k-m+1)}{\binom{k+1}{m}} = 1.$$

Guo and Qi (2014b) established the following formula for computing a two-parameter Euler polynomials, $E_n(x; \alpha, \lambda)$ in terms of Stirling numbers of the second kind, $S(n, k)$. Davis (2013a) defined the partial Stirling numbers, $T_n(k)$

for integers $n > 0$ and k with n positive as

$$T_n(k) = \sum_{i\ odd} \binom{n}{i} i^k.$$

Certain results related with $T_n(k)$ can be seen from the work; Lundell (1978), Davis (1990), Clarke (1995), Young (2003), and Sun and Davis (2007).

Nijenhuis and Wilf (1987) proved that $s(n, k)$ is divisible by the odd part of $n-1$ if $n+k$ is odd. Later, Howard (1990a) improved the result by showing that $s(n, k)$ is divisible by $\binom{n}{2}$ if $n + k$ is odd. Howard (1990b) obtained the following congruences for $0 < 2r < 2p - 2$ and $0 < m < 2p - 2$:

$$s(n, n-2r) \equiv \frac{-n}{2r}\binom{n-1}{2r} B_{2r} \pmod{p^{2v_p(n)}},$$

$$s(n, n-2r-1) \equiv \frac{-n^2(2r+1)}{4r}\binom{n-1}{2r+1} B_{2r} \pmod{p^{3v_p(n)}},$$

$$s(n+m, n) \equiv \frac{n}{m}\binom{n+m}{m}(-1)^m B_m^{(m)} \pmod{p^{2v_p(n)}},$$

where B_n is the n^{th} Bernoulli number and $B_m^{(m)}$ is a higher order Bernoulli number. For a polynomial $f(x)$ with integral coefficients and positive integers n and m, Cao and Pan (2008) obtained

$$v_p\left(\sum_{k\equiv r \pmod{p-1}} s(n, k) f(k) a^k\right) \geq v_p(n!) - log_p\left(\binom{n}{l}\right),$$

where $l = \min(\deg\ f,\ \lfloor \frac{n}{p} \rfloor)$, a and r are arbitrary integers. Lengyel (2015) investigated the problem of p-adic properties of Stirling numbers of the first kind and obtained the following valuation; for integers a, b, and k such that $a \geq 1$ with $(a, p) = 1$ and $2 \geq k + 1 \geq b$,

$$v_p(s(ap^n + b, ap^n + b - k)) = v_p(s(b, b-k))$$

if n is sufficiently large. He also presented a conjecture that for any integer $a \geq 1$

with $(a, p) = 1$, $k \geq 3$ odd, and $n \geq n_1$ with some sufficiently large n_1,

$$v_p(s(ap^n, ap^n - k)) = v_p(s(ap^{n_1}, ap^{n_1} - k)) + 2(n - n_1).$$

Hong and Qiu (2020) partially proved the preceding conjecture for $p \geq 5$ by giving a restrictions on n_1 such that $n_1 > 3\log_p(k-1) + \log_p a$ and $v_p(s(ap^{n_1}, ap^{n_1} - k + 1)) < n_1$.

Leonetti and Sanna (2017) defined $H(n, k) = \sum \frac{1}{(i_1 i_2 \cdots i_k)}$, where the sum extended over all positive integers $i_1 < i_2 < \cdots < i_n \leq n$, and showed a relation with the Stirling numbers of the first kind by $H(n, k) = \frac{s(n+1,k+1)}{n!}$. If $k \geq 2$, they proved that the p-adic valuation of $H(n, k)$ is strictly greater than $-(k-1)(\log_p(\frac{n}{k-1}) - 1)$, for all positive integers $n \in [(k-1)p, x]$ whose base p representation starts with the base p representation of $k - 1$, but at most $3x^{0.835}$ exceptions. For a non-negative integer k and $n = kp^r + m$ such that $0 \leq m < p^r$, Komatsu and Young (2017) proved that

$$v_p(s(n + 1, k + 1)) = v_p(n!) - v_p(k!) - kr.$$

Adelberg (2018) confirmed the result that, if $p - 1 \mid n - k$ and $p \nmid \binom{k-1}{r}$, where $r = \frac{n-k}{p-1}$, then

$$v_p(s(n, k)) = \frac{s_p(k-1) - s_p(n-1)}{p-1}.$$

Qiu and Hong (2019) obtained the following valuations for arbitrary integers n, m, and k such that $2 \leq m \leq n$ and $2 \leq k \leq 2^{m-1} + 1$;

$$v_2(s(2^n, 2^m - k)) = 2^n - 2^m - (n - m)(2^m - \left\lfloor \frac{k}{2} \right\rfloor) + m - 2 - v_2(\left\lfloor \frac{k}{2} \right\rfloor) + (n-1)\epsilon_k,$$

where $\epsilon_k = 0$ if k is even and $\epsilon_k = 1$ if k is odd.

Bell (1939) obtained the following results using the generalized Stirling numbers as $\zeta_n^{(k,r)}$ with $\zeta_n^{(k,1)} = S(n,k)$ and $\zeta_n^{(k,-1)} = s(n,k)$:

$$\zeta_p^{(k,r)} \equiv 0 \pmod p,\ \text{if } 1 < k < p; \quad \zeta_{p+1}^{(k,r)} \equiv 0 \pmod p,\ \text{if } 2 < k < p+1;$$

$$\zeta_{p+1}^{(k,r+1)} - \zeta_{p+1}^{(k,r)} \equiv \zeta_2^{(k,r+1)} \pmod p; \quad \zeta_{p^2}^{(k,r)} \equiv \delta_{p^2,k} + r\delta_{p,k} \pmod p,\ \text{if } k > 1;$$

$$\zeta_{2p}^{(k,r)} \equiv \zeta_{2p}^{(k,r+1)} - 2\zeta_{p+1}^{(k,r+1)} + \zeta_2^{(k,r+1)} \pmod p,$$

where $\delta_{i,j}$ denotes the usual Kronecker delta function. Becker and Riordan (1948) studied the arithmetic properties of Bell and Stirling numbers and proved the following results using $S(n,k)$ as $S(k,n)$:

$$S(p+r,c) \equiv S(r+1,c) + S(r,c-p) \pmod p;$$

$$S(c+ip+r+j(p-1), c+ip) \equiv \binom{i+j}{i} S(c+r,c) \pmod p;$$

$$S(r+p^i,c) \equiv S(r+1,c) + S(r,c-p) + \cdots + S(r,c-p^i) \pmod p.$$

Lundell (1978) evaluated the p-adic valuations of $g.c.d.(k!S(n,k) : m \le k \le n)$ which have some applications to certain problems in algebraic topology concerned with the calculation of e-invariants and formulas relating different characteristic classes in K-theory. Sagan (1985) obtained the following congruence using group action on abelian groups;

$$\sum_{i=0}^{r}(-1)^i \binom{r}{i} \sum_{j=0}^{i} S(i,j)[-p(p-1)]^{i-j} \sum_{l=0}^{j} \binom{j-1+\delta_{ij}}{l}$$
$$\times S(n+(r-i)p+l, k-(j-l)p) \equiv 0 \pmod{p^r}.$$

Nijenhuis and Wilf (1987) used the generating function to show that $S(n,k)$ is divisible by the odd part of k if $n+k$ is odd. Howard (1990a) improved the result by showing that $S(n,k)$ is divisible by $\binom{k+1}{2}$ if $n+k$ is odd. Sved (1988) obtained the following congruence, of the Lucas congruence type, for Stirling numbers of the second kind; if $p \nmid k$ and $n' = \lfloor \frac{np-p\lfloor \frac{k}{p} \rfloor -1}{p-1} \rfloor$, then

$$S(n,k) \equiv \binom{a_h}{b_h}\binom{a_{h-1}}{b_{h-1}} \cdots \binom{a_1}{b_1} S(a_0,b_0) \pmod{p},$$

where a_i and b_i are the $(i+1)^{th}$ p-adic digits of n' and k respectively. Tsumura (1991) obtained the following congruence for integers n, m, and k such that $n \geq m \geq 0$, $n \equiv m \pmod{(p-1)p^e}$, and $N(k) = Min\{m, e+1\} > 0$:

$$S(n,k) \equiv S(m,k) \pmod{p^{N(k)}}.$$

Lengyel (1994) conjectured that $v_2(S(2^n,k)) = s_2(k) - 1$ if $1 \leq k \leq 2^n$ and confirmed that there exists a function $f(k)$ such that $v_2(S(c2^n,k)) = s_2(k) - 1$ if $n \geq f(k)$ and c is odd. This conjecture was later confirmed by Wannemacker (2005) using the Identity (1.32). Lengyel (2009) extended the same result to $v_2(S(c2^n,k)) = s_2(k)-1$ for any integer c and $1 \leq k \leq 2^n$. Hong *et al.* (2012) also proved that $v_2(S(2^n+1,k+1)) = s_2(k) - 1$ if $1 \leq k \leq 2^n$. Clarke (1995) applied a version of Hensel's lemma to analytic functions on the p-adic integers and the results were used to determine the divisibility properties of Stirling numbers of the second kind.

For $n = a(p-1)p^q$, $p \nmid a$ and $1 \leq k \leq n$, Gessel and Lengyel (2001) proved that

$$v_p(S(n,k)) = \left\lfloor \frac{k-1}{p-1} \right\rfloor + \tau_p(k),$$

if q is sufficiently large, $\frac{k}{p}$ is not an odd integer and $\tau_p(k)$ is a non-negative integer which vanishes if k is a multiple of $p-1$. Cao and Pan (2008) proved that for any positive integer α;

$$v_p(k!S(n,k)) \geq v_p\left(\left\lfloor \frac{n}{p^{\alpha-1}} \right\rfloor !\right) - \left\lfloor \frac{n-k}{(p-1)p^{\alpha-1}} \right\rfloor.$$

Amdeberhan *et al.* (2008a) analyzed the 2-adic properties of $S(n,k)$ and found the following exact valuations;

$$v_2(S(n,3)) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ 1, & \text{if } n \text{ is even,} \end{cases}$$

$$v_2(S(n,4)) = \begin{cases} 1, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

They introduced the concept of m-level dividing the set $\mathbb{N}$ into 2^m classes so that the classes, $C_{m,j} = \{2^m i + j : i \in \mathbb{N}\}$ form a partition of $\mathbb{N}$ into different classes modulo 2^m. They also proposed a conjecture that the class $C_{5,7}$ is exceptional and $v_2(S(4i,5)) \neq v_2(S(4i+3,5))$ if i belongs to class $C_{5,7}$. This conjecture was proved by Hong *et al.* (2012). Davis (2008) determined the set of integers n satisfying the valuation

$$v_2(S(2^L + n - 1, n)) = \left\lfloor \frac{n-1}{2} \right\rfloor,$$

where $L = n - 1 + \lfloor n/2 \rfloor$. Chan and Manna (2010) used a rational generating function to obtain the congruences for $S(n, kp^m) \pmod{p^m}$. Zhao *et al.* (2014) obtained the following result for 2-adic valuation; for positive integers a, c, n with c odd, $n \geq 2$ and $1 \leq a \leq 2^n$,

$$v_2(S(c2^n, (c-1)2^n + a)) = s_2(a) - 1.$$

Zhao *et al.* (2015) proved that if c is odd and $2 \leq m \leq n$, then

$$v_2(S(c2^{n+1}, 2^m - 1) - S(c2^n, 2^m - 1)) = n + 1 \tag{1.61}$$

except when $m = n = 2$ and $c = 1$, in which case $v_2(S(8,3) - S(4,3)) = 6$. This settled Lengyel's conjecture (Lengyel, 2009, Conj. 2.). Miska (2018) proved, for any prime p,

$$v_p(S(n,k)) = v_p(S(a + p^{m_0-1}(p-1), k)) + v_p(n-a) - m_0 + 1,$$

where m_0, n, k, and a are integers such that $a < k < p$ and $n \equiv a \pmod{p^{m_0-1}(p-1)}$. Adelberg (2018) defined minimum zero case (MZC) as the case when $p-1$ divides $n-k$, and p does not divide $\binom{n+\frac{n-k}{p-1}}{n}$ and obtained the following important results using higher order Bernoulli number:

(i) If $n \geq k$, then

$$v_p(S(n,k)) \geq \left\lceil \frac{s_p(k) - s_p(n)}{p-1} \right\rceil. \tag{1.62}$$

(ii) If $S(n,k)$ is a minimum zero case, then

$$v_p(S(n,k)) = \frac{s_p(k) - s_p(n)}{p-1}. \tag{1.63}$$

(iii) If $S(n,k)$ is a minimum zero case, then so is $S(np,kp)$ and

$$v_p(S(n,k)) = v_p(S(np,kp)). \tag{1.64}$$

Feng and Qiu (2020) confirmed that the formula, $v_p(S(n,n-k))$ depends on the value of $S_2(i,i-k)$ for $k+2 \leq i \leq 2k$, where $S_r(n,k)$ is the r-associated Stirling number of the second kind. They also gave the formula to compute $v_p(S(n,n-k))$, which enables to show $v_p((n-k)!S(n,n-k)) < n$ for $0 \leq k \leq min\{7, n-1\}$ and $p \geq 3$. Adelberg (2021) concentrated on a 2-adic analysis of $S(n,k)$ and classified the results by the following cases; Minimum zero case (MZC), Almost minimum zero case ($AMZC$), Shifted minimum zero case ($SMZC$), and Shifted almost minimum zero case ($SAMZC$). More improved results for the same cases and extension to odd primes and $s(n,k)$ can be found in Adelberg and Lengyel (2022). Some interesting results about divisibility properties of Stirling numbers of the second kind are available in Carlitz (1953), Carlitz (1955), Polya *et al.* (1980), Clarke (1981), Peele (1988), Davis (1990), Howard (1990b), Young (1999), Sun

(2007), Demaio (2008), Berrizbeitia *et al.* (2010), and Davis (2013b).

1.8 Conclusion

In this chapter, we have presented basic definitions of divisibility, congruence, p-adic valuations and Stirling numbers of the first and second kinds. We have also presented the periodicity, applications of Stirling numbers and review of literature.

Chapter 2

Divisibility of Certain Classes of Stirling Numbers of the Second Kind[1]

2.1 Introduction

Various approaches and techniques have been appearing in the literature to formulate the p-adic valuation of Stirling numbers of the second kind. An interesting formula to evaluate $v_2(S(c2^n, k))$ is given by Lengyel (2009)

$$v_2(S(c2^n, k)) = s_2(k) - 1$$

for any positive integer n, c, and $1 \leq k \leq 2^n$. The immediate consequence of this formula is to find whether this pattern still holds for an odd prime, p.

This chapter deals with some interesting results of the p-adic valuations of $S(n, k)$, including the case when n is a power of prime, p. We have developed an alternate formula for evaluating Stirling numbers of the second kind and also prove certain results like $v_p(S(p^2, kp)) \geq 2$, $v_p(S(p^n, kp)) \geq 2$, $v_p(S(2p, p)) \geq 2$, $v_p(S(2p, p-1)) = 1$, $v_p(S(2p, p-1)) \geq 2$ and $v_p(S(2p, p+2)) \geq 1$. Primality of p using $S(p, k)$ and divisibility of $S(n, p)$ are also discussed.

[1]***Journal of Combinatorics and Number Theory***, **12(2)**, 63-–77 (2022)

2.2 Tools and Identity of $S(n,k)$

In order to formulate $S(n,k)$, we divide partitions into different classes based on the number of subsets with same cardinality in the partitions. Let $\{n_i : 1 \leq i \leq t\}$ and $\{e_i : 1 \leq i \leq t\}$ be two sets of positive integer such that $\sum_{i=1}^{t} n_i e_i = n$ and $\sum_{i=1}^{t} e_i = k$, where $n_i's$ are distinct and $e_i's$ need not to be distinct. We define $s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)})$ as the number of those partitions of n objects into k non-empty subsets containing exactly e_i subsets with cardinality n_i. So, we introduce

$$S(n,k) = \sum_{\sum e_i = k, \sum n_i e_i = n} s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}). \tag{2.1}$$

In the partition of 6 objects into 3 non-empty subsets, we see

$$S(6,3) = s(1^{(2)}, 4^{(1)}) + s(1^{(1)}, 2^{(1)}, 3^{(1)}) + s(2^{(3)}),$$

where $s(1^{(2)}, 4^{(1)})$ counts the number of those partitions containing exactly two singleton subsets and one subset with four elements, $s(1^{(1)}, 2^{(1)}, 3^{(1)})$ counts those partitions containing exactly one singleton subset, one subset with two elements and one subset with three elements and $s(2^{(3)})$ is the number of those partitions containing exactly three subsets with two elements.

Kwong (1989a) proved that the sequence of Stirling numbers of the second kind $S(n,k)$ modulo M for any positive integer $M > 1$ is cyclic and gave the minimum periods for different values of k and M. One of the interesting result that he mentioned is

$$\pi(k; p^N) = (p-1)p^{N+b-2} \qquad \text{if} \quad p^{b-1} < k \leq p^b, \tag{2.2}$$

where $\pi(k; p^N)$ denotes the minimum period of $\{S(n,k) \pmod{p^N}\}_{n \geq 1}$ for an odd prime p. Adelberg (2018) obtained the following important results:

1. If $n \geq k$, then
$$v_p(S(n,k)) \geq \left\lceil \frac{s_p(k) - s_p(n)}{p-1} \right\rceil. \tag{2.3}$$
2. If $S(n,k)$ is a minimum zero case, i.e., $(p-1)|(n-k)$ and $p \not| \binom{n+\frac{n-k}{p-1}}{n}$, then
$$v_p(S(n,k)) = \frac{s_p(k) - s_p(n)}{p-1}. \tag{2.4}$$
3. If $S(n,k)$ is a minimum zero case, then so is $S(np,kp)$ and
$$v_p(S(n,k)) = v_p(S(np,kp)). \tag{2.5}$$

The above results about minimum zero case gives an exact p-adic valuations for a large class of $S(n,k)$.

2.3 Results

In this section, we introduce an alternate formula to find the Stirling numbers of the second kind and p-adic valuations of some classes of $S(n,k)$. Some of these results have been generalized using minimum periods.

Lemma 2.3.1. *If n and k are two positive integers, then*
$$s(n^{(k)}) = \prod_{i=0}^{k-1} \binom{n(k-i)-1}{n-1}.$$

Proof. The case for $n = 1$ is trivial.

We provide the proof for $n > 1$ by using induction hypothesis on k.

We know that $s(n^{(k)})$ counts the number of partitions of nk objects into k subsets such that each k subsets contains exactly n objects.

The case for $k = 1$ is trivial since $s(n^{(1)}) = 1$.

Assume that the theorem holds for every positive integer less than k. Let

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_n \\ a_{n+1} & a_{n+2} & a_{n+3} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{(k-1)n+1} & a_{(k-1)n+2} & a_{(k-1)n+3} & \cdots & a_{kn} \end{array}$$

be the nk objects. The order of the subsets in the partition does not count as each subsets have the same cardinality. We can now safely assume that the first object a_1 always belongs to the first subset of the partition. Thus, the number of choices for the first subset is equal to the number of choices of the remaining $n-1$ objects from $nk-1$, *i.e.*, $\binom{nk-1}{n-1}$. Now, the remaining $nk-n = n(k-1)$ objects are partition into $k-1$ subsets each containing n elements. The number of such partitions are $s(n^{(k-1)})$ and hence

$$s(n^{(k)}) = \binom{nk-1}{n-1} s(n^{(k-1)}).$$

By induction hypothesis, we get

$$s(n^{(k-1)}) = \prod_{i=0}^{k-2} \binom{n(k-1-i)-1}{n-1}.$$

It follows that

$$s(n^{(k)}) = \prod_{i=0}^{k-1} \binom{n(k-i)-1}{n-1}.$$

Using the binomial coefficients in terms of factorials, the above result may be written as

$$s(n^{(k)}) = \frac{(nk)!}{k!(n!)^k}.$$

This completes the proof. □

Theorem 2.3.1. *Let* $\{n_i : 1 \le i \le t\}$ *and* $\{e_i : 1 \le i \le t\}$ *be two sets of positive* *integers and $n_i's$ are distinct. If $\sum_{i=1}^{t} n_i e_i = n$ and $\sum_{i=1}^{t} e_i = k$, then*

$$s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) = \prod_{j=1}^{t} \binom{n-\sum_{i=0}^{j-1} n_i e_i}{n_j e_j} s(n_j^{(e_j)}) \qquad \textit{(if } n_0 = e_0 = 0\textit{)}$$
$$= \frac{n!}{\prod_{j=1}^{t} e_j!(n_j!)^{e_j}}.$$

Proof. We first choose n_1e_1 objects from n objects and the number of such choices is $\binom{n}{n_1e_1}$. These n_1e_1 objects are then partition into e_1 subsets containing n_1 objects each. The total number of such partitions is

$$s({n_1}^{(e_1)}) = \frac{(n_1e_1)!}{e_1!(n_1!)^{e_1}}.$$

Now we partition the remaining $n-n_1e_1$ objects into $k-e_1$ subsets such that each partition contains e_i number of subsets with cardinality n_i for each $2 \le i \le t$. The total number of such partitions is $s({n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)})$. Thus,

$$s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) = \binom{n}{n_1e_1} s({n_1}^{(e_1)}) s({n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}).$$

Similarly, we can see that

$$s({n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) = \binom{n-n_1e_1}{n_2e_2} s({n_2}^{(e_2)}) s({n_3}^{(e_3)}, {n_4}^{(e_4)}, ..., {n_t}^{(e_t)}).$$

Therefore,

$$s({n_1}^{(e_1)}, {n_2}^{(e_2)}, ..., {n_t}^{(e_t)}) = \binom{n}{n_1e_1}\binom{n-n_1e_1}{n_2e_2} s({n_1}^{(e_1)}) s({n_2}^{(e_2)})$$
$$\times\, s({n_3}^{(e_3)}, {n_4}^{(e_4)}, ..., {n_t}^{(e_t)}).$$

Repeating the same process over and over, we get

$$s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) = \binom{n}{n_1e_1}\binom{n-n_1e_1}{n_2e_2}\cdots\binom{n-\sum_{i=1}^{t-2} n_ie_i}{n_{t-1}e_{t-1}} \times s({n_1}^{(e_1)})\cdots s({n_t}^{(e_t)}) \tag{2.6}$$

$$= s({n_t}^{(e_t)})\prod_{j=1}^{t-1}\binom{n-\sum_{i=0}^{j-1} n_ie_i}{n_je_j} s({n_j}^{(e_j)})$$

$$= \prod_{j=1}^{t}\binom{n-\sum_{i=0}^{j-1} n_ie_i}{n_je_j} s({n_j}^{(e_j)}), \tag{2.7}$$

since $n - \sum_{i=0}^{t-1} n_ie_i = n_te_t$ when $n_0 = e_0 = 0$.

The above expression may be expressed as

$$s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) = \frac{n!}{(n_1e_1)!(n_2e_2)!\cdots(n_te_t)!}\prod_{j=1}^{t} s({n_j}^{(e_j)})$$

$$= n!\prod_{j=1}^{t}\frac{s({n_j}^{(e_j)})}{(n_je_j)!}.$$

By using the results of Lemma 2.3.1, we have

$$\frac{s({n_j}^{(e_j)})}{(n_je_j)!} = \frac{1}{e_j!(n_j!)^{e_j}}.$$

It follows that

$$s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) = n!\prod_{j=1}^{t}\frac{1}{e_j!(n_j!)^{e_j}}$$

$$= \frac{n!}{\prod_{j=1}^{t} e_j!(n_j!)^{e_j}}. \tag{2.8}$$

Hence the theorem follows. □

We come to an alternate formula for evaluation of $S(n, k)$ with the help of (2.1) and (2.7).

Corollary 2.3.1. *Let n and k are two positive integers such that $n \geq k$, then*

$$S(n, k) = \sum_{\sum e_i = k, \sum n_ie_i = n} \frac{n!}{\prod e_i!(n_i!)^{e_i}},$$

where the sum runs over every pair of sets of positive integer $\{n_i\}$ and $\{e_i\}$ with same cardinality satisfying $\sum e_i = k$ and $\sum n_i e_i = n$ provided $n_i's$ are distinct.

It is easy to verify from the above theorem that the p-adic valuations of $S(p,k)$ is always greater than or equal to 1 if p is an odd prime and k lies between 2 and $p-1$.

Now we introduce some results about divisibility of Stirling numbers[2]:

Theorem 2.3.2. *A positive integer n is a prime if and only if $n \mid S(n,k)$ for all $2 \leq k \leq n-1$.*

Proof. The generating function of $S(n,k)$ in terms of falling powers is given by

$$x^n = \sum_{k=o}^{n} S(n,k)\{x\}_k \tag{2.9}$$

for any non-negative integer n.

If n is a positive integer such that $n|S(n,k)$ for all $2 \leq k \leq n-1$, put $x = n$ in Equation (2.9)

$$\begin{aligned} n^n &= \sum_{k=o}^{n} S(n,k)\{n\}_k \\ &= \{n\}_n + \{n\}_1 + \sum_{k=2}^{n-1} S(n,k)\{n\}_k \\ &= n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 + n + \sum_{k=2}^{n-1} n(n-1)\cdots(n-(k-1))S(n,k). \end{aligned}$$

It follows that

$$n^{n-1} = (n-1)(n-2)\cdots 3\cdot 2\cdot 1 + 1 + \sum_{k=2}^{n-1}(n-1)(n-2)\cdots(n-(k-1))S(n,k)$$

Since $n|S(n,k)$ for all $2 \leq k \leq n-1$, we get

$$0 \equiv (n-1)! + 1 \pmod{n}$$

[2]On the p-adic valuations of Stirling numbers of the second kind, *Contemporary Mathematics*, **12(2)**, 63-–77(2022)

or

$$(n-1)! \equiv -1 \pmod{n}.$$

Hence n is prime.

The proof of the converse is straight forward. □

The next result is a congruence relation on binomial coefficients and will be used in the proof of the subsequent theorem.

Lemma 2.3.2. *If p is a prime, then*

$$v_p\left(\binom{p-1}{i} - (-1)^i\right) \geq 1 \quad or \quad v_p\left(\binom{p-1}{i}\right) = 0.$$

Proof. For $i = 0$, the case is trivial.

We assume that $i > 0$. The binomial coefficient $\binom{p-1}{i}$ is given by

$$\binom{p-1}{i} = \frac{(p-1)!}{(p-1-i)!i!}.$$

Therefore,

$$\begin{aligned} i!\binom{p-1}{i} &= (p-1)(p-2)...(p-i+2)(p-i+1)(p-i) \\ &\equiv (-1)(-2)...(-i) \pmod{p} \\ &\equiv (-1)^i i! \pmod{p}. \end{aligned}$$

Since $0 < i < p$, $gcd(p,i) = 1$. Then,

$$\binom{p-1}{i} \equiv (-1)^i \pmod{p}.$$

□

Theorem 2.3.3. *Let p be an odd prime. For any positive integer $n \geq p$,*

$$v_p(S(n,p)) = 0$$

if and only if $(p-1)|(n-1)$.

Proof. Using the above Lemma, we have

$$p!S(n,p) = \sum_{i=1}^{p} \binom{p}{i} (-1)^{p-i} i^n$$
$$\equiv \sum_{i=1}^{p} \binom{p}{i} (-1)^{p-i} i^n \pmod{p}.$$

Since $\binom{p}{i} = \binom{p-1}{i-1}\frac{p}{i}$, we get

$$(p-1)!S(n,p) \equiv \sum_{i=1}^{p-1} (-1)^{i-1}(-1)^{p-i} i^{n-1}.$$

Using Wilson's theorem, the preceding congruence reduces to

$$S(n,p) \equiv \sum_{i=1}^{p-1} i^{n-1} \pmod{p},$$

as p is odd.

Now, we use the following well known results

$$\sum_{i=1}^{p-1} i^{n-1} \equiv \begin{cases} 0 \pmod{p}, & \text{if } (p-1) \nmid (n-1) \\ -1 \pmod{p}, & \text{if } (p-1) | (n-1). \end{cases}$$

Hence the theorem follows. □

Theorem 2.3.4. *Let p be an odd prime and c be a positive integer such that $1 \leq c \leq p-1$. Then, for positive integers n and k such that $k \leq n$,*

$$v_p(S(cp^n, cp^k)) = 0.$$

Proof. The theorem is a special case of (Adelberg, 2018, Th. 2.2).

We have

$$cp^n - cp^k = c(p^n - p^k) = c(p-1)\sum_{j=0}^{n-k-1} p^{j+k}$$

which implies that $cp^n - cp^k$ is divisible by $p-1$. We also have $1 \leq c \leq p-1$ and $1 \leq cp^k \leq cp^n$. It follows that $S(cp^n, cp^k)$ is a minimum zero case and hence we have

$$v_p(S(cp^n, cp^k)) = \frac{s_p(cp^k) - s_p(cp^n)}{p-1} = 0, \tag{2.10}$$

since $s_p(cp^n) = s_p(cp^k) = s_p(c) = c$. $\square$

Theorem 2.3.5. *Let p be an odd prime, then*

$$v_p(S(p^n, 2p)) \geq n$$

for every integer $n \geq 2$.

Proof. Using identity (1.32)

$$(2p)!S(p^n, 2p) = \sum_{i=0}^{2p} \binom{2p}{i} (-1)^{2p-i} i^{p^n}$$

which can also be written as

$$\begin{aligned}(2p)!S(p^n, 2p) &= \sum_{i=0}^{2p} \binom{2p}{2p-i} (-1)^{i} (2p-i)^{p^n} \\ &= \sum_{i=0}^{2p} \binom{2p}{i} (-1)^{2p-i} (2p-i)^{p^n}.\end{aligned}$$

Since $\binom{m}{i} = \binom{m}{m-i}$ for every integers $0 \leq i \leq m$ and $2p - i \equiv i \pmod 2$, we have

$$2(2p)!S(p^n, 2p) = \sum_{i=0}^{2p} \binom{2p}{i} (-1)^{2p-i} (i^{p^n} + (2p-i)^{p^n}). \tag{2.11}$$

If $p \not| i$ for $0 \leq i \leq 2p$, then

$$2p - i \equiv -i \pmod p,$$

which also yields the congruence

$$(2p-i)^{p^n} \equiv -(i)^{p^n} \pmod{p^{n+1}}.$$

It follows that

$$\binom{2p}{i} (-1)^{2p-i} ((2p-i)^{p^n} + (i)^{p^n}) \equiv 0 \pmod{p^{n+2}}, \quad \text{since } p \mid \tbinom{2p}{i}. \tag{2.12}$$

Thus, each term of the right-hand side of Equation (2.11) is divisible by p^{n+2} and hence

$$(2p)!S(p^2, 2p) \equiv 0 \pmod{p^{n+2}}.$$

Therefore

$$v_p(2(2p)!S(p^2, 2p)) \geq n + 2$$
$$v_p(S(p^2, 2p)) \geq n.$$

Hence the theorem follows. □

Theorem 2.3.6. *Let p be a prime and n and k be two positive integers with $k \leq p - 1$, then there exists a positive integer m in $1 \leq m < p - 1$ such that*

$$S(n,k) \equiv \begin{cases} S(m,k) \pmod{p}, & \text{if } n \not\equiv 0 \pmod{p-1}, \\ (p-1-k)! \pmod{p}, & \text{if } n \equiv 0 \pmod{p-1}. \end{cases}$$

Proof. By division algorithm, we have

$$n = (p-1)q + m$$

where q is the quotient and m is the remainder such that $0 \leq m < p - 1$.
Now

$$\begin{aligned} k!S(n,k) &= \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} i^n \\ &= \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} i^{(p-1)q+m} \\ &\equiv \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} i^m \pmod{p} \end{aligned}$$

since $i^{p-1} \equiv 1 \pmod{p}$ for $1 \leq i \leq k \leq p - 1$ by Fermat's little theorem.
If $m \neq 0$, we have

$$k!S(n,k) \equiv k!S(m,k) \pmod{p}.$$

Since k is less than p, it follows that $p \nmid k!$ which results

$$S(n,k) \equiv S(m,k) \pmod{p},$$

for every n such that $n \not\equiv 0 \pmod{p-1}$.

Next, if $m = 0$, we have

$$\begin{aligned} k!S(n,k) &\equiv \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} \pmod{p} \\ &\equiv \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} - (-1)^k \pmod{p} \\ &\equiv (-1)^{k+1} \pmod{p}. \end{aligned}$$

We also know that

$$\binom{p-1}{k} \equiv (-1)^k \pmod{p}$$

yields

$$\frac{1}{k!} \equiv (-1)^{k+1}(p-1-k)! \pmod{p},$$

which implies that

$$S(n,k) \equiv (p-1-k)! \pmod{p},$$

which completes the proof. □

From the above theorem, we see that, if $1 \le m < k$,

$$S(n,k) \equiv 0 \pmod{p} \quad \text{since } S(m,k) = 0.$$

However, the case for $m = k$ results

$$S(n,k) \equiv 1 \pmod{p}.$$

Corollary 2.3.2. *Let p be an odd prime and k be a positive integer less than p,* *then*

$$S(n,k) \equiv \begin{cases} 1 \pmod{p}, & \textit{if } n \equiv k \pmod{p-1}, \\ 0 \pmod{p}, & \textit{if } n \equiv i \pmod{p-1} \textit{ for } 1 \leq i \leq k-1. \end{cases}$$

If we apply the above theorem and corollary to the special cases for $k = p-1, p-2$ and $p-3$, we get

$$S(n,p-1) \equiv \begin{cases} 1 \pmod{p}, & \text{if } n \equiv 0 \pmod{p-1}, \\ 0 \pmod{p}, & \text{otherwise.} \end{cases}$$

$$S(n,p-2) \equiv \begin{cases} 1 \pmod{p}, & \text{if } n \equiv 0, p-2 \pmod{p-1}, \\ 0 \pmod{p}, & \text{otherwise.} \end{cases}$$

$$S(n,p-3) \equiv \begin{cases} 2 \pmod{p}, & \text{if } n \equiv 0 \pmod{p-1}, \\ 3 \pmod{p}, & \text{if } n \equiv p-2 \pmod{p-1}, \\ 1 \pmod{p}, & \text{if } n \equiv p-3 \pmod{p-1}, \\ 0 \pmod{p}, & \text{if otherwise.} \end{cases}$$

We have calculated the values of $v_p(S(p^2, kp))$ for different values of (p, k) within the range $2 \leq p \leq 100$ and $2 \leq k \leq p-1$. The following values are obtained;

$$v_p(S(p^2,kp)) = \begin{cases} 7, & \text{if } (p,k) = (7,4) \\ 6, & \text{if } (p,k) = (37,4), (59,14), (67,8) \\ 3, & \text{if } k = p-1 \text{ and } (p,k) = (37,5), (59,15), (67,9) \\ 5, & \text{if } k \text{ is even and } (p,k) \neq (7,4), (37,4), (59,14), (67,8) \\ 2, & \text{if } k \text{ is odd and } (p,k) \neq (37,5), (59,15), (67,9). \end{cases} \tag{2.13}$$

Based on these calculations, we propose the following conjecture:

Conjecture 2.3.1. *If k is an integer such that $1 < k < p-1$, then*

$$v_p(S(p^2,kp)) = \begin{cases} 5 \text{ or } 6, & \text{if } k \text{ is even;} \\ 2 \text{ or } 3, & \text{if } k \text{ is odd,} \end{cases} \tag{2.14}$$

for any prime $p > 7$.

Theorem 2.3.7. *Let p be an odd prime and k be an integer such that $2 \leq k \leq p-1$, then*

$$v_p(S(p^2,kp)) \geq 2.$$

Proof. We know (due to (2.1))

$$S(p^2,kp) = \sum_{\sum e_i = pk, \sum n_i e_i = p^2} s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}).$$

To prove the theorem, we divide each term of the sum over the partitions containing e_i subsets with cardinality n_i into the following cases depending on the divisibility of $n_i e_i$ by p.

Case 1: $p \not| n_i e_i$ **for some** i**,** $1 \leq i \leq t$

If $p \not| n_i e_i$, re-arrange the index by interchanging i and 1 so that $p \not| n_1 e_1$.

Using Equation (2.7), we have

$$s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) = \binom{p^2}{n_1 e_1} s({n_1}^{(e_1)}) \prod_{j=2}^{t} \binom{n - \sum_{i=0}^{j-1} n_i e_i}{n_j e_j} s({n_j}^{(e_j)})$$

which implies that

$$\binom{p^2}{n_1 e_1} | s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}).$$

We also know that $p^2 | \binom{p^2}{n_1 e_1}$ if $p \not| n_1 e_1$. It follows that

$$p^2 | s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}).$$

Therefore,

$$v_p(s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)})) \geq 2$$

if $p \not| n_i e_i$ for some i, $1 \leq i \leq t$.

Case 2: $p|n_i e_i$ **for every** i**,** $1 \leq i \leq t$

In this case, either $p|n_i$ or $p|e_i$ for all $1 \leq i \leq t$. We divide this case into two sub-cases, where the first sub-case deals with $p|e_i$ for all $1 \leq i \leq t$ and the second sub-case deals with $p \not| e_i$ for some i, $1 \leq i \leq t$.

Case 2.1: $p|e_i$ **for every** i**,** $1 \leq i \leq t$

It is clear that there exists a positive integer a_i for each $1 \leq i \leq t$ such that $e_i = pa_i$. By the given condition, we have

$$\sum_{i=1}^{t} e_i = kp,$$

which implies that

$$\sum_{i=1}^{t} a_i = k.$$

Now, we have

$$s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) = \frac{p^2!}{\prod_{i=1}^{t} e_i!(n_i!)^{e_i}},$$

which yields

$$\begin{aligned} v_p(s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)})) &= v_p(p^2!) - v_p(\prod_{i=1}^{t} e_i!(n_i!)^{e_i}) \\ &= p + 1 - \sum_{i=1}^{t} v_p(e_i!) - \sum_{i=1}^{t} e_i v_p(n_i!). \quad (2.15) \end{aligned}$$

Since $\sum_{i=1}^{t} n_i e_i = p^2$ and by replacing $e_i = pa_i$, we get

$$\sum_{i=1}^{t} n_i a_i = p,$$

which implies that $1 \leq n_i < p$ for every $1 \leq i \leq t$ since $\sum_{i=1}^{t} a_i = k \geq 2$. It follows that

$$v_p(n_i!) = 0.$$

We also have

$$\begin{aligned} v_p(e_i!) &= v_p((a_i p)!) \\ &= a_i. \end{aligned}$$

Now Equation (2.15) reduces to

$$\begin{aligned} v_p(s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)})) &= p + 1 - \sum_{i=1}^{t} a_i \\ &= p + 1 - k \\ &\geq p + 1 - (p-1) \quad \text{since } k \leq p-1 \\ &= 2. \end{aligned}$$

Thus, it follows that $v_p(s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)})) \geq 2$ if $p|e_i \;\; \forall \;\; 1 \leq i \leq t$.

Case 2.2: $p \nmid e_i$ **for some** i**,** $1 \leq i \leq t$

Let α be the number of $e_i's$ which are divisible by p.

Then, $0 \leq \alpha < t$.

If $\alpha = 0$, then each $e_i's$ are not divisible by p which means p divides each n_i and we can write $n_i = pm_i$ for each i. Therefore

$$\sum pm_i e_i = p^2 \Rightarrow \sum m_i e_i = p,$$

which implies $\sum e_i \leq p$ as each $m_i's$ are positive integers. This is a contradiction as $\sum e_i = kp$ with $k \geq 2$. Thus, we must have $\alpha > 0$.

Now, we re-arrange the index in such a manner that $p|e_i$ if $1 \leq i \leq \alpha$ and $p \nmid e_i$ if $\alpha < i \leq t$, which implies that $e_i = pb_i$ for some positive integer b_i for all $1 \leq i \leq \alpha$.

We also have $n_i = pm_i$ for some positive integer m_i and for all $\alpha + 1 \leq i \leq t$. It follows that

$$\begin{aligned} kp &= \sum_{i=1}^{t} e_i \\ &= \sum_{i=1}^{\alpha} e_i + \sum_{i=\alpha+1}^{t} e_i \\ &= \sum_{i=1}^{\alpha} pb_i + \sum_{i=\alpha+1}^{t} e_i, \end{aligned}$$

which implies that $p|\sum_{i=\alpha+1}^{t} e_i$. Since $\alpha < t$, and $e_i's$ are positive integers, we must have

$$\sum_{i=\alpha+1}^{t} e_i \geq p.$$

We also have

$$p^2 = \sum_{i=1}^{t} n_i e_i = \sum_{i=1}^{\alpha} n_i e_i + \sum_{i=\alpha+1}^{t} n_i e_i = p \sum_{i=1}^{\alpha} n_i b_i + p \sum_{i=\alpha+1}^{t} m_i e_i,$$

which implies that

$$\begin{aligned} p &= \sum_{i=1}^{\alpha} n_i b_i + \sum_{i=\alpha+1}^{t} m_i e_i \\ &\geq \sum_{i=1}^{\alpha} n_i b_i + \sum_{i=\alpha+1}^{t} e_i \qquad \text{(since } m_i's \text{ are positive)} \\ &\geq \sum_{i=1}^{\alpha} n_i b_i + p. \end{aligned}$$

Thus, we get

$$\sum_{i=1}^{\alpha} n_i b_i \leq 0,$$

which is a contradiction as each term is positive and $\alpha \neq 0$. Therefore, this case cannot happen.

We conclude that p^2 divides $s(n_1{}^{(e_1)}, n_2{}^{(e_2)}, n_3{}^{(e_3)}, ..., n_t{}^{(e_t)})$ for each case where $\sum_{i=1}^{t} n_i e_i = p^2$ and $\sum_{i=1}^{t} e_i = kp$.

So,

$$p^2|S(p^2, kp) \qquad \text{if } 2 \leq k \leq p-1.$$

□

The preceding theorem confirms that the lower bound of $v_p(S(p^2, kp))$ for $2 \leq k < p-1$ is 2, as mentioned in the Conjecture 2.3.1. The next theorem is a generalization of the above theorem.

Theorem 2.3.8. *Let p be an odd prime and k be an integer $2 \leq k \leq p-1$, then*

$$v_p(S(p^n, kp)) \geq 2$$

for any integer $n \geq 2$.

Proof. Replace $N = 2$ in Equation (5.4), we get

$$\pi(kp; p^2) = (p-1)p^b \qquad \text{if} \quad p^{b-1} < kp \leq p^b.$$

Since $2 \leq k \leq p-1$, we also have $p < kp < p^2$ and hence $b = 2$. Therefore,

$$\pi(kp; p^2) = (p-1)p^2.$$

It follows that

$$S(a + d(p-1)p^2, kp) \equiv S(a, kp) \pmod{p^2} \tag{2.16}$$

for every positive integer a and d.

Now, we prove the theorem by induction on n. The previous theorem states that our hypothesis is true for $n = 2$, i.e.,

$$v_p(S(p^2, kp)) \geq 2,$$

which can be written as

$$S(p^2, kp) \equiv 0 \pmod{p^2}.$$

Assume that the theorem holds for all $n \leq m$ for some positive integer $m \geq 2$ so that

$$v_p(S(p^n, kp)) \geq 2 \qquad \text{for all } 2 \leq n \leq m,$$

which implies

$$S(p^m, kp) \equiv 0 \pmod{p^2}.$$

Putting $a = p^m$ and $d = p^{m-2}$ in Equation (2.16), we get

$$S(p^{m+1}, kp) \equiv 0 \pmod{p^2}.$$

Thus the theorem is also true for $n = m + 1$.

It follows that the theorem is true for every integer $n \geq 2$. □

Theorem 2.3.9. *Let p an odd prime, then*

$$v_p(S(2p, p)) \geq 2.$$

Proof. Using Equation (2.1) and Theorem 2.3.1, we have

$$S(2p, p) = \sum_{\sum e_j = p, \sum n_j e_j = 2p} s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) \tag{2.17}$$

and

$$s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) = \frac{2p!}{\prod_{j=1}^{t} e_j!(n_j!)^{e_j}}$$

for some positive integer t.

It follows that

$$v_p(s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)})) = 2 - \sum_{j=1}^{t} v_p(e_j!) - \sum_{j=1}^{t} e_j v_p(n_j!).$$

Now we consider the following cases in Equation (2.17):

Case 1: $n_j < p$ and $e_j < p$ for every j

It is easy to see that $v_p(s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)})) = 2$ if each $e_j's$ and $n_j's$

are less than p and we get

$$s(n_1{}^{(e_1)}, n_2{}^{(e_2)}, n_3{}^{(e_3)}, ..., n_t{}^{(e_t)}) \equiv 0 \pmod{p^2} \tag{2.18}$$

if both e_j and n_j are less than p.

Case 2: $e_j \geq p$ **for some** j

We know that $\sum e_j = p$ which implies each $e_j's$ are less than p unless for the case $t = 1$, $e_1 = p$ so that $n_1 e_1 = 2p$ or $n_1 = 2$. In this case, the term is

$$s(2^{(p)}) = \frac{(2p)!}{p!(2!)^p}$$

and can be written as

$$\begin{aligned} \frac{s(2^{(p)})}{p} &= \frac{(p+1)(p+2)\cdots(p+p-1)}{2^{p-1}} \\ &\equiv (p-1)! \equiv -1 \pmod{p}, \end{aligned}$$

or

$$s(2^{(p)}) \equiv -p \pmod{p^2}. \tag{2.19}$$

Case 3: $n_j \geq p$ **for some** j

If $n_j \geq p$ for some j, then $e_j = 1$ due to $\sum e_j = p$ and $\sum n_j e_j = 2p$. The upper bound for the value of n_j is $p + 1$ since the remaining $2p - n_j$ objects cannot fill the remaining empty $p - 1$ subsets if $n_j > p + 1$.

Case 3.1: $n_j = p + 1$ **for some** j

If $n_j = p + 1$ for some j, all the remaining $p - 1$ subsets must contain a single object and the corresponding term for this case is $s((p+1)^{(1)}, 1^{(p-1)})$, i.e., $t = 2$, $n_1 = p + 1$, $e_1 = 1 = n_2$ and $e_2 = p - 1$. Then

$$s((p+1)^{(1)}, 1^{(p-1)}) = \frac{(2p)!}{(p-1)!(p+1)!},$$

which can also write as

$$\frac{s((p+1)^{(1)}, 1^{(p-1)})}{p} \equiv 2 \pmod p$$

or

$$s((p+1)^{(1)}, 1^{(p-1)}) \equiv 2p \pmod{p^2}. \tag{2.20}$$

Case 3.2: $n_j = p$ **for some** j

In this case, one subset contains p elements, one another subset contains two elements and remaining $p-2$ subsets must contain a single object. The corresponding term for this case is $s(p^{(1)}, 2^{(1)}, 1^{(p-2)})$, i.e., $t = 3$, $n_1 = p$, $e_1 = 1 = e_2 = n_3$, $n_2 = 2$ and $e_3 = p - 2$.

Using (2.8), we have

$$s(p^{(1)}, 2^{(1)}, 1^{(p-2)}) = \frac{(2p)!}{(p-2)!p!2!}$$

which reduces to

$$\frac{s(p^{(1)}, 2^{(1)}, 1^{(p-2)})}{p} \equiv -1 \pmod p$$

or

$$s(p^{(1)}, 2^{(1)}, 1^{(p-2)}) \equiv -p \pmod{p^2}. \tag{2.21}$$

Combining the results in (2.17), (2.18), (2.19), (2.20) and (2.21), we get

$$S(2p, p) \equiv 0 \pmod{p^2}.$$

This completes the proof. □

Theorem 2.3.10. *For any prime* $p \geq 5$*,*

$$v_p(S(2p, p-1)) = 1$$

or more specifically

$$S(2p, p-1) \equiv \frac{1}{6}p \pmod{p^2}.$$

Proof. We look into the following cases where p^2 does not divide $s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)})$ as in the preceding theorem:

1. $n_i = p$ for some i
2. $n_i = p + 1$ for some i
3. $n_i = p + 2$ for some i.

In the first case, there are two possible terms namely, $s(p^{(1)}, 3^{(1)}, 1^{(p-3)})$ and $s(p^{(1)}, 2^{(2)}, 1^{(p-4)})$. So

$$s(p^{(1)}, 3^{(1)}, 1^{(p-3)}) = \frac{(2p)!}{(p-3)!p!3!} \equiv \frac{2}{3}p \pmod{p^2}$$

and

$$s(p^{(1)}, 2^{(2)}, 1^{(p-4)}) = \frac{(2p)!}{2!(p-4)!p!(2!)^2} \equiv -\frac{3}{2}p \pmod{p^2}.$$

For the second case, the only possible term is $s((p+1)^{(1)}, 2^{(1)}, 1^{(p-3)})$ and

$$s((p+1)^{(1)}, 2^{(1)}, 1^{(p-3)}) = \frac{(2p)!}{(p-3)!(p+1)!2!} \equiv 2p \pmod{p^2}.$$

The final case also contains only one term, $s((p+2)^{(1)}, 1^{(p-2)})$ and

$$s((p+2)^{(1)}, 1^{(p-2)}) = \frac{(2p)!}{(p-2)!(p+2)!} \equiv -p \pmod{p^2}.$$

Thus, we have

$$\begin{aligned} S(2p, p-1) &\equiv \frac{2}{3}p - \frac{3}{2}p + 2p - p \pmod{p^2} \\ &\equiv \frac{1}{6}p \pmod{p^2}. \end{aligned}$$

This completes the proof. □

Using the results of minimum periods in Equation (5.4) and exploiting the same technique as in the proof of Theorem 2.3.8, we generalize Theorem 2.3.9 and Theorem 2.3.10 as

Theorem 2.3.11. *Let p an odd prime, then*

$$v_p(S(2p^n, p)) \geq 2.$$

Theorem 2.3.12. *For any prime $p \geq 5$,*

$$v_p(S(2p^n, p-1)) = 1$$

or more specifically

$$S(2p^n, p-1) \equiv \frac{1}{6}p \pmod{p^2}.$$

The proofs of Theorems 2.3.11 and 2.3.12 are similar to the proofs of Theorems 2.3.9 and 2.3.10 respectively.

Theorem 2.3.13. *For any odd prime p,*

$$v_p(S(2p, p+1)) = 0 \tag{2.22}$$

or

$$S(2p, p+1) \equiv 2 \pmod{p^2}. \tag{2.23}$$

Equation (2.22) is a special case of (2.4) since $S(2p, p+1)$ is a minimum zero case. Hence

$$v_p(S(2p, p+1)) = \frac{s_p(p+1) - s_p(2p)}{p-1} = 0,$$

where $s_p(n)$ is the sum of p-adic digits of n.

Using Equation (2.5), we can also say that

$$v_p(S(2p^{n+1}, (p+1)p^n)) = 0$$

for any positive integer n.

The second result (2.23) can be obtained using the same method as in Theorem 2.3.10.

Theorem 2.3.14. *For any odd prime p,*

$$v_p(S(2p, p+2)) \geq 1$$

or

$$S(2p, p+2) \equiv 2^p - 2 \pmod{p^2}.$$

Proof. There are two cases where p^2 does not divide $s({n_1}^{(e_1)}, {n_2}^{(e_2)}, \cdots, {n_t}^{(e_t)})$. The first case is $s(1^{(p)}, i^{(1)}, (p-i)^{(1)})$ for $2 \leq i \leq (p-1)/2$ and

$$s(1^{(p)}, i^{(1)}, (p-i)^{(1)}) = \frac{(2p)!}{p!i!(p-i)!} \equiv 2\binom{p}{i} \pmod{p^2}.$$

It follows that

$$\sum_{i=2}^{\frac{p-1}{2}} s(1^{(p)}, i^{(1)}, (p-i)^{(1)}) \equiv 2^p - 2 - 2p \pmod{p^2}.$$

The second case is $s(1^{(p+1)}, (p-1)^{(1)})$ and

$$s(1^{(p+1)}, (p-1)^{(1)}) = \frac{(2p)!}{(p+1)!(p-1)!} \equiv 2p \pmod{p^2}.$$

Now, we have

$$S(2p, p+2) \equiv 2^p - 2 - 2p + 2p \equiv 2^p - 2 \pmod{p^2}.$$

This completes the proof. □

It is well-known that $2^p - 2$ is always divisible by p using Fermat's theorem. The result for mod p^2 is, however, not known in general. Numerical evidence suggests that there are some primes p greater than 1000 where p^2 divides $2^p - 2$. So, this leads to an interesting problem in finding out those primes p such that $v_p(S(2p, p+2)) \neq 1$ or, equivalently, $p^2 \nmid 2^p - 2$.

2.4 Conclusions

This chapter introduces an alternate formula for evaluating Stirling numbers of the second kind, $S(n,k)$. This formula is used to determine the lower bound of the p-adic valuations of Stirling numbers of the second kind of the class $S(p^2, kp)$, where p is an arbitrary odd prime and k is a positive integer such that $2 \leq k \leq p-1$. Some generalized results for the p-adic valuation of $S(p^n, kp)$, $S(2p^{n+1}, (p+1)p^n)$ and $S(2p^n, p)$ are also proved using minimum periods. The estimated values of the p-adic valuation for $S(2p, p-1)$, $S(2p, p)$, $S(2p, p+1)$ and $S(2p, p+2)$ are also obtained.

Chapter 3

Some Congruence Properties of Stirling Numbers of the Second Kind[2]

3.1 Introduction

An interesting congruence relation between Stirling numbers of the second kind and binomial coefficients is given by Chan and Manna (2010) as

$$\begin{aligned} S(n, a2^m) \equiv\ & a2^{m-1}\binom{\lfloor \frac{n-1}{2} \rfloor - a2^{m-2} - 1}{\lfloor \frac{n-1}{2} \rfloor - a2^{m-1}} \\ & + \frac{1+(-1)^n}{2}\binom{\frac{n}{2} - a2^{m-2} - 1}{\frac{n}{2} - a2^{m-1}} \pmod{2^m}, \end{aligned}$$

where n, a, and m are positive integers such that $m \geq 3$ and $n \geq a2^m + 1$. From the above congruence, we can easily verify that $v_2(S(4n+3, k)) \geq v_2(k)$ for any two positive integers n and k such that $k < 4n+3$ and $v_2(k) \geq 3$. The generalizations of the result to modulo p^m for odd prime p are discussed in the second and third sections of this chapter.

In this chapter, we obtain the p-adic valuations of Stirling numbers of the second kind using congruence property. The main results include the congruence

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recursion of $S(n,k) \pmod p$ for different cases of k, equivalence of $S(n,kp^m)$ in terms of binomial coefficient for opposite parity of n and k, congruence recurrence relation of $S(n+kp,kp)$ for different conditions of n and the lower bounds of $v_p(S(p^n-1,kp))$ and $v_p(S(p^n,kp))$. We confirm the results that $S(p^n,k) \equiv S(p,k) \pmod{p^2}$ if $1 \le k \le p$ and $S(p^2,k) \equiv \binom{p}{k_1} S(p-k_1,k_0) \pmod{p^2}$ if $k = k_1p + k_0$ and $k_0 \neq 0$.

3.2 Preliminaries

In this section, we provide the necessary background material to state and prove our main results in the next section. Throughout this chapter, p denotes an odd prime number. Whenever $p-1$ divides $n-k$, we denote the binomial coefficient $\binom{\frac{n-k}{p-1}-1}{\frac{n-kp}{p-1}}$ by $A_{k,n,p}$ for convenience. If m, n, and k are positive integers such that $p \nmid k$ and $n \ge kp^m$, then the following holds (Chan and Manna, 2010)

$$S(n,kp^m) \equiv \begin{cases} A_{kp^{m-1},n,p} \pmod{p^m}, & \text{if } n \equiv k \pmod{p-1}; \\ 0 \pmod{p^m}, & \text{otherwise.} \end{cases} \tag{3.1}$$

Sagan (1985) obtained the following congruence using group action on abelian groups:

$$S(n+2p,k) \equiv \sum_{i=0}^{1} S(n+p+i,k+(i-1)p) - \sum_{i=0}^{2} \binom{2}{i} S(n+i,k+(i-2)p) + p(p-1)S(n,k-p) \pmod{p^2};\ n>0,\ n+2p \ge k. \tag{3.2}$$

Observe that on eliminating the terms containing $k-p$ and $k-2p$ in (3.2), we have

$$S(n,k) \equiv 2S(n-p+1,k) - S(n-2p+2,k) \pmod{p^2};\ k \le p,\ n > 2p. \tag{3.3}$$

Consequently, using induction, one arrives at the following:

$$S(n,k) \equiv (r+1)S(n-r(p-1),k) - rS(n-(r+1)(p-1),k) \pmod{p^2} \quad (3.4)$$

if $n-(r+1)(p-1) > 2$.

From Equations (3.3) and (3.4), we get

$$S(n,k) \equiv \begin{cases} 2S(n-p+1,k) \pmod{p^2}, & \text{if } 2p < n < 2p+k-2; \\ (r+1)S(n-r(p-1),k) \pmod{p^2}, & \text{if } 2 < n-(r+1)(p-1) < k. \end{cases} \quad (3.5)$$

Feng and Qiu (2020) employed a combinatorial approach and proved the following result:

$$v_p(S(n,n-k)) = v_p\left(\binom{n}{k+1}\right) + t_p(n,k); \; n \geq k+1, \quad (3.6)$$

where

$$t_p(n,k) = \begin{cases} 0, & \text{if } k=1; \\ v_p(3n-5) - v_p(4), & \text{if } k=2; \\ v_p(n^2-5n+6) - v_p(2), & \text{if } k=3; \\ v_p(15n^3-150n^2+485n-502) - v_p(48), & \text{if } k=4; \\ v_p(3n^4-50n^3+305n^2-802n+760) - v_p(16), & \text{if } k=5; \\ v_p(63n^5-1575n^4+1543n^3-73801n^2+171150n & \\ \qquad -156296) - v_p(576), & \text{if } k=6. \end{cases} \quad (3.7)$$

3.3 Main Results

This section is divided into various cases. We first divide into divisibility of $S(n,k)$ by p and p^n in general. We further divide into divisibility of k by p. We begin by providing the following results;

Theorem 3.3.1. *For an odd prime p and an integer n, we have*

a) $v_p(S(p^n-1, kp-1)) \geq 2$; $2 \leq k < p-1$; $v_p(S(p^n-1,(p-1)p-1)) = 1$.

b) $S(p+n, k) \equiv S(n+1, k) + S(n, k-p) \pmod{p}$.

Proof. (a) Using Equation (1.33) and the fact that p^2 divides $S(p^n, kp)$, we have

$$S(p^n-1, kp-1) = S(p^n, kp) - kpS(p^n-1, kp) \equiv kpS(p^n-1, kp) \pmod{p^2}. \quad (3.8)$$

So, it is enough to prove that p divides $S(p^n-1, kp)$. Taking $m = 1$, $n = p^n - 1$ in (3.1), we get

$$S(p^n-1, kp) \equiv \begin{cases} A_{kp,p^n,p} \pmod{p}, & \text{if } k = p-1; \\ 0 \pmod{p}, & \text{if } 2 \leq k \leq p-2, \end{cases} \quad (3.9)$$

where from (3.9), the result (a) follows when $2 \leq k \leq p-2$. To prove (a) for the case when $k = p-1$, we observe using Lucas congruence for $n \geq 2$ that $A_{(p-1)p,p^n-1,p} \equiv -1 \pmod{p}$. Consequently, from (3.9), we have

$$S(p^n-1, (p-1)p) \equiv -1 \pmod{p}. \quad (3.10)$$

Equation (3.10), in view of (3.8), proves that $v_p(S(p^n-1,(p-1)p-1)) = 1$.

(b) From Equation (1.40), we get

$$S(p+n, k) = \sum_{i=0}^{k}\sum_{j=i}^{k}\binom{j}{i}\frac{(k-i)!}{(k-j)!}S(p, k-i)S(n, j). \quad (3.11)$$

The terms within the summation in (3.11), except those with indices such that

$$(i, j) \in \{(k-1, k-1), (k-1, k), (k-p, k-p)\},$$

are all divisible by p. This observation, along with (1.33), gives

$$\begin{aligned} S(p+n, k) &\equiv S(n, k-1) + kS(n, k) + S(n, k-p) \pmod{p} \\ &\equiv S(n+1, k) + S(n, k-p) \pmod{p}. \end{aligned}$$

Thus, the result (b) follows. □

3.3.1 Divisibility of $S(n, k)$ by p

Chan and Manna (2010) obtained a congruence for $S(n, k)$ when k is divisible by p and not divisible by p. The result when k is divisible by p is simple for acquiring the divisibility of $S(n, k)$. We further look into the case when k is not a multiple of p, say $k = cp^m + b$, where $b \neq 0$ and $p \nmid b$.

We will utilize the following result while proving Theorem 3.3.2.

Lemma 3.3.1. *Let p be a prime and n and k be two positive integers such that $n > 0$ and $k \leq p - 1$, then there exists a positive integer $1 \leq m < p - 1$ such that*

$$S(n,k) \equiv \begin{cases} S(m,k) \pmod{p}, & \text{if } n \not\equiv 0 \pmod{p-1}; \\ (p-1-k)! \pmod{p}, & \text{if } n \equiv 0 \pmod{p-1}, \end{cases} \tag{3.12}$$

where m is the remainder when n is divided by p.

The following theorem is a generalization of Lemma 3.3.1 in which k is restricted to an integer less than or equal to $p - 1$ for a given prime p. This theorem, however, provides the congruence for $S(n, k)$ modulo p for any integer k less than or equal to n.

Theorem 3.3.2. *For an odd prime p and integer k with $p \nmid k$, let b be the last p-adic digit of k. Let $m = v_p(k - b)$, $c = (k - b)p^{-m}$, and a is the remainder when $n - k$ is divided by $p - 1$. Then*

$$S(n,k) \equiv \begin{cases} S(a,b)\dbinom{\lfloor \frac{n-k}{p-1} \rfloor + cp^{m-1}}{cp^{m-1}} \pmod{p}, & \text{if } n \not\equiv c \pmod{p-1}; \\ (p-1-b)!\dbinom{\lfloor \frac{n-k}{p-1} \rfloor + cp^{m-1}}{cp^{m-1}} \pmod{p}, & \text{otherwise.} \end{cases}$$

Proof. The result for $k < p$ is trivial since $c = 0$. Due to Chan and Manna (2010, Theorem 5.3), we have for $m \geq 1$ and $n \geq cp^m + b$ that

$$\begin{aligned} S(n, cp^m + b) &\equiv \sum_{i \equiv c \pmod{p-1}}^{n} S(i, cp^m) S(n-i, b) \pmod{p^m} \\ &\equiv \sum_{i \equiv c \pmod{p-1}}^{n} \binom{\frac{i - cp^{m-1}}{p-1} - 1}{\frac{i - cp^m}{p-1}} S(n-i, b) \pmod{p^m}. \end{aligned} \tag{3.13}$$

The index i in the last summation runs through $i \equiv c \pmod{p-1}$; so $i = c + (p-1)j$ for some j with $cp^m \leq c + (p-1)j$ and $b \leq n - c - (p-1)j$. If we define $A = \lfloor \frac{n - cp^m - b}{p-1} \rfloor$, then (3.13) reduces to the form:

$$\begin{aligned} S(n, cp^m + b) &\equiv \sum_{i=0}^{A} \binom{\frac{cp^m + i(p-1) - cp^{m-1}}{p-1} - 1}{\frac{cp^m + i(p-1) - cp^m}{p-1}} S(n - cp^m - i(p-1), b) \pmod{p^m} \\ &\equiv \sum_{i=0}^{A} \binom{cp^{m-1} + i - 1}{i} S(n - cp^m - i(p-1), b) \pmod{p^m}. \end{aligned} \tag{3.14}$$

If $1 \leq b \leq p-1$, then by Lemma 3.3.1, there exists an integer a such that

$$S(n - cp^m - i(p-1), b) \equiv \begin{cases} S(a, b) \pmod{p}, \text{ if } n \not\equiv c \pmod{p-1}; \\ (p-1-b)! \pmod{p}, \text{ otherwise.} \end{cases} \tag{3.15}$$

Here, a is the remainder when $n - cp^m - i(p-1)$ is divided by $p-1$, that is, the remainder when $n - c$ is divided by $p-1$. So, for $n \not\equiv c \pmod{p-1}$, we have

$$S(n, cp^m + b) \equiv \sum_{i=0}^{A} \binom{cp^{m-1} + i - 1}{i} S(a, b) \equiv S(a, b) \binom{A + cp^{m-1}}{cp^{m-1}} \pmod{p},$$

and for the other case, that is, when $n \equiv c \pmod{p-1}$, we have

$$S(n, cp^m + b) \equiv (p-1-b)! \binom{A + cp^{m-1}}{cp^{m-1}} \pmod{p},$$

as desired. □

Remark 3.3.1. *Taking $n = p^2$ in the proof of Theorem 3.3.2, we see that $A =$*

$p-c$. Hence, for an odd prime p and $k > p$ with $p \nmid k$, we have $v_p(S(p^2, k)) \geq 1$, which can also be deduced using Equation (1.62)*.*

3.3.2 Divisibility of $S(n, k)$ by p^m with $p \mid k$

The following theorem extends the result of Chan and Manna (2010, Theorem 5.2) when n and k of $S(n, kp^m)$ are of opposite parity.

Theorem 3.3.3. *If p is an odd prime and n and k are of opposite parity, then*

$$S(n, kp^m) \equiv \begin{cases} (-1)^{n-1}\dfrac{nk}{2}A_{kp^{m-1},n-1,p}p^m \pmod{p^{2m}}, & \text{if } n-1 \equiv k \pmod{p-1}; \\ 0 \pmod{p^{2m}}, & \text{otherwise.} \end{cases} \tag{3.16}$$

Proof. Using Equation (1.32) and the hypothesis of parity and $k \equiv kp^m \pmod 2$, we have

$$\begin{aligned}
2(kp^m)!S(n, kp^m) &= \sum_{i=0}^{kp^m} \binom{kp^m}{i}(-1)^i \left((-1)^k i^n + (kp^m - i)^n\right) \\
&= \sum_{i=0}^{kp^m} \binom{kp^m}{i}(-1)^i \Big\{(-1)^k i^n + (-1)^n i^n \\
&\quad + \sum_{j=1}^{n} \binom{n}{j}(-1)^{n-j}k^j i^{n-j}p^{mj}\Big\} \\
&= \sum_{i=0}^{kp^m} \binom{kp^m}{i}(-1)^i \sum_{j=1}^{n} \binom{n}{j}(-1)^{n-j}k^j i^{n-j}p^{mj} \\
&= \sum_{j=1}^{n} \binom{n}{j}(-1)^{n-j}k^j p^{mj}(kp^m)!S(n-j, kp^m),
\end{aligned} \tag{3.17}$$

where we have used $(-1)^k i^n + (-1)^n i^n = 0$. Thus, $S(n, kp^m) \equiv 0 \pmod{p^m}$ if n and k are opposite parity. It then follows from (3.17) that

$$2S(n, kp^m) \equiv \sum_{j=1}^{t} \binom{n}{j}(-1)^{n-j}k^j p^{mj} S(n-j, kp^m) \pmod{p^{m(t+1)}} \tag{3.18}$$

holds for $1 \leq t \leq n$. Since n and k are of opposite parity, so are $n-2$ and

k. Consequently, $S(n-2, kp^m) \equiv 0 \pmod{p^m}$. This observation together with (3.18) for $t = 2$ gives us the following;

$$\begin{aligned} 2S(n, kp^m) &\equiv (-1)^{n-1} nkp^m S(n-1, kp^m) \\ &+ \binom{n}{2}(-1)^{n-2}k^2p^{2m}S(n-2, kp^m) \pmod{p^{3m}} \\ &\equiv (-1)^{n-1} nkp^m S(n-1, kp^m) \pmod{p^{3m}}, \end{aligned} \tag{3.19}$$

which is also true for modulo p^{2m}. Thus, applying Equation (3.1) to $S(n-1, kp^m)$ and combining it with Equation (3.19) produces Equation (3.16). □

Corollary 3.3.1. *For an odd prime p and two positive integers n and k,*

(a) If k is even and $p^n \geq kp^m$, then $S(p^n, kp^m) \equiv 0 \pmod{p^{2m}}$.

(b) If n and k are of opposite parity such that $s_p(kp^{m-1} + \alpha - 1) = s_p(k) + s_p(\alpha - 1)$; $n - 1 \equiv k \pmod{p-1}$; and $\alpha = \frac{n-1-kp^m}{p-1}$, then $v_p(S(n, kp^m)) = 2m - v_p(\alpha) - 1$.

Proof. (a) Follows from Equation (3.16).

(b) If $s_p(x + y) = s_p(x) + s_p(y)$, then by Kummer's theorem (see Mihet, 2010), $v_p\left(\binom{x+y}{y}\right) = v_p(x) - v_p(y + 1)$, which in view of (3.16) proves (b). □

Remark 3.3.2. *If k is even, then replacing n by p^n in (3.18), we get*

$$2S(p^n, kp^m) = \sum_{i=1}^{p^n} \binom{p^n}{i}(-1)^{p^n-i}k^ip^{mi}S(p^n - i, kp^m). \tag{3.20}$$

The i-th term within the summation in (3.20) is divisible by p^{n+mi} if $p \nmid i$. However, if $p \mid i$, then the corresponding term within the summation in (3.20) is divisible by $p^{n-v_p(i)+mi}$. Then for all t with $1 \leq t \leq p^n$ with $p \nmid (t + 1)$, we have the following key congruence:

$$2S(p^n, kp^m) \equiv \sum_{i=1}^{t} \binom{p^n}{i}(-1)^{p^n-i}k^ip^{mi}S(p^n - i, kp^m) \pmod{p^{(t+1)m+n}}. \tag{3.21}$$

Theorem 3.3.4. *Let p be an odd prime. Let m, n, and k be positive integers such that $n > m$, k is even, and $p \nmid k$. Then $v_p(S(p^n, kp^m)) \geq n + 2m$, unless $m = 1$ and $k = p - 1$, in which case $v_p(S(p^n, (p-1)p)) = n + 1$.*

Proof. Taking $t = 1$ in (3.21), we have for even k that

$$2S(p^n, kp^m) \equiv kp^{n+m} S(p^n - 1, kp^m) \pmod{p^{2m+n}}, \tag{3.22}$$

where we have

$$S(p^n - 1, kp^m) \equiv \begin{cases} A_{kp^{m-1}, p^n-1, p} \pmod{p^m}, & \text{if } k = p - 1; \\ 0 \pmod{p^m}, & \text{if } 1 \leq k \leq p - 2. \end{cases} \tag{3.23}$$

The binomial coefficient on the right-hand side follows

$$A_{kp^{m-1}, p^n-1, p} \equiv \begin{cases} -1 \pmod{p}, & \text{if } m = 1 < n; \\ 0 \pmod{p^m}, & \text{if } 2 \leq m < n. \end{cases} \tag{3.24}$$

Using (3.24) in (3.23) and then (3.23) in (3.22) proves the desired assertion. □

Theorem 3.3.5. *If $p > 3$ is prime; n, m, k are positive integers with $m < n$ and $k < p$, then*

$$S(p^n, kp^m) \equiv \begin{cases} \frac{kp^{n+1}}{2} S(p^n - 1, kp) - \frac{9p^{n+3}}{4} \pmod{p^{n+4}}, & \text{if } m = 1,\ k = p - 3; \\ \frac{kp^{n+m}}{2} S(p^n - 1, kp^m) \pmod{p^{4m+n}}, & \text{otherwise.} \end{cases} \tag{3.25}$$

Proof. Taking $t = 2$ in (3.21) for even k, we get

$$\begin{aligned} 2S(p^n, kp^m) &\equiv kp^{n+m} S(p^n - 1, kp^m) \\ &\quad - p^{n+2m} \binom{p^n - 1}{2} k^2 S(p^n - 2, kp^m) \pmod{p^{3m+n}}. \end{aligned}$$

Since $p^n - 2$ and k are opposite parity, $S(p^n - 2, kp^m) \equiv 0 \pmod{p^m}$, and so

$$2S(p^n, kp^m) \equiv kp^{n+m} S(p^n - 1, kp^m) \pmod{p^{3m+n}}. \tag{3.26}$$

Similarly, on taking $t = 3$ in (3.21), we have for $p > 3$ that

$$2S(p^n, kp^m) \equiv kp^{n+m}S(p^n - 1, kp^m) - \frac{1}{2}(p^n - 1)k^2p^{n+2m}S(p^n - 2, kp^m)$$
$$+ \frac{1}{6}(p^n - 1)(p^n - 2)k^3p^{n+3m}S(p^n - 3, kp^m) \pmod{p^{n+4m}}. \quad (3.27)$$

If k is even, $k \neq p - 3$, and $1 \leq k \leq p - 1$, then by Theorem 3.3.3 and (3.27), we have

$$S(p^n, kp^m) \equiv \frac{k}{2}p^{n+m}S(p^n - 1, kp^m) \pmod{p^{4m+n}}. \quad (3.28)$$

For the case $k = p - 3$, we have from Theorem 3.3.3 that

$$S(p^n - 2, (p-3)p^m) \equiv \begin{cases} -3p \pmod{p^2}, & \text{if } m = 1; \\ 0 \pmod{p}^{2m}, & \text{if } m > 1. \end{cases} \quad (3.29)$$

Also, from (3.1), we have

$$S(p^n - 3, (p-3)p^m) \equiv \begin{cases} -1 \pmod{p}, & \text{if } m = 1; \\ 0 \pmod{p^m}, & \text{if } m > 1. \end{cases} \quad (3.30)$$

Combining (3.27)–(3.30), we get (3.25). □

Remark 3.3.3. *Identity* (1.39) *gives rise to the following relation;*

$$\binom{kp}{p}S(kp + n, kp) = \sum_{i=0}^{n} \binom{kp + n}{(k-1)p + i}$$
$$\times S((k-1)p + i, (k-1)p)S(n + p - i, p). \quad (3.31)$$

If $n = t(p - 1)$, *then the* i*-th term within the summation in* (3.31) *is divisible by* p^2 *in case* $i \not\equiv 0 \pmod{p-1}$ *since* p *divides both* $S((k-1)p + i, (k-1)p)$ *and* $S(n + p - i, p)$. *On the other hand, if* $i \equiv 0 \pmod{p-1}$ *and* $0 \neq i \neq n$, *then* p *divides* $\binom{kp+n}{(k-1)p+i}$. *Using Equation* (3.1) *for* $i \equiv 0 \pmod{p-1}$, *we get the* *following equations:*

$$\begin{aligned} S((k-1)p+i,(k-1)p) &\equiv \binom{k-2+\frac{i}{p-1}}{\frac{i}{p-1}} \pmod p, \\ S(n+p-i(p-1),p) &\equiv 1 \pmod p. \end{aligned}$$

We also know that $\binom{kp}{p} \equiv k \pmod{p^2}$ *due to Equation* (1.27)*. It follows that*

$$\begin{aligned} kS(kp+n,kp) \equiv & \binom{kp+n}{(k-1)p} S(n+p,p) \\ & + \binom{kp+n}{(k-1)p+n} S((k-1)p+n,(k-1)p) \\ & + \sum_{i=1}^{t-1} \binom{(k+t)p-t}{(t-i)p+p-t+i} \binom{k-2+i}{i} \pmod{p^2} \quad (3.32) \end{aligned}$$

for $n = t(p-1)$*.*

Theorem 3.3.6. *If* p *is an odd prime,* $0 \le k+t < p$*,* $n = tp+j$*, and* $0 \le j < p-t-1$*, then*

$$\begin{aligned} S(kp+n,kp) \equiv & \frac{r}{k}\binom{k+t}{k-1} S(n+p,p) \\ & + \frac{\binom{k+t}{r}}{\binom{k}{r}} S((k-r)p+n,(k-r)p) \pmod{p^2} \quad (3.33) \end{aligned}$$

for $1 \le r \le k-1$*.*

Proof. We analyze (3.31) for the case when $(p-1) \nmid n$ and $t(p-1) < n < (t+1)(p-1)$. If $i \equiv 0 \pmod{p-1}$, then $S(n+p-i,p) \equiv 0 \pmod p$ but $S((k-1)p+i,(k-1)p) \equiv \binom{k-2+\frac{i}{p-1}}{\frac{i}{p-1}} \pmod p$. On the other hand, if $n \equiv i \pmod{p-1}$, then $S((k-1)p+i,(k-1)p) \equiv 0 \pmod p$ and $S(n+p-i,p) \equiv 1 \pmod p$. The rest of the terms where $i \not\equiv 0 \pmod{p-1}$ and $n \not\equiv i \pmod{p-1}$

are divisible by p^2. It follows that

$$\binom{kp}{p} S(kp+n, kp) \equiv \sum_{i=0}^{t} \binom{kp+n}{(k-1)p+i(p-1)} \binom{k-2+i}{i} S(n+p-i(p-1), p)$$
$$+ \sum_{i=0}^{t} \binom{kp+n}{(k-1)p+n-i(p-1)} S((k-1)p+n-i(p-1), (k-1)p) \pmod{p^2}. \tag{3.34}$$

Further, we restrict n to $n = tp + j$, $0 \leq j < p - t - 1$, and $0 \leq k + t < p$. In this case, the binomial coefficients in both sums of the right-hand side of (3.34) are divisible by p because of Lucas congruence. So, all the terms except when $i = 0$ in both summations are divisible by p^2. Equation (3.34) thus reduces to the form

$$\binom{kp}{p} S(kp+n, kp) \equiv \binom{kp+n}{(k-1)p} S(n+p, p)$$
$$+ \binom{kp+n}{(k-1)p+n} S((k-1)p+n, (k-1)p) \pmod{p^2}. \tag{3.35}$$

If we also apply Lucas congruence to the binomial coefficients, we have

$$kS(kp+n, kp) \equiv \binom{k+t}{k-1} S(n+p, p)$$
$$+ (k+t) S((k-1)p+n, (k-1)p) \pmod{p^2}. \tag{3.36}$$

The theorem follows by using induction on r together with Equation (3.36). □

Corollary 3.3.2. *If p is an odd prime, $0 \leq k + t < p$, $n = tp + j$, and $0 \leq j < p - t - 1$, then*

$$S(kp+n, kp) \equiv \binom{t+k}{t+1} S(p+n, p) \pmod{p^2}. \tag{3.37}$$

Proof. Take $r = k - 1$ in (3.31). □

Theorem 3.3.7. *Let p be an odd prime, k, t, and n be positive integers with*

$4 \le k+2 \le k+t < p$ and $n = tp-1$. Then for $1 \le r \le k-1$, we have

$$S(kp+n,kp) \equiv \frac{r}{k}\binom{k-1+t}{k-1}S(n+p,p) + \frac{\binom{k+t-1}{r}}{\binom{k}{r}}S((k-1)p+n,(k-1)p)$$
$$+\frac{2rk-r(r+1)}{k(k-1)}\binom{k-1+t}{k-2}\sum_{i=1}^{t}(-1)^{i-1}\binom{t+1}{i}S(n+p-i(p-1),p) \pmod{p^2}. \tag{3.38}$$

Proof. We analyze (3.34) for the case when $n = tp-1$, $t+k \le p$, and $t \ge 2$ so that $(p-1) \nmid (tp-1)$. In this case, $S(n+p-i(p-1),p) \equiv 0 \pmod p$ and

$$\binom{kp+n}{(k-1)p+i(p-1)} \equiv (-1)^{i-1}\binom{k-1+t}{k-2+i} \pmod p,\ i \ne 0,$$
$$\binom{kp+n}{(k-1)p} \equiv \binom{k-1+t}{k-1} \pmod p \text{ when } i=0.$$

Also, $S((k-1)p+n-i(p-1),(k-1)p) \equiv 0 \pmod p$, $\binom{kp+n}{(k-1)p+n-i(p-1)} \equiv (-1)^{i-1}\binom{k-1+t}{i} \pmod p$ if $i \ne 0$ and $\binom{kp+n}{(k-1)p+n} \equiv k+t-1 \pmod p$ for $i=0$. Consequently, (3.34) reduces to

$$kS(kp+n,kp) \equiv \binom{k-1+t}{k-1}S(n+p,p) + (k+t-1)S((k-1)p+n,(k-1)p)$$
$$+\sum_{i=1}^{t}(-1)^{i-1}\binom{k-1+t}{k-2+i}\binom{k-2+i}{i}S(n+p-i(p-1),p)$$
$$+\sum_{i=1}^{t}(-1)^{i-1}\binom{k-1+t}{i}S((k-1)p+n-i(p-1),(k-1)p) \pmod{p^2}. \tag{3.39}$$

By Corollary 3.3.2, we have

$$S((k-1)p+n-i(p-1),(k-1)p) \equiv \binom{k-1+t-i}{t-i+1}S(p+n-i(p-1),p) \pmod{p^2}. \tag{3.40}$$

Combining (3.39) and (3.40), we get

$$S(kp+n,kp) \equiv \frac{1}{k}\binom{k-1+t}{k-1}S(n+p,p) + \frac{k+t-1}{k}S((k-1)p+n,(k-1)p) + \frac{2}{k}\binom{k-1+t}{k-2}\sum_{i=1}^{t}(-1)^{i-1}\binom{t+1}{i}S(n+p-i(p-1),p) \pmod{p^2}. \tag{3.41}$$

The theorem follows by using induction on r and utilizing the preceding congruence (3.41). □

Corollary 3.3.3. *Let p be an odd prime, k, t, and n be positive integers such that $4 \le k+2 \le k+t < p$ and $n = tp-1$. Then*

$$S(kp+n,kp) \equiv \binom{t+k-1}{k-1}S(p+n,p) + \sum_{i=1}^{t} b_i S(p+n-i(p-1),p) \pmod{p^2}, \tag{3.42}$$

where $b_i = (-1)^{i-1}\frac{(t+k-1)!}{(t-i+1)!(k-2)!i!}$.

Proof. Take $r = k-1$ in (3.41). □

Theorem 3.3.8. *If $p > 3$ is a prime and $2 \le k < p-1$, where k is even, then $v_p(S(p^2-1,kp)) \ge 2$.*

Proof. For an even integer k with $2 \le k < p-1$, letting $n = p^2-kp-1$ and $t = p-k$ in Theorem 3.3.7 generates the following relation:

$$S(p^2-1,kp) \equiv (-1)^{k-1}S(p^2+p-kp-1,p) - \sum_{i=1}^{p-k}\binom{p-1-i}{k-2}S(p^2+p-kp-1-i(p-1),p) \pmod{p^2}. \tag{3.43}$$

Replacing k, n, and r by p, $p+p^2-kp-1$, and $p-k-1$, respectively, in (3.4), we get

$$S(p+p^2-kp-1,p) \equiv (p-k)S(3p-k-2,p) \\ -(p-k-1)S(2p-k-1,p) \pmod{p^2}. \quad (3.44)$$

Similarly, for $1 \leq i < p-k-1$, we have

$$S(p+p^2-kp-1-i(p-1),p) \equiv (p-k-i)S(3p-k-2,p) \\ -(p-k-i-1)S(2p-k-1,p) \pmod{p^2}. \quad (3.45)$$

The preceding three congruences together lead to the following:

$$\begin{aligned} S(p^2-1,kp) \equiv & (-1)^{k-1}[(p-k)S(3p-k-2,p)-(p-k-1)S(2p-k-1,p)] \\ & -(k-1)S(2p-k-1,p)-\binom{k}{2}S(3p-k-2,p) \\ & -\sum_{i=1}^{p-k-2}\binom{p-1-i}{k-2}(p-k-i)S(3p-k-2,p) \\ & +\sum_{i=1}^{p-k-2}\binom{p-1-i}{k-2}(p-k-i-1)S(2p-k-1,p) \pmod{p^2}. \end{aligned} \quad (3.46)$$

Now using the identities $\sum_{i=0}^{y}\binom{x+i}{x}=\binom{x+y+1}{x+1}$ and $\binom{p-1-i}{k-2}\equiv\binom{k+i-2}{k-2} \pmod{p}$ for even k, it follows that

$$\sum_{i=1}^{p-k}\binom{p-1-i}{k-2}(p-k-i) \equiv k+1 \pmod{p}, \quad (3.47)$$

$$\sum_{i=1}^{p-k}\binom{p-1-i}{k-2}(p-k-i-1) \equiv k+3 \pmod{p}. \quad (3.48)$$

From Equations (3.46) and (3.47), we get

$$S(p^2-1,kp) \equiv 2S(2p-k-1,p)-S(3p-k-2,p) \pmod{p^2}. \quad (3.49)$$

Since $2p < 3p - k - 2 < 3p - 2$, we obtain from Equation (3.5) that

$$S(3p - k - 2, p) \equiv 2S(2p - k - 1, p) \pmod{p^2}, \tag{3.50}$$

and the theorem follows. □

The following theorem settles the lower bound of $v_p(S(p^2, kp))$ for even k in Conjecture 2.3.1.

Theorem 3.3.9. *If $p > 3$ is a prime and k is even with $2 \leq k < p - 1$, then*

$$v_p(S(p^2, kp)) \geq 5. \tag{3.51}$$

Proof. Taking $n = 2$ and $m = 1$ in (3.25), we get

$$S(p^2, 2kp) \equiv \begin{cases} \binom{\frac{p-3}{2}} p^3 S(p^2 - 1, (p-3)p) - \dfrac{9p^5}{4} \pmod{p^6}, & \text{if } k = \frac{p-3}{2}; \\ kp^3 S(p^2 - 1, 2kp) \pmod{p^6}, & \text{otherwise.} \end{cases} \tag{3.52}$$

From (3.52), we have the following weaker congruence:

$$S(p^2, 2kp) \equiv kp^3 S(p^2 - 1, 2kp) \pmod{p^5}. \tag{3.53}$$

The theorem follows from (3.53) and Theorem 3.3.8. □

3.3.3 Divisibility of $S(p^n, k)$ when $p \nmid k$

From Equation (1.62), we get

$$v_p(S(p^n, k)) \geq \left\lceil \frac{s_p(k) - 1}{p - 1} \right\rceil, \tag{3.54}$$

which shows that $S(p^n, k)$ is divisible by p unless k is a power of p (i.e., $k = p^m$ for some positive integer m). Now we have the following result.

Theorem 3.3.10. *Let p be an odd prime and $1 \leq k \leq p-1$. Then for any positive integer n,*

$$S(p^n, k) \equiv S(p, k) \pmod{p^2}. \text{ So, } v_p(S(p^n, k) - S(p, k)) \geq 2. \tag{3.55}$$

Proof. Taking $n = p^2$ and $r = p - 1$ in (3.4), we get

$$S(p^2, k) \equiv pS(2p-1, k) - (p-1)S(p, k) \pmod{p^2}. \tag{3.56}$$

Using the minimum period from Equation (5.4) for $S(2p-1, k)$, the theorem follows at once. □

Theorem 3.3.11. *Let p be an odd prime and $p < k < p^2$. If $k = k_1p + k_0$ and $k_0 \neq 0$, then*

$$S(p^2, k) \equiv \binom{p}{k_1} S(p - k_1, k_0) \pmod{p^2}. \tag{3.57}$$

Proof. From Equation (1.39), we get

$$\begin{aligned}\binom{k_1p + k_0}{k_0} S(p^2, k_1p + k_0) &= \sum_{i=0}^{p^2} \binom{p^2}{i} S(i, k_1p) S(p^2 - i, k_0) \\ &\equiv \sum_{i=k_1}^{p-1} \binom{p^2}{ip} S(ip, k_1p) S(p^2 - ip, k_0) \pmod{p^2}\end{aligned} \tag{3.58}$$

since $v_p(\binom{p^n}{i}) = n - v_p(i)$.

Since p divides $\binom{p^2}{ip}$ for $k_1 \leq i \leq p-1$ and p divides $S(ip, k_1p)$ unless $i = k_1$, we have

$$\binom{k_1p + k_0}{k_0} S(p^2, k_1p + k_0) \equiv \binom{p^2}{k_1p} S(p^2 - k_1p, k_0) \pmod{p^2}. \tag{3.59}$$

Also, $S(p^2, k_1p + k_0) \equiv 0 \pmod{p}$ and $\binom{k_1p+k_0}{k_0} \equiv 1 \pmod{p}$. Moreover, $\binom{p^2}{k_1p} \equiv \binom{p}{k_1} \pmod{p^2}$. Consequently,

$$S(p^2, k_1p + k_0) \equiv \binom{p}{k_1} S(p^2 - k_1p, k_0) \pmod{p^2}. \tag{3.60}$$

Now using the minimum periods on the preceding congruence, the theorem follows. $\square$

Remark 3.3.4. *Theorem 3.3.11 gives an exact p-adic valuation for some special cases.*

Corollary 3.3.4. *Let p be an odd prime and $k = k_1p + k_0 > p$, where k_1 and k_0 are the p-adic digits of k. If $s_p(k) = p$ or $s_p(k) < p$ and $t_p(p - k_1, p - s_p(k)) = 0$, then*

$$v_p(S(p^2, k)) = 1. \tag{3.61}$$

Different values of t_p are mentioned in Equation (3.7). The following theorem is a generalization of Theorem 3.3.11.

Theorem 3.3.12. *If k is a positive integer not divisible by an odd prime p with $p < k < p^n$ for some positive integer $n \geq 2$, then*

$$S(p^n, k) \equiv \begin{cases} 0 \pmod{p^2}, & \text{if } s_p(k) > p; \\ \binom{p}{k_{n_\tau}, \cdots, k_{n_1}, p - \sum_{r=1}^{\tau-1} k_{n_r}} S(p - \sum_{r=0}^{\tau-1} k_{n_{\tau-r}}, k_0) \pmod{p^2}, & \text{otherwise,} \end{cases} \tag{3.62}$$

where $k_{n_\tau}, k_{n_{\tau-1}}, \cdots, k_{n_1}$ are the non zero p-adic digits of k.

Proof. For $n = 2$, the result follows from Theorem 3.3.11. So, let $n > 2$. Let $k = \sum_{r=0}^{t} k_r p^r$ such that for each $1 \leq r \leq t$, k_r is the r-th p-adic digit of k and $k_0 \neq 0 \neq k_t$. Removing all the zero digits from the expression of k, we can re-write

$$k = \sum_{r=0}^{\tau} k_{n_r} p^{n_r}, \tag{3.63}$$

where $k_{n_r} \neq 0$ for every r with $1 \leq r \leq \tau$. Then from (1.39), we have

$$\binom{k}{k_{n_\tau} p^{n_\tau}} S(p^n, k) = \sum_{i=0}^{p^n} \binom{p^n}{i} S(i, k_{n_\tau} p^{n_\tau}) S(p^n - i, \sum_{r=0}^{\tau-1} k_{n_r} p^{n_r}). \tag{3.64}$$

The binomial coefficients on the right-hand side in (3.64) are divisible by p^2 unless

i is divisible by p^{n-1} and hence the preceding equation reduces to

$$S(p^n,k) \equiv \sum_{i=0}^{p} \binom{p^n}{ip^{n-1}} S(ip^{n-1}, k_{n_\tau}p^\tau) S(p^n - ip^{n-1}, \sum_{r=0}^{\tau-1} k_{n_r}p^{n_r}) \pmod{p^2}. \tag{3.65}$$

Here, $\binom{p^n}{ip^{n-1}} \equiv \binom{p}{i} \pmod{p^2}$ and each of these binomial coefficients are divisible by p unless $i = 0$ or p. Using (3.1), we observe that $S(ip^{n-1}, k_{n_\tau}p^\tau)$ is divisible by p^{n_τ} if $i \neq k_{n_\tau}$. Moreover, $S(k_{n_\tau}p^{n-1}, k_{n_\tau}p^{n_\tau}) \equiv 1 \pmod{p}$, and so,

$$S(p^n,k) \equiv \binom{p}{k_{n_\tau}} S(p^n - k_{n_\tau}p^{n-1}, \sum_{r=0}^{\tau-1} k_{n_r}p^{n_r}) \pmod{p^2}. \tag{3.66}$$

Let $U_j = S((p - \sum_{r=0}^{j} k_{n_{\tau-r}})p^{n-1}, \sum_{r=0}^{\tau-j-1} k_{n_r}p^{n_r})$, we then have

$$\binom{\sum_{r=0}^{\tau-1} k_{n_r}p^{n_r}}{k_{n_{\tau-1}}p^{n_{\tau-1}}} U_0 = \sum_{i=0}^{(p-k_{n_\tau})p^{n-1}} \binom{(p-k_{n_\tau})p^{n-1}}{i} S(i, k_{n_{\tau-1}}p^{n_{\tau-1}}) \times S((p-k_{n_\tau})p^{n-1} - i, \sum_{r=0}^{\tau-2} k_{n_r}p^{n_r}). \tag{3.67}$$

Using the same technique as in the proof of Theorem 3.3.11, we obtain

$$U_0 \equiv \sum_{i=0}^{p-k_{n_\tau}} \binom{(p-k_{n_\tau})p^{n-1}}{ip^{n-1}} S(ip^{n-1}, k_{n_{\tau-1}}p^{n_{\tau-1}}) \times S((p-k_{n_\tau})p^{n-1} - ip^{n-1}, \sum_{r=0}^{\tau-2} k_{n_r}p^{n_r}) \pmod{p} \tag{3.68}$$

which also yields the following:

$$U_0 \equiv \begin{cases} 0 \pmod{p}, & \text{if } k_{n_\tau} + k_{n_{\tau-1}} \geq p; \\ \binom{p-k_{n_\tau}}{k_{n_{\tau-1}}} U_1 \pmod{p}, & \text{otherwise.} \end{cases} \tag{3.69}$$

Combining (3.66) with the preceding congruence, we get

$$S(p^n,k) \equiv \begin{cases} 0 \pmod{p^2}, & \text{if } k_{n_\tau} + k_{n_{\tau-1}} \geq p; \\ \binom{p}{k_{n_\tau}, k_{n_{\tau-1}}, p-k_{n_\tau}-k_{n_{\tau-1}}} U_1 \pmod{p^2}, & \text{otherwise.} \end{cases}$$

We employ the same technique recursively to get the congruence

$$S(p^n,k)\equiv\begin{cases}0 \pmod{p^2}, & \text{if } \sum_{r=0}^{\tau-1} k_{n_{\tau-r}} \geq p;\\ \binom{p}{k_{n_\tau},\cdots,k_{n_1},p-\sum_{r=1}^{\tau-1}k_{n_r}}U_{\tau-1} \pmod{p^2}, & \text{otherwise.}\end{cases} \tag{3.70}$$

Now applying the minimum period on $U_{\tau-1}=S((p-\sum_{r=0}^{\tau-1}k_{n_{\tau-r}})p^{n-1},k_0)$, we get the desired result. □

Corollary 3.3.5. *Let p be an odd prime.*

(a) Let $k=\sum_{i=0}^{t}k_ip^i>p$ be the p-adic expansion of k with $k_0\neq 0$. If $s_p(k)\leq p$ and $t_p(p-s_p(k)+k_0,p-s_p(k))=0$, then $v_p(S(p^2,k))=1$.

(b) If $n\geq 2$, $s_p(k)<p$, and $1<kp+1<p^n$, then $v_p(S(p^n,kp+1))=1$.

(c) If $p\nmid k$ and $k<p^m\leq p^n$, then $S(p^n,k)\equiv S(p^m,k) \pmod{p^2}$.

Remark 3.3.5. *From Corollary 3.3.5(a), we observe that $t_p(p-s_p(k)+k_0,p-s_p(k))=0$ when $s_p(k)=p-1$; $s_p(k)=p-2$ and $v_p(3p-3k_1-5)=0$; $s_p(k)=p-3$; $s_p(k)=p-4$ and $v_p(15n^3-150n^2+485n-502)=0$ and so on.*

3.4 Conclusions

We study the congruence properties of Stirling numbers of the second kind to obtain their p-adic valuations. We extend the results of Chan and Manna (2010) (for $S(n,kp^m) \pmod{p^m}$) to a higher congruence for some special cases. If n and k are opposite parity, we find out that $v_p(S(n,kp^m))$ is always greater than or equal to m and the estimates of the valuation gets doubled when $p-1\mid n-1-k$, i.d., $v_p(S(n,kp^m))\geq 2m$. We prove that for even integer k, $v_p(S(p^n,kp^m))\geq n+2m$ unless $k=p-1$ and $m=1$, in which case the p-adic valuation is exactly $n+1$; the same result is then strengthened to congruence modulo p^{4m+n}. We establish a congruence recurrence for $S(n+kp,kp)$ in k for different classes of n. The recurrence for the case $n = tp - 1$ with $t = p - k$ is used to evaluate the p-adic valuation $v_p(S(p^2, kp)) \geq 5$, this confirms the lower bound of Conjecture 2.3.1 is true.

Chapter 4

Congruence Relation Between Stirling Numbers of the First and Second Kind[3]

4.1 Introduction

Stirling numbers of the second kind are known to have various relations with the first kind. Stenlund (2019) derived some interesting relations, which include the following:

$$N = \sum_{m=1}^{N}\sum_{j=1}^{m} s(m,j)\sum_{k=1}^{j} S(j,k) = \sum_{m=1}^{N}\sum_{j=1}^{m} S(m,j)\sum_{k=1}^{j} s(j,k)$$

and

$$(S(N+1,m) - S(N,m-1))s(m,n) = (s(m,n-1) - s(m+1,n))S(N,m).$$

This chapter studies the congruence relations between Stirling numbers of the first and second kind. The main results include the congruences for $S(n,kp^m)$ and $s(kp^m,n)$ explicitly in terms of binomial coefficient when $n \equiv k \pmod{p-1}$. We further obtain congruences for $S(n,k)$ and $s(k,a)$ in modulo p, p^m and p^n where $m = \lfloor log_p(k) \rfloor$ and $n \geq m$. We express the congruences to more simpler

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forms in certain cases. The exact values of $v_p(S(n,k))$ and $v_p(s(n,k))$ for some special cases are obtained. We also discuss the cases of same congruence property for $S(n,k)$ and $S(n-1,k-1)$.

4.2 Preliminaries

The generating functions of $S(n,k)$ and $s(n,k)$ play a significant role in obtaining their congruence properties. We will assume p as an odd prime unless stated otherwise. We use the congruence property of the polynomial (Chan and Manna, 2010)

$$\prod_{i=1}^{kp^m}(1-ix) \equiv (1-x^{p-1})^{kp^{m-1}} \pmod{p^m}. \tag{4.1}$$

If we replace x with $1/y$, we get

$$\prod_{i=1}^{kp^m}(1-ix) = \frac{1}{y^{kp^m}}\prod_{i=1}^{kp^m}(y-i) \ \text{ and } \ (1-x^{p-1})^{kp^{m-1}} = \frac{1}{y^{k(p-1)p^{m-1}}}(y^{p-1}-1)^{p^{m-1}}.$$

It follows that

$$\prod_{i=1}^{kp^m}(y-i) \equiv y^{kp^{m-1}}(y^{p-1}-1)^{kp^{m-1}} \pmod{p^m}. \tag{4.2}$$

We can also write this result as

$$x^{\underline{kp^m}} \equiv x^{kp^{m-1}}(x^{p-1}-1)^{kp^{m-1}} \pmod{p^m}. \tag{4.3}$$

Replacing x by $-x$ in Equations (4.1) and (4.3), we obtain

$$\prod_{i=1}^{kp^m}(1+ix) \equiv (1-x^{p-1})^{kp^{m-1}} \pmod{p^m} \tag{4.4}$$

and

$$x^{\overline{kp^m}} \equiv (-1)^k x^{kp^{m-1}}(x^{p-1}-1)^{kp^{m-1}} \pmod{p^m}. \tag{4.5}$$

Davis and Webb (1993) proved, for a prime $p > 3$, that

$$\binom{np}{kp} \equiv \binom{n}{k} \pmod{p^e}, \tag{4.6}$$

where $e = 3 + v_p(n) + v_p(k) + v_p(n-k) + v_p(\binom{n}{k})$. With the help of Equation (4.6), it is easy to confirm

$$(1-x)^{p^n} \equiv (1-x^{p^{n-m}})^m \pmod{p^{m+1}} \tag{4.7}$$

for any integers n and m such that $0 \leq m \leq n$.

4.3 Main Results

In this section, we prove the main results of this chapter which are presented in theorems and corollaries. The first theorem gives the congruence relations between Stirling numbers of the first kind and Binomial coefficients.

Theorem 4.3.1. *If p is an odd prime and k is a positive integer not divisible by p, then for any positive integer m, the following congruences hold:*

a) $s(kp^m, kp^m - b) \equiv (-1)^k s(kp^m, kp^{m-1} + b) \pmod{p^m}$,

b) $s(kp^m, kp^m - b) \equiv \binom{kp^{m-1}}{\frac{b}{p-1}} (-1)^{\frac{b}{p-1}} \pmod{p^m}$,

if $b \equiv 0 \pmod{p-1}$ *and* $b \leq k(p-1)p^{m-1}$.

c) $s(kp^m, b) \equiv 0 \pmod{p^m}$,

if $b \leq kp^{m-1}$ *or* $kp^{m-1} - b \not\equiv 0 \pmod{p-1}$.

d) $s(kp^m, b) \equiv s(kp^m + 1, b + 1) \pmod{p^m}$,

for any integer b such that $1 \leq b \leq kp^m - 1$.

Proof. We know that

$$\prod_{i=1}^{kp^m-1}(1-ix) = \sum_{i=0}^{kp^m-1} s(kp^m, kp^m-i)x^i$$

$$x^{\underline{kp^m}} = \sum_{i=0}^{kp^m} s(kp^m, i)x^i.$$

Due to Equations (4.1) and (4.3), we get

$$\prod_{i=1}^{kp^m-1}(1-ix) \equiv (1-x^{p-1})^{kp^{m-1}} \pmod{p^m};$$

$$x^{\underline{kp^m}} \equiv x^{kp^{m-1}}(x^{p-1}-1)^{kp^{m-1}} \pmod{p^m}.$$

It follows that

$$\sum_{i=0}^{kp^m-1} s(kp^m, kp^m-i)x^i \equiv \sum_{j=0}^{kp^{m-1}} \binom{kp^{m-1}}{j}(-1)^j x^{j(p-1)} \pmod{p^m};$$

$$\sum_{i=0}^{kp^m} s(kp^m, i)x^i \equiv x^{kp^{m-1}} \sum_{j=0}^{kp^{m-1}} \binom{kp^{m-1}}{j}(-1)^{k-j} x^{j(p-1)} \pmod{p^m}.$$

Comparing the coefficients of x^b, we get the first three results, and the last result is obtained from the congruence

$$\prod_{i=1}^{kp^m}(1-ix) \equiv \prod_{i=1}^{kp^m-1}(1-ix) \pmod{p^m}. \tag{4.8}$$

Hence, the theorem follows. □

Corollary 4.3.1. *Let $n = kp^m$, $m \geq 1$, be an integer with only one non-zero p-adic digit, and $p > 3$ be a prime. Then, $s(n, a)$ is divisible by p if and only if $n \not\equiv a \pmod{(p-1)p^{m-1}}$. Further, $s(n, a)$ is divisible by p^{t+1}, $0 \leq t \leq m-1$, if and only if $n - a < kp^{m-1}$ or $n \not\equiv a \pmod{(p-1)p^{m-1-t}}$.*

Proof. Follow the proof of Theorem 4.3.1 and use Equation (4.7). □

Corollary 4.3.2. *For an odd prime p and integers k and m such that $p \nmid k$ and*

$m \geq 1$, we have

$$v_p(s(kp^m+1,a+1)) = v_p(s(kp^m,a)) = m-1-v_p(a), \tag{4.9}$$

whenever $m-1 > v_p(a)$, $kp^{m-1} \leq a \leq kp^m$, and $k \equiv a \pmod{p-1}$.

Proof. The proof is based on the equality $v_p(\binom{ap^n}{bp^m}) = n-m$ when $p \nmid a$, $p \nmid b$, and $n > m$. □

Remark 4.3.1. *If $n \geq m$, the p-adic valuation of the binomial coefficient $\binom{ap^n}{bp^m}$ is equal to $v_p(\binom{a}{bp^{m-n}})$. It follows that if $m-1 \leq v_p(a)$ and $v_p(\binom{k}{kp-ap^{1-m}}) \leq m-1$, we have*

$$v_p(s(kp^m+1,a+1)) = v_p(s(kp^m,a)) = v_p\left(\binom{k}{kp-ap^{1-m}}\right). \tag{4.10}$$

It is also trivial from Theorem 4.3.1 that

$$Min\{v_p(s(kp^m+1,a+1)),\ v_p(s(kp^m,a))\} \geq m \tag{4.11}$$

when $a < kp^{m-1}$ or $kp^{m-1}-a \not\equiv 0 \pmod{p-1}$.

The following theorem is a generalization of Theorem 4.3.1.

Theorem 4.3.2. *For an odd prime p and positive integers k, m, a, and b, the following congruences hold;*

$$s(kp^m+a,kp^{m-1}+b) \equiv \sum_i (-1)^{k-i}\binom{kp^{m-1}}{i} s(a,b-i(p-1)) \pmod{p^m} \tag{4.12}$$

if $b \leq a+k(p-1)p^{m-1}$, and

$$s(kp^m+a,b) \equiv 0 \pmod{p^m}. \tag{4.13}$$

if $b \leq kp^{m-1}$.

Proof. We have

$$x^{\underline{kp^m+a}} = \sum_{i=0}^{kp^m+a} s(kp^m+a,i)x^i \tag{4.14}$$

and

$$x^{\underline{kp^m+a}} = \prod_{i=0}^{kp^m}(x-i)\prod_{i=1}^{a-1}(x-(kp^m+i))$$
$$\equiv x^{kp^{m-1}}(x^{p-1}-1)^{kp^{m-1}}x^{\underline{a}} \pmod{p^m}$$
$$\equiv x^{kp^{m-1}}\sum_{i=0}^{kp^{m-1}}\binom{kp^{m-1}}{i}(-1)^{k-i}x^{i(p-1)}\sum_{j=0}^{a}s(a,j)x^j \pmod{p^m}. \quad (4.15)$$

Comparing the coefficients of $x^{kp^{m-1}+b}$ in the RHS of Equations (4.14) and (4.15), we obtain

$$s(kp^m+a, kp^{m-1}+b) \equiv \sum_{i(p-1)+j=b}(-1)^{k-i}\binom{kp^{m-1}}{i}s(a,j) \pmod{p^m}. \quad (4.16)$$

Changing the index j to $b-i(p-1)$ confirms the first result of the theorem. The coefficient of x^n on the right-hand side of Equation (4.15) vanishes if $n \leq kp^{m-1}$; hence, the second result follows. □

The following corollaries are special cases of the preceding theorem.

Corollary 4.3.3. *For an odd prime p and positive integers k, m, a, and b;*

(i) $\lambda_{a,b}^{k,p^m} \equiv (-1)^k s(a,b) \pmod{p^m}$ *if* $b \leq p-1$,

(ii) $$v_p(\lambda_{p-1,b}^{k,p^m}) = \begin{cases} m-1-v_p(\lfloor \frac{b}{p-1}\rfloor), & \text{if } (p-1)\nmid b \text{ and } \lfloor\frac{b}{p-1}\rfloor < p^{m-1}; \\ m-1-v_p(\frac{b}{p-1}-1), & \text{if } (p-1)\mid b \text{ and } \frac{b}{p-1} < p^{m-1}, \end{cases}$$

where $\lambda_{a,b}^{k,p^m} = s(kp^m+a, kp^{m-1}+b)$.

Proof. On observation of Equation (4.16), we can see that;

(i) If $b \leq p-1$, then the only solution of $i(p-1)+j=b$ for (i,j) is $(0,b)$, unless $b = p-1$, in which case there are two solutions, namely $(0, p-1)$ and $(1,0)$. The corresponding term for the solution $(1,0)$ vanishes as $s(a,0)=0$. Hence, (i) follows.

(ii) Let $b = q(p-1)+r$ such that $0 \leq r < p-1$. The only solution of $i(p-1)+j =$

$q(p-1)+r$ with $j \leq p-1$ is $(i,j)=(q,r)$, if $r \neq 0$. Thus, we get

$$s(kp^m+p-1, kp^{m-1}+b) \equiv (-1)^{k-q}\binom{kp^{m-1}}{q} s(p-1,r) \pmod{p^m}. \tag{4.17}$$

Now, we obtain the congruence for $s(p-1,b)$:

We have

$$x^{\underline{p-1}} = \frac{x^{\underline{p}}}{x-p+1} \equiv (x^p - x)(1-x+x^2-\cdots) \pmod{p}.$$

It follows that

$$\sum_{i=0}^{p-1} s(p-1,i)x^i \equiv -x+x^2-x^3+\cdots+x^{p-1} \pmod{p}$$

and

$$s(p-1,i) \equiv (-1)^i \pmod{p} \tag{4.18}$$

if $1 \leq i \leq p-1$.

Therefore, the valuation of the binomial coefficient $\binom{kp^{m-1}}{q}$ is $m-1-v_p(q)$ and $s(p-1,r)$ is not divisible by p. Thus, the first case of (ii) follows.

On the other hand, if $r=0$ or $b=q(p-1)$, then there are two solutions of $i(p-1)+j=q(p-1)$, namely $(q,0)$ and $(q-1,p-1)$. The corresponding term for the index $(q,0)$ is zero since $s(p-1,0)=0$. Following the proof of the first result, we get the second case of (ii). □

Corollary 4.3.4. *For an odd prime p and positive integers k, m, a, and b;*

$$s(kp^m+a, kp^{m-1}+b) \equiv (-1)^q \binom{kp^{m-1}}{q} s(a,r) \pmod{p^m}$$

if $a < p-1$ and $b = q(p-1)+r$ with $0 \leq r < p-1$.

Proof. Given Equation (4.16), the only solution of $i(p-1)+j=q(p-1)+r$ is $(i,j)=(q,r)$. Hence the result follows. □

Remark 4.3.2. *The p-adic valuations of large classes of Stirling numbers of the first kind can be obtained using Theorem 4.3.1, Corollaries 4.3.3, and 4.3.4. The* *first result of Corollary 4.3.4 yields the following exact p-adic valuation,*

$$v_p(s(kp^m + a, kp^{m-1} + b)) = m - 1 - v_p\Big(\Big\lfloor \frac{b}{p-1} \Big\rfloor\Big), \tag{4.19}$$

if $b \equiv 1, a, a-1$ *or* $a-3 \pmod{p-1}$, *assuming conditions of the corollary apply.*

Theorem 4.3.3. *Let* p *be an odd prime and* $k,\ a,\ b,\ m,$ *and* t *be positive integers such that* $max\{k,\ a\} \le p-1$, $t < m$, *and* $b \le ap^t$. *Then,*

$$\eta^{kp^m}_{ap^t,b} \equiv \begin{cases} s(ap^t, b) \pmod{p^m}, & \text{if } a \not\equiv b \pmod{p-1} \text{ or } b < ap^{t-1}; \\ s(ap^t, b) \pmod{p^{m-v_p(b)-1}}, & \text{otherwise,} \end{cases} \tag{4.20}$$

where $\eta^k_{a,b} = s(k+a, k+b)$.

Proof. Replace a and b in Equation (4.16) with ap^t and $k(p-1)p^{m-1} + b$, respectively; we obtain

$$s(kp^m + ap^t, kp^m + b) \equiv \sum_{i(p-1)+j=k(p-1)p^{m-1}+b} (-1)^{k-i} \binom{kp^{m-1}}{i} s(ap^t, j) \pmod{p^m}.$$

If we replace the index j with $(kp^{m-1} - i)(p-1) + b$, we get

$$s(kp^m + a, kp^m + b) \equiv \sum_i (-1)^{k-i} \binom{kp^{m-1}}{i} s(a, (kp^{m-1} - i)(p-1) + b) \pmod{p^m}.$$

By reversing the index, we get

$$s(kp^m + ap^t, kp^m + b) \equiv \sum_i (-1)^i \binom{kp^{m-1}}{i} s(ap^t, b + i(p-1)) \pmod{p^m}. \tag{4.21}$$

Using Theorem 4.3.1, $s(ap^t, b + i(p-1))$ is divisible by p^t if $a \not\equiv b \pmod{p-1}$ or $b < ap^{t-1}$. The valuation of $\binom{kp^{m-1}}{i}$ is $m - 1 - v_p(i)$ unless $i = 0$. Thus, the valuation of the $i-$th terms, $i \ne 0$, of the right-hand side of Equation (4.21) is greater than or equal to $m - 1 + t - v_p(i)$. The range of the index i is determined by the inequality $b \le b + i(p-1) \le ap^t$, which implies that $0 \le i \le a\sum_{r=0}^{t-1} p^r$. It follows that $v_p(i) \le t-1$ and consequently $m - 1 + t - v_p(i) \ge m$. Hence, p^m divides all the $i-th$ terms except the term with $i = 0$. Thus, the first case of the theorem follows.

Now, we assume that $a \equiv b \pmod{p-1}$ and $ap^{t-1} \leq b \leq ap^t$. Therefore, we can express b as $ap^t - q(p-1)$ with $0 \leq q < ap^{t-1}$. Equation (4.21) becomes

$$s(kp^m + ap^t, kp^m + b) \equiv \sum_i (-1)^i \binom{kp^{m-1}}{i} s(ap^t, ap^t - (q-i)(p-1)) \pmod{p^m}. \tag{4.22}$$

Given Theorem 4.3.1, the valuation of the i-th term on the RHS of the preceding congruence is $m + t - 2 - v_p(i) - v_p(q-i)$ if $i > 0$. Therefore, two sub cases arise, namely $v_p(q) < v_p(i)$ and $v_p(i) \leq v_p(q)$;

If $v_p(q) < v_p(i)$, we have $v_p(q-i) = v_p(q)$ and

$$m + t - 2 - v_p(i) - v_p(q-i) = m - 1 - v_p(q) + (t - 1 - v_p(i)) \geq m - 1 - v_p(q),$$

since $v_p(i) \leq t - 1$.

If $v_p(i) \leq v_p(q)$, we get

$$m + t - 2 - v_p(i) - v_p(q-i) \geq m - 1 - v_p(q) + (t - 1 - v_p(q-i)) \geq m - 1 - v_p(q),$$

since $v_p(q-i) \leq t - 1$.

The equality $b = ap^t - q(p-1)$ also implies that $v_p(b) = v_p(q)$ for the given condition. It follows that all the terms except when $i = 0$ are divisible by $p^{m-1-v_p(b)}$. Hence, the second case of the theorem follows. □

4.3.1 Valuations of $S(n, kp^m)$

Using Equations (4.38) and (4.1), we have the following result (see Chan and Manna (2010));

$$\sum_{n=0}^{\infty} S(n + kp^m, kp^m) x^n \equiv \sum_{j=0}^{\infty} \binom{kp^{m-1} + j - 1}{j} x^{j(p-1)} \pmod{p^m}, \tag{4.23}$$

which implies $S(n, kp^m) \equiv 0 \pmod{p^m}$ if $n \not\equiv k \pmod{p-1}$; otherwise,

$$S(kp^m + a, kp^m) \equiv \binom{kp^{m-1} + \frac{a}{p-1} - 1}{kp^{m-1} - 1} \pmod{p^m} \tag{4.24}$$

for any non-negative integer a with $a \equiv 0 \pmod{p-1}$.

Due to Equation (4.8), we also have the congruence

$$S(kp^m + a - 1, kp^m - 1) \equiv S(kp^m + a, kp^m) \pmod{p^m}. \tag{4.25}$$

The following theorem is a consequence of Equations (4.23), (4.24), and (4.25).

Theorem 4.3.4. *Let p be an odd prime and k be an integer not divisible by p. For any positive integer n such that $n \equiv k \pmod{p-1}$ and $n < kp^m + (p-1)p^{m-1}$ with $m > 1$;*

$$v_p(S(n-1, kp^m - 1)) = v_p(S(n, kp^m)) = m - 1 - v_p(n). \tag{4.26}$$

If $n \not\equiv k \pmod{p-1}$, then

$$min\{v_p(S(n, kp^m)),\ v_p(S(n-1, kp^m - 1))\} \geq m. \tag{4.27}$$

Proof. The second result is trivial from Equations (4.23) and (4.25).

We assume $n \geq kp^m$ and let $n = kp^m + b(p-1)$ with $b < p^{m-1}$. From Equations (4.24) and (4.25), we have

$$\begin{aligned} S(kp^m + b(p-1) - 1, kp^m - 1) &\equiv S(kp^m + b(p-1), kp^m) \pmod{p^m} \\ &\equiv \binom{kp^{m-1} + b - 1}{b} \pmod{p^m}. \end{aligned}$$

The p-adic valuation of the above binomial coefficient is given as

$$v_p\left(\binom{kp^{m-1} + b - 1}{b}\right) = \frac{s_p(b) + s_p(kp^{m-1} - 1) - s_p(kp^{m-1} + b - 1)}{p - 1}, \tag{4.28}$$

and we have

$$\begin{aligned} s_p(kp^{m-1} - 1) &= s_p((k-1)p^{m-1} + p^{m-1} - 1) \\ &= s_p((k-1)p^{m-1}) + s_p(p^{m-1} - 1) \\ &= s_p(k-1) + (m-1)(p-1). \end{aligned} \tag{4.29}$$

Further, k is not divisible by p and then $s_p(k-1) = s_p(k) - 1$. Therefore, Equation (4.29) reduces to

$$s_p(kp^m - 1) = s_p(k) + (m-1)(p-1) - 1. \tag{4.30}$$

Since $m > 1$ and $b < p^{m-1}$, we have

$$\begin{aligned} s_p(kp^{m-1} + b - 1) &= s_p(kp^{m-1}) + s_p(b-1) \\ &= s_p(k) + s_p(b-1). \end{aligned} \tag{4.31}$$

Let $b = b'p^{v_p(b)}$, $p \nmid b'$, for some positive integer b'. Replacing kp^m with b in Equation (4.30), we get

$$s_p(b-1) = s_p(b) - 1 + v_p(b)(p-1) \tag{4.32}$$

since $s_p(b) = s_p(b')$. Therefore, combining Equations (4.28), (4.30), (4.31), and (4.32), we get

$$v_p\Big(\binom{kp^{m-1} + b - 1}{b}\Big) = m - 1 - v_p(b).$$

It is also trivial from our assumption of b that $v_p(b) = v_p(n)$. Hence the theorem follows. □

Definition 4.3.1. *Let a be a positive integer whose p-adic expansion is given by*

$$a = a_0 + a_1p + a_2p^2 + \cdots + a_tp^t.$$

For a fixed integer k, $1 \leq k \leq p-1$, we define $\rho_{p,k,m}(a)$ as

$$\rho_{p,k,m}(a) = \begin{cases} 0 & \text{if } k + a_m < p, \\ 1+n & \text{if } k + a_m \geq p \text{ and } a_{m+1} = \cdots = a_{m+n} = p-1 \neq a_{m+n+1}. \end{cases} \tag{4.33}$$

Here, $\rho_{p,k,m}(a)$ is the number of carries when adding a and kp^m in base p. Using Kummer's theorem, we can see that $\rho_{p,k,m}(a)$ is, in fact, $v_p(\binom{a+kp^m}{a})$.

The preceding theorem restricts the value of n to less than some particular value. The following theorem gives an alternate result of Theorem 4.3.4 when there is no restriction on the values of n but restrict $k < p$.

Theorem 4.3.5. *Let p be an odd prime and k be a positive integer less than p. For positive integers m and n such that $n \equiv k \pmod{p-1}$;*

$$v_p(S(n-1, kp^m - 1)) = v_p(S(n, kp^m)) = \rho_{p,k-1,m-1}\Big(\frac{n - kp^m}{p-1}\Big), \tag{4.34}$$

if $\rho_{p,k-1,m-1}(\frac{n-kp^m}{p-1}) \leq m-1 \leq v_p(n)$. However, if $\rho_{p,k,m-1}(\frac{n-kp^m}{p-1}) \leq v_p(n) < m-1$, then

$$v_p(S(n-1, kp^m - 1)) = v_p(S(n, kp^m)) = m - 1 - v_p(n) + \rho_{p,k,m-1}\Big(\frac{n - kp^m}{p-1}\Big). \tag{4.35}$$

Proof. Since $n \equiv k \pmod{p-1}$, we can write $n = kp^m + a(p-1)$. Let $a = \sum_{i=0}^{q} a_i p^i$ be the p-adic expansion of $a = \frac{n-kp^m}{p-1}$ for some positive integer q. To prove the theorem, it is enough to obtain $v_p(\binom{kp^{m-1}+a-1}{a})$ for both cases. For the first case, we have

$$a = a'p^t$$

for some positive integers a' and t such that $p \nmid a'$ and $t \geq m-1$. Therefore,

$$s_p(kp^{m-1} + a'p^t - 1) = s_p(k + a'p^{t-m+1} - 1) + s_p(p^{m-1} - 1)$$

and

$$s_p(kp^{m-1}-1) = k - 1 + s_p(p^{m-1}-1).$$

Thus, the valuation of the binomial coefficient is

$$v_p\Big(\binom{kp^{m-1}+a-1}{a}\Big) = \frac{s_p(a') - s_p(k-1+a'p^{t-m+1}) + k - 1}{p-1}. \tag{4.36}$$

Suppose $t > m-1$, the sum of the digits $s_p(k-1+a'p^{t-m+1})$ can be split into the sum $s_p(k-1)+s_p(a')$ since we assume $1 \le k \le p-1$. In this case, the valuation of the binomial coefficient becomes zero, which is also equal to $\rho_{p,k,m-1}(a)$, since the $(m-1)$-th p-adic digit of a is zero and $k + a_{m-1} = k < p$. Now, we assume that $v_p(a) = m-1$, which means that $t = m-1$ and $a' = a_{m-1} + a_m p + a_{m+1}p^2 + \cdots$. It follows that if $k - 1 + a_{m-1} < p$, then

$$s_p(kp^{m-1}+a-1) = s_p(a) + s_p(kp^{m-1}-1)$$

and $v_p(\binom{kp^{m-1}+a-1}{a}) = 0 = \rho_{p,k-1,m-1}(a)$.

If $k - 1 + a_{m-1} > p$, then

$$a = a_{m-1}p^{m-1} + (p-1)\sum_{i=m}^{m-2+\rho_{p,k-1,m-1}(a)} p^i + \sum_{i=m-1+\rho_{p,k-1,m-1}(a)}^{q} a_i p^i,$$

$$\begin{aligned} kp^{m-1}+a-1 &= (a_{m-1}+k-1-p)p^{m-1} \\ &\quad + (a_{m-1+\rho_{p,k-1,m-1}(a)}+1)p^{m-1+\rho_{p,k-1,m-1}(a)} \\ &\quad + \sum_{i=m+\rho_{p,k-1,m-1}(a)}^{q} a_i p^i, \end{aligned}$$

which implies

$$s_p(a) = a_{m-1} + (p-1)(\rho_{p,k-1,m-1}(a)-1) + \sum_{i=m+\rho_{p,k-1,m-1}(a)}^{q} a_i,$$

$$s_p(kp^{m-1}+a-1) = k-p+a_{m-1} + \sum_{i=m+\rho_{p,k-1,m-1}(a)}^{q} a_i$$
$$= s_p(a) - (p-1)\rho_{p,k-1,m-1}(a) + k - 1.$$

Therefore, we get

$$v_p\Big(\binom{kp^{m-1}+a-1}{a}\Big) = \frac{s_p(a) - s_p(k-1+a) + k - 1}{p-1}$$
$$= \rho_{p,k-1,m-1}(a).$$

Using this valuation in Equation (4.24), the first result of the theorem follows. The second result of the theorem can be obtained through the same method. □

Remark 4.3.3. *If there is no restriction on the value of k and $v_p(a) < m-1$, we can write a as $a = cp^{m-1} + b$, where $b < p^{m-1}$. Therefore,*

$$v_p\Big(\binom{kp^{m-1}+a-1}{a}\Big) = m - 1 - v_p(b) + v_p\Big(\binom{k+c}{c}\Big).$$

For $v_p(a) \geq m-1$, we have $a = cp^{m-1}$ for some integer c and hence

$$v_p\Big(\binom{kp^{m-1}+a-1}{a}\Big) = v_p\Big(\binom{k+c-1}{c}\Big).$$

If the valuations obtained are less than $m-1$ for both cases, then they are the valuations of $S(kp^m + a(p-1), kp^m)$ for both cases.

Theorem 4.3.6. *Let p be a prime greater than 3 and n, k, and m be positive integers. Then, p divides $S(n, kp^m)$ if $n \not\equiv k \pmod{(p-1)p^{m-1}}$. More precisely, p^{t+1}, $0 \leq t \leq m-1$, divides $S(n, kp^m)$ if $n \not\equiv k \pmod{(p-1)p^{m-1-t}}$.*

Proof. The theorem can be proved using Equations (4.38), (4.1), and (4.7). □

4.3.2 Congruence relation between $S(n,k)$ and $s(n,k)$

Now, we establish congruence relations between Stirling numbers of the first and the second kind using their generating functions. The next theorem gives the results for $S(n,k)$ modulo p. We use the notation $[n]$ to denote the set $\{0,\ 1,\ 2,\ \cdots,\ n\}$ for simplicity.

Theorem 4.3.7. *Let p be an odd prime and n, k, a, and d are positive integers such that $a < p$ and $0 \le d < p-1$, then*

$$S(n, kp+a) \equiv (-1)^d s(p-a, p-a-d) \binom{\frac{n-k-a-d}{p-1}}{k} \pmod{p}$$

if $n-k-a \equiv d \pmod{p-1}$ for some $d \in [p-1-a]$, and

$$S(n, kp+a) \equiv 0 \pmod{p}$$

if $n-k-a \not\equiv d \pmod{p-1}$ for any $d \in [p-1-a]$.

Proof. We have

$$\frac{1}{\prod_{i=1}^{kp+a}(1-ix)} = \frac{\prod_{i=kp+a+1}^{(k+1)p}(1-ix)}{\prod_{i=1}^{(k+1)p}(1-ix)},$$

and we get the following congruences:

$$\frac{1}{\prod_{i=1}^{kp+a}(1-ix)} \equiv \frac{\prod_{i=0}^{p-a-1}(1+ix)}{(1-x^{p-1})^{k+1}} \pmod{p},$$
$$\sum_{n=0}^{\infty} S(n+a', a')x^n \equiv \sum_{i=0}^{p-a-1} s_i^{p-a}(-1)^i x^i \sum_{j=0}^{\infty} \binom{k+j}{j} x^{j(p-1)} \pmod{p},$$

where $a' = kp+a$ and $s_i^{p-a} = s(p-a, p-a-i)$.

It follows that if $n \equiv d \pmod{p-1}$ for $d \in [p-a-1]$, then

$$S(n+kp+a, kp+a) \equiv (-1)^d \binom{k+\frac{n-d}{p-1}}{k} s(p-a, p-a-d) \pmod{p}, \quad (4.37)$$

and if $n \not\equiv d \pmod{p-1}$ for any $d \in [p-a-1]$, then

$$S(n+kp+a, kp+a) \equiv 0 \pmod{p}. \quad (4.38)$$

If we replace $n+kp+a$ with n in Equations (4.37) and (4.38), we get the required results. □

Remark 4.3.4. *The following results are consequences of Theorem 4.3.7, specifically for the prime $p = 3$ and $p = 5$:*
For $p = 3$, we have two classes (since $p - 1 = 2$) for each $a \in \{0,\ 1,\ 2\}$. We get the following congruences:

$$S(n, 3k) \equiv \begin{cases} \binom{\frac{n-k}{2}}{k} \pmod 3, & \text{if } n-k \text{ is even;} \\ 0 \pmod 3, & \text{if } n-k \text{ is odd,} \end{cases}$$

$$S(n, 3k+1) \equiv \begin{cases} \binom{\frac{n-k-1}{2}}{k} \pmod 3, & \text{if } n-k \text{ is odd;} \\ \binom{\frac{n-k-2}{2}}{k} \pmod 3, & \text{if } n-k \text{ is even,} \end{cases}$$

$$S(n, 3k+2) \equiv \begin{cases} \binom{\frac{n-k-2}{2}}{k} \pmod 3, & \text{if } n-k \text{ is even;} \\ 0 \pmod 3, & \text{if } n-k \text{ is odd.} \end{cases}$$

For $p = 5$, we have four classes (since $p - 1 = 4$) for each $a \in \{0,\ 1,\ 2,\ 3,\ 4\}$. We get the following congruences:

$$S(n, 5k) \equiv \begin{cases} \binom{\frac{n-k}{4}}{k} \pmod 5, & \text{if } n \equiv k \pmod 4; \\ 0 \pmod 5, & \text{if } n \not\equiv k \pmod 4, \end{cases}$$

$$S(n, 5k+1) \equiv \begin{cases} s(4,4)\binom{\frac{n-k-1}{4}}{k} \equiv \binom{\frac{n-k-1}{4}}{k} \pmod 5, & \text{if } n \equiv k+1 \pmod 4; \\ s(4,3)\binom{\frac{n-k-2}{4}}{k} \equiv \binom{\frac{n-k-2}{4}}{k} \pmod 5, & \text{if } n \equiv k+2 \pmod 4; \\ s(4,2)\binom{\frac{n-k-3}{4}}{k} \equiv \binom{\frac{n-k-3}{4}}{k} \pmod 5, & \text{if } n \equiv k+3 \pmod 4; \\ s(4,1)\binom{\frac{n-k-4}{4}}{k} \equiv \binom{\frac{n-k-4}{4}}{k} \pmod 5, & \text{if } n \equiv k \pmod 4, \end{cases}$$

$$S(n,5k+2) \equiv \begin{cases} s(3,3)\binom{\frac{n-k-2}{4}}{k} \equiv \binom{\frac{n-k-2}{4}}{k} \pmod 5, & \textit{if } n \equiv k+2 \pmod 4; \\ s(3,2)\binom{\frac{n-k-3}{4}}{k} \equiv 3\binom{\frac{n-k-3}{4}}{k} \pmod 5, & \textit{if } n \equiv k+3 \pmod 4; \\ s(3,1)\binom{\frac{n-k-4}{4}}{k} \equiv 2\binom{\frac{n-k-4}{4}}{k} \pmod 5, & \textit{if } n \equiv k \pmod 4; \\ 0 \pmod 5, & \textit{if } n \equiv k+1 \pmod 4, \end{cases}$$

$$S(n,5k+3) \equiv \begin{cases} s(2,2)\binom{\frac{n-k-3}{4}}{k} \equiv \binom{\frac{n-k-3}{4}}{k} \pmod 5, & \textit{if } n \equiv k+3 \pmod 4; \\ s(2,1)\binom{\frac{n-k-4}{4}}{k} \equiv \binom{\frac{n-k-4}{4}}{k} \pmod 5, & \textit{if } n \equiv k \pmod 4; \\ 0 \pmod 5, & \textit{if } n \equiv k+1 \textit{ or } k+2 \pmod 4, \end{cases}$$

$$S(n,5k+4) \equiv \begin{cases} \binom{\frac{n-k-4}{4}}{k} \pmod 5, & \textit{if } n \equiv k \pmod 4; \\ 0 \pmod 5, & \textit{if } n \equiv k+1,\ k+2,\ \textit{or } k+3 \pmod 4. \end{cases}$$

In the case of $S(n,5k)$, the multiplier $s(5,5-d)$ where $d \in \{0,\ 1,\ 2,\ 3\}$ is divisible by 5 except when $d=0$. It is also easy to see that the binomial coefficients on the RHS of the above equations reduce to 1 if $k=0$. This observation leads us to acquire the following exact p-adic valuations:

For a prime $p=3$ and any positive integer n,

a) $v_3(S(2n,2)) = 0$,

b) $v_3(S(6n+3,3)) = v_3(S(6n+5,3)) = 0$,

c) $v_3(S(6n,4)) = v_3(S(6n+1,4)) = v_3(S(6n+4,4)) = v_3(S(6n+5,4)) = 0$,

d) $v_3(S(6n+1,5)) = v_3(S(6n+1,5)) = 0$,

e) $v_3(S(6n,6)) = 0$,

f) $v_3(S(6n+1,7)) = v_3(S(6n+2,7)) = 0$,

g) $v_3(S(6n,6)) = 0$.

For a prime $p = 5$, we have the following p-adic valuations:

a) $v_5(S(4n, 2)) = v_5(S(4n + 2, 2)) = v_5(S(4n + 3, 2)) = 0,$

b) $v_5(S(4n, 3)) = v_5(S(4n + 3, 3)) = 0,$

c) $v_5(S(4n, 4)) = 0,$

d) $v_5(S(20n + r, 5)) = 0, \quad \text{if } r \in \{5, 9, 13, 17\},$

e) $v_5(S(20n + r, 6)) = 0, \quad \text{if } r \in [19] \setminus \{2, 3, 4, 5\},$

f) $v_5(S(20n + r, 7)) = 0, \quad \text{if } r \in [19] \setminus \{2, 3, 4, 5, 6, 10, 14, 18\},$

g) $v_5(S(20n + r, 8)) = 0, \quad \text{if } r \in \{0, 1, 8, 9, 12, 13, 16, 17\},$

h) $v_5(S(20n + r, 9)) = 0, \quad \text{if } r \in \{1, 9, 13, 17\}.$

The following theorem gives a generalization of Theorem 4.3.7 from modulo p to modulo p^m.

Theorem 4.3.8. *For an odd prime p and positive integers k, b, m, and n such that $b < p^{m-1}$, the following congruences hold;*

$$S(n + b', b') \equiv \sum_j (-1)^n \binom{(k+1)p^{m-1} + j - 1}{j} s(c, c - n + j(p-1)) \pmod{p^m} \tag{4.39}$$

and

$$s(b', b' - n) \equiv \sum_j (-1)^{n+j} \binom{(k+1)p^{m-1}}{j} S(n - j(p-1) + c, c) \pmod{p^m}, \tag{4.40}$$

where $c = p^m - b$ and $b' = kp^m + b$.

Proof. We have

$$\frac{1}{\prod_{i=1}^{kp^m+b}(1 - ix)} = \frac{\prod_{i=kp^m+b+1}^{(k+1)p^m}(1 - ix)}{\prod_{i=1}^{(k+1)p^m}(1 - ix)}$$

and

$$\prod_{i=1}^{kp^m+b-1}(1-ix)=\frac{\prod_{i=1}^{(k+1)p^m}(1-ix)}{\prod_{i=kp^m+b}^{(k+1)p^m}(1-ix)}.$$

We obtain the following two congruences

$$\frac{1}{\prod_{i=1}^{kp^m+b}(1-ix)}\equiv\frac{\prod_{i=1}^{p^m-b-1}(1+ix)}{(1-x^{p-1})^{(k+1)p^{m-1}}}\pmod{p^m},\tag{4.41}$$

and

$$\prod_{i=1}^{kp^m+b-1}(1-ix)\equiv\frac{(1-x^{p-1})^{(k+1)p^{m-1}}}{\prod_{i=0}^{p^m-b}(1+ix)}\pmod{p^m}.\tag{4.42}$$

Equations (4.41) and (4.42) generate Equations (4.39) and (4.40), respectively. Hence the theorem follows. □

Theorem 4.3.9. *For an odd prime p and positive integers k, a, m, and n such that $kp^m < p^n$ and $m+1 \le n$, the following congruences hold;*

$$S(a+kp^m,kp^m)\equiv(-1)^a\sum_j\binom{p^{n-1}+j-1}{j}s(bp^m,bp^m-a+j(p-1))\pmod{p^n}\tag{4.43}$$

and

$$s(kp^m,kp^m-a)\equiv\sum_j(-1)^{a+j}\binom{p^{n-1}}{j}S(a-j(p-1)+bp^m,bp^m)\pmod{p^n},\tag{4.44}$$

where $b=p^{n-m}-k$.

Proof. Using the same technique as in the proof of Theorem 4.3.8, we obtain the following two congruences;

$$\frac{1}{\prod_{i=1}^{kp^m}(1-ix)}\equiv\frac{\prod_{i=1}^{(p^{n-m}-k)p^m-1}(1+ix)}{(1-x^{p-1})^{p^{n-1}}}\pmod{p^n}\tag{4.45}$$

and

$$\prod_{i=1}^{kp^m-1}(1-ix)\equiv\frac{(1-x^{p-1})^{p^{n-1}}}{\prod_{i=0}^{(p^{n-m}-k)p^m}(1+ix)}\pmod{p^n}.\tag{4.46}$$

These two congruences generate the required results. Hence the theorem follows. □

Remark 4.3.5. *In the second result of the preceding theorem, the generating function on the RHS of Equation* (4.46) *generates an infinite term. In contrast, the LHS generates* $kp^m - 1$ *terms only. It follows that the sum vanishes when* $a \geq kp^m$*, i.e.,*

$$\sum_j (-1)^{s_i^{p-a}} \equiv (-1)^a [S(a + bp^m, bp^m) - S(a - p^m(p-1) + bp^m, bp^m)] \pmod{p^{m+1}}, \tag{4.47}$$

whenever $p - 1 \nmid a$*. Moreover, if* $a < p^m(p-1)$ *and* $p - 1 \nmid a$*, we obtain*

$$s(kp^m, kp^m - a) \equiv (-1)^a S(a + bp^m, bp^m) \pmod{p^{m+1}}. \tag{4.48}$$

We can utilize Theorem 4.3.9 to obtain some values of $v_p(S(n, kp^m))$*, which are greater than or equal to* m*. Although Theorem 4.3.5 deals with* $v_p(S(n, kp^m))$*, the theorem is restricted to valuations less than or equal to* m *since the key congruences used in Theorem 4.3.5 are in modulo* p^m*. The congruences obtained in Theorem 4.3.9 are in modulo* p^n *for arbitrary* n*, usually greater than or equal to* m *of* $S(n, kp^m)$*; the next theorem is an application of such congruence.*

Theorem 4.3.10. *Let* p *be an odd prime and* a, u, k, *and* m *be positive integers such that* $p \nmid a$ *and* $a \equiv 0 \pmod{p-1}$*. If* $p^{m-1} \leq u = \frac{a}{p-1} < p^m$ *and* $\frac{u}{p^{m-1}} + k \geq p$*, then*

$$v_p(S(kp^m + a, kp^m)) = m. \tag{4.49}$$

Proof. Replace n and a in the first result of Theorem 4.3.9 with $m+1$ and $u(p-1)$, respectively; we get

$$S(a+kp^m, kp^m) \equiv \sum_j \binom{p^m+j-1}{j} s(bp^m, bp^m-(u-j)(p-1)) \pmod{p^{m+1}} \tag{4.50}$$

where $b = p-k$.

Let $u = \sum_{i=0}^{m-1} u_i p^i$ be the p-adic expansion of u. We know that $s(bp^m, bp^m-(u-j)(p-1))$ is divisible by p^m if $bp^m-(u-j)(p-1) < bp^{m-1}$ or $j < \sum_{i=0}^{m-1} u_i p^i - bp^{m-1} = \alpha$ (*say*). We can also confirm that p divides $\binom{p^m+j-1}{j}$ if $j < \alpha$, unless $j = 0$, in which case $s(bp^m, bp^m-(u-j)(p-1)) = 0$ since $bp^m - u(p-1) < 0$. Thus, all the j-th terms with $0 \le j < \alpha$ are divisible by p^{m+1}, and we obtain

$$S(a+kp^m, kp^m) \equiv \sum_{j=\alpha}^{u} \binom{p^m+j-1}{j} s(bp^m, bp^m-(u-j)(p-1)) \pmod{p^{m+1}}. \tag{4.51}$$

From Theorem 4.3.1(b), we have

$$s(bp^m, bp^m-(u-j)(p-1)) \equiv \binom{bp^{m-1}}{u-j}(-1)^{u-j} \pmod{p^m}. \tag{4.52}$$

It follows that the p-adic valuation of each term on the RHS of Equation (4.51) is $m - v_p(j) + m - 1 - v_p(u-j)$. If $p \nmid j$ and $j \neq u$, then the valuation is $2m-1-v_p(u-j)$, which is greater than or equal to $m+1$ unless $v_p(u-j) = m-1$. On the other hand, if $p \mid j$, then $p \nmid (u-j)$, and the valuation becomes $2m-1-v_p(j)$, which is greater than or equal to $m+1$ unless $v_p(j) = m-1$. If $v_p(u-j) = m-1$, then $u-j = rp^{m-1}$ for some r, $p \nmid r$. If $v_p(j) = m-1$, then $j = tp^{m-1}$ for some t, $p \nmid t$. The only remaining term whose p-adic valuation is less than $m+1$ is the term with $j=u$. Thus, Equation (4.51) reduces to

$$\begin{aligned} S(a+kp^m,kp^m) &\equiv \sum_{r=1}^{b} \binom{p^m+u-rp^{m-1}-1}{p^m-1} s(bp^m, bp^m-r(p-1)p^{m-1}) \\ &+ \binom{p^m+u-rp^{m-1}-1}{p^m-1} s(bp^m, bp^m-r(p-1)p^{m-1}) \\ &+ \sum_{t=u_{m-1}-b+1}^{u_{m-1}} \binom{p^m+tp^{m-1}-1}{p^m-1} s(bp^m, bp^m-(u-tp^{m-1})(p-1)) \pmod{p^{m+1}}, \end{aligned} \tag{4.53}$$

which can be written as

$$\begin{aligned} S(a+kp^m,kp^m) &\equiv \sum_{r=0}^{b} \binom{p^m+u-rp^{m-1}-1}{p^m-1} s(bp^m, bp^m-r(p-1)p^{m-1}) \\ &+ \sum_{t=u_{m-1}-b+1}^{u_{m-1}} \binom{p^m+tp^{m-1}-1}{p^m-1} s(bp^m, bp^m-(u-tp^{m-1})(p-1)) \pmod{p^{m+1}}. \end{aligned} \tag{4.54}$$

Now, we have the following congruences

$$\begin{aligned} \binom{p^m+u-rp^{m-1}-1}{p^m-1} &= \frac{p^m}{p^m+u-rp^{m-1}} \binom{p^m+u-rp^{m-1}}{p^m} \\ &\equiv \frac{p^m}{u} \pmod{p^{m+1}}, \end{aligned} \tag{4.55}$$

$$s(bp^m, bp^m-rp^{m-1}(p-1)) \equiv \binom{bp^{m-1}}{rp^{m-1}}(-1)^r \equiv \binom{b}{r}(-1)^r \pmod{p}, \tag{4.56}$$

$$\begin{aligned} \binom{p^m+tp^{m-1}-1}{p^m-1} &= \frac{p^m}{p^m+tp^{m-1}} \binom{p^m+tp^{m-1}}{p^m} \\ &\equiv \frac{p}{t} \pmod{p^2}, \end{aligned} \tag{4.57}$$

and

$$\begin{aligned} s(bp^m,\theta) &\equiv \binom{bp^{m-1}}{u-tp^{m-1}}(-1)^{u-t} \pmod{p^m} \\ &\equiv \frac{bp^{m-1}}{u-tp^{m-1}}\binom{bp^{m-1}-1}{u-tp^{m-1}-1}(-1)^{u-t} \pmod{p^m} \\ &\equiv \frac{bp^{m-1}}{u}\binom{b-1}{u_{m-1}-t}(-1)^{s_p(u)-u_{m-1}-1}(-1)^{u-t} \pmod{p^m} \\ &\equiv \frac{bp^{m-1}}{u}\binom{b-1}{u_{m-1}-t}(-1)^{t-u_{m-1}-1} \pmod{p^m}, \end{aligned} \tag{4.58}$$

where $\theta = bp^m - (u - tp^{m-1})(p-1)$.

Let

$$X = \sum_{r=0}^{b} \binom{p^m + u - rp^{m-1} - 1}{p^m - 1} s(bp^m, bp^m - r(p-1)p^{m-1}) \tag{4.59}$$

and

$$Y = \sum_{t=u_{m-1}-b+1}^{u_{m-1}} \binom{p^m + tp^{m-1} - 1}{p^m - 1} s(bp^m, bp^m - (u - tp^{m-1})(p-1)). \tag{4.60}$$

From Equations (4.55), (4.56), and (4.59), we get

$$\begin{aligned} X &\equiv \frac{p^m}{u}\sum_{r=1}^{b}\binom{b}{r}(-1)^r \pmod{p^{m+1}} \\ &\equiv 0 \pmod{p^{m+1}}. \end{aligned} \tag{4.61}$$

From Equations (4.57), (4.58), and (4.60), we get

$$\begin{aligned} Y &\equiv \frac{p^m}{u}\sum_{t=u_{m-1}-b+1}^{u_{m-1}} \frac{b}{t}\binom{b-1}{u_{m-1}-t}(-1)^{t-u_{m-1}-1} \pmod{p^{m+1}} \\ &\equiv \frac{p^m}{u}\sum_{t=0}^{b-1}\frac{b}{u_{m-1}-t}\binom{b-1}{t}(-1)^{t-1} \pmod{p^{m+1}} \\ &\equiv \frac{p^m}{u}\sum_{t=0}^{b-1}\frac{b-t}{u_{m-1}-t}\binom{b}{t}(-1)^{t-1} \pmod{p^{m+1}}. \end{aligned} \tag{4.62}$$

Using partial fraction decomposition, we get

$$\sum_{t=0}^{b-1} \frac{b-t}{u_{m-1}-t}\binom{b}{t}(-1)^{b-t-1} = \frac{1}{\binom{u_{m-1}}{b}}. \tag{4.63}$$

From Equations (4.54) and (4.59) – (4.63), we obtain

$$S(a+kp^m, kp^m) \equiv \frac{(-1)^b p^m}{u\binom{u_{m-1}}{b}} \pmod{p^{m+1}}. \tag{4.64}$$

Since $p \nmid u\binom{u_{m-1}}{b}$,

$$v_p(S(a+kp^m, kp^m)) = m. \tag{4.65}$$

Hence, the theorem holds. □

The following theorem gives a generalization of Theorem 4.3.8 to congruence modulo p^n for any positive integer n greater than m.

Theorem 4.3.11. *For an odd prime p and positive integers a, u, and n such that $a \leq p^n$, the following two congruences holds;*

$$S(u+a,a) \equiv (-1)^u \sum_j \binom{p^{n-1}+j-1}{j} s(p^n-a, p^n-a-u+j(p-1)) \pmod{p^n} \tag{4.66}$$

and

$$s(a,a-u) \equiv \sum_j (-1)^{u+j}\binom{p^{n-1}}{j} S(u-j(p-1)+p^n-a, p^n-a) \pmod{p^n}. \tag{4.67}$$

Proof. The proof is similar to the proof of Theorems 4.3.8 and 4.3.9. □

Remark 4.3.6. *It follows from Theorem 4.3.11 that if $0 \leq u < p-1$, then the index j has only one possible value, which is $j = 0$. Therefore,*

$$S(u+a,a) \equiv (-1)^u s(p^n-a, p^n-a-u) \pmod{p^n} \tag{4.68}$$

and

$$s(a,a-u) \equiv (-1)^u S(u+p^n-a, p^n-a) \pmod{p^n}. \tag{4.69}$$

4.4 Conclusions

We study the congruence relationship between Stirling numbers of the first and second kinds using their generating functions. We have developed some powerful congruences for both $S(n, k)$ and $s(n, k)$ separately. The congruence connecting the two numbers are also obtained. One of the interesting result is that the congruence properties of $S(n-1, kp^m - 1)$ and $S(n, kp^m)$ are the same. We also find that some congruences developed in this chapter are effective in finding their p-adic valuations. It is evident from the results that the congruence obtained are more effective in evaluating p-adic valuations of $S(n, k)$ when $v_p(k)$ is non-zero ($v_p(n)$ in the case of $s(n, k)$). We even establish that a super congruence in modulo p^m for any $m \in \mathbb{Z}^+$ can be obtained for any $S(n, k)$ in terms of a sum containing products of Stirling numbers of the first kind and binomial coefficient and vice versa.

Chapter 5

Periodicity and Divisibility of Stirling Numbers of the Second Kind[4]

5.1 Introduction

The sequence of Stirling numbers of the second kind, $S(n,k)$ for a fixed k, is periodic in modulo p^N for a positive integer N and a prime p. Generally, $\pi(k;p^N)$ denotes the minimum period of the sequence $\{S(n,k) \pmod{p^N}\}_{n\geq 0}$. Trivially, $\pi(1,p^N)=1$ for any $N\geq 1$. We denote $\gamma(k;p^N)$ as the greatest positive integer such that $S(\gamma(k;p^N)-1,k) \not\equiv S(\gamma(k;p^N)-1+\pi(k;p^N),k) \pmod{p^N}$ and $\gamma(k;p^N)=0$ if there exists no such positive integer. Carlitz (1955) proved that the period of $Bell(r,s) \pmod{p^k}$ is a divisor of

$$p^{k-l}(p^{p^m}-1), \qquad p^{m-1}\leq s<p^m;$$

for $s=1$, there is a slightly better period $p^k(p^p-1)/(p-1)$ corresponding to the known result $(p^p-1)/(p-1)$ in the case $k=1$. Kwong (1989a) confirmed the

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following minimum period:

$$1)\ \ \pi(1;2^N) = \pi(2;2^N) = 1, \tag{5.1}$$

$$2)\ \ \pi(3,2^N) = \pi(4,2^N) = \begin{cases} 2, & \text{if } N = 1 \text{ or } 2; \\ 2^{N-1}, & \text{if } N \geq 3, \end{cases} \tag{5.2}$$

$$3)\ \ \pi(k;p^N) = 2^{N+b-2} \quad \text{if } 2^{b-1} < k \leq 2^b,\ b \geq 3, \tag{5.3}$$

$$4)\ \ \pi(k;p^N) = (p-1)p^{N+b-2} \quad \text{if } k > p > 2 \text{ and } p^{b-1} < k \leq p^b. \tag{5.4}$$

In this chapter, we use the periodicity properties and partial Stirling numbers to obtain results about the divisibility of Stirling numbers of the second kind. The main results include different congruence results of $S(n,k) \pmod{p^N}$, $N \in \mathbb{Z}^+$, where n and k are classified into different cases. We classify n base on its divisibility relation with $(p-1)p^{N-1}$ or $(p-1)p^N$. On the other hand, different cases of k are $1 \leq k \leq p$, $k = p$, $p \leq k < 2p$ and $k > p$. The main results also include evaluation of different values of $v_p(S(n,k))$ in terms of $s_p(k)$ for different classes of n and k. We present some applications of the results for primes $p = 2$ and $p = 3$.

5.2 Materials and Methods

Definition 5.2.1. *For any prime p and positive integer k, the partial Stirling numbers $\alpha_p(n,k)$ and $\beta_p(n,k)$ are defined as*

$$\alpha_p(n,k) = \sum_{p|i} \binom{k}{i} (-1)^i i^n$$

and

$$\beta_p(n,k) = \sum_{p\nmid i} \binom{k}{i} (-1)^i i^n.$$

Thus,

$$(-1)^k k! S(n,k) = \alpha_p(n,k) + \beta_p(n,k),$$

which follows

$$\alpha_p(n,k) \equiv 0 \pmod{p^m} \tag{5.5}$$

and

$$\beta_p(n,k) \equiv (-1)^k k! S(n,k) \pmod{p^m}, \tag{5.6}$$

whenever $m \geq n$.

Guo and Zhang (2014) proved the following identity

$$\sum_{k=-\infty}^{\infty} \binom{2n}{n+3k} (-1)^k = 2 \cdot 3^{n-1}. \tag{5.7}$$

Bach (1968) generalized the above identity as

$$\sum_{k=-\infty}^{\infty} \binom{2n+r}{n+3k} (-1)^k = 2 \cdot 3^{n-1+\frac{r}{2}} cos \frac{r\pi}{6} \tag{5.8}$$

for positive integers n and r.

Theorem 5.2.1. *(Lundell, 1978) Let p be an odd prime. For positive integers r and k such that $r < k$,*

$$\sum_{i \geq 0} (-1)^i \binom{k}{ip} (ip)^r \equiv 0 \pmod{p^{max\{\lfloor \frac{k-r-1}{p-1} \rfloor + v_p(k), r\}}}. \tag{5.9}$$

The notation $\lfloor x \rfloor$ denotes the greatest integer function of x. A stronger result for the above result with a restriction on k such that $r - (p-1)\lfloor \frac{r}{p-1} \rfloor - 1 \leq k - (p-1)\lfloor \frac{k}{p-1} \rfloor$ and $k > r > p$ is

$$\sum_{i \geq 0} (-1)^i \binom{k}{ip} (ip)^r \equiv 0 \pmod{p^{max\{\lfloor \frac{k}{p-1} \rfloor - \lfloor \frac{r}{p-1} \rfloor + v_p(k), r\}}}. \tag{5.10}$$

Another result analogous to Theorem 5.2.1 with restricted k to $k - (p-1)\lfloor \frac{k}{p-1} \rfloor < r - (p-1)\lfloor \frac{r}{p-1} \rfloor - 1$ is

Theorem 5.2.2. *(Lundell, 1978) Let $k = q(p-1) + a = up + b$, $0 \leq b < p$, and*

$1 \leq r < p-1$. *If* $0 \leq a < r-1$, *then*

i) for $b = 0$ *or* $b > a+1$*:*

$$\sum_{i \geq 0} (-1)^i \binom{k}{ip} (ip)^r \equiv 0 \pmod{p^{q-1+v_p(k)}}. \tag{5.11}$$

ii) for $b = a+1$*:*

$$\sum_{i \geq 0} (-1)^i \binom{k}{ip} (ip)^r \equiv (-1)^a (-p)^{q-1} [(a+1)!(S(r+1, a+1) - bS(r, a+1))] \pmod{p^q}. \tag{5.12}$$

iii) for $b = a > 0$*:*

$$\sum_{i \geq 0} (-1)^i \binom{k}{ip} (ip)^r \equiv (-1)^a (-p)^{q-1} a! S(r, a) \pmod{p^q}. \tag{5.13}$$

iv) for $0 < b < a$*:*

$$\sum_{i \geq 0} (-1)^i \binom{k}{ip} (ip)^r \equiv 0 \pmod{p^q}. \tag{5.14}$$

Theorem 5.2.3. *(Gessel and Lengyel, 2001) Let* p *be an odd prime and* m *be an integer with* $0 < m < min\{k, p\}$ *such that* $r = \frac{k-m}{p-1}$ *is an integer. We set* $r \equiv r'$ $\pmod p$ *with* $1 \leq r' \leq p$. *If* $r' > m$, *then for any integer* t

$$\sum_{i \equiv t \pmod p} \binom{k}{i} (-1)^i i^m \equiv (-1)^{m + \frac{k-m}{p-1} - 1} m! \binom{k}{m} p^{\frac{k-m}{p-1} - 1} \pmod{p^{\frac{k-m}{p-1} + v_p(m! \binom{k}{m})}}. \tag{5.15}$$

The above results are employed in the next section to determine the p-adic valuations of $S(n, k)$.

5.3 Results and Discussion

This section presents our main results in theorems. We begin with the divisibility of $S(n, k)$ by power of a prime, p when $k < p$.

Theorem 5.3.1. *Let p be an odd prime and k be a positive integer such that $k < p$. For any positive integer N, the following congruence holds:*

i) If $n \equiv 0 \pmod{(p-1)p^{N-1}}$ with $n > 0$, then

$$S(n,k) \equiv \frac{(-1)^{k-1}}{k!} \pmod{p^N}. \tag{5.16}$$

ii) If $n \equiv m \pmod{(p-1)p^{N-1}}$ for some integer m such that $1 \leq m < k$, then

$$S(n,k) \equiv 0 \pmod{p^N}. \tag{5.17}$$

iii) If $n \equiv k \pmod{(p-1)p^{N-1}}$ with $n > m$, then

$$S(n,k) \equiv 1 \pmod{p^N}. \tag{5.18}$$

Proof. Using Equation (1.32), we have

$$S(m,k) = \frac{1}{k!}\sum_{i=1}^{k}\binom{k}{i}(-1)^{k-i}i^m.$$

Since $k < p$, for any i, $1 \leq i \leq k$, we have

$$i^{m+(p-1)p^{N-1}} \equiv i^m \pmod{p^N}$$

for any positive integer N and m. It follows that

$$S(m,k) \equiv \frac{1}{k!}\sum_{i=1}^{k}\binom{k}{i}(-1)^{k-i}i^{m+(p-1)p^{N-1}} \equiv S(m+(p-1)p^{N-1},k) \pmod{p^N}.$$

If $1 \leq m < k$, $S(m,k) = 0$, and the second result follows.

If we put $m = k$, we obtain the third result.

The first result is the consequence of the following congruence

$$\begin{aligned}\frac{1}{k!}\sum_{i=1}^{k}\binom{k}{i}(-1)^{k-i}i^{(p-1)p^{N-1}} &\equiv \frac{1}{k!}\sum_{i=1}^{k}\binom{k}{i}(-1)^{k-i} \pmod{p^N}\\ &\equiv \frac{1}{k!}[(1-1)^k-(-1)^k] \equiv \frac{(-1)^{k-1}}{k!} \pmod{p^N}.\end{aligned}$$

□

Remark 5.3.1. *We know that* $(p-1)p^{N-1}$ *is the period of the sequence* $\{S(n,k) \pmod{p^N}\}$ *when* $k \le p$. *However, the minimum period of the sequence is the least common multiple of the orders of* i *modulo* p^N *for* $1 \le i \le k$. *Theorem 5.3.1 still holds if we replace* $(p-1)p^{N-1}$ *with* $\pi(k;p^N)$. *From the theorem, we can observe that* $\gamma(k;p^N) = 1$ *if* $k \le p$.

The next theorem presents the divisibility properties of $S(n,p)$ for an odd prime p.

Theorem 5.3.2. *Let* p *be an odd prime and* n *be an integer such that* $n \ge p$. *The following congruences hold:*
i) If $n \equiv 0 \pmod{(p-1)p^{N-1}}$ *for any positive integer* N, *then*

$$S(n,p) \equiv 0 \pmod{p^N}. \tag{5.19}$$

ii) If $1 \le m < p$, *then*

$$S(n,p) \equiv p^{m-1} \pmod{p^m}, \tag{5.20}$$

for any integer n *with* $n \equiv m \pmod{(p-1)p^{m-1}}$ *and* $n > m$. *Consequently,*

$$v_p(S(n,p)) = m-1. \tag{5.21}$$

Proof. For any positive integer m, we have

$$S(m,p) = \frac{1}{p!}\sum_{i=1}^{p}\binom{p}{i}(-1)^{p-i}i^m = \frac{p^m}{p!} + \frac{1}{p!}\sum_{i=1}^{p-1}\binom{p}{i}(-1)^{p-i}i^m.$$

Let N be any positive integer. Then

$$S(m,p) \equiv \frac{p^m}{p!} + \frac{1}{p!}\sum_{i=1}^{p-1}\binom{p}{i}(-1)^{p-i}i^{m+(p-1)p^{N-1}} \pmod{p^N}.$$

Since $p > 2$, $N \le m + (p-1)p^{N-1}$ for any positive integer m, we obtain

$$S(m,p) \equiv \frac{p^m}{p!} + S(m+(p-1)p^{N-1}, p) \pmod{p^N}.$$

Replacing N with m in the preceding equation, we get the second result.

To obtain the first result, we have

$$\begin{aligned} S((p-1)p^{N-1}, p) &= \frac{1}{p!}\sum_{i=1}^{p}\binom{p}{i}(-1)^{p-i} i^{(p-1)p^{N-1}} \\ &\equiv \frac{1}{p!}\sum_{i=1}^{p-1}\binom{p}{i}(-1)^{p-i} \pmod{p^N} \\ &\equiv 0 \pmod{p^N}. \end{aligned}$$

Hence the theorem follows. □

Remark 5.3.2. *The proof of the preceding theorem also confirms that* $\gamma(p; p^N) = N$.

We now discuss the divisibility of $S(n, k)$ when k is greater than a given odd prime p.

Lemma 5.3.1. *Let* p *be an odd prime,* k *and* N *be positive integers such that* $k > p$*, then*

$$N + v_p(k) \leq \pi(k; p^N). \tag{5.22}$$

Proof. The proof is straightforward. □

Theorem 5.3.3. *For an odd prime* p *and non negative-integers* $n,\ k,\ m,$ *and* N *such that* $k > p$ *and* $min\{N,\ m\} \geq 1$*, the following congruence holds:*

$$S(n+m\pi(k;p^N), k) \equiv \begin{cases} \frac{(-1)^k}{k!}\beta_p(n,k) \pmod{p^N}, & \text{if } n < N + v_p(k!); \\ S(n,k) \pmod{p^N} & \text{if } n \geq N + v_p(k!). \end{cases} \tag{5.23}$$

If $n < k$*, then*

$$S(n+m\pi(k;p^N), k) \equiv \begin{cases} \frac{(-1)^{k-1}}{k!}\alpha_p(n,k) \pmod{p^N}, & \text{if } n < N + v_p(k!); \\ 0 \pmod{p^N} & \text{if } n \geq N + v_p(k!). \end{cases} \tag{5.24}$$

Proof. We have

$$S(n,k) = \frac{1}{k!}\sum_{i=1}^{k}\binom{k}{i}(-1)^{k-i}i^n.$$

Let $v_p(k!) = t$, then

$$p^t S(n,k) = \frac{p^t}{k!}\sum_{i=1}^{k}\binom{k}{i}(-1)^{k-i}i^n.$$

If $p \nmid i$, it is well known that

$$i^{n+p^r(p-1)} \equiv i^n \pmod{p^{r+1}}.$$

However, if $p \mid i$, the preceding congruence holds only when $n \geq r+1$. Thus, we get

$$\begin{aligned} p^t S(n,k) &\equiv \frac{p^t}{k!}\sum_{p\nmid i}\binom{k}{i}(-1)^{k-i}i^{n+p^r(p-1)} + \frac{p^t}{k!}\sum_{p\mid i}\binom{k}{i}(-1)^{k-i}i^n \pmod{p^{r+1}} \\ &\equiv p^t S(n+p^r(p-1),k) + \frac{p^t}{k!}\sum_{p\mid i}\binom{k}{i}(-1)^{k-i}i^n \pmod{p^{r+1}}. \end{aligned}$$

Choose r such that $r+1 = v_p(k!) + N$ for some positive integer N. Therefore,

$$S(n,k) \equiv S(n+\mu,k) + \frac{(-1)^k}{k!}\alpha_p(n,k) \pmod{p^N}, \tag{5.25}$$

where $\mu = (p-1)p^{t+N-1}$ and $\alpha_p(n,k) = \sum_{p\mid i}\binom{k}{i}(-1)^{k-i}i^n$. The term $\frac{1}{k!}\alpha_p(n,k)$ vanishes if $n \geq N + v_p(k!)$, and it follows

$$\gamma(k;p^N) \leq N + v_p(k!). \tag{5.26}$$

Using the concept of minimum periods and the fact that $\gamma(k;p^N) \leq N + v_p(k!) \leq \pi(k;p^N)$, it is easy to confirm that

$$S(n+\mu,k) \equiv S(n+m\pi(k;p^N),k) \pmod{p^N}, \tag{5.27}$$

for any positive integers m and n. Hence, Equations (5.25) and (5.27) confirm the first result of the theorem.

If we restrict the value of n strictly less than k, then $S(n,k) = 0$ and the second result follows immediately. $\square$

Remark 5.3.3. *Observing the theorem, it can be seen that*

$$v_p(S(n,k)) = v_p(S(m,k)) \tag{5.28}$$

whenever $n \equiv m \pmod{\pi(k; p^{1+v_p(S(n,k))})}$.

Equation (5.25) *confirms that* $\gamma(k;p^N)$ *is the greatest non-negative integer* n *such that*

$$\frac{1}{k!}\alpha_p(n-1,k) \not\equiv 0 \pmod{p^N}.$$

Equation (5.24) *confirms that for any positive integer* N,

$$v_p(S(n,k)) \geq N \tag{5.29}$$

if $n \equiv r \pmod{\pi(k;p^N)}$ *for some positive integer* r *such that* $N + v_p(k!) \leq r < k$.

Theorem 5.3.4. *For an odd prime* p *and integers* n, m, N, *and* k *such that* $p \leq k < 2p$ *and* $1 \leq m < N+1$,

$$S(n,k) \equiv \frac{(-1)^k p^{m-1}}{(p-1)!(k-p)!} \pmod{p^N}, \tag{5.30}$$

whenever $n \equiv m \pmod{(p-1)p^N}$ *with* $n > m$. *Hence, the corresponding exact* p*-adic valuation is*

$$v_p(S(n,k)) = m-1, \tag{5.31}$$

whenever $n \equiv m \pmod{(p-1)p^m}$ *with* $n > m$.

Proof. Here, $v_p(k!) = 1$. Let m be a positive integer such that $m < N+1 = N + v_p(k!)$ and $m < k$, then using Theorem 5.3.3, we have

$$S(n,k) \equiv \frac{(-1)^{k-1}}{k!}\alpha_p(m,k) \pmod{p^N} \tag{5.32}$$

for any positive integer n satisfying $n \equiv m \pmod{\pi(k;p^N)}$ with $\pi(k;p^N) =$

$(p-1)p^N$. Since $p < k < 2p$, we have

$$\alpha_p(m,k) = \binom{k}{p}(-1)^p p^m. \tag{5.33}$$

It follows that

$$\begin{aligned} S(n,k) &\equiv \frac{(-1)^k p^m}{p!(k-p)!} \pmod{p^N} \\ &\equiv \frac{(-1)^k p^{m-1}}{(p-1)!(k-p)!} \pmod{p^N}. \end{aligned} \tag{5.34}$$

Hence, the first result holds. If we replace $N = m$, we get the second result. □

Theorem 5.3.5. *Let p be an odd prime, k and m be positive integers such that $m \geq k$ and $p > k$, then for any integers a and $n \neq m$ with $n \equiv m \pmod{(p-1)p^{m-k+1}}$,*

$$S(n, kp+a) \equiv \frac{(-1)^{k+a-1}p^m}{(kp+a)!} k! S(m,k) \pmod{p^{m-k+1}}. \tag{5.35}$$

Furthermore, if $p \nmid S(m,k)$, then

$$v_p(S(n, kp+a)) = m - k. \tag{5.36}$$

Proof. In this case, $v_p((kp+a)!) = k$. Replace N with $m-k+1$ and k with $kp+a$ in Theorem 5.3.3, then we get $m < v_p((kp+a)!) + N = k + m - k + 1 = m + 1$ and

$$S(n,k) \equiv \frac{(-1)^{kp+a-1}}{(kp+a)!} \alpha_p(m, kp+a) \pmod{p^{m-k+1}} \tag{5.37}$$

if $n \equiv m \pmod{\pi(kp+a; p^{m-k+1})}$, where $\pi(kp+a; p^{m-k+1}) = (p-1)p^{m-k+1}$.

Now, we have

$$\begin{aligned}\frac{(-1)^{kp+a-1}}{(kp+a)!}\alpha_p(m,kp+a) &= \frac{(-1)^{k+a-1}}{(kp+a)!}\sum_{i=1}^{k}\binom{kp+a}{ip}(-1)^{ip}(ip)^m \\ &\equiv \frac{(-1)^{k+a-1}p^m}{(kp+a)!}\sum_{i=1}^{k}\binom{k}{i}(-1)^i(i)^m \pmod{p^{m-k+1}} \\ &\equiv \frac{(-1)^{k+a-1}p^m}{(kp+a)!}k!S(m,k) \pmod{p^{m-k+1}}. \qquad (5.38)\end{aligned}$$

Combining Equations (5.37) and (5.38), the theorem follows. □

Theorem 5.3.6. *Let p be an odd prime, N, m, and k be integers such that $m < \lfloor \frac{k-m-1}{p-1} \rfloor + v_p(k) = N$, $k > m$, and $s_p(k-1) > m$. Then*

$$v_p(S(n,k)) \geq \left\lfloor \frac{s_p(k-1)-m}{p-1} \right\rfloor \qquad (5.39)$$

for any positive integer $n > m$, such that $n \equiv m \pmod{\pi(k;p^N)}$.

Proof. Taking $N = \lfloor \frac{k-n-1}{p-1} \rfloor + v_p(k)$, $m < N$, and $m < k$, it is trivial that $m < N + v_p(k!)$. Therefore, using Theorem 5.3.3, we have

$$S(n,k) \equiv \frac{(-1)^{k-1}}{k!}\alpha_p(m,k) \pmod{p^N}, \qquad (5.40)$$

if $n \equiv m \pmod{\pi(k;p^N)}$.

From Theorem 5.2.1, we also have

$$\alpha_p(m,k) \equiv 0 \pmod{p^N} \qquad (5.41)$$

since we assume $N > m$.

Combining Equations (5.40) and (5.41), we get

$$S(n,k) \equiv 0 \pmod{p^{N-v_p(k)}}, \qquad (5.42)$$

assuming $N > v_p(k!)$. It follows that

$$v_p(S(n,k)) \geq N - v_p(k!), \qquad (5.43)$$

with the given condition of n.

Now, we have

$$\begin{aligned} N - v_p(k!) &= \left\lfloor \frac{k-m-1}{p-1} \right\rfloor + v_p(k) - v_p(k!) \\ &= \left\lfloor \frac{k-m-1}{p-1} \right\rfloor - v_p((k-1)!) \\ &= \left\lfloor \frac{s_p(k-1)-m}{p-1} \right\rfloor. \end{aligned}$$

Hence the theorem follows. □

Theorem 5.3.7. *Let p be an odd prime, N, k, a, b are non-negative integers such that $k > p$, $k \equiv a \pmod{p-1}$, $k \equiv b \pmod{p}$, $0 \le b < p$, and $1 \le r < p-1$. If $0 \le a < r-1 < N + v_p(k!) - 1$, $n > r$ and $n \equiv r \pmod{\pi(k; p^N)}$, then the following results hold:*

i) for $b = 0$ or $b > a+1$, and $N = \lfloor \frac{k}{p-1} \rfloor - 1 + v_p(k)$, then

$$v_p(S(n,k)) \ge \left\lfloor \frac{s_p(k-1)+1}{p-1} - 1 \right\rfloor. \tag{5.44}$$

ii) for $b = a+1$, and $N = \lfloor \frac{k}{p-1} \rfloor$, then

$$v_p(S(n,k)) = \left\lfloor \frac{s_p(k)}{p-1} \right\rfloor - 1 \tag{5.45}$$

if $S(r+1,b) \not\equiv bS(r,b) \pmod{p}$.

iii) for $b = a > 0$, and $N = \lfloor \frac{k}{p-1} \rfloor$, then

$$v_p(S(n,k)) = \left\lfloor \frac{s_p(k)}{p-1} \right\rfloor - 1 \tag{5.46}$$

if $p \nmid S(r,a)$.

iv) for $0 < b < a$, and $N = \lfloor \frac{k}{p-1} \rfloor$, then

$$v_p(S(n,k)) \ge \left\lfloor \frac{s_p(k)}{p-1} \right\rfloor. \tag{5.47}$$

Proof. We use Theorems 5.2.2 and 5.3.3 to prove the theorem.

i) Following the proof of Theorem 5.3.6, we have

$$\begin{aligned} N - v_p(k!) &= \left\lfloor \frac{k}{p-1} \right\rfloor - 1 + v_p(k) - v_p(k!) \\ &= \left\lfloor \frac{k}{p-1} - v_p((k-1)!) \right\rfloor - 1 \\ &= \left\lfloor \frac{1 + s_p(k-1)}{p-1} \right\rfloor - 1, \end{aligned}$$

where $N = \lfloor \frac{k}{p-1} \rfloor - 1 + v_p(k)$.

ii), iii), and iv). If $N = \lfloor \frac{k}{p-1} \rfloor$, then

$$\begin{aligned} N - v_p(k!) &= \left\lfloor \frac{k}{p-1} \right\rfloor - v_p(k!) \\ &= \left\lfloor \frac{k}{p-1} - v_p(k!) \right\rfloor \\ &= \left\lfloor \frac{s_p(k)}{p-1} \right\rfloor - 1. \end{aligned}$$

□

Theorem 5.3.8. *Let p be an odd prime, k and m be integers with $k > p$ and $0 < m < p$ such that $k \equiv m \pmod{p-1}$ and $\frac{k-m}{p-1} \equiv r \pmod{p}$ with $1 \le r \le p$. If $r > m$, then*

$$v_p(S(n,k)) = \left\lfloor \frac{s_p(k-m)}{p-1} \right\rfloor \tag{5.48}$$

for any integer n satisfying $n \equiv m \pmod{\pi(k; p^N)}$, where $N = \frac{k-m}{p-1} + v_p(m!\binom{k}{m})$.

Proof. We use Theorems 5.2.3 and 5.3.3 to prove the theorem. Following the proof of Theorem 5.3.6 and the preceding theorem, it is easy to show that

$$N - v_p(k!) = \left\lfloor \frac{s_p(k-m)}{p-1} \right\rfloor . \tag{5.49}$$

Hence the theorem follows. □

5.3.1 The case for prime $p = 3$

In this case, we can classify k into six different equivalent classes, namely, $6m$, $6m+1, \cdots, 6m+5$. Using Equations (5.7) and (5.8), we obtain the following results for $m > 0$,

$$\alpha_3(0, 6m+r) = \begin{cases} (-1)^m 2 \cdot 3^{3m-1}, & \text{if } r = 0; \\ (-1)^m 3^{3m}, & \text{if } r = 1 \textit{ or } 2; \\ 0, & \text{if } r = 3; \\ (-1)^{m+1} 3^{3m+1}, & \text{if } r = 4; \\ (-1)^{m+1} 3^{3m+2}, & \text{if } r = 5. \end{cases} \tag{5.50}$$

Using the binomial identity of the form

$$\binom{k}{ip}(ip) = k\left[\binom{k}{ip} - \binom{k-1}{ip}\right],$$

we get the following recursion relation for $\alpha_p(n,k)$ as

$$\sum (-1)^i \binom{k}{ip}(ip)^n = k\left[\sum (-1)^i \binom{k}{ip}(ip)^{n-1} - \sum (-1)^i \binom{k-1}{ip}(ip)^{n-1}\right].$$

The above identity can also be written as

$$\alpha_p(n,k) = k[\alpha_p(n-1,k) - \alpha_p(n-1,k-1)]. \tag{5.51}$$

Using Equations (5.50) and (5.51), we obtain the following tables for the values of $\alpha_3(n, 6m+r)$ within the range $n \in \{1,\ 2,\ 3\}$ and $0 \le r \le 5$. The entry (r,n) gives the value of $\frac{(-1)^m \alpha_3(n,6m+r)}{3^{3m-1}}$.

Table-I: $(r,n) \longrightarrow \frac{(-1)^m \alpha_3(n,6m+r)}{3^{3m-1}}$

	$n=1$	$n=2$	$n=3$
$r=0$	$6m$	$4m(3m+1)$	$36m^2$
$r=1$	$(6m+1)$	$(6m+1)$	$-(6m+1)(12m^2-2m-1)$
$r=2$	0	$-(6m+1)(6m+2)$	$-3(6m+1)(6m+2)(2m+1)$
$r=3$	$-9(2m+1)$	$-27(2m+1)^2$	$-3(2m+1)(72m^2+90m+25)$
$r=4$	$-9(6m+4)$	$-9(6m+4)(4m+3)$	$-9(6m+4)(12m^2+22m+9)$
$r=5$	$-18(6m+5)$	$-54(6m+5)(m+1)$	$-9(6m+5)(12m^2+32m+18)$

If we set $N = v_3(\alpha_3(n, 6m+r)) - v_3((6m+r)!) + 1$, then $n < N + v_3((6m+r)!)$ for $m \geq 1$, $n = 1,\ 2,\ or\ 3$, and $0 \leq r \leq 5$ but $(r, n) \neq (2, 1)$. It follows that if $u \equiv n \pmod{\pi(6m+r; p^N)}$, the exact 3-adic valuations can be expressed by

$$\begin{aligned} v_3(S(u, 6m+r)) &= v_3(\alpha_3(n, 6m+r)) - v_3((6m+r)!). \\ &= \frac{s_3(6m+r) - r}{2} + f_{r,n}(m), \end{aligned} \tag{5.52}$$

for some function $f_{r,n}$. The values of $f_{r,n}(m)$ for $n = 1,\ 2$, and 3 are given in the following table:

Table-II: $(r, n) \longrightarrow f_{r,n}(m)$

	$n=1$	$n=2$	$n=3$
$r=0$	$v_3(m)$	$v_3(m)-1$	$2v_3(m)+1$
$r=1$	-1	-1	$v_3(12m^2-2m-1)-1$
$r=2$	NA	-1	$v_3(2m+1)$
$r=3$	$v_3(2m+1)+1$	$2v_3(2m+1)+2$	$v_3(2m+1)$
$r=4$	1	$v_3(4m+3)+1$	$v_3(12m^2+22m+9)+1$
$r=5$	1	$v_3(m+1)+2$	$v_3(12m^2+32m+18)+1$

From Table-I, we can see that $\alpha_3(1, 6m+2) = 0$. It follows that

$$v_3(S(u, 6m+2)) \geq N, \tag{5.53}$$

whenever $u \equiv 1 \pmod{\pi(6m+2; p^N)}$ for any positive integer N and $u \geq N + v_3((6m+2)!)$.

5.3.2 The case for prime $p = 2$

The minimum periods, for this case, are obtained by Kwong (1989a) as

$$\pi(k; 2^N) = \begin{cases} 1, & \text{if } k = 1 \textit{ or } 2; \\ 2, & \text{if } k = 3 \textit{ or } 4 \text{ and } N = 1 \textit{ or } 2; \\ 2^{N-1}, & \text{if } k = 3 \textit{ or } 4 \text{ and } N > 2; \\ 2^{N+b-2} & \text{if } 2^{b-1} < k \leq 2^b \text{ and } b \geq 3. \end{cases} \tag{5.54}$$

Unlike the above case, the sum $\alpha_2(n, k)$ does not have the alternating sign as

$$\alpha_2(n, k) = \sum_{2|i} \binom{k}{i} i^n,$$

which gives

$$\alpha_2(0, k) = \sum_{2|i} \binom{k}{i} = 2^{k-1}. \tag{5.55}$$

Using Equation (5.51), we obtain the following identities:

$$\alpha_2(1, k) = k2^{k-2}, \quad (k > 2) \tag{5.56}$$

$$\alpha_2(2, k) = k(k+1)2^{k-3}, \quad (k \geq 3) \tag{5.57}$$

$$\alpha_2(3, k) = k^2(k+3)2^{k-4}, \quad (k \geq 4) \tag{5.58}$$

$$\alpha_2(4, k) = k(k+1)(k^2+5k-2)2^{k-5}, \quad (k \geq 5) \tag{5.59}$$

$$\alpha_2(5, k) = k^2(k^3+10k^2+15k-10)2^{k-6}, \quad (k \geq 6) \tag{5.60}$$

$$\alpha_2(6, k) = k(k+1)(k^4+14k^3+31k^2-46k+16)2^{k-7}, \quad (k \geq 7) \tag{5.61}$$

and so on. It follows that $\alpha_2(n,k)$, where $k \geq n$, can be written in the form

$$\alpha_2(n,k) = g_n(k)2^{k-n-1}, \tag{5.62}$$

where $g_n(k)$ is a polynomial over $\mathbb{Z}$ of degree n. The polynomial g_n can be generated by the recursion

$$g_{n+1}(k) = k[2g_n(k) - g_n(k-1)]. \tag{5.63}$$

with initial polynomial $g_1(k) = k$.

Let us take $k = 3$ and $k = 4$. The second result of Theorem 5.3.3 is also valid for an even prime $p = 2$ if we add a condition $n + m\pi(k; p^N) \geq v_p(k!) + N$. In case, $k = 4$ and $N = 2$, we get

$$v_2(k!) + N = 5.$$

We also know that $\pi(4; 2^2) = 2$ and it follows that

$$S(n,4) \equiv \frac{(-1)^3}{4!}\alpha_2(0,4) \pmod 4 \equiv -\frac{2^3}{4!} \pmod 4 \equiv 1 \pmod 4$$

if $n \geq 5$ and $n \equiv 0 \pmod 2$. On the other hand, if $n \equiv 1 \pmod 2$ or n is odd and $n \geq 5$, then

$$S(n,4) \equiv \frac{(-1)^3}{4!}\alpha_2(1,4) \pmod 4 \equiv -\frac{4 \cdot 2^2}{4!} \pmod 4 \equiv 2 \pmod 4.$$

It follows that

$$v_2(S(n,4)) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases} \tag{5.64}$$

The case for $k = 3$ can be tackled similarly as

$$S(n,3) = \begin{cases} 1 \pmod 4, & \text{if } n \text{ is odd;} \\ 2 \pmod 4, & \text{if } n \text{ is even.} \end{cases} \tag{5.65}$$

Thus, the exact p-adic valuation is given by

$$v_2(S(n,3)) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases} \tag{5.66}$$

If we take $N = 3$, then the minimum period is 4 for both $k = 3$ and 4. The periodicity starts from $n = 5$, i.e., $\gamma(4;8) = 5$ and the cycle of the sequence $\{S(n,4) \pmod 8\}_{n\geq 0}$ is $(2,\ 1,\ 6,\ 5)$. Thus, the sequence looks like

$$\{0,\ 0,\ 0,\ 0,\ 1,\ 2,\ 1,\ 6,\ 5,\ 2,\ 1,\ 6,\ 5,\cdots\}.$$

The sequence for $\{S(n,3) \pmod 8\}_{n\geq 0}$ takes the following form

$$\{0,\ 0,\ 0,\ 1,\ 6,\ 1,\ 2,\ 5,\ 6,\ 1,\ 2,\ 5,\cdots\}.$$

The periodicity starts from $n = 4 = N + v_2(3!)$, and the cycle of the sequence is $(6,\ 1,\ 2,\ 5)$.

From Equation (5.56) for $k > 4$, we get

$$v_2(S(n,k)) = s_2(k-1) - 1 \tag{5.67}$$

if $n \equiv 1 \pmod{\pi(k; 2^N)}$ with $N = s_2(k-1)$ and $n \geq v_2(k!) + N$.

The generalization of Equation (5.67) can be obtained using Equation (5.62) as

$$v_2(S(n,k)) = s_2(k-1) - m + v_2(\frac{g_m(k)}{k}), \tag{5.68}$$

where $k > 4$, $n \geq v_2(k!) + N > m$, $N = v_2(\frac{g_m(k)}{k}) + s_2(k-1) - m + 1$ and $n \equiv m \pmod{\pi(k; 2^N)}$. Looking into the fact in Equations (5.56) to (5.61), we confirm that $g_n(k)$ is always even if $2 \leq n \leq 6$.

5.4 Conclusions

The partial sum of Stirling numbers $\alpha_p(n,k)$ plays a key role in obtaining $v_p(S(n,k))$. The minimum periods help us determine a class of Stirling numbers of the second kind, $\{S(m,k)\}_{m\in\wedge}$, for some indexing set $\wedge$, which share the same p-adic valuation. We have found that the periodicity of the sequence, $\{S(n,k) \pmod{p^N}\}_{n\geq 0}$ sometimes starts before the k-th term, i.e., $\gamma(k;p^N) < k$. We have proved that $\gamma(k;p^N) \leq N + v_p(k!)$ when $\gamma(k;p^N) < k$, the first $k - \gamma(k;p^N)$ entries of the cycle of $\{S(n,k) \pmod{p^N}\}_{n\geq 0}$ are zeros.

Chapter 6

Summary and Conclusions

We study the p-adic valuations of Stirling numbers of the second kind through different approaches. The existing results in the literature mostly account for 2-adic valuations and certain results for the odd prime p. The exact p-adic valuations of $S(n, k)$ estimate and lower bound for $v_p(S(n, k))$ are obtained for various classes of n and k. These results are usually accompanied with congruences modulo prime power. Some of the results are the generalization and extension of existing theorems and lemmas. We introduce alternate approach and material to obtain certain results. We also develop a new formula to count $S(n, k)$ which is very useful in obtaining the divisibility properties of $S(n, k)$.

The first chapter is the general introduction of the thesis, which includes a review of classical results, basic definitions, a brief introduction of p-adic analysis, applications of p-adic valuations and Stirling numbers, and a review of the literature.

Chapter 2 introduces a new formula for $S(n, k)$ using a combinatorial approach. The new formula is utilized to obtain some p-adic valuations and congruence properties of $S(n, k)$ for different classes of n and k. These results are then extended and generalized using minimum periods. We have also proved that the lower bound of $v_p(S(p^2, kp))$, $2 \leq k \leq p-1$, is 2, which confirms a part of

Conjecture 2.3.1.

Chapter 3 presents the p-adic valuations of $S(n,k)$ when n is a power of a prime. We find that the results when k is divisible by p (or p^m) are quite different from the ones where k is not divisible by p. We have proved that $v_p(S(p^2,kp)) \geq 5$ when k is even, which confirms the lower bound as in Conjecture 2.3.1. Furthermore, we find that the values of $v_p(S(n,kp^m))$ are affected by the parity of n and k. In fact, if n and k are opposite in parity, i.e., $n-k$ is odd, then $v_p(S(n,kp^m)) \geq 2m$ when $(p-1) \nmid (n-k)$ and $v_p(S(n,kp^m)) \geq m$ when $(p-1) \mid (n-k)$. However, if the parity of n and k are the same, i.e., $n-k$ is even, then $v_p(S(n,kp^m)) \geq m$ when $(p-1) \nmid (n-k)$. We further investigate the divisibility of $S(p^n,k)$ when p does not divide k and we have found that the divisibility depends on the sum of the p-adic digits of k.

In Chapter 4, we study a series of congruence relations between Stirling numbers of the first kind and the second kind. Certain congruence properties are obtained from the rational generating functions. We also obtain congruences for $S(n,k)$ and $s(k,n)$ in terms of certain sums involving binomial coefficients. We even introduce a super congruence modulo p^n for the Stirling numbers. These congruences are also utilized to determine the p-adic valuations of some classes of Stirling numbers of the first and second kinds. The application of these results for specific primes, $p=3$ and $p=5$ are also presented.

Chapter 5 develops a method to utilize the periodicity and minimum periods of the sequence $\{S(n,k) \pmod{p^N}\}_{n\geq 0}$ to obtain the p-adic valuations of some classes of Stirling numbers of the second kind. We obtain p-adic valuations of Stirling numbers of the second kind using partial Stirling numbers. We also find that some specific terms of the cycle of the periodic sequence $\{S(n,k) \pmod{p^N}\}_{n\geq 0}$ are always zero which confirm N as the lower bound of the p-adic valuations of the Stirling numbers. These results are confirmed for the case of $p=2$ and $p=3$.

Chapter 6 is a summary and conclusion.

Finally, a list of references is given at the end.

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Brief Bio-Data

Personal Information

Name	: A Lalchhuangliana, Assistant Professor Government Champhai College, Champhai, Mizoram
Father's name	: CVL Tanpuia
Mother's name	: Lalchhawnthangi
Date of Birth	: 04.08.1985
Nationality	: Indian
Gender	: Male
Marital Status	: Married
Present Address	: House No: T-39, Tlabung, Lunglei-796 751, Mizoram
e-mail	: abchhak@gmail.com

Academic Records

Examination	Board/University	Year
High School Leaving Certificate (HSLC)	MBSE, Mizoram	2001
Higher Secondary School Leaving Certificate	MBSE, Mizoram	2003
B.Sc.(Math)	Mizoram University	2006
M.Sc.(Math)	NEHU, Shillong	2009
M.Phil.(Math)	NEHU, Shillong	2011
JRF(NET)	CSIR-UGC	2009

(A LALCHHUANGLIANA)

List of Publications

1. SS Singh, A Lalchhuangliana and PK Saikia(2021). On the p-adic valuations of Stirling numbers of the second kind, *Contemporary Mathematics*, 2(1), 24-–36 (Scopus).

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3. A Lalchhuangliana and SS Singh(2023). Periodicity and divisibility of Stirling numbers of the second kind, *Science and Technology Journal*, 11(01), 82–89 (UGC-Care List).

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Research Article

On the P-Adic Valuations of Stirling Numbers of the Second Kind

S. S. Singh[1*], A. Lalchhuangliana[1], P. K. Saikia[2]

[1]Department of Mathematics and Computer Science, Mizoram University, Aizawl, Mizoram, India
[2]Department of Mathematics, North Eastern Hill University, Shillong, Meghalaya, India
E-mail: sssanasam@yahoo.com

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Abstract: In this paper, we introduced certain formulas for p-adic valuations of Stirling numbers of the second kind $S(n, k)$ denoted by $v_p(S(n, k))$ for an odd prime p and positive integers k such that $n \geq k$. We have obtained the formulas, $v_p(S(n, n - a))$ for $a = 1, 2, 3$ and $v_p(S(cp^n, cp^k))$ for $1 \leq c \leq p - 1$ and primality test of positive integer n. We have presented the results of $v_p(S(p^2, kp))$ for $2 \leq k \leq p - 1$, $2 < p < 100$ and a table of $v_p(S(p, k))$. We have posed the following conjectures from our analysis:

1. Let $p \neq 7$ be an odd prime and k be an even integer such that $0 < k < p - 1$. Then

$$v_p(S(p^2, kp)) - v_p(S(p^2, p(k+1)) = 3.$$

2. If k be an integer such that $1 < k < p - 1$, then the p-adic valuations satisfy

$$v_p(S(p^2, kp)) = \begin{cases} 5 \text{ or } 6, & if \;\; k \text{ is even} \\ 2 \text{ or } 3, & if \;\; k \text{ is odd} \end{cases}$$

for any prime $p > 7$.

3. For any primes p and positive integer k such that $2 \leq k \leq p - 1$, then

$$v_p(S(p, k)) \leq 2.$$

Keywords: p-adic valuations, stirling numbers of the second kind, congruence, primes, minimum period

MSC: 05A18,11A51,11B73, 11E95

1. Introduction

Stirling numbers of the first and second kinds were introduced by James Stirling [1]. These numbers have been found to be of great utility in various branches of Mathematics such as combinatorics, number theory, calculus of finite differences, theory of algorithms, etc. The p-adic valuations of Stirling numbers of the second kind appear frequently

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in algebraic topology by Davis [2] to obtain new results related to James numbers, v_1-periodic homotopy groups and exponents of $SU(n)$. More details of Stirling numbers of the second kind may be seen on Comtet [3] and Graham et al. [4].

Stirling numbers of the second kind are more interesting than the first kind by their intrinsic nature. There are many interesting results of 2-adic valuations of Stirling numbers of the second kind in the open literature. Recently, Wannemacker's proof [5] of Lengyel's conjecture [6], results of $v_2(k!S(c-2n+u, k))$ for $c > 0$ by Lengyel [7], the proof of Wannemacker's conjecture by Hong [8], the works of Amdeberhan et al. [9] and Zhao et al. [10] are other notable results of 2-adic valuation. Gessel and Lengyel [11] proved that for an arbitrary prime p and $n = a(p-1)p^q$, $1 \le k \le n$

$$v_p(k!S(n,k)) = \left\lfloor \frac{k-1}{p-1} + \tau(k) \right\rfloor,$$

where a and q are positive integers such that $(a, p) = 1$, q is sufficiently large, $\frac{k}{p}$ is an odd integer and $\tau(p)$ is a non-negative integer.

Strauss [12] and Pan [13] discussed the problems of 3-adic valuations and 2-adic valuations of certain sums of binomial coefficients respectively. Sun [14] also presented the results of p-adic valuations for multinomial coefficients. Friedland [15] used 2-adic valuations of certain ratios of factorials to prove a conjecture of Falikman-Friedland-Lowery on the parity of degrees of projective varieties of $n \times n$ complex symmetric matrices of rank at most k. Some more results of p-adic valuations are also given in Gouvea [16], Koblitz [17] and Adelberg [18].

This paper consists of some interesting results about p-adic valuations for a few class of Stirling numbers of the second kind $S(n, k)$. This number $v_p(S(n, k))$, where either n or k is related to p, has been obtained independently for some values of p, n and k. The values of $v_p(S(n, k))$ are computed by using GP/PARI software and they are presented in Table 1.

2. Materials and methods

Definition 2.1 Let p be a prime. For any non-zero integer a, the p-adic valuation of a, denoted by $v_p(a)$, is defined as the exponent of the highest power of p dividing a.

It may be noted that $v_p(0) = \infty$ and $v_p(a)$ for a non-zero integer a, is a non-negative integer.

So, $v_3(25) = 0$, $v_5(25) = 2$.

Note that, for any prime p, $v_p(\pm 1) = 0$. For a given prime p and any two integers a and b, we have

$$v_p(a+b) \ge min\{v_p(a), v_p(b)\}, \quad v_p(ab) = v_p(a) + v_p(b).$$

The p-adic valuation v_p can further be extended to the field of rational numbers, $r = \frac{a}{b}$, $a, b \in \mathbb{Z}$ and $b \ne 0$ as

$$v_p(r) = v_p(a) - v_p(b).$$

Definition 2.2 Given two non-negative integers n and k, not both zero, the Stirling number of the second kind $S(n, k)$ is defined as the number of ways one can partition a set with n elements into exactly k non-empty subsets.

Example 2.1 All partitions of the set {1, 2, 3, 4} into 2 non-empty subsets are {1}, {2, 3, 4}; {2}, {1, 3, 4}; {3}, {1, 2, 4}; {4}, {1, 2, 3}; {1, 2}, {3, 4}; {1, 3}, {2, 4} and {1, 4}, {2, 3}. Hence, $S(4, 2) = 7$.

By convention, we set $S(0, 0) = 1$ and $S(0, k) = 0$ for $k \ge 1$. Thus, $S(n, k)$ is the number of ways of distributing n distinct balls into k indistinguishable boxes (the order of the boxes does not count) such that no box is empty.

It is clear that $S(n, k) = 0$ if $1 \le n < k$ and $S(n, n) = 1$ for all $n \ge 0$.

We use the following properties to prove the results of $v_p(S(n, k))$:

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i} i^n, \tag{1}$$

which gives

$$S(n,2)=2^{n-1}-1,\ S(n,1)=1,\ S(n,0)=0. \tag{2}$$

It is easy to derive the following specific identities of $S(n, k)$ using the results of ([19] p. 115-116).

$$S(n,n-1)=\binom{n}{2}\ \ if\ n\geq 2, \tag{3}$$

$$S(n,n-2)=\binom{n}{3}+3\binom{n}{4}\ \ if\ n\geq 4, \tag{4}$$

$$S(n,n-3)=\binom{n}{4}+10\binom{n}{5}+15\binom{n}{6}\ \ if\ n\geq 6. \tag{5}$$

3. Results

In this section, we present some basic results of the p-adic valuations of Stirling numbers starting with $S(n, n-1)$ for $n > 1$.

Proposition 3.1 For any positive integer $n > 1$ and an odd prime p

$$v_p\left(S\left(n,n-1\right)\right)=v_p\left(n\right)+v_p\left(n-1\right).$$

Proof. Using the identity (3), we have

$$S(n,n-1)=\binom{n}{2}=\frac{n(n-1)}{2}.$$

The multiplicative property of $v_p(a)$ implies that

$$v_p(S(n, n-1)) = v_p(n) + v_p(n-1) - v_p(2)$$

$$= v_p(n) + v_p(n-1)$$

as $v_p(2) = 0$, p being odd.

Applying Kummer's theorem [20] to the binomial coefficient $\binom{n}{2} = S(n, n-1)$, the above result can be put in the following form

$$v_p(S(n,n-1))=\frac{s_p(n-2)-s_p(n)+2}{p-1}, \tag{6}$$

where $s_p(n)$ denotes the sum of the p-adic digits of n.

Corollary 3.1 Let p be an odd prime. For any positive integer n and c such that $gcd(p, c) = 1$,

$$v_p(S(cp^n,cp^n-1))=n.$$

Proof. By the proposition, we have

$$v_p(S(cp^n, cp^n-1)) = v_p(cp^n) + v_p(cp^n-1).$$

Since $v_p(cp^n - 1) = 0$ and using the multiplicative property of $v_p(a)$, we can obtain

$$\begin{aligned} v_p(S(cp^n, cp^n-1)) &= v_p(cp^n) \\ &= n + v_p(c). \end{aligned}$$

As $gcd(p, c) = 1$, it is clear that $v_p(c) = 0$. This completes the proof.

Proposition 3.2 For any positive integer $n \geq 2$ and an odd prime p,

$$v_p(S(n,n-2)) = \begin{cases} v_p(n) + v_p(n-1) + v_p(n-2) + v_p(3n-5), & \text{if } p > 3, \\ v_3(n) + v_3(n-1) + v_3(n-2) - 1, & \text{if } p = 3. \end{cases}$$

These results can be proved in the similar manner.

Corollary 3.2 For any positive integer n and an odd prime p,

$$v_p(S(cp^n, cp^n-2)) = \begin{cases} n, & \text{if } p > 5, \\ n+1, & \text{if } p = 5 \text{ and } n > 1, \\ n-1, & \text{if } p = 3, \end{cases}$$

if c is a positive integer not divisible by p.

Proposition 3.3 Let p be an odd prime. For any positive integer $n \geq 6$,

$$v_p(S(n,n-3)) = \begin{cases} v_p(n) + v_p(n-1) + 2v_p(n-2) + 2v_p(n-3), & \text{if } p \geq 5, \\ v_p(n) + v_p(n-1) + 2v_p(n-2) + 2v_p(n-3) - 1, & \text{if } p = 3. \end{cases}$$

Proof. Using the identity (5), we have

$$S(n,n-3) = \binom{n}{4} + 10\binom{n}{5} + 15\binom{n}{6}, \text{ if } n \geq 6.$$

It can also be expressed as

$$\begin{aligned} S(n,n-3) &= \binom{n}{4}\left[\frac{n^2-5n+6}{2}\right] \\ &= \binom{n}{4}\left[\frac{(n-2)(n-3)}{2}\right] \\ &= \left[\frac{n(n-1)(n-2)^2(n-3)^2}{2^4 \cdot 3}\right] \end{aligned}$$

The multiplicative property of $v_p(-)$ implies that

$$v_p\left(S(n,n-3)\right) = v_p(n) + v_p(n-1) + 2v_p(n-2) + 2v_p(n-3) - v_p(3)$$

as $v_p(2) = 0$ and p being odd.

Using Kummer's theorem [20] to $\binom{n}{4}$, we get the following result,

$$v_p(S(n,n-3)) = \frac{s_p(n-4) - s_p(n) + s_p(4)}{p-1} + v_p(n-2) + v_p(n-3). \tag{7}$$

where $s_p(n)$ denotes the sum of the p-adic digits of n. This completes the proof.

Corollary 3.3 For any positive integer n and odd prime p, the following result holds

$$v_p(S(cp^n, cp^n - 3)) = \begin{cases} n, & if \quad p > 3, \\ n+1, & if \quad p = 3, \end{cases}$$

if p does not divides c (provided $cp^n \neq 3$ if $p = 3$).

Proof. By the proposition, we have

$$v_p(S(cp^n, cp^n - 3)) = v_p(cp^n) + v_p(cp^n - 1) + v_p(cp^n - 2) + 2v_p(cp^n - 3) - v_p(3).$$

Since $v_p(cp^n - 1) = v_p(cp^n - 2) = v_p(cp^n - 3) = v_p(3) = 0$ if $p \geq 5$, we get

$$v_p(\mathrm{S}(cp^n, cp^n - 3)) = v_p(cp^n)$$

$$= n + v_p(c).$$

As $gcd(p, c) = 1$, it is clear that $v_p(c) = 0$.

For the case $p = 3$, $2v_3(c3^n - 3) - v_3(3) = 1$ and $v_3(c3^n - 1) = v_3(c3^n - 2) = 0$ and hence

$$v_3(S(c3^n, c3^n - 3)) = v_p(c3^n) + 1$$

$$= n + 1$$

This completes the proof.

Now, we give an alternate proof of the primality of integer n by divisibility of $S(n, k)$ given by Deamio and Touset [21]. The proof of corollary 2 in their paper is not correct if we take $n = 4$ and $p = 2$, then $S(4, 3) = 6 \not\equiv 1 \bmod 2$ and $2 \mid S(4, 3)$. We tackled this problem, in this paper, more simpler manner. This problem with an alternate solution also appears in Pólya et al. [22].

Theorem 3.1 If p is an odd prime, then $p \mid S(n, k)$ if $s_p(k) > s_p(n)$.

The above theorem is an immediate consequence of ([18], Lemma 2.1) which states that

$$v_p(S(n,k)) \geq \frac{s_p(k) - s_p(n)}{p-1}. \tag{8}$$

Replacing n by an odd prime p in the above theorem, we get the following results.

Corollary 3.4 If p is an odd prime, then $p \mid S(p, k)$ if $2 \leq k \leq p - 1$.

The problem in the above Corollary 3.4 appears in Graham et al. [4] and proof was given by Demaio and Touset [21].

Theorem 3.2 A positive integer n is a prime if and only if $n \mid S(n, k)$ for all $2 \leq k \leq n - 1$.

Proof. The generating function of $S(n, k)$ in terms of falling powers is given by

$$x^n = \sum_{k=o}^{n} S(n,k)\{x\}_k \tag{9}$$

for any non-negative integer n.

If n is a positive integer such that $n \mid S(n, k)$ for all $2 \le k \le n-1$, put $x = n$ in Equation (9)

$$n^n = \sum_{k=o}^{n} S(n,k)\{n\}_k$$

$$= \{n\}_n + \{n\}_1 + \sum_{k=2}^{n-1} S(n,k)\{n\}_k$$

$$= n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 + n + \sum_{k=2}^{n-1} n(n-1)\cdots(n-(k-1))S(n,k).$$

It follows that

$$n^{n-1} = (n-1)(n-2)\cdots 3\cdot 2\cdot 1 + 1 + \sum_{k=2}^{n-1}(n-1)(n-2)\cdots(n-(k-1))S(n,k)$$

Since $n \mid S(n, k)$ for all $2 \le k \le n-1$, we get

$$0 \equiv (n-1)! + 1 \mod n$$

or

$$(n-1)! \equiv -1 \mod n.$$

Hence, n is prime.

The converse follows from Corollary 3.4.

Lemma 3.1 If p is a prime, then

$$v_p\left(\binom{p-1}{i} - (-1)^i\right) \ge 1 \text{ or } v_p\left(\binom{p-1}{i}\right) = 0.$$

Proof. For $i = 0$, the case is trivial.

We assume that $i > 0$. The binomial coefficient $\binom{p-1}{i}$ is given by

$$\binom{p-1}{i} = \frac{(p-1)!}{(p-1-i)!i!}.$$

Therefore,

$$i!\binom{p-1}{i} = (p-1)(p-2)\ldots(p-i+2)(p-i+1)(p-i)$$

$$\equiv (-1)(-2)\ldots(-i) \mod p$$

$$\equiv (-1)^i i! \mod p.$$

Since $0 < i < p$, $gcd(p, i) = 1$. Then,

$$\binom{p-1}{i} \equiv (-1)^i \mod p.$$

Theorem 3.3 Let p be an odd prime. For any positive integer $n \geq p$,

$$v_p(S(n, p)) = 0$$

if and only if $(p-1) \mid (n-1)$.

Proof. Using the above Lemma 3.1, we have

$$\begin{aligned} p!S(n,p) &= \sum_{i=1}^{p} \binom{p}{i} (-1)^{p-i} i^n \\ &\equiv \sum_{i=1}^{p} \binom{p}{i} (-1)^{p-i} i^n \mod p. \end{aligned}$$

Since $\binom{p}{i} = \binom{p-1}{i-1}\frac{p}{i}$, we get

$$(p-1)!S(n,p) \equiv \sum_{i=1}^{p-1} (-1)^{i-1}(-1)^{p-i} i^{n-1}.$$

Using Wilson's theorem, the preceding congruence reduces to

$$S(n,p) \equiv \sum_{i=1}^{p-1} i^{n-1} \mod p,$$

as p is odd.

Now, we use the following well-known results

$$\sum_{i=1}^{p-1} i^{n-1} \equiv \begin{cases} 0 \mod p, & \text{if} \quad (p-1) \nmid (n-1) \\ -1 \mod p, & \text{if} \quad (p-1) \mid (n-1). \end{cases}$$

Hence, the theorem follows.

Theorem 3.4 Let p be an odd prime and c be a positive integer such that $1 \leq c \leq p-1$. Then, for positive integers n and k such that $k \leq n$,

$$v_p(S(cp^n, cp^k)) = 0.$$

Proof. The theorem is a special case of ([18], Th. 2.2).

We have

$$cp^n - cp^k = c(p^n - p^k) = c(p-1)\sum_{j=0}^{n-k-1} p^{j+k}$$

which implies that $cp^n - cp^k$ is divisible by $p-1$. We also have $1 \leq c \leq p-1$ and $1 \leq cp^k \leq cp^n$.

It follows that $S(cp^n, cp^k)$ is a minimum zero case and hence we have

$$v_p(S(cp^n, cp^k)) = \frac{s_p(cp^k) - s_p(cp^n)}{p-1} = 0, \quad (10)$$

since $s_p(cp^n) = s_p(cp^k) = s_p(c) = c$.

Theorem 3.5 Let p be an odd prime, then

$$v_p(S(p^n, 2p)) \geq n$$

for every integer $n \geq 2$.

Proof. Using identity (1)

$$(2p)!S(p^n, 2p) = \sum_{i=0}^{2p} \binom{2p}{i} (-1)^{2p-i} i^{p^n}$$

which can also be written as

$$(2p)!S(p^n, 2p) = \sum_{i=0}^{2p} \binom{2p}{2p-i} (-1)^i (2p-i)^{p^n}$$

$$= \sum_{i=0}^{2p} \binom{2p}{i} (-1)^{2p-i} (2p-i)^{p^n}.$$

Since $\binom{m}{i} = \binom{m}{m-i}$ for every integers $0 \leq i \leq m$ and $2p - i \equiv i \bmod 2$, we have

$$2(2p)!S(p^n, 2p) = \sum_{i=0}^{2p} \binom{2p}{i} (-1)^{2p-i} (i^{p^n} + (2p-i)^{p^n}). \quad (11)$$

If $p \nmid i$ for $0 \leq i \leq 2p$, then

$$2p - i \equiv -i \bmod p,$$

which also yields the congruence

$$(2p-i)^{p^n} \equiv -(i)^{p^n} \bmod p^{n+1}.$$

It follows that

$$\binom{2p}{i} (-1)^{2p-i} ((2p-i)^{p^n} + (i)^{p^n}) \equiv 0 \bmod p^{n+2}, \text{ since } p \mid \binom{2p}{i}. \quad (12)$$

Thus, each terms of the right hand side of (11) is divisible by p^{n+2} and hence

$$(2p)!S(p^2, 2p) \equiv 0 \bmod p^{n+2}$$

Therefore

$$v_p(2(2p)!S(p^2, 2p)) \geq n+2$$

$$v_p(S(p^2, 2p)) \geq n$$

Hence, the theorem follows.

Theorem 3.6 Let p be a prime and n and k be two positive integers with $k \leq p - 1$, then there exists a positive integer m in $1 \leq m < p - 1$ such that

$$S(n,k) \equiv \begin{cases} S(m,k) \bmod p, & \text{if } n \not\equiv 0 \pmod{p-1}, \\ (p-1-k)! \bmod p, & \text{if } n \equiv 0 \pmod{p-1}. \end{cases}$$

Proof. By division algorithm, we have

$$n = (p-1)q + m$$

where q is the quotient and m is the remainder such that $0 \leq m < p - 1$.

Now

$$k!S(n,k) = \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} i^n$$

$$= \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} i^{(p-1)q+m}$$

$$\equiv \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} i^m \bmod p$$

since $i^{p-1} \equiv 1 \bmod p$ for $1 \leq i \leq k \leq p - 1$ by Fermat's little theorem.

If $m \neq 0$, we have

$$k!S(n, k) \equiv k!S(m, k) \bmod p.$$

Since k is less than p, it follows that $p \nmid k!$ which results

$$S(n, k) \equiv S(m, k) \bmod p.$$

for every n such that $n \not\equiv 0 \bmod p - 1$.

Next, if $m = 0$, we have

$$k!S(n,k) \equiv \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} \bmod p$$

$$\equiv \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} - (-1)^k \bmod p$$

$$\equiv (-1)^{k+1} \bmod p,$$

We also know that

$$\binom{p-1}{k} \equiv (-1)^k \bmod p \text{ or}$$

$$\frac{(p-1)!}{(p-1-k)!k!} \equiv (-1)^k \mod p \text{ or}$$

$$\frac{1}{k!} \equiv (-1)^{k+1}(p-1-k)! \mod p$$

which implies that

$$S(n, k) \equiv (p-1-k)! \mod p,$$

which completes the proof.

From the above theorem, we see that if $1 \le m < k$

$$S(n, k) \equiv 0 \mod p \text{ since } S(m, k) = 0.$$

However, the case for $m = k$ results

$$S(n, k) \equiv 1 \mod p.$$

We can write the following results

Corollary 3.5 Let p be an odd prime and k be a positive integer less than p, then

$$S(n,k) \equiv \begin{cases} 1 \mod p, & \text{if } n \equiv k \mod (p-1), \\ 0 \mod p, & \text{if } n \equiv i \mod (p-1) \text{ for } 1 \le i \le k-1. \end{cases}$$

If we applied the above theorem and corollary to the special cases for $k = p-1$, $p-2$ and $p-3$, we get

$$S(n, p-1) \equiv \begin{cases} 1 \mod p, & \text{if } n \equiv 0 \mod (p-1), \\ 0 \mod p, & \text{otherwise.} \end{cases}$$

$$S(n, p-2) \equiv \begin{cases} 1 \mod p, & \text{if } n \equiv 0, p-2 \mod (p-1), \\ 0 \mod p, & \text{otherwise.} \end{cases}$$

$$S(n, p-3) \equiv \begin{cases} 2 \mod p, & \text{if } n \equiv 0 \mod (p-1), \\ 3 \mod p, & \text{if } n \equiv p-2 \mod (p-1), \\ 1 \mod p, & \text{if } n \equiv p-3 \mod (p-1), \\ 0 \mod p, & \text{if otherwise.} \end{cases}$$

assuming $p \ne 3$ for the last two cases.

4. Discussions

We have computed $v_p(S(p^2, kp))$ for primes $3 \le p \le 100$ and $2 \le k \le p-1$ using PARI/GP software.

Table 1. (p, k) such that $v_p(S(p, k)) = 2$ for $3 \le p \le 1000$ and $2 \le k \le p - 1$

(p, k)	(p, k)	(p, k)	(p, k)	(p, k)	(p, k)	(p, k)
(5, 3)	(167, 7)	(307, 12)	(463, 340)	(653, 429)	(857, 592)	(947, 204)
(13, 5)	(167, 103)	(307, 146)	(467, 278)	(659, 457)	(859, 300)	(947, 478)
(19, 14)	(173, 52)	(317, 188)	(499, 63)	(661, 417)	(859, 357)	(953, 391)
(29, 14)	(181, 166)	(331, 20)	(499, 320)	(677, 367)	(859, 558)	(977, 476)
(31, 16)	(193, 23)	(337, 261)	(509, 324)	(683, 271)	(863, 712)	(991, 953)
(41, 13)	(193, 45)	(353, 162)	(521, 169)	(683, 401)	(877, 77)	(997, 786)
(42, 12)	(197, 85)	(359, 96)	(521, 180)	(691, 468)	(877, 204)	
(47, 12)	(211, 62)	(359, 316)	(521, 479)	(709, 330)	(877, 542)	
(53, 5)	(211, 159)	(373, 230)	(523, 343)	(709, 371)	(881, 63)	
(53, 41)	(223, 61)	(379, 253)	(523, 483)	(709, 669)	(881, 72)	
(53, 45)	(227, 187)	(383, 323)	(569, 123)	(733, 47)	(881, 408)	
(59, 35)	(229, 25)	(397, 27)	(569, 348)	(743, 23)	(881, 625)	
(73, 8)	(233, 7)	(397, 78)	(569, 363)	(751, 744)	(887, 149)	
(79, 14)	(239, 134)	(401, 198)	(577, 119)	(761, 54)	(887, 208)	
(89, 32)	(239, 219)	(409, 45)	(577, 434)	(773, 143)	(887, 443)	
(89, 34)	(241, 15)	(409, 80)	(593, 498)	(773, 262)	(907, 611)	
(107, 16)	(251, 233)	(419, 133)	(601, 303)	(787, 228)	(911, 560)	
(127, 8)	(251, 247)	(419, 256)	(601, 515)	(797, 290)	(919, 163)	
(139, 28)	(257, 131)	(419, 310)	(607, 173)	(809, 119)	(929, 347)	
(149, 5)	(269, 98)	(431, 25)	(607, 242)	(811, 733)	(929, 469)	
(151, 50)	(271, 211)	(431, 112)	(607, 518)	(821, 533)	(929, 801)	
(151, 58)	(283, 91)	(431, 116)	(617, 209)	(827, 257)	(937, 528)	
(157, 45)	(283, 201)	(433, 91)	(647, 117)	(827, 765)	(941, 342)	
(163, 101)	(293, 76)	(439, 308)	(647, 309)	(839, 50)	(947, 85)	
(163, 127)	(293, 162)	(461, 341)	(653, 369)	(839, 744)	(947, 116)	

The obtained values of $v_p(S(p^2, kp))$ for different values of (p, k) are

$$v_p(S(p^2, kp)) = \begin{cases} 7, & if \quad (p,k) = (7,4) \\ 6, & if \quad (p,k) = (37,4),(59,14),(67,8) \\ 3, & if \quad k = p-1 \text{ and } (p,k) = (37,5),(59,15),(67,9) \\ 5, & if \quad k \text{ is even and } (p,k) \ne (7,4),(37,4),(59,14),(67,8) \\ 2, & if \quad k \text{ is odd and } (p,k) \ne (37,5),(59,15),(67,9). \end{cases} \tag{13}$$

We also provide in Table 1, the pairs of p and k where $v_p(S(p, k)) = 2$ for $3 \le p \le 1000$ and $2 \le k \le p - 1$. It should be noted that $v_p(S(p, k)) = 1$ for all the remaining pairs (p, k).

After a closed examinations of the output, we have observed that the arrays of $v_p(S(p^2, kp))$ follow certain patterns which interpret as conjectures.

1. Let $p > 7$ be an odd prime and k be an even integer such that $0 < k < p - 1$. Then

$$v_p(S(p^2, kp)) - v_p(S(p^2, p(k+1)) = 3.$$

2. If k be an integer such that $1 < k < p - 1$, then the p-adic valuations satisfy

$$v_p(S(p^2, kp) = \begin{cases} 5 \text{ or } 6, & if\ k \text{ is even} \\ 2 \text{ or } 3, & if\ k \text{ is odd} \end{cases}$$

for any prime $p > 7$.

3. For any odd prime p and a positive integer k such that $2 \leq k \leq p - 1$,

$$v_p(S(p, k)) \leq 2.$$

5. Conclusions

This paper deals with some results of p-adic valuations of Stirling number of the second kind, $S(n, k)$ for odd prime p. We have derived the formulas for $v_p(S(n, n-1))$, $v_p(S(cp^n, cp^n - 1))$, $v_p(S(n, n-2))$, $v_p(S(p^n, p^n - 2))$, $v_p(S(n, n-3))$ and $v_p(S(p^n, p^n - 3))$. It has been shown the primality test of n using divisibility of n to $S(n, k)$, $1 < k < n$. We have obtained the results that $v_p(S(n, p))$ depends on the divisibility of $n - 1$ by $p - 1$ and $v_p(S(cp^n, cp^k)) = 0$ for every integer $n \geq k \geq 1$ and $p - 1 \geq c \geq 1$. We also posed three conjectures after analyzing Table 1 and computational results of (13).

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DIVISIBILITY OF CERTAIN CLASSES OF STIRLING NUMBERS OF THE SECOND KIND

S. S. Singh[1,*] **and A. Lalchhuangliana**[2,†]
[1]Department of Mathematics & Computer Science, Mizoram University, Aizawl, Mizoram, India
[2]Department of Mathematics, Govt. Champhai College, Champhai, Mizoram, India

Abstract

In this paper, we introduce and prove an alternate formula of Stirling numbers of the second kind as

$$S(n,k) = \sum_{\sum e_i = k,\, \sum n_i e_i = n} \frac{n!}{\prod_{i=1}^{t} e_i!(n_i!)^{e_i}}.$$

We have obtained some results of p-adic valuations of $S(p^2, kp)$ for $2 \le k \le p-1$, $S(2p, p-1)$, $S(2p, p)$, $S(2p, p+1)$ and $S(2p, p+2)$ and also expressed in terms of congruence mod p^2. The generalization of the p-adic valuations and congruences mod p^2 of $S(p^n, kp)$ for $2 \le k \le p-1$ and $S(2p^n, k)$ for $k = p-1, p$ are also presented.

Keywords: congruence, divisibility, primes, p-adic valuations, Stirling numbers

2020 Mathematics Subject Classification: 11A07, 11B73, 11E95, 05A18

1. INTRODUCTION

Stirling numbers of the second kind $S(n,k)$ counts the number of partitions of n objects into k non-empty distinct subsets as [S]

$$S(n,k) = \frac{1}{k!} \sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} j^n.$$

It has a well known recurrence relation

$$S(n+1, k+1) = S(n,k) + (k+1) S(n, k+1).$$

Various results and patterns of p-adic valuations of Stirling numbers of the second kind have been developed through recent years. Lengyel [L] proved that there exists a function f such

*E-mail address: saratcha32@yahoo.co.uk.
†E-mail address: abchhak@gmail.com.

that $v_2(S(c2^n, k)) = s_2(k) - 1$ for $n > f(k)$ and positive integer c, where $v_p(r)$ with p prime is $\sup\{a \geqslant 0 : p^a \mid r\}$ and $s_2(k)$ is the sum of the digits in the binary representation of k. He proposed a conjecture stating that it holds for every positive integer n and for $c = 1$. This conjecture was later proved by Wannemacker [W] using the induction hypothesis. Some notable problems about 2-adic valuation of Striling numbers of the second kind are Amdeberhan *et al.* [AMM], Davis [D], Friedland and Krattenthaler [FK], Hong *et al.* [HZZ] and Zhao *et al.* [ZHZ].

The results of 2-adic valuations of $S(n, k)$, where n is a power of 2, tempted many researchers to look into the results for odd primes p. A few results of $v_p(S(n, k))$ have been developed for some specific classes and found that the patterns and results are quite different from the case when $p = 2$. Gessel and Lengyel [GL] proved that the order of divisibility by prime p of $k!S(a(p-1)p^q, k)$ does not depend on a and q is sufficiently large and k/p is not an odd integer. Recently, Singh *et al.* [SLS] proposed a conjecture which states that for $k = 2, \ldots, p-2$ we have

$$v_p(S(p^2, kp)) = \begin{cases} 2 \text{ or } 3 & \text{if } k \text{ is odd,} \\ 5 \text{ or } 6 & \text{if } k \text{ is even.} \end{cases}$$

Miska [M] proved, for any prime p,

$$v_p(S(n, k)) = v_p(S(a + p^{m_0-1}(p-1), k)) + v_p(n-a) - m_0 + 1$$

for any positive integers m_0, n, k and a such that $a < k < p$ and $n \equiv a \mod (p^{m_0-1}(p-1))$.

Feng and Qiu [FQ] concluded that the formula of $v_p(S(n, n-k))$ depends on the value of $S_2(i, i-k)$, where $k+2 \leq i \leq 2k$ (The r-associated Stirling number of the second kind denoted by $S_r(n, k)$ is defined as the number of ways to partition a set of n elements into k non-empty subsets such that each of the k subsets has at least r elements). They also give a formula to compute $v_p(S(n, n-k))$, which enables to show $v_p((n-k)!S(n, n-k)) < n$ with $0 \leq k \leq min\{7, n-1\}$ and $p \geq 3$. Certain useful results of Stirling numbers may be explored in Clarke [C], Comtet [Co], Graham *et al.* [GKP], Nijenhuis [NW], Sun [Su] Tsumura [T], Young [Y] and Zhao [ZZH].

This paper deals with some interesting results of p-adic valuations of $S(n, k)$ for n as a power of prime p. We have developed an alternate formula for evaluating Stirling number of second kind and also proved certain results like $v_p(S(p^2, kp)) \geq 2$, $v_p(S(p^n, kp)) \geq 2$, $v_p(S(2p, p)) \geq 2$, $v_p(S(2p, p-1)) = 1$, $v_p(S(2p, p-1)) \geq 2$ and $v_p(S(2p, p+2)) \geq 1$.

2. Tools and Identity of $S(n, k)$

In order to formulate $S(n, k)$, we divide partitions into different classes based on the number of subsets with same cardinality in the partitions. Let $\{n_i : 1 \leq i \leq t\}$ and $\{e_i : 1 \leq i \leq t\}$ be two sets of positive integer such that $\sum_{i=1}^{t} n_i e_i = n$ and $\sum_{i=1}^{t} e_i = k$, where $n_i's$ are distinct and $e_i's$ need not to be distinct. We define $s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)})$ as the number of those partitions of n objects into k non-empty subsets containing exactly e_i

subsets with cardinality n_i. So, we introduce

$$S(n,k) = \sum_{\sum e_i = k, \sum n_i e_i = n} s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}). \tag{2.1}$$

In the partition of 6 objects into 3 non-empty subsets, we see

$$S(6,3) = s(1^{(2)}, 4^{(1)}) + s(1^{(1)}, 2^{(1)}, 3^{(1)}) + s(2^{(3)}),$$

where $s(1^{(2)}, 4^{(1)})$ counts the number of those partitions containing exactly two singleton subsets and one subset with four elements, $s(1^{(1)}, 2^{(1)}, 3^{(1)})$ counts those partitions containing exactly one singleton subset, one subset with two elements and one subset with three elements and $s(2^{(3)})$ is the number of those partitions containing exactly three subsets with two elements.

Kwong [K] proved that the sequence of Stirling numbers of the second kind $S(n,k)$ modulo M for any positive integer $M > 1$ is cyclic and gave the minimum periods for different values of k and M. One of the interesting result that he mentioned is

$$\pi(k; p^N) = (p-1)p^{N+b-2} \qquad \text{if} \quad p^{b-1} < k \le p^b, \tag{2.2}$$

where $\pi(k; p^N)$ denotes the minimum period of the sequence $\{S(n,k) \mod p^N\}_{n\ge 1}$ for an odd prime p. Adelberg [A] obtained the following important results:

1. If $n \ge k$, then
$$v_p(S(n,k)) \ge \left\lceil \frac{s_p(k) - s_p(n)}{p-1} \right\rceil.$$

2. If $S(n,k)$ is a minimum zero case, i.e., $(p-1)|(n-k)$ and $p \nmid \binom{n+\frac{n-k}{p-1}}{n}$, then
$$v_p(S(n,k)) = \frac{s_p(k) - s_p(n)}{p-1}. \tag{2.3}$$

3. If $S(n,k)$ is a minimum zero case, then so is $S(np,kp)$ and
$$v_p(S(n,k)) = v_p(S(np,kp)). \tag{2.4}$$

The above results about minimum zero case gives an exact p-adic valuations for a large class of $S(n,k)$.

3. RESULTS

In this section, we introduce an alternate formula to find the Stirling numbers of the second kind and p-adic valuations of some classes of $S(n,k)$. Some of these results have been generalized using minimum periods.

Lemma 3.1. *If n and k are two positive integers, then*

$$s(n^{(k)}) = \prod_{i=0}^{k-1} \binom{n(k-i)-1}{n-1}.$$

Proof. The case for $n = 1$ is trivial.

We provide the proof for $n > 1$ by using induction hypothesis on k.

We know that $s(n^{(k)})$ counts the number of partitions of nk objects into k subsets such that each k subsets contains exactly n objects.

The case for $k = 1$ is trivial since $s(n^{(1)}) = 1$.

Assume that the theorem holds for every positive integer less than k. Let

$$\begin{matrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_{n+1} & a_{n+2} & a_{n+3} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{(k-1)n+1} & a_{(k-1)n+2} & a_{(k-1)n+3} & \cdots & a_{kn} \end{matrix}$$

be the nk objects. The order of the subsets in the partition does not count as each subsets have the same cardinality. We can now safely assume that the first object a_1 always belongs to the first subset of the partition. Thus, the number of choices for the first subset is equal to the number of choices of the remaining $n - 1$ objects from $nk - 1$, $i.e.$, $\binom{nk-1}{n-1}$. Now, the remaining $nk - n = n(k - 1)$ objects are partition into $k - 1$ subsets each containing n elements. The number of such partitions are $s(n^{(k-1)})$ and hence

$$s(n^{(k)}) = \binom{nk-1}{n-1} s(n^{(k-1)}).$$

By induction hypothesis, we get

$$s(n^{(k-1)}) = \prod_{i=0}^{k-2} \binom{n(k-1-i)-1}{n-1}.$$

It follows that

$$s(n^{(k)}) = \prod_{i=0}^{k-1} \binom{n(k-i)-1}{n-1}.$$

Using the binomial coefficients in terms of factorials, the above result may be written as

$$s(n^{(k)}) = \frac{(nk)!}{k!(n!)^k}.$$

This completes the proof. □

Theorem 3.1. *Let* $\{n_i : 1 \le i \le t\}$ *and* $\{e_i : 1 \le i \le t\}$ *be two sets of positive integers and* $n_i' s$ *are distinct. If* $\sum_{i=1}^{t} n_i e_i = n$ *and* $\sum_{i=1}^{t} e_i = k$*, then*

$$\begin{aligned} s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) &= \prod_{j=1}^{t} \binom{n - \sum_{i=0}^{j-1} n_i e_i}{n_j e_j} s(n_j^{(e_j)}) \quad (\textit{if } n_0 = e_0 = 0) \\ &= \frac{n!}{\prod_{j=1}^{t} e_j!(n_j!)^{e_j}}. \end{aligned}$$

Proof. We first choose n_1e_1 objects from n objects and the number of such choices is $\binom{n}{n_1e_1}$. These n_1e_1 objects are then partition into e_1 subsets containing n_1 objects each. The total number of such partitions is

$$s({n_1}^{(e_1)}) = \frac{(n_1e_1)!}{e_1!(n_1!)^{e_1}}.$$

Now we partition the remaining $n - n_1e_1$ objects into $k - e_1$ subsets such that each partition contains e_i number of subsets with cardinality n_i for each $2 \le i \le t$. The total number of such partitions is $s({n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)})$. Thus,

$$s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) = \binom{n}{n_1e_1} s({n_1}^{(e_1)}) s({n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}).$$

Similarly, we can see that

$$s({n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) = \binom{n - n_1e_1}{n_2e_2} s({n_2}^{(e_2)}) s({n_3}^{(e_3)}, {n_4}^{(e_4)}, ..., {n_t}^{(e_t)}).$$

Therefore,

$$\begin{aligned} s({n_1}^{(e_1)}, {n_2}^{(e_2)}, ..., {n_t}^{(e_t)}) = &\binom{n}{n_1e_1}\binom{n - n_1e_1}{n_2e_2} s({n_1}^{(e_1)}) s({n_2}^{(e_2)}) \\ &\times s({n_3}^{(e_3)}, {n_4}^{(e_4)}, ..., {n_t}^{(e_t)}). \end{aligned}$$

Repeating the same process over and over, we get

$$\begin{aligned} s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) =&\binom{n}{n_1e_1}\binom{n - n_1e_1}{n_2e_2} \cdots \binom{n - \sum_{i=1}^{t-2} n_ie_i}{n_{t-1}e_{t-1}} \\ &\times s({n_1}^{(e_1)}) \cdots s({n_t}^{(e_t)}) &(3.1) \\ =& s({n_t}^{(e_t)}) \prod_{j=1}^{t-1} \binom{n - \sum_{i=0}^{j-1} n_ie_i}{n_je_j} s({n_j}^{(e_j)}) \\ =& \prod_{j=1}^{t} \binom{n - \sum_{i=0}^{j-1} n_ie_i}{n_je_j} s({n_j}^{(e_j)}), &(3.2) \end{aligned}$$

since $n - \sum_{i=0}^{t-1} n_ie_i = n_te_t$ when $n_0 = e_0 = 0$.

The above expression may be expressed as

$$\begin{aligned} s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) =&\frac{n!}{(n_1e_1)!(n_2e_2)! \cdots (n_te_t)!} \prod_{j=1}^{t} s({n_j}^{(e_j)}) \\ =& n! \prod_{j=1}^{t} \frac{s({n_j}^{(e_j)})}{(n_je_j)!}. \end{aligned}$$

By using the results of Lemma 3.1, we have

$$\frac{s({n_j}^{(e_j)})}{(n_je_j)!} = \frac{1}{e_j!(n_j!)^{e_j}}.$$

It follows that

$$\begin{aligned} s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) =& n! \prod_{j=1}^{t} \frac{1}{e_j!(n_j!)^{e_j}} \\ =& \frac{n!}{\prod_{j=1}^{t} e_j!(n_j!)^{e_j}}. \end{aligned} \tag{3.3}$$

Hence the theorem follows. □

We come to an alternate formula for evaluation of $S(n, k)$ with the help of (2.1) and (3.3).

Corollary 3.1. *Let n and k are two positive integers such that $n \geq k$, then*

$$S(n, k) = \sum_{\sum e_i = k, \sum n_i e_i = n} \frac{n!}{\prod e_i!(n_i!)^{e_i}},$$

where the sum runs over every pair of sets of positive integer $\{n_i\}$ and $\{e_i\}$ with same cardinality satisfying $\sum e_i = k$ and $\sum n_i e_i = n$ provided $n_i's$ are distinct.

It is easy to verify from the above theorem that the p-adic valuations of $S(p, k)$ is always greater than or equal to 1 if p is an odd prime and k lies between 2 and $p - 1$. We can also state the following results:

Theorem 3.2. *Let p be an odd prime and k be an integer such that $2 \leq k \leq p - 1$, then*

$$v_p(S(p^2, kp)) \geq 2.$$

Proof. We know (due to (2.1))

$$S(p^2, kp) = \sum_{\sum e_i = pk, \sum n_i e_i = p^2} s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}).$$

To prove the theorem, we divide each terms of the sum over the partitions containing e_i subsets with cardinality n_i into the following cases depending on the divisibility of $n_i e_i$ by p.

Case 1: $p \nmid n_i e_i$ for some $1 \leq i \leq t$.

If $p \nmid n_i e_i$, re-arrange the index by interchanging i and 1 so that $p \nmid n_1 e_1$. Using Eq. (3.2), we have

$$s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) = \binom{p^2}{n_1 e_1} s({n_1}^{(e_1)}) \prod_{j=2}^{t} \binom{n - \sum_{i=0}^{j-1} n_i e_i}{n_j e_j} s({n_j}^{(e_j)})$$

which implies that

$$\binom{p^2}{n_1 e_1} | s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}).$$

We also know that $p^2 | \binom{p^2}{n_1 e_1}$ if $p \nmid n_1 e_1$. It follows that

$$p^2 | s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}).$$

Therefore,

$$v_p(s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)})) \geq 2$$

if $p \nmid n_i e_i$ for some i, $1 \leq i \leq t$.

Case 2: $p | n_i e_i$ for every $1 \leq i \leq t$.

In this case, either $p|n_i$ or $p|e_i$ for all $1 \leq i \leq t$. We divide this case into two sub-cases, where the first sub-case deals with $p|e_i$ for all $1 \leq i \leq t$ and the second sub-case deals with $p \nmid e_i$ for some i, $1 \leq i \leq t$.

Case 2.1: $p|e_i$ for every $1 \leq i \leq t$.

It is clear that there exists a positive integer a_i for each $1 \leq i \leq t$ such that $e_i = pa_i$. By the given condition, we have

$$\sum_{i=1}^{t} e_i = kp$$

which implies that

$$\sum_{i=1}^{t} a_i = k.$$

Now, we have

$$s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) = \frac{p^2!}{\prod_{i=1}^{t} e_i!(n_i!)^{e_i}}$$

which yields

$$\begin{aligned} v_p(s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)})) &= v_p(p^2!) - v_p(\prod_{i=1}^{t} e_i!(n_i!)^{e_i}) \\ &= p + 1 - \sum_{i=1}^{t} v_p(e_i!) - \sum_{i=1}^{t} e_i v_p(n_i!). \end{aligned} \tag{3.4}$$

Since $\sum_{i=1}^{t} n_i e_i = p^2$ and by replacing $e_i = pa_i$, we get

$$\sum_{i=1}^{t} n_i a_i = p$$

which implies that $1 \leq n_i < p$ for every $1 \leq i \leq t$ since $\sum_{i=1}^{t} a_i = k \geq 2$. It follows that

$$v_p(n_i!) = 0.$$

We also have

$$\begin{aligned} v_p(e_i!) &= v_p((a_i p)!) \\ &= a_i. \end{aligned}$$

Now Equation (3.4) reduces to

$$\begin{aligned} v_p(s(n_1{}^{(e_1)}, n_2{}^{(e_2)}, n_3{}^{(e_3)}, ..., n_t{}^{(e_t)})) &= p + 1 - \sum_{i=1}^{t} a_i \\ &= p + 1 - k \\ &\geq p + 1 - (p-1) \quad \text{since } k \leq p-1 \\ &= 2. \end{aligned}$$

Thus, it follows that $v_p(s(n_1{}^{(e_1)}, n_2{}^{(e_2)}, n_3{}^{(e_3)}, ..., n_t{}^{(e_t)})) \geq 2$ if $p|e_i \ \ \forall \ \ 1 \leq i \leq t$.

Case 2.2: $p \nmid e_i$ for some $1 \leq i \leq t$.

Let α be the number of $e_i's$ which are divisible by p. Then, $0 \leq \alpha < t$. If $\alpha = 0$, then each $e_i's$ are not divisible by p which means p divides each n_i and we can write $n_i = pm_i$ for each i. Therefore

$$\sum pm_ie_i = p^2 \Rightarrow \sum m_ie_i = p$$

which implies $\sum e_i \leq p$ as each $m_i's$ are positive integers. This is a contradiction as $\sum e_i = kp$ with $k \geq 2$. Thus, we must have $\alpha > 0$.

Now, we re-arrange the index in such a manner that $p|e_i$ if $1 \leq i \leq \alpha$ and $p \nmid e_i$ if $\alpha < i \leq t$, which implies that $e_i = pb_i$ for some positive integer b_i for all $1 \leq i \leq \alpha$. We also have $n_i = pm_i$ for some positive integer m_i and for all $\alpha + 1 \leq i \leq t$. It follows that

$$\begin{aligned} kp &= \sum_{i=1}^{t} e_i \\ &= \sum_{i=1}^{\alpha} e_i + \sum_{i=\alpha+1}^{t} e_i \\ &= \sum_{i=1}^{\alpha} pb_i + \sum_{i=\alpha+1}^{t} e_i \end{aligned}$$

which implies that $p|\sum_{i=\alpha+1}^{t} e_i$. Since $\alpha < t$, and $e_i's$ are positive integers, we must have

$$\sum_{i=\alpha+1}^{t} e_i \geq p.$$

We also have

$$p^2 = \sum_{i=1}^{t} n_ie_i = \sum_{i=1}^{\alpha} n_ie_i + \sum_{i=\alpha+1}^{t} n_ie_i = p\sum_{i=1}^{\alpha} n_ib_i + p\sum_{i=\alpha+1}^{t} m_ie_i,$$

which implies that

$$\begin{aligned} p &= \sum_{i=1}^{\alpha} n_i b_i + \sum_{i=\alpha+1}^{t} m_i e_i \\ &\geq \sum_{i=1}^{\alpha} n_i b_i + \sum_{i=\alpha+1}^{t} e_i \qquad \text{(since } m_i's \text{ are positive)} \\ &\geq \sum_{i=1}^{\alpha} n_i b_i + p. \end{aligned}$$

Thus, we get

$$\sum_{i=1}^{\alpha} n_i b_i \leq 0,$$

which is a contradiction as each term is positive and $\alpha \neq 0$. Therefore, this case cannot happen.

We conclude that p^2 divides $s(n_1{}^{(e_1)}, n_2{}^{(e_2)}, n_3{}^{(e_3)}, ..., n_t{}^{(e_t)})$ for each case, where $\sum_{i=1}^{t} n_i e_i = p^2$ and $\sum_{i=1}^{t} e_i = kp$. So,

$$p^2 | S(p^2, kp) \qquad \text{if } 2 \leq k \leq p-1.$$

□

The preceding theorem confirms that the lower bound of $v_p(S(p^2, kp))$ for $2 \leq k < p-1$ is 2 as mentioned in the conjecture of Singh [SLS] is true. The next theorem is a generalization of the above theorem.

Theorem 3.3. *Let p be an odd prime and k be an integer $2 \leq k \leq p-1$, then*

$$v_p(S(p^n, kp)) \geq 2$$

for any integer $n \geq 2$.

Proof. Replacing $N = 2$ in Eq. (2.2), we get

$$\pi(kp; p^2) = (p-1)p^b \qquad \text{if} \quad p^{b-1} < kp \leq p^b.$$

Since $2 \leq k \leq p-1$, we also have $p < kp < p^2$ and hence $b = 2$. Therefore,

$$\pi(kp; p^2) = (p-1)p^2.$$

It follows that

$$S(a + d(p-1)p^2, kp) \equiv S(a, kp) \mod p^2 \tag{3.5}$$

for every positive integer a and d.

Now, we prove the theorem by induction on n. The previous theorem states that our hypothesis is true for $n = 2$, i.e.,

$$v_p(S(p^2, kp)) \geq 2$$

which can be written as

$$S(p^2, kp) \equiv 0 \pmod{p^2}.$$

Assume that the theorem holds for all $n \leq m$ for some positive integer $m \geq 2$ so that

$$v_p(S(p^n, kp)) \geq 2 \qquad \text{for all } 2 \leq n \leq m$$

which implies

$$S(p^m, kp) \equiv 0 \pmod{p^2}.$$

Putting $a = p^m$ and $d = p^{m-2}$ in Eq. (3.5), we get

$$S(p^{m+1}, kp) \equiv 0 \pmod{p^2}.$$

Thus the theorem is also true for $n = m + 1$. It follows that the theorem is true for every integer $n \geq 2$. □

Theorem 3.4. *Let* p *be an odd prime, then*

$$v_p(S(2p, p)) \geq 2.$$

Proof. Using Eq. (2.1) and Theorem 3.1, we have

$$S(2p, p) = \sum_{\sum e_j = p, \sum n_j e_j = 2p} s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}), \tag{3.6}$$

and

$$s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) = \frac{2p!}{\prod_{j=1}^{t} e_j!(n_j!)^{e_j}}$$

for some positive integer t. It follows that

$$v_p(s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)})) = 2 - \sum_{j=1}^{t} v_p(e_j!) - \sum_{j=1}^{t} e_j v_p(n_j!)).$$

Now we consider the following cases in Eq. (3.6).

Case 1: $n_j < p$ and $e_j < p$ for every j.

It is easy to see that $v_p(s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)})) = 2$ if each $e_j's$ and $n_j's$ are less than p and we get

$$s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)}) \equiv 0 \pmod{p^2} \tag{3.7}$$

if both e_j and n_j are less than p.

Case 2: $e_j \geq p$ for some j.

We know that $\sum e_j = p$ which implies each $e_j's$ are less than p unless for the case $t = 1$, $e_1 = p$ so that $n_1 e_1 = 2p$ or $n_1 = 2$. In this case, the term is

$$s(2^{(p)}) = \frac{(2p)!}{p!(2!)^p}$$

and can be written as

$$\frac{s(2^{(p)})}{p} = \frac{(p+1)(p+2)\cdots(p+p-1)}{2^{p-1}}$$
$$\equiv (p-1)! \equiv -1 \pmod{p}.$$

or

$$s(2^{(p)}) \equiv -p \pmod{p^2}. \tag{3.8}$$

Case 3: $n_j \geq p$ for some j.

If $n_j \geq p$ for some j, then $e_j = 1$ due to $\sum e_j = p$ and $\sum n_j e_j = 2p$. The upper bound for the value of n_j is $p+1$ since the remaining $2p - n_j$ objects cannot fill the remaining empty $p-1$ subsets if $n_j > p+1$.

Case 3.1: $n_j = p+1$ for some j.

If $n_j = p+1$ for some j, all the remaining $p-1$ subsets must contain a single object and the corresponding term for this case is $s((p+1)^{(1)}, 1^{(p-1)})$, i.e., $t = 2$, $n_1 = p+1$, $e_1 = 1 = n_2$ and $e_2 = p-1$. Then

$$s((p+1)^{(1)}, 1^{(p-1)}) = \frac{(2p)!}{(p-1)!(p+1)!},$$

which can also write as

$$\frac{s((p+1)^{(1)}, 1^{(p-1)})}{p} \equiv 2 \pmod{p}$$

or

$$s((p+1)^{(1)}, 1^{(p-1)}) \equiv 2p \pmod{p^2}. \tag{3.9}$$

Case 3.2: $n_j = p$ for some j.

In this case, one subset contains p elements, one another subset contains two elements and remaining $p-2$ subsets must contain a single object. The corresponding term for this case is $s(p^{(1)}, 2^{(1)}, 1^{(p-2)})$, i.e., $t = 3$, $n_1 = p$, $e_1 = 1 = e_2 = n_3$, $n_2 = 2$ and $e_3 = p-2$. Using (3.3), we have

$$s(p^{(1)}, 2^{(1)}, 1^{(p-2)}) = \frac{(2p)!}{(p-2)!p!2!}$$

which reduces to

$$\frac{s(p^{(1)}, 2^{(1)}, 1^{(p-2)})}{p} \equiv -1 \pmod{p}$$

or

$$s(p^{(1)}, 2^{(1)}, 1^{(p-2)}) \equiv -p \pmod{p^2}. \tag{3.10}$$

Combining the results in (3.6), (3.7), (3.8), (3.9) and (3.10), we get

$$S(2p, p) \equiv 0 \pmod{p^2}.$$

This completes the proof. □

Theorem 3.5. *For any prime $p \geq 5$, we have*

$$v_p(S(2p, p-1)) = 1$$

or more specifically

$$S(2p, p-1) \equiv \frac{1}{6}p \pmod{p^2}.$$

Proof. We look into the following cases where p^2 does not divide $s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)})$ as in the preceding theorem:

1. $n_i = p$ for some i
2. $n_i = p+1$ for some i
3. $n_i = p+2$ for some i.

In the first case, there are two possible terms namely, $s(p^{(1)}, 3^{(1)}, 1^{(p-3)})$ and $s(p^{(1)}, 2^{(2)}, 1^{(p-4)})$. So

$$s(p^{(1)}, 3^{(1)}, 1^{(p-3)}) = \frac{(2p)!}{(p-3)!p!3!} \equiv \frac{2}{3}p \pmod{p^2}$$

and

$$s(p^{(1)}, 2^{(2)}, 1^{(p-4)}) = \frac{(2p)!}{2!(p-4)!p!(2!)^2} \equiv -\frac{3}{2}p \pmod{p^2}.$$

For the second case, the only possible term is $s((p+1)^{(1)}, 2^{(1)}, 1^{(p-3)})$ and

$$s((p+1)^{(1)}, 2^{(1)}, 1^{(p-3)}) = \frac{(2p)!}{(p-3)!(p+1)!2!} \equiv 2p \pmod{p^2}.$$

The final case also contains only one term, $s((p+2)^{(1)}, 1^{(p-2)})$ and

$$s((p+2)^{(1)}, 1^{(p-2)}) = \frac{(2p)!}{(p-2)!(p+2)!} \equiv -p \pmod{p^2}.$$

Thus, we have

$$\begin{aligned} S(2p, p-1) &\equiv \frac{2}{3}p - \frac{3}{2}p + 2p - p \pmod{p^2} \\ &\equiv \frac{1}{6}p \pmod{p^2}. \end{aligned}$$

This completes the proof. □

Using the results of minimum periods in Eq. (2.2) and exploiting the same technique as in the proof of Theorem 3.3, we generalize Theorem 3.4 and Theorem 3.5 as follows.

Theorem 3.6. *Let p be an odd prime, then*

$$v_p(S(2p^n, p)) \geq 2.$$

Theorem 3.7. *For any prime $p \geq 5$, we have*

$$v_p(S(2p^n, p-1)) = 1$$

or more specifically

$$S(2p^n, p-1) \equiv \frac{1}{6}p \pmod{p^2}.$$

Theorem 3.8. *For any odd prime p, we have*

$$v_p(S(2p, p+1)) = 0; \tag{3.11}$$

moreover,

$$S(2p, p+1) \equiv 2 \pmod{p^2}. \tag{3.12}$$

Eq. (3.11) is a special case of (2.3) since $S(2p, p+1)$ is a minimum zero case. Hence

$$v_p(S(2p, p+1)) = \frac{s_p(p+1) - s_p(2p)}{p-1} = 0,$$

where $s_p(n)$ is the sum of p-adic digits of n.
Using Eq. (2.4), we can also say that

$$v_p(S(2p^{n+1}, (p+1)p^n)) = 0$$

for any positive integer n.

The second result (3.12) can be obtained using the same method as in Theorem 3.5.

Theorem 3.9. *For any odd prime p,*

$$v_p(S(2p, p+2)) \geq 1$$

or

$$S(2p, p+2) \equiv 2^p - 2 \pmod{p^2}.$$

Proof. There are two cases where p^2 does not divide $s({n_1}^{(e_1)}, {n_2}^{(e_2)}, {n_3}^{(e_3)}, ..., {n_t}^{(e_t)})$.

The first case is $s(1^{(p)}, i^{(1)}, (p-i)^{(1)})$ for $2 \leq i \leq (p-1)/2$ and

$$s(1^{(p)}, i^{(1)}, (p-i)^{(1)}) = \frac{(2p)!}{p!i!(p-i)!} \equiv 2\binom{p}{i} \pmod{p^2}.$$

It follows that

$$\sum_{i=2}^{\frac{p-1}{2}} s(1^{(p)}, i^{(1)}, (p-i)^{(1)}) \equiv 2^p - 2 - 2p \pmod{p^2}.$$

The second case is $s(1^{(p+1)}, (p-1)^{(1)})$ and

$$s(1^{(p+1)}, (p-1)^{(1)}) = \frac{(2p)!}{(p+1)!(p-1)!} \equiv 2p \pmod{p^2}.$$

Now, we have

$$S(2p, p+2) \equiv 2^p - 2 - 2p + 2p \equiv 2^p - 2 \pmod{p^2}.$$

This completes the proof. □

It is known that $2^p - 2$ is always divisible by p using Fermat's theorem. The result for mod p^2 is however not known in general. There are some primes p especially greater than 1000 where p^2 divides $2^p - 2$. So, this leads to an interesting problem as to find out those primes p such that $v_p(S(2p, p+2)) \neq 1$ which is equivalent to $p^2 \nmid 2^p - 2$.

4. CONCLUSION

This paper introduce an alternate formula for evaluation of Stirling numbers of the second kind $S(n, k)$. This formula is used to determine the lower bound of the p-adic valuations of Stirling numbers of the second kind of the classes $S(p^2, kp)$, where p is an arbitrary odd prime and k is a positive integer such that $2 \leq k \leq p - 1$. Some generalized results of p-adic evaluation of $S(p^n, kp)$, $S(2p^{n+1}, (p+1)p^n)$ and $S(2p^n, p)$ are also proved using minimum periods. The results of p-adic valuation for $S(2p, p-1)$, $S(2p, p)$, $S(2p, p+1)$ and $S(2p, p+2)$ are also obtained.

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Periodicity and Divisibility of Stirling Numbers of the Second Kind

A. Lalchhuangliana[1,2] and S.S. Singh[1]
[2]Department of Mathematics & Computer Science, Mizoram University
Aizawl - 796 004, Mizoram, India
[1]Department of Mathematics, Government Champhai College
Champhai - 796 321, Mizoram, India
E-Mail: [1,2]abchhak@gmail.com, [1]saratcha32@yahoo.co.uk

Abstract—This paper introduces certain periodic properties of Stirling numbers of the second kind, $S(n, k)$. We have investigated the relationship between the minimum periods and the p-adic valuations of $S(n, k)$ for an odd prime. Some exact values and estimates of $v_p(S(n, k))$ are obtained. It has been found that some values of $v_p(S(n, k))$ depend entirely on the p-adic valuations of the partial Stirling numbers.

Keywords: Congruence; minimum period; p-adic valuation; prime; sequence.
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INTRODUCTION

A sequence $\{x_n\}$ is said to be periodic if there exist positive integers μ and π such that $x_n = x_{n+\pi}$ for every $n \geq \mu$. The number π is called a period of the sequence $\{x_n\}$, which may not necessary to be unique. In fact, any multiple of a period is also a period, and the smallest possible value of such period is called a minimum period. The minimum period of a given sequence is unique and divides other periods of the sequence.

The sequence of Stirling numbers of the second kind $S(n, k)$, for a fixed k, is periodic in modulo p^N for some positive integer N and a prime p. We denote $\pi(k; p^N)$ for the minimum period of the sequence $\{S(n, k) \pmod{p^N}\}_{n\geq 0}$. It is trivial that $\pi(1, p^N) = 1$ for any $N \geq 1$. Given any positive integers k and n, the explicit formula of $S(n, k)$ can be expressed as

$$S(n,k) = \frac{1}{k!}\sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} i^n. \quad (1.1)$$

For a prime p, the p-adic valuation of an integer n denoted $v_p(n)$ is the highest exponent of p that divides n. If $a = \frac{b}{c}$ for $a, c \in \mathbb{Z}$, $c \neq 0$, then $v_p(a) = v_p(b) - v_p(c)$. Every integer a has a unique p-adic expansion of the form, $a = a_0 + a_1 p + \cdots + a_t p^t$ for some positive integer, t and a_i such that $0 \leq i \leq t$. Here, a_i's are called p-adic digits of a and $s_p(a)$ denotes the sum of the p-adic digits of a.

Carlitz (1955) showed that if $k > p > 2$ and $p^{b-1} < k \leq p^b$ for $b \geq 2$ and $(p-1)p^{N+b-2}$ is a period for $\{S(n, k) \pmod{p^N}\}_{n\geq 0}$. Kwong (1989) later confirmed that the periods obtained by Carlitz (1955) are in fact the minimum periods and may be expressed as

$$\pi(k; p^N) = (p-1)p^{N+b-2} \text{ if } k > p > 2 \text{ and } p^{b-1} < k \leq p^b. \quad (1.2)$$

Kwong also stated that if $1 < k \leq p$, $\pi(k; p^N)$ is the least common multiple of the orders of i modulo p^N for $1 \leq i \leq k$.

The 2-adic valuation of the partial Stirling numbers is an interesting topic in the field of Algebraic topology, see Bendersky and Davis (1991), Crabb and Knapp (1988), and Davis (2012). Davis (2013) obtained some 2-adic valuations

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of the partial Stirling numbers including the following important result $v_2\left(\sum_i \binom{n}{2i+1} i^k\right) \geq v_2\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$.

For more details about different approaches and results for the divisibility of Stirling numbers, one may explore from Lengyel (1994), Wannemacker (2005), Amdeberhan *et al.* (2008), Adelberg (2018), Feng and Qiu (2020), Adelberg (2021), Singh et al. (2021) and Singh and Lalchhuangliana (2022).

In this paper, we use the periodicity properties and the partial Stirling numbers to obtain the p-adic valuations of Stirling numbers of the second kind. The necessary material and essential results are provided in the second section. The third section presents the main results and discusses some results in the Remarks. The application of our results for specific primes are presented at the end of the third section. The final section deals with some observations and conclusions of the paper.

MATERIALS AND METHODS

Definition 2.1. *Let $\gamma(k; p^N)$ be the smallest positive integer such that $S(n + \pi(k; p^N), k) \equiv S(n, k)$ (mod p^N) for every integer $n \geq \gamma(k; p^N)$.*

Definition 2.2. *For any prime p and positive integer k, the partial Stirling numbers $\alpha_p(n, k)$ and $\beta_p(n, k)$ are defined as* $\alpha_p(n,k) = \sum_{p|i} \binom{k}{i} (-1)^i i^n$ and $\beta_p(n,k) = \sum_{p\nmid i} \binom{k}{i} (-1)^i i^n$.

Thus, $(-1)^k k!\, S(n, k) = \alpha_p(n, k) + \beta_p(n, k)$, which follows

$$\alpha_p(n, k) \equiv 0 (\text{mod } p^m) \tag{2.1}$$

and

$$\beta_p(n, k) \equiv (-1)^k k!\, S(n, k) \ (\text{mod } p^m), \tag{2.2}$$

whenever $m \geq n$.

Guo and Zhang (2014) proved the following identity

$$\sum_{k=-\infty}^{\infty} \binom{2n}{n+3k} (-1)^k = 2 \cdot 3^{n-1}. \tag{2.3}$$

Bachraoui (2020) generalized the above identity as

$$\sum_{k=-\infty}^{\infty} \binom{2n+r}{n+3k} (-1)^k = 2 \cdot 3^{n-1+\frac{r}{2}} cos\, \frac{r\pi}{6} \tag{2.4}$$

for positive integers n and r.

Theorem 2.1. (Lundell, 1978) *Let p be an odd prime. For positive integers r and k such that r < k,*

$$\sum_{i\geq 0} (-1)^i \binom{k}{ip} (ip)^r \equiv 0 \ (\text{mod } p^{\max\left\{\left\lfloor \frac{k-r-1}{p-1} \right\rfloor + v_p(k), r\right\}}). \tag{2.5}$$

The notation $\lfloor x \rfloor$ denotes the greatest integer function of x. A stronger result for the above result with a restriction on k such that $r-(p-1)\left\lfloor \frac{r}{p-1} \right\rfloor - 1 \leq k-(p-1)\left\lfloor \frac{k}{p-1} \right\rfloor$ and $k > r > p$ is

$$\sum_{i\geq 0} (-1)^i \binom{k}{ip} (ip)^r \equiv 0 \ (\text{mod } p^{\max\left\{\left\lfloor \frac{k}{p-1} \right\rfloor - \left\lfloor \frac{r}{p-1} \right\rfloor + v_p(k), r\right\}}). \tag{2.6}$$

Another result analogous to Theorem 2.1 with restricted k to $k-(p-1)\left\lfloor \frac{k}{p-1} \right\rfloor < r-(p-1)\left\lfloor \frac{r}{p-1} \right\rfloor - 1$ is

Theorem 2.2. (Lundell, 1978) *Let $k = q(p-1) + a = up + b$, $0 \leq b < p$, and $1 \leq r < p-1$.*

If $0 \leq a < r-1$, then

(i) *for $b = 0$ or $b > a + 1$:*

$$\sum_{i\geq 0} (-1)^i \binom{k}{ip} (ip)^r \equiv 0 \ (\text{mod } p^{q-1+v_p(k)}). \tag{2.7}$$

(ii) *for $b = a + 1$:*

$$\sum_{i\geq 0} (-1)^i \binom{k}{ip} (ip)^r \equiv (-1)^a (-p)^{q-1}$$
$$\left[(a+1)!\left(S(r+1, a+1) - bS(r, a+1)\right)\right] (\text{mod } p^q). \tag{2.8}$$

(iii) *for $b = a > 0$:*

$$\sum_{i\geq 0} (-1)^i \binom{k}{ip} (ip)^r \equiv (-1)^a (-p)^{q-1} a!\, S(r, a) \ (\text{mod } p^q). \tag{2.9}$$

(iv) *for $0 < b < a$:*

$$\sum_{i\geq 0} (-1)^i \binom{k}{ip} (ip)^r \equiv 0 \ (\text{mod } p^q). \tag{2.10}$$

Gessel and Lengyel (2001) introduced the following theorem:

Theorem 2.3. *Let p be an odd prime and m be an integer with $0 < m < min\{k, p\}$ such that $r = \frac{k-m}{p-1}$ is an integer. We set $r \equiv r'$ (mod p) with $1 \leq r' \leq p$. If $r' > m$, then for any integer t*

$$\sum_{i\equiv t \text{ mod } p} \binom{k}{i} (-1)^i i^m \equiv (-1)^{m+\frac{k-m}{p-1}-1} m!$$
$$\binom{k}{m} p^{\frac{k-m}{p-1}-1} \left(\text{mod } p^{\frac{k-m}{p-1}+v_p\left(m!\binom{k}{m}\right)}\right). \tag{2.11}$$

The above results are employed in the next section to determine the p-adic valuations of $S(n, k)$.

RESULTS AND DISCUSSION

This section presents our main results in theorems. We begin with the divisibility of $S(n, k)$ by power of a prime, p when $k < p$.

Theorem 3.1. *Let p be an odd prime and k be a positive integer such that $k < p$. For any positive integer N, the following congruence holds:*

(i) *If $n \equiv 0 \ (mod\ (p-1)p^{N-1})$ with $n > 0$, then*

$$S(n,k) \equiv \frac{(-1)^{k-1}}{k!} (\text{mod } p^N). \quad (3.1)$$

(ii) *If $n \equiv m \ (mod\ (p-1)p^{N-1})$ for some integer m such that $1 \le m < k$, then*

$$S(n,k) \equiv 0 \ (\text{mod } p^N) \quad (3.2)$$

(iii) *If $n \equiv k \ (mod\ (p-1)p^{N-1})$ with $n > m$, then*

$$S(n,k) \equiv 1 \ (\text{mod } p^N) \quad (3.3)$$

Proof. Using Equation (1.1), we have

$$S(m,k) = \frac{1}{k!}\sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} i^m.$$

Since $k < p$, for any i, $1 \le i \le k$, we have

$$i^{m+(p-1)p^{N-1}} \equiv i^m \ (\text{mod } p^N)$$

for any positive integer N and m. It follows that

$$S(m,k) \equiv \frac{1}{k!}\sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} i^{m+(p-1)p^{N-1}}$$
$$\equiv S(m+(p-1)p^{N-1},k) \ (\text{mod } p^N).$$

If $1 \le m < k$, $S(m,k) = 0$, and the second result follows.

If we put $m = k$, we obtain the third result.

The first result is the consequence of the following congruence

$$\frac{1}{k!}\sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} i^{(p-1)p^{N-1}} \equiv \frac{1}{k!}\sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} (\text{mod } p^N)$$
$$\equiv \frac{1}{k!}[(1-1)^k - (-1)^k] \equiv \frac{(-1)^{k-1}}{k!} \ (\text{mod } p^N).$$

Remark 3.1. We know that $(p-1)p^{N-1}$ is the period of the sequence $\{S(n,k) \ mod\ p^N\}$ when $k \le p$. However, the minimum period of the sequence is the least common multiple of the orders of *i modulo p^N for $1 \le i \le k$*. Theorem 3.1 still holds if we replace $(p-1)p^{N-1}$ with $\pi(k; p^N)$. From the theorem, we can observe that $\gamma(k; p^N) = 1$ *if* $k \le p$.

The next theorem presents the divisibility properties of $S(n,p)$ for an odd prime p.

Theorem 3.2. *Let p be an odd prime and n be an integer such that $n \ge p$. The following congruences hold:*

(i) *If $n \equiv 0 \ (mod\ (p-1)p^{N-1})$ for any positive integer N, then*

$$S(n,k) \equiv 0 \ (\text{mod } p^N). \quad 3.4$$

(ii) *If $1 \le m < p$, then*

$$S(n,k) \equiv p^{m-1} \ (\text{mod } p^m), \quad (3.5)$$

for any integer n with $n \equiv m(mod(p-1)p^{m-1})$ and $n > m$. Consequently,

$$v_p(S(n,p)) = m-1. \quad (3.6)$$

Proof. For any positive integer m, we have

$$S(m,p) = \frac{1}{p!}\sum_{i=1}^{p} \binom{p}{i} (-1)^{p-i} i^m = \frac{p^m}{p!} + \frac{1}{p!}\sum_{i=1}^{p-1} \binom{p}{i} (-1)^{p-i} i^m.$$

Let N be any positive integer. Then

$$S(m,p) \equiv \frac{p^m}{p!} + \frac{1}{p!}\sum_{i=1}^{p-1} \binom{p}{i} (-1)^{p-i} i^{m+(p-1)p^{N-1}} \ (\text{mod } p^N).$$

Since $p > 2$, $N \le m + (p-1)p^{N-1}$ for any positive integer m, we obtain

$$S(m,p) \equiv \frac{p^m}{p!} + S(m+(p-1)p^{N-1},p) \ (\text{mod } p^N).$$

Replacing N with m in the preceding equation, we get the second result.

To obtain the first result, we have

$$S((p-1)p^{N-1},p) = \frac{1}{p!}\sum_{i=1}^{p} \binom{p}{i} (-1)^{p-i} i^{(p-1)p^{N-1}} \equiv \frac{1}{p!}\sum_{i=1}^{p-1} \binom{p}{i} (-1)^{p-i} (\text{mod } p^N)$$
$$\equiv 0 \ (\text{mod } p^N).$$

Hence the theorem follows.

Remark 3.2. From the proof of the preceding theorem, it is clear that $\gamma(p; p^N) = N$.

We now discuss the divisibility of $S(n,k)$ when k is greater than a given odd prime p.

Lemma 3.1. *Let p be an odd prime, k and N be positive integers such that $k > p$, then*

$$N + v_p(k) \le \pi(k; p^N). \quad (3.7)$$

Proof. The proof is straightforward

Theorem 3.3. *For an odd prime p and non negative-integers $n, k, m,$ and N such that $k > p$ and $\min\{N, m\} \ge 1$, the following congruence holds:*

$$S(n+m\pi(k;p^N),k) \equiv \begin{cases} \frac{(-1)^k}{k!}\beta_p(n,k)(\text{mod } p^N), & if\ n < N + v_p(k!); \\ S(n,k)(\text{mod } p^N) & if\ n \ge N + v_p(k!). \end{cases} \quad (3.8)$$

If $n < k$, then

$$S(n+m\pi(k;p^N),k) \equiv \begin{cases} \frac{(-1)^{k-1}}{k!}\alpha_p(n,k) \pmod{p^N}, & \text{if } n < N+v_p(k!); \\ 0 \pmod{p^N} & \text{if } n \geq N+v_p(k!). \end{cases} \quad (3.9)$$

Proof. We have

$$S(n,k) = \frac{1}{k!}\sum_{i=1}^{k}\binom{k}{i}(-1)^{k-i}i^n.$$

Let $v_p(k!) = t$, then

$$p^t S(n,k) = \frac{p^t}{k!}\sum_{i=1}^{k}\binom{k}{i}(-1)^{k-i}i^n.$$

If $p \nmid i$, it is well known that

$$i^{n+p^r(p-1)} \equiv i^n \pmod{p^{r+1}}.$$

However, if $p|i$, the preceding congruence holds only when $n \geq r+1$. Thus, we get

$$p^t S(n,k) \equiv \frac{p^t}{k!}\sum_{p\nmid i}\binom{k}{i}(-1)^{k-i}i^{n+p^r(p-1)} + \frac{p^t}{k!}\sum_{p|i}\binom{k}{i}(-1)^{k-i}i^n \pmod{p^{r+1}}$$

$$\equiv p^t S(n+p^r(p-1),k) + \frac{p^t}{k!}\sum_{p|i}\binom{k}{i}(-1)^{k-i}i^n \pmod{p^{r+1}}.$$

Choose r such that $r+1 = v_p(k!) + N$ for some positive integer N. Therefore

$$S(n,k) \equiv S(n+\mu,k) + \frac{(-1)^k}{k!}\alpha_p(n,k) \pmod{p^N}, \quad (3.10)$$

where $\mu = (p-1)p^{t+N-1}$ and a $\alpha_p(n,k) = \sum_{p|i}\binom{k}{i}(-1)^{k-i}i^n$. The term $\frac{1}{k!}\alpha_p(n,k)$ vanishes if $n \geq N + v_p(k!)$, and it follows

$$\gamma(k;p^N) \leq N + v_p(k!). \quad (3.11)$$

Using the concept of minimum periods and the fact that $\gamma(k;p^N) \leq N + v_p(k!) \leq \pi(k;p^N)$, it is easy to confirm that

$$S(n+\mu,k) \equiv S(n+m\pi(k;p^N),k) \pmod{p^N}, \quad (3.12)$$

for any positive integers m and n. Hence, Equations (3.10) and (3.12) confirm the first result of the theorem.

If we restrict the value of n strictly less than k, then $S(n,k) = 0$ and the second result follows immediately.

Remark 3.3. Observing the theorem, it can be seen that

$$v_p(S(n,k)) = v_p(S(m,k)) \quad (3.13)$$

whenever $n \equiv m \pmod{\pi(k;p^{1+v_p(S(n,k))})}$.

Equation (3.10) confirms that $\gamma(k;p^N)$ is the greatest non-negative integer n such that

$$\frac{1}{k!}\alpha_p(n-1,k) \not\equiv 0 \pmod{p^N}.$$

Equation (3.9) confirms that for any positive integer N,

$$v_p(S(n,k)) \geq N \quad (3.14)$$

if $n \equiv r \pmod{\pi(k;p^N)}$ for some positive integer r such that $N + v_p(k!) \leq r < k$.

Theorem 3.4. *For an odd prime p and integers n, m, N, and k such that $p \leq k < 2p$* and $1 \leq m < N+1$,

$$S(n,k) \equiv \frac{(-1)^k p^{m-1}}{(p-1)!(k-p)!} \pmod{p^N}, \quad (3.15)$$

whenever $n \equiv m(mod(p-1)p^N)$ with $n > m$. Hence, the corresponding exact p-adic valuation is

$$v_p(S(n,k)) = m-1, \quad (3.16)$$

whenever $n \equiv m(mod(p-1)p^m)$ with $n > m$.

Proof. Here, $v_p(k!) = 1$. Let m be a positive integer such that $m < N+1 = N + v_p(k!)$ and $m < k$, then using Theorem 3.3, we have

$$S(n,k) \equiv \frac{(-1)^{k-1}}{k!}\alpha_p(m,k) \pmod{p^N} \quad (3.17)$$

for any positive integer n satisfying $n \equiv m \pmod{\pi(k;p^N)}$ with $\pi(k;p^N) = (p-1)p^N$. Since $p < k < 2p$, we have

$$\alpha_p(m,k) = \binom{k}{p}(-1)^p p^m. \quad (3.18)$$

It follows that

$$S(n,k) \equiv \frac{(-1)^k p^m}{p!(k-p)!} \pmod{p^N} \equiv \frac{(-1)^k p^{m-1}}{(p-1)!(k-p)!} \pmod{p^N}. \quad (3.19)$$

Hence, the first result holds. If we replace $N = m$, we get the second result.

Theorem 3.5. *Let p be an odd prime, k and m be positive integers such that $m \geq k$ and $p > k$, then for any integers a and $n \neq m$ with $n \equiv m \pmod{(p-1)p^{m-k+1}}$,*

$$S(n,kp+a) \equiv \frac{(-1)^{k+a-1}p^m}{(kp+a)!}k!\,S(m,k) \pmod{p^{m-k+1}}. \quad (3.20)$$

Furthermore, if $p \nmid S(m,k)$, then

$$v_p(S(n,kp+a)) = m-k. \quad (3.21)$$

Proof. In this case, $v_p((kp+a)!) = k$. Replace N with $m-k+1$ and k with $kp+a$ in Theorem 3.3, then we get $m < v_p((kp+a)!) + N = k + m - k + 1 = m+1$ and

$$S(n,k) \equiv \frac{(-1)^{kp+a-1}}{(kp+a)!}\alpha_p(m,kp+a) \pmod{p^{m-k+1}} \quad (3.22)$$

if $n \equiv m \pmod{\pi(kp + a;\, p^{m-k+1})}$, where $\pi(kp + a;\, p^{m-k+1}) = (p - 1)p^{m-k+1}$.

Now, we have

$$\frac{(-1)^{kp+a-1}}{(kp+a)!}\acute{a}_p(m,kp+a) = \frac{(-1)^{k+a-1}}{(kp+a)!}\sum_{i=1}^{k}\binom{kp+a}{ip}(-1)^{ip}(ip)^m$$
$$\equiv \frac{(-1)^{k+a-1}p^m}{(kp+a)!}\sum_{i=1}^{k}\binom{k}{i}(-1)^i(i)^m \pmod{p^{m-k+1}}$$
$$\equiv \frac{(-1)^{k+a-1}p^m}{(kp+a)!}k!\,S(m,k) \pmod{p^{m-k+1}}. \tag{3.23}$$

Combining Equations (3.22) and (3.23), the theorem follows.

Theorem 3.6. *Let p be an odd prime, N, m, and k be integers such that* $m < \lfloor\frac{k-m-1}{p-1}\rfloor + v_p(k) = N$, $k > m$, *and* $s_p(k-1) > m$. *Then*

$$v_p(S(n,k)) \geq \left\lfloor\frac{s_p(k-1)-m}{p-1}\right\rfloor \tag{3.24}$$

for any positive integer $n > m$, *such that* $n \equiv m \pmod{\pi(k;\, p^N)}$.

Proof. Taking $N = \left\lfloor\frac{k-n-1}{p-1}\right\rfloor + v_p(k)$, $m < N$, and $m < k$, it is trivial that $m < N + v_p(k!)$.

Therefore, using Theorem 3.3, we have

$$S(n,k) \equiv \frac{(-1)^{k-1}}{k!}\alpha_p(m,k) \pmod{p^N}, \tag{3.25}$$

if $n \equiv m \pmod{\pi(k;\, p^N)}$.

From Theorem 2.1, we also have

$$\alpha_p(m,k) \equiv 0 \pmod{p_N} \tag{3.26}$$

since we assume $N > m$.

Combining Equations (3.25) and (3.26), we get

$$S(n,k) \equiv 0 \left(\text{mod } p^{N-v_p(k)}\right), \tag{3.27}$$

assuming $N > v_p(k!)$. It follows that

$$v_p(S(n,k)) \geq N - v_p(k!), \tag{3.28}$$

with the given condition of n.

Now, we have

$$N - v_p(k!) = \left\lfloor\frac{k-m-1}{p-1}\right\rfloor + v_p(k) - v_p(k!) = \left\lfloor\frac{k-m-1}{p-1}\right\rfloor - v_p((k-1)!)$$
$$= \left\lfloor\frac{s_p(k-1)-m}{p-1}\right\rfloor.$$

Hence the theorem follows.

Theorem 3.7. *Let p be an odd prime, N, k, a, b are non-negative integers such that* $k > p$, $k \equiv a \pmod{p-1}$, $k \equiv b \pmod{p}$, $0 \leq b < p$, *and* $1 \leq r < p - 1$. *If* $0 \leq a < r - 1 < N + v_p(k!) - 1$, $n > r$ *and* $n \equiv r \pmod{\pi(k;\, p^N)}$, *then the following results hold:*

(i) *for* $b = 0$ *or* $b > a + 1$, *and* $N = \left\lfloor\frac{k}{p-1}\right\rfloor - 1 + v_p(k)$, *then*

$$v_p(S(n,k)) \geq \left\lfloor\frac{s_p(k-1)+1}{p-1} - 1\right\rfloor. \tag{3.29}$$

(ii) *for* $b = a + 1$, *and* $N = \left\lfloor\frac{k}{p-1}\right\rfloor$, *them*

$$v_p(S(n,k)) = \left\lfloor\frac{s_p(k)}{p-1}\right\rfloor - 1 \tag{3.30}$$

if $S(r+1,b) \not\equiv bS(r,b) \pmod{p}$.

(iii) *for* $b = a > 0$, *and* $N = \left\lfloor\frac{k}{p-1}\right\rfloor$, *them*

$$v_p(S(n,k)) = \left\lfloor\frac{s_p(k)}{p-1}\right\rfloor - 1 \tag{3.31}$$

if $p \nmid S(r,a)$.

(iv) *for* $0 < b < a$, *and* $N = \left\lfloor\frac{k}{p-1}\right\rfloor$, *them*

$$v_p(S(n,k)) \geq \left\lfloor\frac{s_p(k)}{p-1}\right\rfloor. \tag{3.32}$$

Proof. We use Theorems 2.2 and 3.3 to prove the theorem.

(i) Following the proof of Theorem 3.6, we have

$$N - v_p(k!) = \left\lfloor\frac{k}{p-1}\right\rfloor - 1 + v_p(k) - v_p(k!) = \left\lfloor\frac{k}{p-1} - v_p((k-1)!)\right\rfloor - 1$$
$$= \left\lfloor\frac{1+s_p(k-1)}{p-1}\right\rfloor - 1,$$

where $N = \left\lfloor\frac{k}{p-1}\right\rfloor - 1 + v_p(k)$.

(ii), (iii), and (iv). If $N = \left\lfloor\frac{k}{p-1}\right\rfloor$

$$N - v_p(k!) = \left\lfloor\frac{k}{p-1}\right\rfloor - v_p(k!) = \left\lfloor\frac{k}{p-1} - v_p(k!)\right\rfloor = \left\lfloor\frac{s_p(k)}{p-1}\right\rfloor - 1.$$

Theorem 3.8. *Let p be an odd prime, k and m be integers with* $k > p$ *and* $0 < m < p$ *such that* $k \equiv m \pmod{p-1}$ *and* $\frac{k-m}{p-1} \equiv r \pmod{p}$ *with* $1 \leq r \leq p$. *If* $r > m$, *then*

$$v_p(S(n,k)) = \left\lfloor\frac{s_p(k-m)}{p-1}\right\rfloor \tag{3.33}$$

for any integer n satisfying $n \equiv m \pmod{\pi(k;\, p^N)}$, *where* $N = \frac{k-m}{p-1} + v_p\left(m!\binom{k}{m}\right)$.

Proof. We use Theorems 2.3 and 3.3 to prove the theorem. Following the proof of Theorem 3.6 and the preceding theorem, it is easy to show that

$$N - v_p(k!) = \left\lfloor\frac{s_p(k-m)}{p-1}\right\rfloor. \tag{3.34}$$

Hence the theorem follows.

The Case for Prime $p = 3$

In this case, we can classify k into six different equivalent classes, namely, $6m$, $6m + 1$, $\cdots$, $6m + 5$. Using Equations (2.3) and (2.4), we obtain the following results for $m > 0$,

$$\alpha_3(0,6m+r) = \begin{cases} (-1)^m 2 \cdot 3^{3m-1}, & if\ r = 0; \\ (-1)^m 3^{3m}, & if\ r = 1\ or\ 2; \\ 0, & if\ r = 3; \\ (-1)^{m+1} 3^{3m+1}, & if\ r = 4; \\ (-1)^{m+1} 3^{3m+2}, & if\ r = 5. \end{cases} \quad (3.35)$$

Using the binomial identity of the form

$$\binom{k}{ip}(ip) = k\left[\binom{k}{ip} - \binom{k-1}{ip}\right],$$

we get the following recursion relation for $\alpha_p(n, k)$ as

$$\sum (-1)^i \binom{k}{ip}(ip)^n = k \left[\sum (-1)^i \binom{k}{ip}(ip)^{n-1} - \sum (-1)^i \binom{k-1}{ip}(ip)^{n-1}\right].$$

The above identity can also be written as

$$\alpha_p(n, k) = k[\alpha_p(n - 1, k) - \alpha_p(n - 1, k - 1)]. \quad (3.36)$$

Using Equations (3.35) and (3.36), we obtain the following tables for the values of $\alpha_3(n, 6m + r)$ within the range $n \in \{1, 2, 3\}$ and $0 \le r \le 5$. The entry (r, n) gives the value of $(-1)^m \alpha_3(n, 6m + r)$.

Table 1: $(r, n) \to (-1)^m \alpha_3(n, 6m + r)$

	$n = 1$	$n = 2$	$n = 3$
$r = 0$	$2m\, 3^{3m}$	$4m(3m + 1)3^{3m-1}$	$4m^2\, 3^{3m+1}$
$r = 1$	$(6m + 1)3^{3m-1}$	$(6m + 1)3^{3m-1}$	$-(6m + 1)(12m^2 - 2m - 1)3^{3m-1}$
$r = 2$	0	$-(6m + 1)(6m + 2)3^{3m-1}$	$-(6m + 1)(6m + 2)(2m + 1)3^{3m}$
$r = 3$	$-(2m + 1)3^{3m+1}$	$-(2m + 1)^2 3^{3m+2}$	$-(2m + 1)(72m^2 + 90m + 25)3^{3m}$
$r = 4$	$-(6m + 4)3^{3m+1}$	$-(6m + 4)(4m + 3)3^{3m+1}$	$-(6m + 4)(12m^2 + 22m + 9)33^{m+1}$
$r = 5$	$-2(6m + 5)3^{3m+1}$	$-2(6m + 5)(m + 1)3^{3m+2}$	$-(6m + 5)(12m^2 + 32m + 18)33^{m+1}$

If we set $N = v_3(\alpha_3(n, 6m + r)) - v_3((6m + r)!) + 1$, then $n < N + v_3((6m + r)!)$ for $m \ge 1$, $n = 1, 2, or\ 3$, and $0 \le r \le 5$ but $(r, n) \ne (2,1)$. It follows that if $u \equiv n \pmod{\pi(6m + r; p^N)}$, the exact 3-adic valuations can be expressed by

$$v_3(S(u, 6m + r)) = v_3(\alpha_3(n, 6m + r)) - v_3((6m + r)!). \\ = \frac{s_3(6m+r)-r}{2} + f_{r,n}(m), \quad 3.37$$

for some function $f_{r,n}$. The values of $f_{r,n}(m)$ for $n = 1$, 2, and 3 are given in the following table:

Table 2: $(r, n) \to f_{r,n}(m)$

	$n = 1$	$n = 2$	$n = 3$
$r = 0$	$v_3(m)$	$v_3(m) - 1$	$2v_3(m) + 1$
$r = 1$	-1	-1	$v_3(12m^2 - 2m - 1) - 1$
$r = 2$	NA	-1	$v_3(2m + 1)$
$r = 3$	$v_3(2m + 1) + 1$	$2v_3(2m + 1) + 2$	$v_3(2m + 1)$
$r = 4$	1	$v_3(4m + 3) + 1$	$v_3(12m^2 + 22m + 9) + 1$
$r = 5$	1	$v_3(m + 1) + 2$	$v_3(12m_2 + 32m + 18) + 1$

From Table-I, we can see that $\alpha_3(1,6m + 2) = 0$. It follows that

$$v_3(S(u, 6m + 2)) \ge N, \quad (3.38)$$

whenever $u \equiv 1 \pmod{\pi(6m + 2; p^N)}$ for any positive integer N and $u \ge N + v_3((6m + 2)!)$.

The Case for Prime $p = 2$

The minimum periods, for this case, are obtained by Kwong (1989) as

$$\pi(k; 2^N) = \begin{cases} 1, & if\ k = 1\ or\ 2; \\ 2, & if\ k = 3\ or\ 4\ and\ N = 1\ or\ 2; \\ 2^{N-1}, & if\ k = 3\ or\ 4\ and\ N > 2; \\ 2^{N+b-2}, & if\ 2^{b-1} < k \le 2^b\ and\ b \ge 3. \end{cases} \quad (3.39)$$

Unlike the above case, the sum $\alpha_2(n, k)$ does not have the alternating sign as

$$\alpha_2(n, k) = \sum_{2|i} \binom{k}{i} i^n,$$

which gives

$$\alpha_2(0, k) = \Sigma_{2|i} \binom{k}{i} = 2^{k-1}. \quad (3.40)$$

Using Equation (3.36), we obtain the following identity:

$$\alpha_2(1, k) = k2^{k-2}, (k > 2) \quad (3.41)$$

$$\alpha_2(2, k) = k(k + 1)2^{k-3}, (k \ge 3) \quad (3.42)$$

$$\alpha_2(3, k) = k_2(k + 3)2^{k-4}, (k \ge 4) \quad (3.43)$$

$$\alpha_2(4, k) = k(k + 1)(k^2 + 5k - 2)2^{k-5}, (k \ge 5) \quad (3.44)$$

$$\alpha_2(5, k) = k^2(k^3 + 10k^2 + 15k - 10)2^{k-6}, (k \ge 6) \quad (3.45)$$

$$\alpha_2(6,k) = k(k+1)(k^4+14k^3+31k^2-46k + 16)2^{k-7}, (k \geq 7) \quad (3.46)$$

and so on. It follows that $\alpha_2(n, k)$, where $k \geq n$, can be written in the form

$$\alpha_2(n,k) = g_n(k)2^{k-n-1}, \quad (3.47)$$

where $g_n(k)$ is a polynomial over $\mathbb{Z}$ of degree n. The polynomial g_n can be generated by the recursion

$$g_{n+1}(k) = k[2g_n(k) - g_n(k-1)], \quad (3.48)$$

with initial polynomial $g_1(k) = k$.

Let us take $k = 3$ and $k = 4$. The second result of Theorem 3.3 is also valid for an even prime $p = 2$ if we add a condition $n + m\pi(k; p^N) \geq v_p(k!) + N$. In case, $k = 4$ and $N = 2$, we get

$$v_2(k!) + N = 5.$$

We also know that $\pi(4; 2^2) = 2$ and it follows that

$$S(n,4) \equiv \frac{(-1)^3}{4!}\alpha_2(0,4)(\text{mod } 4) \equiv -\frac{2^3}{4!}(\text{mod } 4) \equiv 1 \ (\text{mod } 4)$$

if $n \geq 5$ and $n \equiv 0$ (mod 2). On the other hand, if $n \equiv 1$ (mod 2) or n is odd and $n \geq 5$, then

$$S(n,4) \equiv \frac{(-1)^3}{4!}\alpha_2(1,4)(\text{mod } 4) \equiv -\frac{4 \cdot 2^2}{4!}(\text{mod } 4) \equiv 2 \ (\text{mod } 4).$$

It follows that

$$v_2(S(n,4)) = \begin{cases} 0, & if\ n\ is\ even, \\ 1, & if\ n\ is\ odd. \end{cases} \quad (3.49)$$

The case for $k = 3$ can be tackled similarly as

$$S(n,3) = \begin{cases} 1\ (mod\ 4), & if\ n\ is\ odd; \\ 2\ (mod\ 4), & if\ n\ is\ even. \end{cases} \quad (3.50)$$

Thus, the exact p-adic valuation is given by

$$v_2(S(n,3)) = \begin{cases} 0, & if\ n\ is\ odd, \\ 1, & if\ n\ is\ even. \end{cases} \quad (3.51)$$

If we take $N = 3$, then the minimum period is 4 for both $k = 3$ and 4. The periodicity starts from $n = 5$, i.e., $\gamma(4; 8) = 5$ and the cycle of the sequence $\{S(n, 4)\ (\text{mod } 8)\}_{n\geq 0}$ is (2, 1, 6, 5). Thus, the sequence looks like {0, 0, 0, 0, 1, 2, 1, 6, 5, 2, 1, 6, 5, ⋯}.

The sequence for $\{S(n, 3)\ (\text{mod } 8)\}_{n\geq 0}$ takes the following form

$$\{0, 0, 0, 1, 6, 1, 2, 5, 6, 1, 2, 5, \cdots\}.$$

The periodicity starts from $n = 4 = N + v_2(3!)$, and the cycle of the sequence is (6, 1, 2, 5).

From Equation (3.41) for $k > 4$, we get

$$v_2(S(n,k)) = s_2(k-1) - 1, \quad (3.52)$$

if $n \equiv 1$ (mod $\pi(k; 2^N)$) with $N = s_2(k - 1)$ and $n \geq v_2(k!) + N$.

The generalization of Equation (3.52) can be obtained using Equation (3.47) as

$$v_2(S(n,k)) = s_2(k-1) - m + v_2\left(\frac{g_m(k)}{k}\right), \quad (3.53)$$

where $k > 4$, $n \geq v_2(k!) + N > m$, $N = v_2(\frac{g_m(k)}{k}) + s_2(k - 1) - m + 1$ and $n \equiv m$ (mod $\pi(k; 2^N)$). Looking into the fact in Equations (3.41) to (3.46), we confirm that $g_n(k)$ is always even if $2 \leq n \leq 6$.

CONCLUSION

The partial sum of Stirling numbers $\alpha_p(n, k)$ plays a key role in obtaining $v_p(S(n, k))$. The minimum periods help us determine a class of Stirling numbers of the second kind, $\{S(m, k)\}_{m\in\wedge}$, for some indexing set $\wedge$, which share the same p-adic valuation. We have found that the periodicity of the sequence, $\{S(n, k)\ (\text{mod } p^N)\}_{n\geq 0}$ sometimes starts before the k-th term, i.e., $\gamma(k; p^N) < k$. We have proved that $\gamma(k; p^N) \leq N + v_p(k!)$ when $\gamma(k; p^N) < k$, the first $k - \gamma(k; p^N)$ entries of the cycle of $\{S(n, k)\ (\text{mod } p^N)\}_{n\geq 0}$ are zeros.

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SOME CONGRUENCE PROPERTIES OF STIRLING NUMBERS OF THE SECOND KIND

A. LALCHHUANGLIANA, S. S. SINGH, JITENDER SINGH

Abstract. This paper establishes certain formulas for p-adic valuation of Stirling numbers of the second kind $S(p^n, k)$ where p is a prime and some related classes. The parity of k also affects the p-adic valuation of $S(n, k)$ if k is divisible by p. In fact, $v_p(S(p^2, kp)) \geq 5$ if k is even. The congruence properties of $S(p^n, k) \pmod{p^2}$ depend on the sum of the p-adic digits of k when k is not a multiple of p.

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1. Introduction

For a prime p and integer a, the p-adic valuation of a is the highest exponent $v_p(a)$ of p such that $p^{v_p(a)}$ divides a. For non-negative integers, n and k, Stirling numbers of the second kind denoted by $S(n, k)$ (see [6, p. 204] and [11, p. 265]) are defined by $S(0, 0) = 1$; $S(n, 0) = 0$ for $n > 0$;

$$S(n,k) = \frac{1}{k!}\sum_{j=1}^{k}\binom{k}{j}(-1)^{k-j}j^n, \text{ for } k > 0, \tag{1.1}$$

where $\binom{k}{j}$ is the binomial coefficient, x^j's coefficient in the expansion of $(1+x)^k$. Stirling numbers of the second kind are known to satisfy the recursion

$$S(n+1, k+1) = S(n, k) + (k+1)S(n, k+1). \tag{1.2}$$

Besides, there are two other important identities; the first one (see Wannemacker [19]) is the following:

$$S(m+n, k) = \sum_{i=0}^{k}\sum_{j=i}^{k}\binom{j}{i}\frac{(k-i)!}{(k-j)!}S(m, k-i)S(n, j). \tag{1.3}$$

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The second identity (see Graham [11, p. 265]) is given by

$$\binom{a+b}{a}S(n,a+b)=\sum_{i=0}^{n}\binom{n}{i}S(i,a)S(n-i,b). \tag{1.4}$$

One of the key highlights of the development regarding 2-adic valuation is Wannemacker's proof [19] of Lengyel's conjecture [14]. In fact, Wannemacker evaluated the 2-adic valuation on each term of identity (1.3) and proved that

$$v_2(S(2^n,k)=s_2(k)-1 \tag{1.5}$$

where $s_p(k)$ is the sum of the digits in the base-p expansion of k for a given prime p.

Recently, Adelberg [2] introduced the concept of Minimum Zero Case (MZC), Almost Minimum Zero Case (AMZC), Shifted Minimum Zero Case (SMZC), and Shifted Almost Minimum Zero Case (SAMZC) for 2-adic valuations of $S(n,k)$. This classification is based on the patterns of non-zero 2-adic digits of $n,\ k$, and $n-k$. For further details about the 2-adic valuation of $S(n,k)$, we refer the reader to consult Amdeberhan *et al.* [3], Davis [7], Friedland and Krattenthaler [9], Hong *et al.* [12], and Zhao *et al.* [20].

For an odd prime p dividing k, Gessel and Lengyel [10] proved that the order of divisibility by prime p of $k!S(a(p-1)p^q,k)$ does not depend upon a for q sufficiently large and k/p odd. The proof is based on divisibility results for p-sected alternating binomial coefficient sums. For $2\le k<p-1$, Singh *et al.* conjectured in [17] that

$$v_p(S(p^2,kp))=\begin{cases}2 \text{ or } 3, & \text{if } k \text{ is odd;}\\ 5 \text{ or } 6, & \text{if } k \text{ is even.}\end{cases} \tag{1.6}$$

For the case when $p-1$ divides $n-k$ and p does not divide $\binom{n+\frac{n-k}{p-1}}{n}$, Adelberg [1] obtained the following formula using a higher-order Bernoulli number:

$$v_p(S(n,k))=\frac{s_p(k)-s_p(n)}{p-1}. \tag{1.7}$$

In this paper, we obtain the p adic valuations of Stirling numbers of the second kind from their congruence properties. The main results are presented in Section 3 and include the following: Theorem 3.1 presents a congruence recursion for $S(n,k) \pmod p$; Theorem 4.2 tackles $S(n,k)$ modulo p for the case when k is not divisible by p; Theorem 4.4 deals with $S(n,kp^m)$ modulo p^{2m} when n and k are opposite parity, and further specific results in Theorems 4.7 and 4.8; Theorems 4.10 and 4.12 establish congruence recursions of $S(n+kp,kp)$ $\pmod{p^2}$ for different conditions of n, and Theorems 4.14 and 4.15 give the lower bounds of $v_p(S(p^n-1,kp))$ and $v_p(S(p^n,kp))$ respectively; Theorem 4.16

confirms that $S(p^n, k) \equiv S(p, k) \pmod{p^2}$ if $1 \le k \le p$; Theorem 4.17 establishes the congruence $S(p^2, k) \equiv \binom{p}{k_1} S(p - k_1, k_0) \pmod{p^2}$ if $k = k_1 p + k_0$, $k_0 \ne 0$ and Theorem 4.20 then generalizes the result of Theorem 4.17.

2. Preliminaries

In this section, we provide the necessary background material to state and prove our main results in the next section. Throughout, p denotes an odd prime number. Whenever $p-1$ divides $n-k$, we denote the binomial coefficient $\binom{\frac{n-k}{p-1}-1}{\frac{n-kp}{p-1}}$ by $A_{k,n,p}$ for convenience. If m, n, and k are positive integers such that $p \nmid k$ and $n \ge kp^m$, then the following holds (see Chan and Manna [5]):

$$S(n, kp^m) \equiv \begin{cases} A_{kp^{m-1},n,p} \pmod{p^m}, & \text{if } n \equiv k \pmod{p-1}; \\ 0 \pmod{p^m}, & \text{otherwise.} \end{cases} \tag{2.1}$$

In [16], Sagan obtained the following two congruences using group action on abelian groups:

$$S(n+2p, k) \equiv \sum_{i=0}^{1} S(n+p+i, k+(i-1)p) - \sum_{i=0}^{2} \binom{2}{i} S(n+i, k+(i-2)p) + p(p-1)S(n, k-p) \pmod{p^2};\ n > 0,\ n+2p \ge k; \tag{2.2}$$

$$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^2}. \tag{2.3}$$

On eliminating the terms containing $k-p$ and $k-2p$ in (2.2), we have

$$S(n, k) \equiv 2S(n_1, k) - S(n_2, k) \pmod{p^2};\ k \le p,\ n > 2p, \tag{2.4}$$

where $n_r = n - r(p-1)$. Consequently, using induction, one arrives at the following:

$$S(n, k) \equiv (r+1)S(n_r, k) - rS(n_{r+1}, k) \pmod{p^2};\ n_{r+1} > 2. \tag{2.5}$$

From Equations (2.4) and (2.5), we get

$$S(n, k) \equiv \begin{cases} 2S(n_1, k) \pmod{p^2}, & \text{if } 2p < n < 2p + k - 2; \\ (r+1)S(n_r, k) \pmod{p^2}, & \text{if } 2 < n_{r+1} < k. \end{cases} \tag{2.6}$$

Bailey [4] obtained a higher congruence version of Equation (2.3) through sums of binomial coefficients that

$$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^3}. \tag{2.7}$$

Singh and Lalchhuangliana [18] generalized the results on p-adic valuation and congruences concerning $S(p^n, kp) \pmod{p^2}$ for $2 \leq k \leq p-1$ and $S(2p^n, k) \pmod{p^2}$ for $k = p-1$ and $k = p$ with the help of minimum periods.
Feng and Qiu [8] employed a combinatorial approach and proved the following result:

$$v_p(S(n, n-k)) = v_p\left(\binom{n}{k+1}\right) + t_p(n, k);\ n \geq k+1, \tag{2.8}$$

$$\text{where } t_p(n,k) = \begin{cases} 0, \text{ if } k = 1; \\ v_p(3n-5) - v_p(4), \text{ if } k = 2; \\ v_p(n^2 - 5n + 6) - v_p(2), \text{ if } k = 3; \\ v_p(15n^3 - 150n^2 + 485n - 502) - v_p(48), \text{ if } k = 4; \\ v_p(3n^4 - 50n^3 + 305n^2 - 802n + 760) - v_p(16), \text{ if } k = 5; \\ v_p(63n^5 - 1575n^4 + 1543n^3 - 73801n^2 + 171150n - 156296) \\ -v_p(576), \text{ if } k = 6. \end{cases} \tag{2.9}$$

Adelberg [1] also gave a rough margin for the lower bound of the p-adic valuation of $S(n, k)$ in terms of the sum of the p-adic digits s_p. More precisely, the following estimate was obtained:

$$v_p(S(n,k)) \geq \left\lceil \frac{s_p(k) - s_p(n)}{p-1} \right\rceil. \tag{2.10}$$

The above estimate tells us that p divides $S(n, k)$ whenever $s_p(n) < s_p(k)$.
For an odd prime p, Kwong [13] obtained the minimum period $\pi(k; p^N)$ of the sequence $\{S(n, k) \pmod{p^N}\}_{n \geq 1}$ using the generating function of $S(n, k)$, which will be used in the sequel and is defined as

$$\pi(k; p^N) = (p-1)p^{N+b-2},\ p^{b-1} < k \leq p^b. \tag{2.11}$$

The techniques used in this article are from number theory and combinatorics. The value of $v_p(S(n, k))$ is estimated through the obtained congruence properties of $S(n, k)$. This approach makes it more simpler than the other techniques.

3. Main Results

This section is divided into various cases. We first divide into divisibility of $S(n, k)$ by p and p^n in general. We further divide into divisibility of k by p. We begin by providing the following results;

Theorem 3.1. For an odd prime p and an integer n, we have

(a) $v_p(S(p^n - 1, kp - 1)) \geq 2$; $2 \leq k < p-1$; $v_p(S(p^n - 1, (p-1)p - 1)) = 1$.
(b) $S(p+n, k) \equiv S(n+1, k) + S(n, k-p) \pmod{p}$.

Proof. (a) Using Equation (1.2) and the fact that p^2 divides $S(p^n, kp)$, we have

$$S(p^n-1, kp-1) = S(p^n, kp) - kpS(p^n-1, kp) \equiv kpS(p^n-1, kp) \pmod{p^2}. \tag{3.1}$$

So, it is enough to prove that p divides $S(p^n-1, kp)$. Taking $m=1$, $n=p^n-1$ in (2.1), we get

$$S(p^n-1, kp) \equiv \begin{cases} A_{kp,p^n,p} \pmod{p}, & \text{if } k = p-1; \\ 0 \pmod{p}, & \text{if } 2 \le k \le p-2, \end{cases} \tag{3.2}$$

where from (3.2), the result (a) follows when $2 \le k \le p-2$. To prove (a) for the case when $k = p-1$, we observe using Lucas congruence for $n \ge 2$ that $A_{(p-1)p,p^n-1,p} \equiv -1 \pmod{p}$. Consequently, from (3.2), we have

$$S(p^n-1, (p-1)p) \equiv -1 \pmod{p}. \tag{3.3}$$

Equation (3.3), in view of (3.1), proves that $v_p(S(p^n-1, (p-1)p-1)) = 1$.
(b) From Equation (1.3), we get

$$S(p+n, k) = \sum_{i=0}^{k} \sum_{j=i}^{k} \binom{j}{i} \frac{(k-i)!}{(k-j)!} S(p, k-i) S(n, j). \tag{3.4}$$

The terms within the summation in (3.4), except those with indices such that

$$(i, j) \in \{(k-1, k-1), (k-1, k), (k-p, k-p)\},$$

are all divisible by p. This observation, along with (1.2), gives

$$\begin{aligned} S(p+n, k) &\equiv S(n, k-1) + kS(n, k) + S(n, k-p) \pmod{p} \\ &\equiv S(n+1, k) + S(n, k-p) \pmod{p}. \end{aligned}$$

Thus, the result (b) follows. □

4. Divisibility of $S(n,k)$ by p

Chan and Manna [5] obtained a congruence for $S(n, k)$ when k is divisible by p and not divisible by p. The result when k is divisible by p is simple for acquiring the divisibility of $S(n, k)$. We further look into the case when k is not a multiple of p, say $k = cp^m + b$, where $b \neq 0$ and $p \nmid b$. We will utilize the following result due to Singh *et al.* [17] while proving Theorem 4.2.

Lemma 4.1. Let p be a prime and n and k be two positive integers such that $n > 0$ and $k \le p-1$, then there exists an integer $1 \le m < p-1$ such that

$$S(n, k) \equiv \begin{cases} S(m, k) \pmod{p}, & \text{if } n \not\equiv 0 \pmod{p-1}; \\ (p-1-k)! \pmod{p}, & \text{if } n \equiv 0 \pmod{p-1}, \end{cases} \tag{4.1}$$

where m is the remainder when n is divided by p.

The following theorem is a generalization of Lemma 4.1 in which k is restricted to an integer less than or equal to $p-1$ for a given prime p. This theorem, however, provides the congruence for $S(n,k)$ modulo p for any integer $k \le n$.

Theorem 4.2. For an odd prime p and integer k with $p \nmid k$, let b be the last p-adic digit of k. Let $m = v_p(k-b)$, $c = (k-b)p^{-m}$, and a is the remainder when $n-k$ is divided by $p-1$. Then

$$S(n,k) \equiv \begin{cases} S(a,b)\binom{\lfloor \frac{n-k}{p-1} \rfloor + cp^{m-1}}{cp^{m-1}} \pmod p, & \text{if } n \not\equiv c \pmod{p-1}; \\ (p-1-b)!\binom{\lfloor \frac{n-k}{p-1} \rfloor + cp^{m-1}}{cp^{m-1}} \pmod p, & \text{otherwise.} \end{cases}$$

Proof. The result for $k < p$ is trivial since $c = 0$. Due to Chan and Manna [5, Theorem 5.3], we have for $m \ge 1$ and $n \ge cp^m + b$ that

$$\begin{aligned} S(n, cp^m + b) &\equiv \sum_{i \equiv c \pmod{p-1}}^{n} S(i, cp^m) S(n-i, b) \pmod{p^m} \\ &\equiv \sum_{i \equiv c \pmod{p-1}}^{n} \binom{\frac{i-cp^{m-1}}{p-1} - 1}{\frac{i-cp^m}{p-1}} S(n-i, b) \pmod{p^m}. \end{aligned} \quad (4.2)$$

The index i in the last summation runs through $i \equiv c \pmod{p-1}$; so $i = c + (p-1)j$ for some j with $cp^m \le c + (p-1)j$ and $b \le n - c - (p-1)j$. If we define $A = \lfloor \frac{n-cp^m-b}{p-1} \rfloor$, then (4.2) reduces to the form:

$$\begin{aligned} S(n, cp^m + b) &\equiv \sum_{i=0}^{A} \binom{\frac{cp^m + i(p-1) - cp^{m-1}}{p-1} - 1}{\frac{cp^m + i(p-1) - cp^m}{p-1}} S_i \pmod{p^m} \\ &\equiv \sum_{i=0}^{A} \binom{cp^{m-1} + i - 1}{i} S_i \pmod{p^m} \end{aligned} \quad (4.3)$$

where $S_i = S(n - cp^m - i(p-1), b)$.

If $1 \le b \le p-1$, then by Lemma 4.1, there exists an integer a such that

$$S_i \equiv \begin{cases} S(a,b) \pmod p, & \text{if } n \not\equiv c \pmod{p-1}; \\ (p-1-b)! \pmod p, & \text{otherwise.} \end{cases} \quad (4.4)$$

Here, a is the remainder when $n - cp^m - i(p-1)$ is divided by $p-1$, that is, the remainder when $n-c$ is divided by $p-1$. So, for $n \not\equiv c \pmod{p-1}$, we have

$$\begin{aligned} S(n, cp^m + b) &\equiv \sum_{i=0}^{A} \binom{cp^{m-1} + i - 1}{i} S(a,b) \pmod p \\ &\equiv S(a,b) \binom{A + cp^{m-1}}{cp^{m-1}} \pmod p, \end{aligned}$$

and for the other case, that is, when $n \equiv c \pmod{p-1}$, we have

$$S(n, cp^m + b) \equiv (p-1-b)!\binom{A + cp^{m-1}}{cp^{m-1}} \pmod{p},$$

as desired. □

Remark 4.3. Taking $n = p^2$ in the proof of Theorem 4.2, we see that $A = p-c$. Hence, for an odd prime p and $k > p$ with $p \nmid k$, we have $v_p(S(p^2, k)) \geq 1$, which can also be deduced using Equation (2.10).

4.1. **Divisibility of $S(n,k)$ by p^m with $p \mid k$.** The following theorem extends the result of Chan and Manna (see [5, Theorem 5.2]) when n and k of $S(n, kp^m)$ are of opposite parity.

Theorem 4.4. If p is an odd prime and n and k are of opposite parity, then

$$S(n, kp^m) \equiv \begin{cases} (-1)^{n-1}\dfrac{nk}{2}A_{kp^{m-1},n-1,p}p^m \pmod{p^{2m}}, & \text{if } p-1 \mid n-1-k; \\ 0 \pmod{p^{2m}}, & \text{otherwise.} \end{cases} \tag{4.5}$$

Proof. Using Equation (1.1) and the hypothesis of parity and $k \equiv kp^m \pmod 2$, we have

$$\begin{aligned} 2(kp^m)!S(n, kp^m) &= \sum_{i=0}^{kp^m}\binom{kp^m}{i}(-1)^i\big((-1)^k i^n + (kp^m - i)^n\big) \\ &= \sum_{i=0}^{kp^m}\binom{kp^m}{i}(-1)^i\Big\{(-1)^k i^n + (-1)^n i^n \\ &+ \sum_{j=1}^{n}\binom{n}{j}(-1)^{n-j}k^j i^{n-j}p^{mj}\Big\} \\ &= \sum_{i=0}^{kp^m}\binom{kp^m}{i}(-1)^i\sum_{j=1}^{n}\binom{n}{j}(-1)^{n-j}k^j i^{n-j}p^{mj} \\ &= \sum_{j=1}^{n}\binom{n}{j}(-1)^{n-j}k^j p^{mj}(kp^m)!S(n-j, kp^m), \end{aligned} \tag{4.6}$$

where we have used $(-1)^k i^n + (-1)^n i^n = 0$. Thus, $S(n, kp^m) \equiv 0 \pmod{p^m}$ if n and k are opposite parity. It then follows from (4.6) that

$$2S(n, kp^m) \equiv \sum_{j=1}^{t}\binom{n}{j}(-1)^{n-j}k^j p^{mj}S(n-j, kp^m) \pmod{p^{m(t+1)}} \tag{4.7}$$

holds for $1 \leq t \leq n$. Since n and k are opposite parity, so are $n-2$ and k. Consequently, $S(n-2, kp^m) \equiv 0 \pmod{p^m}$. This observation together with

(4.7) for $t = 2$ gives us the following;

$$\begin{aligned}2S(n, kp^m) &\equiv (-1)^{n-1} nkp^m S(n-1, kp^m) \\ &\quad + \binom{n}{2}(-1)^{n-2}k^2p^{2m}S(n-2, kp^m) \pmod{p^{3m}} \\ &\equiv (-1)^{n-1}nkp^m S(n-1, kp^m) \pmod{p^{3m}}, \end{aligned} \tag{4.8}$$

which is also true for modulo p^{2m}. Thus, applying Equation (2.1) to $S(n-1, kp^m)$ and combining it with Equation (4.8) produces Equation (4.5). □

Corollary 4.5. For an odd prime p and two positive integers n and k,

(a) If k is even and $p^n \geq kp^m$, then $S(p^n, kp^m) \equiv 0 \pmod{p^{2m}}$.

(b) If n and k are of opposite parity such that $s_p(kp^{m-1} + \alpha - 1) = s_p(k) + s_p(\alpha - 1)$; $n - 1 \equiv k \pmod{p-1}$; and $\alpha = \frac{n-1-kp^m}{p-1}$, then $v_p(S(n, kp^m)) = 2m - v_p(\alpha) - 1$.

Proof. (a) Follows from Equation (4.5).
(b) If $s_p(x+y) = s_p(x) + s_p(y)$, then by Kummer's theorem [15], $v_p\left(\binom{x+y}{y}\right) = v_p(x) - v_p(y+1)$, which in view of (4.5) proves (b). □

Remark 4.6. If k is even, then replacing n by p^n in (4.7), we get

$$2S(p^n, kp^m) = \sum_{i=1}^{p^n} \binom{p^n}{i} (-1)^{p^n - i} k^i p^{mi} S(p^n - i, kp^m). \tag{4.9}$$

The i-th term within the summation in (4.9) is divisible by p^{n+mi} if $p \nmid i$. However, if $p \mid i$, then the corresponding term within the summation in (4.9) is divisible by $p^{n - v_p(i) + mi}$. Then for all t with $1 \leq t \leq p^n$ with $p \nmid (t+1)$, we have the following key congruence:

$$2S(p^n, kp^m) \equiv \sum_{i=1}^{t} \binom{p^n}{i} (-1)^{p^n - i} k^i p^{mi} S(p^n - i, kp^m) \pmod{p^{(t+1)m+n}}. \tag{4.10}$$

Theorem 4.7. Let p be an odd prime. Let m, n, and k be positive integers such that $n > m$, k is even, and $p \nmid k$. Then $v_p(S(p^n, kp^m)) \geq n + 2m$, unless $m = 1$ and $k = p - 1$, in which case $v_p(S(p^n, (p-1)p)) = n + 1$.

Proof. Taking $t = 1$ in (4.10), we have for even k that

$$2S(p^n, kp^m) \equiv kp^{n+m} S(p^n - 1, kp^m) \pmod{p^{2m+n}}, \tag{4.11}$$

where we have

$$S(p^n - 1, kp^m) \equiv \begin{cases} A_{kp^{m-1}, p^n - 1, p} \pmod{p^m}, & \text{if } k = p - 1; \\ 0 \pmod{p^m}, & \text{if } 1 \leq k \leq p - 2. \end{cases} \tag{4.12}$$

The binomial coefficient on the right-hand side follows

$$A_{kp^{m-1},p^n-1,p} \equiv \begin{cases} -1 \pmod{p}, & \text{if } m = 1 < n; \\ 0 \pmod{p^m}, & \text{if } 2 \le m < n. \end{cases} \tag{4.13}$$

Using (4.13) in (4.12) and then (4.12) in (4.11) proves the desired assertion. □

Theorem 4.8. If $p > 3$ is prime; n , m, k are positive integers with $m < n$ and $k < p$, then

$$S(p^n, kp^m) \equiv \begin{cases} \frac{k}{2}p^{n+1}X_1 - \frac{9p^{n+3}}{4} \pmod{p^{n+4}}, & \text{if } m = 1 \text{ and } k = p - 3; \\ \frac{k}{2}p^{n+m}X_m \pmod{p^{4m+n}}, & \text{otherwise} \end{cases} \tag{4.14}$$

where $X_i = S(p^n - 1, kp^i)$.

Proof. Taking $t = 2$ in (4.10) for even k, we get

$$\begin{aligned} 2S(p^n, kp^m) &\equiv kp^{n+m}S(p^n - 1, kp^m) \\ &\quad - p^{n+2m}\Big(\frac{p^n - 1}{2}\Big)k^2S(p^n - 2, kp^m) \pmod{p^{3m+n}}. \end{aligned}$$

Since $p^n - 2$ and k are opposite parity, $S(p^n - 2, kp^m) \equiv 0 \pmod{p^m}$, and so

$$2S(p^n, kp^m) \equiv kp^{n+m}S(p^n - 1, kp^m) \pmod{p^{3m+n}}. \tag{4.15}$$

Similarly, on taking $t = 3$ in (4.10), we have for $p > 3$ that

$$\begin{aligned} 2S(p^n, kp^m) &\equiv kp^{n+m}S(p^n - 1, kp^m) - \frac{1}{2}(p^n - 1)k^2p^{n+2m}S(p^n - 2, kp^m) \\ &\quad + \frac{1}{6}(p^n - 1)(p^n - 2)k^3p^{n+3m}S(p^n - 3, kp^m) \pmod{p^{n+4m}}. \end{aligned} \tag{4.16}$$

If k is even, $k \neq p-3$, and $1 \le k \le p-1$, then by Theorem 4.4 and (4.16), we have

$$S(p^n, kp^m) \equiv \frac{k}{2}p^{n+m}S(p^n - 1, kp^m) \pmod{p^{4m+n}}. \tag{4.17}$$

For the case $k = p - 3$, we have from Theorem 4.4 that

$$S(p^n - 2, (p-3)p^m) \equiv \begin{cases} -3p \pmod{p^2}, & \text{if } m = 1; \\ 0 \pmod{p}^{2m}, & \text{if } m > 1. \end{cases} \tag{4.18}$$

Also, from (2.1), we have

$$S(p^n - 3, (p-3)p^m) \equiv \begin{cases} -1 \pmod{p}, & \text{if } m = 1; \\ 0 \pmod{p^m}, & \text{if } m > 1. \end{cases} \tag{4.19}$$

Combining (4.16)–(4.19), we get (4.14). □

Remark 4.9. Identity (1.4) gives rise to the following relation;

$$\binom{kp}{p}S(kp+n,kp) = \sum_{i=0}^{n}\binom{kp+n}{(k-1)p+i} \times S((k-1)p+i,(k-1)p)S(n+p-i,p). \quad (4.20)$$

If $n = t(p-1)$, then the i-th term within the summation in (4.20) is divisible by p^2 in case $i \not\equiv 0 \pmod{p-1}$ since p divides both $S((k-1)p+i,(k-1)p)$ and $S(n+p-i,p)$. On the other hand, if $i \equiv 0 \pmod{p-1}$ and $0 \neq i \neq n$, then p divides $\binom{kp+n}{(k-1)p+i}$. Using Equation (2.1) for $i \equiv 0 \pmod{p-1}$, we have

$$\begin{aligned} S((k-1)p+i,(k-1)p) &\equiv \binom{k-2+\frac{i}{p-1}}{\frac{i}{p-1}} \pmod{p}, \\ S(n+p-i(p-1),p) &\equiv 1 \pmod{p}. \end{aligned}$$

We also know that $\binom{kp}{p} \equiv k \pmod{p^2}$ is due to Equation (2.3). It follows that

$$\begin{aligned} &kS(kp+n,kp) \\ &\equiv \binom{kp+n}{(k-1)p}S(n+p,p) + \binom{kp+n}{(k-1)p+n}S((k-1)p+n,(k-1)p) \\ &+\sum_{i=1}^{t-1}\binom{(k+t)p-t}{(t-i)p+p-t+i}\binom{k-2+i}{i} \pmod{p^2} \text{ for } n = t(p-1). \quad (4.21) \end{aligned}$$

Theorem 4.10. If p is an odd prime, $0 \leq k+t < p$, $n = tp+j$, and $0 \leq j < p-t-1$, then

$$S(kp+n,kp) \equiv \frac{r}{k}\binom{k+t}{k-1}S(n+p,p) + \frac{\binom{k+t}{r}}{\binom{k}{r}}S((k-r)p+n,(k-r)p) \pmod{p^2} \quad (4.22)$$

for $1 \leq r \leq k-1$.

Proof. We analyze (4.20) for the case when $(p-1) \nmid n$ and $t(p-1) < n < (t+1)(p-1)$. If $i \equiv 0 \pmod{p-1}$, then $S(n+p-i,p) \equiv 0 \pmod{p}$ but $S((k-1)p+i,(k-1)p) \equiv \binom{k-2+\frac{i}{p-1}}{\frac{i}{p-1}} \pmod{p}$. On the other hand, if $n \equiv i \pmod{p-1}$, then $S((k-1)p+i,(k-1)p) \equiv 0 \pmod{p}$ and $S(n+p-i,p) \equiv 1 \pmod{p}$. The rest of the terms where $i \not\equiv 0 \pmod{p-1}$ and $n \not\equiv i \pmod{p-1}$ are divisible by p^2. It follows that

$$\binom{kp}{p}S(kp+n,kp) \equiv \sum_{i=0}^{t}\binom{kp+n}{(k-1)p+i(p-1)}\binom{k-2+i}{i} \times S(n+p-i(p-1),p) + \sum_{i=0}^{t}\binom{kp+n}{(k-1)p+n-i(p-1)}$$

$$\times S((k-1)p+n-i(p-1),(k-1)p) \pmod{p^2}. \quad (4.23)$$

If we restrict n to $n = tp + j$, $0 \le j < p - t - 1$, and $0 \le k + t < p$, then the binomial coefficients in both sums of the right-hand side of (4.23) are divisible by p because of Lucas congruence. So, all the terms except when $i = 0$ in both summations are divisible by p^2. Equation (4.23) thus reduces to the form

$$\binom{kp}{p}S(kp+n,kp) \equiv \binom{kp+n}{(k-1)p}S(n+p,p) + \binom{kp+n}{(k-1)p+n}S((k-1)p+n,(k-1)p) \pmod{p^2}. \quad (4.24)$$

If we also apply Lucas congruence to the binomial coefficients, we have

$$kS(kp+n,kp) \equiv \binom{k+t}{k-1}S(n+p,p) + (k+t)S((k-1)p+n,(k-1)p) \pmod{p^2}. \quad (4.25)$$

The theorem follows by using induction on r together with Equation (4.25). □

Corollary 4.11. If p is an odd prime, $0 \le k + t < p$, $n = tp + j$, and $0 \le j < p - t - 1$, then

$$S(kp+n,kp) \equiv \binom{t+k}{t+1}S(p+n,p) \pmod{p^2}. \quad (4.26)$$

Proof. Take $r = k - 1$ in (4.20). □

Theorem 4.12. Let p be an odd prime, k, t, and n be positive integers with $4 \le k + 2 \le k + t < p$ and $n = tp - 1$. Then for $1 \le r \le k - 1$, we have

$$S(kp+n,kp) \equiv \frac{r}{k}\binom{k-1+t}{k-1}S(n+p,p) + \frac{\binom{k+t-1}{r}}{\binom{k}{r}}S((k-1)p+n,(k-1)p) + \frac{2rk-r(r+1)}{k(k-1)}\binom{k-1+t}{k-2} \times \sum_{i=1}^{t}(-1)^{i-1}\binom{t+1}{i}S(n+p-i(p-1),p) \pmod{p^2}. \quad (4.27)$$

Proof. We analyze (4.23) for the case when $n = tp - 1$, $t + k \le p$, and $t \ge 2$ so that $(p-1) \nmid (tp-1)$. In this case, $S(n+p-i(p-1),p) \equiv 0 \pmod{p}$ and

$$\binom{kp+n}{(k-1)p+i(p-1)} \equiv (-1)^{i-1}\binom{k-1+t}{k-2+i} \pmod{p},\ i \ne 0,$$
$$\binom{kp+n}{(k-1)p} \equiv \binom{k-1+t}{k-1} \pmod{p} \text{ when } i = 0.$$

Also, $S((k-1)p+n-i(p-1),(k-1)p) \equiv 0 \pmod p$, $\binom{kp+n}{(k-1)p+n-i(p-1)} \equiv (-1)^{i-1}\binom{k-1+t}{i} \pmod p$ if $i \neq 0$ and $\binom{kp+n}{(k-1)p+n} \equiv k+t-1 \pmod p$ for $i=0$. Consequently, (4.23) reduces to

$$\begin{aligned} kS(kp+n,kp) &\equiv \binom{k-1+t}{k-1} S(n+p,p) + (k+t-1)S((k-1)p+n,(k-1)p) \\ &+ \sum_{i=1}^{t} (-1)^{i-1} \binom{k-1+t}{k-2+i} \binom{k-2+i}{i} S(n+p-i(p-1),p) \\ &+ \sum_{i=1}^{t} (-1)^{i-1} \binom{k-1+t}{i} S((k-1)p+n-i(p-1),(k-1)p) \pmod{p^2}. \end{aligned} \tag{4.28}$$

By Corollary 4.11, we have

$$\begin{aligned} &S((k-1)p+n-i(p-1),(k-1)p) \\ &\qquad \equiv \binom{k-1+t-i}{t-i+1} S(p+n-i(p-1),p) \pmod{p^2}. \end{aligned} \tag{4.29}$$

Combining (4.28) and (4.29), we get

$$\begin{aligned} S(kp+n,kp) &\equiv \frac{1}{k}\binom{k-1+t}{k-1} S(n+p,p) + \frac{k+t-1}{k} S((k-1)p+n,(k-1)p) \\ &+ \frac{2}{k}\binom{k-1+t}{k-2} \sum_{i=1}^{t} (-1)^{i-1} \binom{t+1}{i} S(n+p-i(p-1),p) \pmod{p^2}. \end{aligned} \tag{4.30}$$

The theorem follows by using induction on r and utilizing the preceding congruence (4.30). □

Corollary 4.13. Let p be an odd prime, k, t, and n be positive integers such that $4 \leq k+2 \leq k+t < p$ and $n = tp-1$. Then

$$S(kp+n,kp) \equiv \binom{t+k-1}{k-1} S(p+n,p) + \sum_{i=1}^{t} b_i S(p+n-i(p-1),p) \pmod{p^2}, \tag{4.31}$$

where $b_i = (-1)^{i-1} \frac{(t+k-1)!}{(t-i+1)!(k-2)!i!}$.

Proof. Take $r = k-1$ in (4.30). □

Theorem 4.14. If $p > 3$ is a prime and $2 \leq k < p-1$, where k is even, then $v_p(S(p^2-1,kp)) \geq 2$.

Proof. For an even integer k with $2 \leq k < p-1$, letting $n = p^2 - kp - 1$ and $t = p - k$ in Theorem 4.12 generates the following relation:

$$S(p^2-1, kp) \equiv (-1)^{k-1} S(p^2+p-kp-1, p) - \sum_{i=1}^{p-k} \binom{p-1-i}{k-2} S(p^2+p-kp-1-i(p-1), p) \pmod{p^2}. \tag{4.32}$$

Replacing k, n, and r by p, $p+p^2-kp-1$, and $p-k-1$, respectively, in (2.5), we get

$$S(p+p^2-kp-1, p) \equiv (p-k) S(3p-k-2, p) - (p-k-1) S(2p-k-1, p) \pmod{p^2}. \tag{4.33}$$

Similarly, for $1 \leq i < p-k-1$, we have

$$S(p+p^2-kp-1-i(p-1), p) \equiv (p-k-i) S(3p-k-2, p) - (p-k-i-1) S(2p-k-1, p) \pmod{p^2}. \tag{4.34}$$

The preceding three congruences together lead to the following:

$$\begin{aligned} S(p^2-1, kp) \equiv{} & (-1)^{k-1}[(p-k)S(3p-k-2,p) - (p-k-1)S(2p-k-1,p)] \\ & - (k-1)S(2p-k-1,p) - \binom{k}{2} S(3p-k-2,p) \\ & - \sum_{i=1}^{p-k-2} \binom{p-1-i}{k-2} (p-k-i) S(3p-k-2,p) \\ & + \sum_{i=1}^{p-k-2} \binom{p-1-i}{k-2} (p-k-i-1) S(2p-k-1,p) \pmod{p^2}. \end{aligned} \tag{4.35}$$

Now using the identities $\sum_{i=0}^{y} \binom{x+i}{x} = \binom{x+y+1}{x+1}$ and $\binom{p-1-i}{k-2} \equiv \binom{k+i-2}{k-2} \pmod{p}$ for even k, it follows that

$$\sum_{i=1}^{p-k} \binom{p-1-i}{k-2} (p-k-i) \equiv k+1 \pmod{p}, \tag{4.36}$$

$$\sum_{i=1}^{p-k} \binom{p-1-i}{k-2} (p-k-i-1) \equiv k+3 \pmod{p}. \tag{4.37}$$

From Equations (4.35) and (4.36), we get

$$S(p^2-1, kp) \equiv 2S(2p-k-1, p) - S(3p-k-2, p) \pmod{p^2}. \tag{4.38}$$

Since $2p < 3p-k-2 < 3p-2$, we obtain from Equation (2.6) that

$$S(3p-k-2, p) \equiv 2S(2p-k-1, p) \pmod{p^2}, \tag{4.39}$$

and the theorem follows. □

The following theorem settles the lower bound of $v_p(S(p^2, kp))$ for even k in the Conjecture of [17], which we mention in Equation (1.6).

Theorem 4.15. If $p > 3$ is a prime and k is even with $2 \leq k < p - 1$, then

$$v_p(S(p^2, kp)) \geq 5. \tag{4.40}$$

Proof. Taking $n = 2$ and $m = 1$ in (4.14), we get

$$S(p^2, 2kp) \equiv \begin{cases} \binom{\frac{p-3}{2}} p^3 S(p^2-1, (p-3)p) - \dfrac{9p^5}{4} \pmod{p^6}, & \text{if } k = \frac{p-3}{2}; \\ kp^3 S(p^2-1, 2kp) \pmod{p^6}, & \text{otherwise.} \end{cases} \tag{4.41}$$

From (4.41), we have the following weaker congruence:

$$S(p^2, 2kp) \equiv kp^3 S(p^2-1, 2kp) \pmod{p^5}. \tag{4.42}$$

The theorem follows from (4.42) and Theorem 4.14. □

4.2. **Divisibility of $S(p^n, k)$ when $p \nmid k$.** From Equation (2.10), we get

$$v_p(S(p^n, k)) \geq \left\lceil \frac{s_p(k) - 1}{p - 1} \right\rceil, \tag{4.43}$$

which shows that $S(p^n, k)$ is divisible by p unless k is a power of p (i.e., $k = p^m$ for some positive integer m). Now we have the following result.

Theorem 4.16. Let p be an odd prime and $1 \leq k \leq p - 1$. Then for any positive integer n,

$$S(p^n, k) \equiv S(p, k) \pmod{p^2}. \text{ So, } v_p(S(p^n, k) - S(p, k)) \geq 2. \tag{4.44}$$

Proof. Taking $n = p^2$ and $r = p - 1$ in (2.5), we get

$$S(p^2, k) \equiv pS(2p-1, k) - (p-1)S(p, k) \pmod{p^2}. \tag{4.45}$$

Using the minimum period from Equation (2.11) for $S(2p - 1, k)$, the theorem follows at once. □

Theorem 4.17. Let p be an odd prime and $p < k < p^2$. If $k = k_1 p + k_0$ and $k_0 \neq 0$, then

$$S(p^2, k) \equiv \binom{p}{k_1} S(p - k_1, k_0) \pmod{p^2}. \tag{4.46}$$

Proof. From Equation (1.4), we get

$$\binom{k_1p+k_0}{k_0}S(p^2,k_1p+k_0)=\sum_{i=0}^{p^2}\binom{p^2}{i}S(i,k_1p)S(p^2-i,k_0)$$
$$\equiv\sum_{i=k_1}^{p-1}\binom{p^2}{ip}S(ip,k_1p)S(p^2-ip,k_0) \pmod{p^2} \tag{4.47}$$

since $v_p(\binom{p^n}{i})=n-v_p(i)$.

Since p divides $\binom{p^2}{ip}$ for $k_1\le i\le p-1$ and p divides $S(ip,k_1p)$ unless $i=k_1$, we have

$$\binom{k_1p+k_0}{k_0}S(p^2,k_1p+k_0)\equiv\binom{p^2}{k_1p}S(p^2-k_1p,k_0) \pmod{p^2}. \tag{4.48}$$

Also, $S(p^2,k_1p+k_0)\equiv 0 \pmod p$ and $\binom{k_1p+k_0}{k_0}\equiv 1 \pmod p$. Moreover, $\binom{p^2}{k_1p}\equiv\binom{p}{k_1} \pmod{p^2}$. Consequently,

$$S(p^2,k_1p+k_0)\equiv\binom{p}{k_1}S(p^2-k_1p,k_0) \pmod{p^2}. \tag{4.49}$$

Now using the minimum periods on the preceding congruence, the theorem follows. □

Remark 4.18. Theorem 4.17 gives an exact p-adic valuation for some special cases.

Corollary 4.19. Let p be an odd prime and $k=k_1p+k_0>p$, where k_1 and k_0 are the p-adic digits of k. If $s_p(k)=p$ or $s_p(k)<p$ and $t_p(p-k_1,p-s_p(k))=0$, then

$$v_p(S(p^2,k))=1. \tag{4.50}$$

Different values of t_p are mentioned in Equation (2.9). The following theorem is a generalization of Theorem 4.17.

Theorem 4.20. If k is a positive integer not divisible by an odd prime p with $p<k<p^n$ for some positive integer $n\ge 2$, then

$$S(p^n,k)$$
$$\equiv\begin{cases}0 \pmod{p^2}, \text{ if } s_p(k)>p;\\ \binom{p}{k_{n_\tau},\cdots,k_{n_1},p-\sum_{r=1}^{\tau-1}k_{n_r}}S(p-\sum_{r=0}^{\tau-1}k_{n_{\tau-r}},k_0) \pmod{p^2}, \text{ otherwise,}\end{cases} \tag{4.51}$$

where $k_{n_\tau},k_{n_{\tau-1}},\cdots,k_{n_1},k_{n_0}=k_0$, are the non zero p-adic digits of k.

Proof. For $n = 2$, the result follows from Theorem 4.17. So, let $n > 2$. Let $k = \sum_{r=0}^{t} k_r p^r$ such that for each $1 \leq r \leq t$, k_r is the r-th p-adic digit of k and $k_0 \neq 0 \neq k_t$. Removing all the zero digits from the expression of k, we can re-write

$$k = \sum_{r=0}^{\tau} k_{n_r} p^{n_r}, \tag{4.52}$$

where $k_{n_r} \neq 0$ for every r with $1 \leq r \leq \tau$. Then from (1.4), we have

$$\binom{k}{k_{n_\tau} p^{n_\tau}} S(p^n, k) = \sum_{i=0}^{p^n} \binom{p^n}{i} S(i, k_{n_\tau} p^{n_\tau}) S(p^n - i, \sum_{r=0}^{\tau-1} k_{n_r} p^{n_r}). \tag{4.53}$$

The binomial coefficients on the right-hand side in (4.53) are divisible by p^2 unless i is divisible by p^{n-1} and hence the preceding equation reduces to

$$S(p^n, k) \equiv \sum_{i=0}^{p} \binom{p^n}{ip^{n-1}} S(ip^{n-1}, k_{n_\tau} p^\tau) S(p^n - ip^{n-1}, \sum_{r=0}^{\tau-1} k_{n_r} p^{n_r}) \pmod{p^2}. \tag{4.54}$$

Here, $\binom{p^n}{ip^{n-1}} \equiv \binom{p}{i} \pmod{p^2}$ and each of these binomial coefficients are divisible by p unless $i = 0$ or p. Using (2.1), we observe that $S(ip^{n-1}, k_{n_\tau} p^\tau)$ is divisible by p^{n_τ} if $i \neq k_{n_\tau}$. Moreover, $S(k_{n_\tau} p^{n-1}, k_{n_\tau} p^{n_\tau}) \equiv 1 \pmod{p}$, and so,

$$S(p^n, k) \equiv \binom{p}{k_{n_\tau}} S(p^n - k_{n_\tau} p^{n-1}, \sum_{r=0}^{\tau-1} k_{n_r} p^{n_r}) \pmod{p^2}. \tag{4.55}$$

We then have

$$\binom{\sum_{r=0}^{\tau-1} k_{n_r} p^{n_r}}{k_{n_{\tau-1}} p^{n_{\tau-1}}} S(p^n - k_{n_\tau} p^{n-1}, \sum_{r=0}^{\tau-1} k_{n_r} p^{n_r}) = \sum_{i=0}^{(p-k_{n_\tau})p^{n-1}} \binom{(p-k_{n_\tau})p^{n-1}}{i}$$
$$\times S(i, k_{n_{\tau-1}} p^{n_{\tau-1}}) S((p-k_{n_\tau})p^{n-1} - i, \sum_{r=0}^{\tau-2} k_{n_r} p^{n_r}). \tag{4.56}$$

Using the same technique as in the proof of Theorem 4.17, we obtain

$$S(p^n - k_{n_\tau} p^{n-1}, \sum_{r=0}^{\tau-1} k_{n_r} p^{n_r}) \equiv \sum_{i=0}^{p-k_{n_\tau}} \binom{(p-k_{n_\tau})p^{n-1}}{ip^{n-1}} S(ip^{n-1}, k_{n_{\tau-1}} p^{n_{\tau-1}})$$
$$\times S((p-k_{n_\tau})p^{n-1} - ip^{n-1}, \sum_{r=0}^{\tau-2} k_{n_r} p^{n_r}) \pmod{p} \tag{4.57}$$

which also yields the following:

$$S(p^n - k_{n_\tau} p^{n-1}, \sum_{r=0}^{\tau-1} k_{n_r} p^{n_r})$$

$$\equiv \begin{cases} 0 \pmod p, \text{ if } k_{n_\tau} + k_{n_{\tau-1}} \geq p; \\ \binom{p-k_{n_\tau}}{k_{n_{\tau-1}}} S((p-k_{n_\tau} - k_{n_{\tau-1}})p^{n-1}, \sum_{r=0}^{\tau-2} k_{n_r} p^{n_r}) \pmod p, \text{ otherwise.} \end{cases} \tag{4.58}$$

Denoting $\lambda_u = S((p - \sum_{r=0}^{u-1} k_{n_{\tau-r}})p^{n-1}, \sum_{r=0}^{\tau-u} k_{n_r} p^{n_r})$ and combining (4.55) with the preceding congruence, we get

$$S(p^n, k) \equiv \begin{cases} 0 \pmod{p^2}, \text{ if } k_{n_\tau} + k_{n_{\tau-1}} \geq p; \\ \binom{p}{k_{n_\tau}, k_{n_{\tau-1}}, p-k_{n_\tau} - k_{n_{\tau-1}}} \lambda_2 \pmod{p^2}, \text{otherwise.} \end{cases} \tag{4.59}$$

We employ the same technique recursively to get the congruence

$$S(p^n, k) \equiv \begin{cases} 0 \pmod{p^2}, \text{ if } \sum_{r=0}^{\tau-1} k_{n_{\tau-r}} \geq p; \\ \binom{p}{k_{n_\tau}, \cdots, k_{n_1}, p-\sum_{r=1}^{\tau-1} k_{n_r}} \lambda_\tau \pmod{p^2}, \text{ otherwise} \end{cases} \tag{4.60}$$

if $n_0 = 0$.

Now applying the minimum period on $S((p - \sum_{r=0}^{\tau-1} k_{n_{\tau-r}})p^{n-1}, k_0)$, we get the desired result. □

Corollary 4.21. Let p be an odd prime.

(a) Let $k = \sum_{i=0}^{t} k_i p^i > p$ be the p-adic expansion of k with $k_0 \neq 0$. If $s_p(k) \leq p$ and $t_p(p - s_p(k) + k_0, p - s_p(k)) = 0$, then $v_p(S(p^2, k)) = 1$.

(b) If $n \geq 2$, $s_p(k) < p$, and $1 < kp + 1 < p^n$, then $v_p(S(p^n, kp+1)) = 1$.

(c) If $p \nmid k$ and $k < p^m \leq p^n$, then $S(p^n, k) \equiv S(p^m, k) \pmod{p^2}$.

Remark 4.22. From Corollary 4.21(a), we observe that $t_p(p - s_p(k) + k_0, p - s_p(k)) = 0$ when $s_p(k) = p - 1$; $s_p(k) = p - 2$ and $v_p(3p - 3k_1 - 5) = 0$; $s_p(k) = p - 3$; $s_p(k) = p - 4$ and $v_p(15n^3 - 150n^2 + 485n - 502) = 0$ and so on.

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A. LALCHHUANGLIANA
abchhak@gmail.com

Department of Mathematics,
Government Champhai College
Champhai - 796 321,
Mizoram, INDIA.

S. S. SINGH
saratcha32@yahoo.co.uk
(Corresponding author)
Department of Maths. and Comp. Sc.,
Mizoram University, Aizawl
Aizawl - 796 004,
Mizoram, INDIA.

JITENDER SINGH
jitender.math@gndu.ac.in
Department of Mathematics,
Guru Nanak Dev University,
Amritsar - 143 005,
Punjab, INDIA.

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ORIGINAL RESEARCH

Congruence relation between Stirling numbers of the first and second kinds

A. Lalchhuangliana · S. S. Singh

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Abstract This paper consists of certain congruence properties of Stirling numbers of the first and second kinds. Some congruence relations between $s(n, k)$ and $S(n, k)$ for different modulo are obtained through their generating functions. We also present some exact p-adic valuations of $s(n, k)$ and $S(n, k)$ for some cases, mainly when $n - k$ is divisible by $p - 1$ for odd prime p. Some estimates of the p-adic valuation of these two numbers are also presented when $p - 1$ does not divide $n - k$.

Keywords Congruence · Generating function · Primes · p-adic valuation · Stirling numbers

Mathematics Subject Classification 11A07 · 11B73 · 11E95.

1 Introduction

The p-adic valuation of an integer a denoted by $v_p(a)$ is the highest exponent of p that divides a, where p is a prime. It follows that $p^{v_p(a)} \mid a$ but $p^{1+v_p(a)} \nmid a$. Every integer has a unique p-adic expansion. If

$$a = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n,$$

where $0 \le a_i \le p - 1$, this expression is unique, and a_i's are called the p-adic digits. The sum of a's p-adic digits is denoted by $s_p(a)$. Thus,

$$s_p(a) = \sum_{i=0}^{n} a_i.$$

This sum is related to the p-adic valuation of binomial coefficients [14] as

$$v_p\left(\binom{n}{k}\right) = \frac{s_p(k) + s_p(n-k) - s_p(n)}{p-1}.$$

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This author contributed equally to this work.

A. Lalchhuangliana
Department of Mathematics and Computer Science, Mizoram University, Tanhril, Aizawl, Mizoram 796004, India
E-mail: abchhak@gmail.com

S. S. Singh
Department of Mathematics, Government Champhai College, Vengthlang North, Champhai, Mizoram 796321, India
E-mail: saratcha32@yahoo.co.uk

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Some important generating functions of Stirling numbers of the second kind $S(n,k)$ and the first kind $s(n,k)$ are

$$x^{\underline{n}} = \sum_{i=0}^{n} s(n,i)x^i, \quad x^{\overline{n}} = \sum_{i=0}^{n} (-1)^{n-i} s(n,i)x^i, \tag{1}$$

$$\prod_{i=1}^{n-1}(1-ix) = \sum_{i=0}^{n-1} s(n,n-i)x^i, \quad \prod_{i=1}^{n-1}(1+ix) = \sum_{i=0}^{n-1}(-1)^i s(n,n-i)x^i, \tag{2}$$

$$\prod_{i=1}^{k}\frac{1}{(1-ix)} = \sum_{n=0}^{\infty} S(n+k,k)x^n, \quad \prod_{i=1}^{k}\frac{1}{(1+ix)} = \sum_{n=0}^{\infty}(-1)^n S(n+k,k)x^n, \tag{3}$$

where the notations $x^{\underline{n}}$ and $x^{\bar{n}}$ stand for the falling factorial and the rising factorial of x, respectively.
Stirling numbers are related to several sequences of numbers, making them interesting to many researchers. In recent years, divisibility properties of Stirling numbers of the second kind for an even prime appeared in Adelberg [2], Hong *et al.* [7], Lengyel [11], Wannemacker [22] and Zhao *et al.* [24].
For positive integers a, k, and a prime p such that $a < k < p$, and for each positive integer $n \equiv a \pmod{p^{m_0-1}(p-1)}$, Miska [15] confirmed that

$$v_p(S(n,k)) = v_p(S(a+(p-1)p^{m_0-1},k)) + v_p(n-a) - m_0 + 1.$$

Singh and Lalchhuangliana [18] proved, using a combinatorial approach and a concept of minimum periods, that

$$v_p(S(p^n,kp)) \geq 2,$$

for any integer $n \geq 2$ and $2 \leq k \leq p-1$.
Chan and Manna [4] obtained a congruence relation for $S(n,kp^m)$ modulo p^m and found that p^m always divides $S(n,kp^m)$ whenever $n \not\equiv k \pmod{p-1}$. Adelberg [1] discussed the concept of Minimum Zero Case (MZC), and if $n = ap^h$, $1 \leq a \leq p-1$, and $(p-1) \mid (n-k)$, then $S(n,k)$ is MZC with exact p-adic valuation

$$v_p(S(n,k)) = \frac{s_p(n) - s_p(k)}{p-1}.$$

For more results about the divisibility properties of Stirling numbers of the second kind, we refer to Davis [5], Sagan [17], Singh *et al.* [19], Tsumura [21], Sun [20], and Young [23].
For any prime p, Lengyel [12] confirmed that

$$v_p(s(ap^n+b, ap^n+b-k)) = v_p(s(b,b-k)),$$

where a, k, b, and n are integers such that $a \geq 1$ with $(a,p) = 1$, $2 \leq k+1 \leq b$ and n is sufficiently large. Qiu and Hong [16] showed that $v_2(s(2^n,k)) = v_2(s(2^n+1,k+1))$ for $1 \leq k \leq 2^n$, and $v_2(s(2n,2n-k)) = 2n-2-v_2(k-1)$ if k is odd and $2 \leq k \leq 2^{n-1}+1$. Komatsu and Young [10] proved that

$$v_p(s(n+1,k+1)) = v_p(n!) - v_p(k!) - kr,$$

where n, k, m, and r are positive integers such that $n = kp^r + m$ and $m < p^r$. More results about the divisibility of Stirling numbers of the first kind can be seen in Cao and Pan [3], Hong and Qiu [8], Howard [9], and Leonetti and Sanna [13].
This paper present the congruences of $S(n,kp^m)$ and $s(kp^m,n)$ explicitly in terms of binomial coefficient when $n \equiv k \pmod{p-1}$. We further obtain congruences for $S(n,k)$ and $s(k,a)$ modulo p, p^m, and p^n, where $m = \lfloor log_p(k) \rfloor$ and $n \geq m$. We reduce the congruences to simpler results for some special cases. The exact values of $v_p(S(n,k))$ and $v_p(s(n,k))$ for some special cases of $n \equiv k \pmod{p-1}$ are obtained. We also discuss the case when $S(n,k)$ and $S(n-1,k-1)$ have the same congruence property.

2 Preliminaries

The generating functions of $S(n, k)$ and $s(n, k)$ play a significant role in obtaining their congruence properties. We will assume p as an odd prime unless stated otherwise. We use the congruence property of the polynomial Chan and Manna [4]

$$\prod_{i=1}^{kp^m}(1 - ix) \equiv (1 - x^{p-1})^{kp^{m-1}} \pmod{p^m}. \tag{4}$$

If we replace x with $1/y$, we get

$$\prod_{i=1}^{kp^m}(1 - ix) = \frac{1}{y^{kp^m}} \prod_{i=1}^{kp^m}(y - i)$$

and

$$(1 - x^{p-1})^{kp^{m-1}} = \frac{1}{y^{k(p-1)p^{m-1}}}(y^{p-1} - 1)^{kp^{m-1}}.$$

It follows that

$$\prod_{i=1}^{kp^m}(y - i) \equiv y^{kp^{m-1}}(y^{p-1} - 1)^{kp^{m-1}} \pmod{p^m}. \tag{5}$$

We can also write this result as

$$x^{\underline{kp^m}} \equiv x^{kp^{m-1}}(x^{p-1} - 1)^{kp^{m-1}} \pmod{p^m}. \tag{6}$$

Replacing x by $-x$ in Equations (4) and (6), we obtain

$$\prod_{i=1}^{kp^m}(1 + ix) \equiv (1 - x^{p-1})^{kp^{m-1}} \pmod{p^m} \tag{7}$$

and

$$x^{\overline{kp^m}} \equiv (-1)^k x^{kp^{m-1}}(x^{p-1} - 1)^{kp^{m-1}} \pmod{p^m}. \tag{8}$$

Davis and Webb [6] proved, for a prime $p > 3$, that

$$\binom{np}{kp} \equiv \binom{n}{k} \pmod{p^e}, \tag{9}$$

where $e = 3 + v_p(n) + v_p(k) + v_p(n - k) + v_p(\binom{n}{k})$. With the help of Equation (9), it is easy to confirm

$$(1 - x)^{p^n} \equiv (1 - x^{p^{n-m}})^m \pmod{p^{m+1}} \tag{10}$$

for any integers n and m such that $0 \leq m \leq n$.

3 Results

In this section, we prove our main results which we present in theorems and corollaries. The first theorem gives the congruence relations between Stirling numbers of the first kind and Binomial coefficients.

Theorem 1 *If p is an odd prime and k is a positive integer not divisible by p, then for any positive integer m, the following congruences hold;*

$$\begin{aligned}
&a)\ \ s(kp^m, kp^m - b) \equiv (-1)^k s(kp^m, kp^{m-1} + b) \pmod{p^m},\\
&b)\ \ s(kp^m, kp^m - b) \equiv \binom{kp^{m-1}}{\frac{b}{p-1}}(-1)^{\frac{b}{p-1}} \pmod{p^m},\\
&\qquad\qquad \textit{if } b \equiv 0 \pmod{p-1} \textit{ and } b \le k(p-1)p^{m-1}.\\
&c)\ \ s(kp^m, b) \equiv 0 \pmod{p^m},\\
&\qquad\qquad \textit{if } b \le kp^{m-1} \textit{ or } kp^{m-1} - b \not\equiv 0 \pmod{p-1}.\\
&d)\ \ s(kp^m, b) \equiv s(kp^m + 1, b+1) \pmod{p^m},
\end{aligned}$$

for any integer b such that $1 \le b \le kp^m - 1$.

Proof We know that

$$\prod_{i=1}^{kp^m-1}(1 - ix) = \sum_{i=0}^{kp^m-1} s(kp^m, kp^m - i)x^i$$

$$x^{\underline{kp^m}} = \sum_{i=0}^{kp^m} s(kp^m, i)x^i.$$

Due to Equations (4) and (6), we get

$$\prod_{i=1}^{kp^m-1}(1 - ix) \equiv (1 - x^{p-1})^{kp^{m-1}} \pmod{p^m};$$

$$x^{\underline{kp^m}} \equiv x^{kp^{m-1}}(x^{p-1} - 1)^{kp^{m-1}} \pmod{p^m}.$$

It follows that

$$\sum_{i=0}^{kp^m-1} s(kp^m, kp^m - i)x^i \equiv \sum_{j=0}^{kp^{m-1}} \binom{kp^{m-1}}{j}(-1)^j x^{j(p-1)} \pmod{p^m};$$

$$\sum_{i=0}^{kp^m} s(kp^m, i)x^i \equiv x^{kp^{m-1}} \sum_{j=0}^{kp^{m-1}} \binom{kp^{m-1}}{j}(-1)^{k-j} x^{j(p-1)} \pmod{p^m}.$$

Comparing the coefficients of x^b, we get the first three results, and the last result is obtained from the congruence

$$\prod_{i=1}^{kp^m}(1 - ix) \equiv \prod_{i=1}^{kp^m-1}(1 - ix) \pmod{p^m}. \tag{11}$$

Hence, the theorem follows. □

Corollary 1 *Let $n = kp^m$, $m \ge 1$, be an integer with only one non-zero p-adic digit, and $p > 3$ be a prime. Then, $s(n, a)$ is divisible by p if and only if $n \not\equiv a \pmod{(p-1)p^{m-1}}$. Further, $s(n, a)$ is divisible by p^{t+1}, $0 \le t \le m - 1$, if and only if $n - a < kp^{m-1}$ or $n \not\equiv a \pmod{(p-1)p^{m-1-t}}$.*

Proof Follow the proof of Theorem 1 and use Equation (10). □

Corollary 2 *For an odd prime p and integers k and m such that $p \nmid k$ and $m \geq 1$, we have*

$$v_p(s(kp^m+1, a+1)) = v_p(s(kp^m, a)) = m - 1 - v_p(a), \tag{12}$$

whenever $m-1 > v_p(a)$, $kp^{m-1} \leq a \leq kp^m$, and $k \equiv a \pmod{p-1}$.

Proof The proof is based on the equality $v_p(\binom{ap^n}{bp^m}) = n - m$ when $p \nmid a$, $p \nmid b$, and $n > m$. □

Remark 1 If $n \geq m$, the p-adic valuation of the binomial coefficient $\binom{ap^n}{bp^m}$ is equal to $v_p(\binom{a}{bp^{m-n}})$. It follows that if $m-1 \leq v_p(a)$ and $v_p(\binom{k}{kp-ap^{1-m}}) \leq m-1$, we have

$$v_p(s(kp^m+1, a+1)) = v_p(s(kp^m, a)) = v_p\left(\binom{k}{kp - ap^{1-m}}\right). \tag{13}$$

It is also trivial from Theorem 1 that

$$Min\{v_p(s(kp^m+1, a+1)),\ v_p(s(kp^m, a))\} \geq m \tag{14}$$

when $a < kp^{m-1}$ or $kp^{m-1} - a \not\equiv 0 \pmod{p-1}$.

The following theorem is a generalization of Theorem 1.

Theorem 2 *For an odd prime p and positive integers k, m, a, and b, the following congruences hold;*

$$s(kp^m + a, kp^{m-1} + b) \equiv \sum_i (-1)^{k-i} \binom{kp^{m-1}}{i} s(a, b - i(p-1)) \pmod{p^m} \tag{15}$$

if $b \leq a + k(p-1)p^{m-1}$, and

$$s(kp^m + a, b) \equiv 0 \pmod{p^m}. \tag{16}$$

if $b \leq kp^{m-1}$.

Proof We have

$$x^{\underline{kp^m+a}} = \sum_{i=0}^{kp^m+a} s(kp^m+a, i)x^i \tag{17}$$

and

$$\begin{aligned} x^{\underline{kp^m+a}} &= \prod_{i=0}^{kp^m}(x-i)\prod_{i=1}^{a-1}(x-(kp^m+i)) \\ &\equiv x^{kp^{m-1}}(x^{p-1}-1)^{kp^{m-1}}x^{\underline{a}} \pmod{p^m} \\ &\equiv x^{kp^{m-1}} \sum_{i=0}^{kp^{m-1}} \binom{kp^{m-1}}{i}(-1)^{k-i}x^{i(p-1)}\sum_{j=0}^{a} s(a,j)x^j \pmod{p^m}. \end{aligned} \tag{18}$$

Comparing the coefficients of $x^{kp^{m-1}+b}$ in the RHS of Equations (17) and (18), we obtain

$$s(kp^m + a, kp^{m-1} + b) \equiv \sum_{i(p-1)+j=b} (-1)^{k-i}\binom{kp^{m-1}}{i} s(a, j) \pmod{p^m}. \tag{19}$$

Changing the index j to $b - i(p-1)$ confirms the first result of the theorem. The coefficient of x^n on the right-hand side of Equation (18) vanishes if $n \leq kp^{m-1}$; hence, the second result follows. □

The following corollaries are special cases of the preceding theorem.

Corollary 3 *For an odd prime p and positive integers k, m, a, and b;*

$$(i)\ \sigma_a \equiv (-1)^k s(a,b) \pmod{p^m} \quad \text{if } b \le p-1,$$
$$(ii)\ v_p(\sigma_{p-1}) = \begin{cases} m-1-v_p(\lfloor \frac{b}{p-1} \rfloor), & \text{if } (p-1) \nmid b \text{ and } \lfloor \frac{b}{p-1} \rfloor < p^{m-1}; \\ m-1-v_p(\frac{b}{p-1}-1), & \text{if } (p-1) \mid b \text{ and } \frac{b}{p-1} < p^{m-1}. \end{cases}$$

where $\sigma_a = s(kp^m + a, kp^{m-1} + b)$.

Proof On observation of Equation (19), we can see that;
(i) If $b \le p-1$, then the only solution of $i(p-1)+j=b$ for (i,j) is $(0,b)$, unless $b=p-1$, in which case there are two solutions, namely $(0,p-1)$ and $(1,0)$. The corresponding term for the solution $(1,0)$ vanishes as $s(a,0)=0$. Hence, (i) follows.
(ii) Let $b=q(p-1)+r$ such that $0 \le r < p-1$. The only solution of $i(p-1)+j=q(p-1)+r$ with $j \le p-1$ is $(i,j)=(q,r)$, if $r \ne 0$. Thus, we get

$$s(kp^m + p-1, kp^{m-1}+b) \equiv (-1)^{k-q}\binom{kp^{m-1}}{q} s(p-1,r) \pmod{p^m}. \tag{20}$$

Now, we obtain the congruence for $s(p-1,b)$:
We have

$$x^{\underline{p-1}} = \frac{x^{\underline{p}}}{x-p+1} \equiv (x^p - x)(1 - x + x^2 - \cdots) \pmod{p}.$$

It follows that

$$\sum_{i=0}^{p-1} s(p-1,i)x^i \equiv -x + x^2 - x^3 + \cdots + x^{p-1} \pmod{p}$$

and

$$s(p-1,i) \equiv (-1)^i \pmod{p} \tag{21}$$

if $1 \le i \le p-1$.
Therefore, the valuation of the binomial coefficient $\binom{kp^{m-1}}{q}$ is $m-1-v_p(q)$ and $s(p-1,r)$ is not divisible by p. Thus, the first case of (ii) follows.
On the other hand, if $r=0$ or $b=q(p-1)$, then there are two solutions of $i(p-1)+j=q(p-1)$, namely $(q,0)$ and $(q-1,p-1)$. The corresponding term for the index $(q,0)$ is zero since $s(p-1,0)=0$. Following the proof of the first result, we get the second case of (ii). □

Corollary 4 *For an odd prime p and positive integers k, m, a, and b;*

$$s(kp^m + a, kp^{m-1}+b) \equiv (-1)^q \binom{kp^{m-1}}{q} s(a,r) \pmod{p^m}$$

if $a < p-1$ *and* $b = q(p-1)+r$ *with* $0 \le r < p-1$.

Proof Given Equation (19), the only solution of $i(p-1)+j=q(p-1)+r$ is $(i,j)=(q,r)$. Hence the result follows. □

Remark 2 The p-adic valuations of large classes of Stirling numbers of the first kind can be obtained using Theorem 1, Corollaries 3, and 4. The first result of Corollary 4 yields the following exact p-adic valuation,

$$v_p(s(kp^m + a, kp^{m-1}+b)) = m-1-v_p\left(\left\lfloor \frac{b}{p-1} \right\rfloor\right), \tag{22}$$

if $b \equiv 1, a, a-1$ or $a-3 \pmod{p-1}$, assuming conditions of the corollary apply.

Theorem 3 *Let p be an odd prime and k, a, b, m, and t be positive integers such that $max\{k, a\} \leq p-1$, $t < m$, and $b \leq ap^t$. Then,*

$$\theta_{ap^t} \equiv \begin{cases} s(ap^t, b) \pmod{p^m}, & \text{if } a \not\equiv b \pmod{p-1} \text{ or } b < ap^{t-1}; \\ s(ap^t, b) \pmod{p^{m-v_p(b)-1}}, & \text{otherwise.} \end{cases} \tag{23}$$

where $\theta_q = s(kp^m + q, kp^m + b)$.

Proof Replace a and b in Equation (19) with ap^t and $k(p-1)p^{m-1} + b$, respectively; we obtain

$$\theta_{ap^t} \equiv \sum_{i(p-1)+j=k(p-1)p^{m-1}+b} (-1)^{k-i} \binom{kp^{m-1}}{i} s(ap^t, j) \pmod{p^m}.$$

If we replace the index j with $(kp^{m-1} - i)(p-1) + b$, we get

$$\theta_{ap^t} \equiv \sum_i (-1)^{k-i} \binom{kp^{m-1}}{i} s(ap^t, (kp^{m-1} - i)(p-1) + b) \pmod{p^m}.$$

By reversing the index, we get

$$\theta_{ap^t} \equiv \sum_i (-1)^i \binom{kp^{m-1}}{i} s(ap^t, b + i(p-1)) \pmod{p^m}. \tag{24}$$

Using Theorem 1, $s(ap^t, b + i(p-1))$ is divisible by p^t if $a \not\equiv b \pmod{p-1}$ or $b < ap^{t-1}$. The valuation of $\binom{kp^{m-1}}{i}$ is $m - 1 - v_p(i)$ unless $i = 0$. Thus, the valuation of the $i-$th terms, $i \neq 0$, of the right-hand side of Equation (24) is greater than or equal to $m - 1 + t - v_p(i)$. The range of the index i is determined by the inequality $b \leq b + i(p-1) \leq ap^t$, which implies that $0 \leq i \leq a \sum_{r=0}^{t-1} p^r$. It follows that $v_p(i) \leq t - 1$ and consequently $m - 1 + t - v_p(i) \geq m$. Hence, p^m divides all the $i - th$ terms except the term with $i = 0$. Thus, the first case of the theorem follows.
Now, we assume that $a \equiv b \pmod{p-1}$ and $ap^{t-1} \leq b \leq ap^t$. Therefore, we can express b as $ap^t - q(p-1)$ with $0 \leq q < ap^{t-1}$. Equation (24) becomes

$$\theta_{ap^t} \equiv \sum_i (-1)^i \binom{kp^{m-1}}{i} s(ap^t, ap^t - (q-i)(p-1)) \pmod{p^m}. \tag{25}$$

Given Theorem 1, the valuation of the i-th term on the RHS of the preceding congruence is $m + t - 2 - v_p(i) - v_p(q-i)$ if $i > 0$. Therefore, two sub cases arise, namely $v_p(q) < v_p(i)$ and $v_p(i) \leq v_p(q)$;
If $v_p(q) < v_p(i)$, we have $v_p(q-i) = v_p(q)$ and

$$m + t - 2 - v_p(i) - v_p(q-i) = m - 1 - v_p(q) + (t - 1 - v_p(i)) \geq m - 1 - v_p(q),$$

since $v_p(i) \leq t - 1$.
If $v_p(i) \leq v_p(q)$, we get

$$m + t - 2 - v_p(i) - v_p(q-i) \geq m - 1 - v_p(q) + (t - 1 - v_p(q-i)) \geq m - 1 - v_p(q),$$

since $v_p(q-i) \leq t - 1$.
The equality $b = ap^t - q(p-1)$ also implies that $v_p(b) = v_p(q)$ for the given condition. It follows that all the terms except when $i = 0$ are divisible by $p^{m-1-v_p(b)}$. Hence, the second case of the theorem follows. □

3.1 Valuations of $S(n, kp^m)$

Using Equations (3) and (4), we have the following result [4];

$$\sum_{n=0}^{\infty} S(n+kp^m, kp^m)x^n \equiv \sum_{j=0}^{\infty} \binom{kp^{m-1}+j-1}{j} x^{j(p-1)} \pmod{p^m}, \tag{26}$$

which implies $S(n, kp^m) \equiv 0 \pmod{p^m}$ if $n \not\equiv k \pmod{p-1}$; otherwise,

$$S(kp^m + a, kp^m) \equiv \binom{kp^{m-1} + \frac{a}{p-1} - 1}{kp^{m-1} - 1} \pmod{p^m} \tag{27}$$

for any non-negative integer a with $a \equiv 0 \pmod{p-1}$.
Due to Equation (11), we also have the congruence

$$S(kp^m + a - 1, kp^m - 1) \equiv S(kp^m + a, kp^m) \pmod{p^m}. \tag{28}$$

The following theorem is a consequence of Equations (26), (27), and (28).

Theorem 4 *Let p be an odd prime and k be an integer not divisible by p. For any positive integer n such that $n \equiv k \pmod{p-1}$ and $n < kp^m + (p-1)p^{m-1}$ with $m > 1$;*

$$v_p(S(n-1, kp^m - 1)) = v_p(S(n, kp^m)) = m - 1 - v_p(n). \tag{29}$$

If $n \not\equiv k \pmod{p-1}$, then

$$min\{v_p(S(n, kp^m)),\ v_p(S(n-1, kp^m - 1))\} \geq m. \tag{30}$$

Proof The second result is trivial from Equations (26) and (28).
We assume $n \geq kp^m$ and let $n = kp^m + b(p-1)$ with $b < p^{m-1}$. From Equations (27) and (28), we have

$$\begin{aligned} S(kp^m + b(p-1) - 1, kp^m - 1) &\equiv S(kp^m + b(p-1), kp^m) \\ &\equiv \binom{kp^{m-1} + b - 1}{b} \pmod{p^m}. \end{aligned}$$

The p-adic valuation of the above binomial coefficient is given as

$$v_p\left(\binom{kp^{m-1} + b - 1}{b}\right) = \frac{s_p(b) + s_p(kp^{m-1} - 1) - s_p(kp^{m-1} + b - 1)}{p-1}, \tag{31}$$

and we have

$$\begin{aligned} s_p(kp^{m-1} - 1) &= s_p((k-1)p^{m-1} + p^{m-1} - 1) \\ &= s_p((k-1)p^{m-1}) + s_p(p^{m-1} - 1) \\ &= s_p(k-1) + (m-1)(p-1). \end{aligned} \tag{32}$$

Further, k is not divisible by p and then $s_p(k-1) = s_p(k) - 1$. Therefore, Equation (32) reduces to

$$s_p(kp^m - 1) = s_p(k) + (m-1)(p-1) - 1. \tag{33}$$

Since $m > 1$ and $b < p^{m-1}$, we have

$$\begin{aligned} s_p(kp^{m-1} + b - 1) &= s_p(kp^{m-1}) + s_p(b-1) \\ &= s_p(k) + s_p(b-1). \end{aligned} \tag{34}$$

Let $b = b'p^{v_p(b)}$, $p \nmid b'$, for some positive integer b'. Replacing kp^m with b in Equation (33); we get

$$s_p(b-1) = s_p(b) - 1 + v_p(b)(p-1) \tag{35}$$

since $s_p(b) = s_p(b')$. Therefore, combining Equations (31), (33), (34), and (35), we get

$$v_p\left(\binom{kp^{m-1} + b - 1}{b}\right) = m - 1 - v_p(b).$$

It is also trivial from our assumption of b that $v_p(b) = v_p(n)$. Hence the theorem follows. □

Definition 1 Let a be a positive integer whose p-adic expansion is given by

$$a = a_0 + a_1 p + a_2 p^2 + \cdots + a_t p^t.$$

For a fixed integer k, $1 \leq k \leq p-1$, we define $\rho_{p,k,m}(a)$ as

$$\rho_{p,k,m}(a) = \begin{cases} 0, & (\text{if } k + a_m < p), \\ 1+n, & (\text{if} k + a_m \geq p, a_{m+1} = \cdots = a_{m+n} = p-1 \\ & \text{and } a_{m+n+1} < p-1). \end{cases} \tag{36}$$

Here, $\rho_{p,k,m}(a)$ is the number of carries when adding a and kp^m in base p. Using Kummer's theorem, we can see that $\rho_{p,k,m}(a)$ is, in fact, $v_p(\binom{a+kp^m}{a})$.
The preceding theorem restricts the value of n to less than some particular value. The following theorem gives an alternate result of Theorem 4 when there is no restriction on the values of n but restrict $k < p$.

Theorem 5 *Let p be an odd prime and k be a positive integer less than p. For positive integers m and n such that $n \equiv k \pmod{p-1}$;*

$$v_p(S(n-1, kp^m - 1)) = v_p(S(n, kp^m)) = \rho_{p,k-1,m-1}\Big(\frac{n-kp^m}{p-1}\Big), \tag{37}$$

if $\rho_{p,k-1,m-1}(\frac{n-kp^m}{p-1}) \leq m-1 \leq v_p(n)$. However, if $\rho_{p,k,m-1}(\frac{n-kp^m}{p-1}) \leq v_p(n) < m-1$, then

$$\begin{aligned} v_p(S(n-1, kp^m - 1)) &= v_p(S(n, kp^m)) \\ &= m - 1 - v_p(n) + \rho_{p,k,m-1}\Big(\frac{n-kp^m}{p-1}\Big). \end{aligned} \tag{38}$$

Proof Since $n \equiv k \pmod{p-1}$, we can write $n = kp^m + a(p-1)$. Let $a = \sum_{i=0}^{q} a_i p^i$ be the p-adic expansion of $a = \frac{n-kp^m}{p-1}$ for some positive integer q. To prove the theorem, it is enough to obtain $v_p(\binom{kp^{m-1}+a-1}{a})$ for both cases. For the first case, we have

$$a = a' p^t$$

for some positive integers a' and t such that $p \nmid a'$ and $t \geq m-1$. Therefore,

$$s_p(kp^{m-1} + a'p^t - 1) = s_p(k + a'p^{t-m+1} - 1) + s_p(p^{m-1} - 1)$$

and

$$s_p(kp^{m-1} - 1) = k - 1 + s_p(p^{m-1} - 1).$$

Thus, the valuation of the binomial coefficient is

$$v_p\Big(\binom{kp^{m-1} + a - 1}{a}\Big) = \frac{s_p(a') - s_p(k - 1 + a'p^{t-m+1}) + k - 1}{p-1}. \tag{39}$$

Suppose $t > m-1$, the sum of the digits $s_p(k-1+a'p^{t-m+1})$ can be split into the sum $s_p(k-1) + s_p(a')$ since we assume $1 \leq k \leq p-1$. In this case, the valuation of the binomial coefficient becomes zero, which is also equal to $\rho_{p,k,m-1}(a)$, since the $(m-1)$-th p-adic digit of a is zero and $k + a_{m-1} = k < p$. Now, we assume that $v_p(a) = m-1$, which means that $t = m-1$ and $a' = a_{m-1} + a_m p + a_{m+1} p^2 + \cdots$. It follows that if $k - 1 + a_{m-1} < p$, then

$$s_p(kp^{m-1} + a - 1) = s_p(a) + s_p(kp^{m-1} - 1)$$

and $v_p(\binom{kp^{m-1}+a-1}{a}) = 0 = \rho_{p,k-1,m-1}(a)$.
If $k - 1 + a_{m-1} > p$, then

$$a = a_{m-1}p^{m-1} + (p-1)\sum_{i=m}^{m-2+\rho_{p,k-1,m-1}(a)} p^i + \sum_{i=m-1+\rho_{p,k-1,m-1}(a)}^{q} a_i p^i,$$

$$\begin{aligned} kp^{m-1} + a - 1 = &(a_{m-1} + k - 1 - p)p^{m-1} \\ &+ (a_{m-1+\rho_{p,k-1,m-1}(a)} + 1)p^{m-1+\rho_{p,k-1,m-1}(a)} \\ &+ \sum_{i=m+\rho_{p,k-1,m-1}(a)}^{q} a_i p^i, \end{aligned}$$

which implies

$$s_p(a) = a_{m-1} + (p-1)(\rho_{p,k-1,m-1}(a) - 1) + \sum_{i=m+\rho_{p,k-1,m-1}(a)}^{q} a_i,$$

$$\begin{aligned} s_p(kp^{m-1} + a - 1) &= k - p + a_{m-1} + \sum_{i=m+\rho_{p,k-1,m-1}(a)}^{q} a_i \\ &= s_p(a) - (p-1)\rho_{p,k-1,m-1}(a) + k - 1. \end{aligned}$$

Therefore, we get

$$\begin{aligned} v_p\left(\binom{kp^{m-1} + a - 1}{a}\right) &= \frac{s_p(a) - s_p(k-1+a) + k - 1}{p-1} \\ &= \rho_{p,k-1,m-1}(a). \end{aligned}$$

Using this valuation in Equation (27), the first result of the theorem follows. The second result of the theorem can be obtained through the same method. □

Remark 3 If there is no restriction on the value of k and $v_p(a) < m - 1$, we can write a as $a = cp^{m-1} + b$, where $b < p^{m-1}$. Therefore,

$$v_p\left(\binom{kp^{m-1} + a - 1}{a}\right) = m - 1 - v_p(b) + v_p\left(\binom{k+c}{c}\right).$$

For $v_p(a) \geq m - 1$, we have $a = cp^{m-1}$ for some integer c and hence

$$v_p\left(\binom{kp^{m-1} + a - 1}{a}\right) = v_p\left(\binom{k+c-1}{c}\right).$$

If the valuations obtained are less than $m-1$ for both cases, then they are the valuations of $S(kp^m + a(p-1), kp^m)$ for both cases.

Theorem 6 *Let p be a prime greater than 3 and n, k, and m be positive integers. Then, p divides $S(n, kp^m)$ if $n \not\equiv k \pmod{(p-1)p^{m-1}}$. More precisely, p^{t+1}, $0 \leq t \leq m-1$, divides $S(n, kp^m)$ if $n \not\equiv k \pmod{(p-1)p^{m-1-t}}$.*

Proof The theorem can be proved using Equations (3), (4), and (10). □

3.2 Congruence relation between S(n,k) and s(n,k)

Now, we establish congruence relations between Stirling numbers of the first and the second kind using their generating functions. The next theorem gives the results for $S(n,k)$ modulo p. We use the notation $[n]$ to denote the set $\{0, 1, 2, \cdots, n\}$ for simplicity.

Theorem 7 *Let p be an odd prime and $n, k, a,$ and d are positive integers such that $a < p$ and $0 \le d < p-1$, then*

$$S(n, kp+a) \equiv (-1)^d s(p-a, p-a-d)\binom{\frac{n-k-a-d}{p-1}}{k} \pmod{p}$$

if $n-k-a \equiv d \pmod{p-1}$ for some $d \in [p-1-a]$, and

$$S(n, kp+a) \equiv 0 \pmod{p}$$

if $n-k-a \not\equiv d \pmod{p-1}$ for any $d \in [p-1-a]$.

Proof We have

$$\frac{1}{\prod_{i=1}^{kp+a}(1-ix)} = \frac{\prod_{i=kp+a+1}^{(k+1)p}(1-ix)}{\prod_{i=1}^{(k+1)p}(1-ix)},$$

and we get the following congruences;

$$\frac{1}{\prod_{i=1}^{kp+a}(1-ix)} \equiv \frac{\prod_{i=0}^{p-a-1}(1+ix)}{(1-x^{p-1})^{k+1}} \pmod{p},$$

$$\sum_{n=0}^{\infty} S(n+kp+a, kp+a)x^n \equiv \sum_{i=0}^{p-a-1} s(p-a, p-a-i)(-1)^i x^i \times \sum_{j=0}^{\infty}\binom{k+j}{j}x^{j(p-1)} \pmod{p}.$$

It follows that if $n \equiv d \pmod{p-1}$ for $d \in [p-a-1]$, then

$$S(n+kp+a, kp+a) \equiv (-1)^d \binom{k+\frac{n-d}{p-1}}{k} s(p-a, p-a-d) \pmod{p}, \tag{40}$$

and if $n \not\equiv d \pmod{p-1}$ for any $d \in [p-a-1]$, then

$$S(n+kp+a, kp+a) \equiv 0 \pmod{p}. \tag{41}$$

If we replace $n+kp+a$ with n in Equations (40) and (41), we get the required results. □

Remark 4 The following results are consequences of Theorem 7, specifically for the prime $p=3$ and $p=5$: **For** p=3, we have two classes (since $p-1=2$) for each $a \in \{0, 1, 2\}$. We get the following congruences:

$$S(n, 3k) \equiv \begin{cases} \binom{\frac{n-k}{2}}{k} \pmod{3}, & \text{if } n-k \text{ is even;} \\ 0 \pmod{3}, & \text{if } n-k \text{ is odd,} \end{cases}$$

$$S(n, 3k+1) \equiv \begin{cases} \binom{\frac{n-k-1}{2}}{k} \pmod{3}, & \text{if } n-k \text{ is odd;} \\ \binom{\frac{n-k-2}{2}}{k} \pmod{3}, & \text{if } n-k \text{ is even,} \end{cases}$$

$$S(n, 3k+2) \equiv \begin{cases} \binom{\frac{n-k-2}{2}}{k} \pmod{3}, & \text{if } n-k \text{ is even;} \\ 0 \pmod{3}, & \text{if } n-k \text{ is odd.} \end{cases}$$

For $p = 5$, we have four classes (since $p - 1 = 4$) for each $a \in \{0,\ 1,\ 2,\ 3,\ 4\}$. We get the following congruences:

$$S(n, 5k) \equiv \begin{cases} \binom{\frac{n-k}{4}}{k} \pmod 5, & \text{if } n \equiv k \pmod 4; \\ 0 \pmod 5, & \text{if } n \not\equiv k \pmod 4, \end{cases}$$

$$S(n, 5k+1) \equiv \begin{cases} \binom{\frac{n-k-1}{4}}{k} \pmod 5, & \text{if } n \equiv k+1 \pmod 4; \\ \binom{\frac{n-k-2}{4}}{k} \pmod 5, & \text{if } n \equiv k+2 \pmod 4; \\ \binom{\frac{n-k-3}{4}}{k} \pmod 5, & \text{if } n \equiv k+3 \pmod 4; \\ \binom{\frac{n-k-4}{4}}{k} \pmod 5, & \text{if } n \equiv k \pmod 4, \end{cases}$$

$$S(n, 5k+2) \equiv \begin{cases} \binom{\frac{n-k-2}{4}}{k} \pmod 5, & \text{if } n \equiv k+2 \pmod 4; \\ 3\binom{\frac{n-k-3}{4}}{k} \pmod 5, & \text{if } n \equiv k+3 \pmod 4; \\ 2\binom{\frac{n-k-4}{4}}{k} \pmod 5, & \text{if } n \equiv k \pmod 4; \\ 0 \pmod 5, & \text{if } n \equiv k+1 \pmod 4, \end{cases}$$

$$S(n, 5k+3) \equiv \begin{cases} \binom{\frac{n-k-3}{4}}{k} \pmod 5, & \text{if } n \equiv k+3 \pmod 4; \\ \binom{\frac{n-k-4}{4}}{k} \pmod 5, & \text{if } n \equiv k \pmod 4; \\ 0 \pmod 5, & \text{if } n \equiv k+1 \text{ or } k+2 \pmod 4, \end{cases}$$

$$S(n, 5k+4) \equiv \begin{cases} \binom{\frac{n-k-4}{4}}{k} \pmod 5, & \text{if } n \equiv k \pmod 4; \\ 0 \pmod 5, & \text{if } n \equiv k+1,\ k+2, \text{ or } k+3 \pmod 4. \end{cases}$$

In the case of $S(n, 5k)$, the multiplier $s(5, 5-d)$ where $d \in \{0,\ 1,\ 2,\ 3\}$ is divisible by 5 except when $d = 0$. It is also easy to see that the binomial coefficients on the RHS of the above equations reduce to 1 if $k = 0$. This observation leads us to acquire the following exact p-adic valuations:
For a prime $p = 3$ and any positive integer n,

$$\begin{aligned} &a)\ \ v_3(S(2n, 2)) = 0, \\ &b)\ \ v_3(S(6n+3, 3)) = v_3(S(6n+5, 3)) = 0, \\ &c)\ \ v_3(S(6n, 4)) = v_3(S(6n+1, 4)) = v_3(S(6n+4, 4)) \\ &\qquad\qquad\qquad\qquad = v_3(S(6n+5, 4)) = 0, \\ &d)\ \ v_3(S(6n+1, 5)) = v_3(S(6n+1, 5)) = 0, \\ &e)\ \ v_3(S(6n, 6)) = 0, \\ &f)\ \ v_3(S(6n+1, 7)) = v_3(S(6n+2, 7)) = 0, \\ &g)\ \ v_3(S(6n, 6)) = 0. \end{aligned}$$

For a prime $p = 5$, we have the following p-adic valuations:

$$\begin{aligned} &a)\ \ v_5(S(4n, 2)) = v_5(S(4n+2, 2)) = v_5(S(4n+3, 2)) = 0, \\ &b)\ \ v_5(S(4n, 3)) = v_5(S(4n+3, 3)) = 0, \\ &c)\ \ v_5(S(4n, 4)) = 0, \\ &d)\ \ v_5(S(20n+r, 5)) = 0, \quad \text{if } r \in \{5, 9, 13, 17\}, \\ &e)\ \ v_5(S(20n+r, 6)) = 0, \quad \text{if } r \in [19] \setminus \{2, 3, 4, 5\}, \\ &f)\ \ v_5(S(20n+r, 7)) = 0, \quad \text{if } r \in [19] \setminus \{2, 3, 4, 5, 6, 10, 14, 18\}, \\ &g)\ \ v_5(S(20n+r, 8)) = 0, \quad \text{if } r \in \{0, 1, 8, 9, 12, 13, 16, 17\}, \\ &h)\ \ v_5(S(20n+r, 9)) = 0, \quad \text{if } r \in \{1, 9, 13, 17\}. \end{aligned}$$

The following theorem gives a generalization of Theorem 7 from modulo p to modulo p^m.

Theorem 8 *For an odd prime p and positive integers k, b, m, and n such that $b < p^{m-1}$, the following congruences hold;*

$$S(n+kp^m+b, kp^m+b) \equiv \sum_j (-1)^n \binom{(k+1)p^{m-1}+j-1}{j} \times s(c, c-n+j(p-1)) \pmod{p^m} \tag{42}$$

and

$$s(kp^m+b, kp^m+b-n) \equiv \sum_j (-1)^{n+j} \binom{(k+1)p^{m-1}}{j} \times S(n-j(p-1)+c, c) \pmod{p^m}, \tag{43}$$

where $c = p^m - b$.

Proof We have

$$\frac{1}{\prod_{i=1}^{kp^m+b}(1-ix)} = \frac{\prod_{i=kp^m+b+1}^{(k+1)p^m}(1-ix)}{\prod_{i=1}^{(k+1)p^m}(1-ix)}$$

and

$$\prod_{i=1}^{kp^m+b-1}(1-ix) = \frac{\prod_{i=1}^{(k+1)p^m}(1-ix)}{\prod_{i=kp^m+b}^{(k+1)p^m}(1-ix)}.$$

We obtain the following two congruences

$$\frac{1}{\prod_{i=1}^{kp^m+b}(1-ix)} \equiv \frac{\prod_{i=1}^{p^m-b-1}(1+ix)}{(1-x^{p-1})^{(k+1)p^{m-1}}} \pmod{p^m}, \tag{44}$$

and

$$\prod_{i=1}^{kp^m+b-1}(1-ix) \equiv \frac{(1-x^{p-1})^{(k+1)p^{m-1}}}{\prod_{i=0}^{p^m-b}(1+ix)} \pmod{p^m}. \tag{45}$$

Equations (44) and (45) generate Equations (42) and (43), respectively. Hence the theorem follows. □

Theorem 9 *For an odd prime p and positive integers k, a, m, and n such that $kp^m < p^n$ and $m+1 \le n$, the following congruences hold;*

$$S(a+kp^m, kp^m) \equiv (-1)^a \sum_j \binom{p^{n-1}+j-1}{j} \times s(bp^m, bp^m-a+j(p-1)) \pmod{p^n} \tag{46}$$

and

$$s(kp^m, kp^m-a) \equiv \sum_j (-1)^{a+j} \binom{p^{n-1}}{j} \times S(a-j(p-1)+bp^m, bp^m) \pmod{p^n}, \tag{47}$$

where $b = p^{n-m} - k$.

Proof Using the same technique as in the proof of Theorem 8, we obtain the following two congruences;

$$\frac{1}{\prod_{i=1}^{kp^m}(1-ix)} \equiv \frac{\prod_{i=1}^{(p^{n-m}-k)p^m-1}(1+ix)}{(1-x^{p-1})^{p^{n-1}}} \pmod{p^n} \tag{48}$$

and

$$\prod_{i=1}^{kp^m-1}(1-ix) \equiv \frac{(1-x^{p-1})^{p^{n-1}}}{\prod_{i=0}^{(p^{n-m}-k)p^m}(1+ix)} \pmod{p^n}. \tag{49}$$

These two congruences generate the required results. Hence the theorem follows. □

Remark 5 In the second result of the preceding theorem, the generating function on the RHS of Equation (49) generates an infinite term. In contrast, the LHS generates $kp^m - 1$ terms only. It follows that the sum vanishes when $a \geq kp^m$, i.e.,

$$\sum_j (-1)^{a+j}\binom{p^{n-1}}{j} S(a-j(p-1)+bp^m, bp^m) \equiv 0 \pmod{p^n}, \tag{50}$$

if $a \geq kp^m$.
If we replace $n = m+1$ in Theorem 9, the second result of the theorem yields the following congruence;

$$\begin{aligned} s(kp^m, kp^m - a) &\equiv (-1)^a[S(a+bp^m, bp^m) \\ &\quad - S(a-p^m(p-1)+bp^m, bp^m)] \pmod{p^{m+1}}, \end{aligned} \tag{51}$$

whenever $p-1 \nmid a$. Moreover, if $a < p^m(p-1)$ and $p-1 \nmid a$, we obtain

$$s(kp^m, kp^m - a) \equiv (-1)^a S(a+bp^m, bp^m) \pmod{p^{m+1}}. \tag{52}$$

We can utilize Theorem 9 to obtain some values of $v_p(S(n, kp^m))$, which are always greater than or equal to m. Although Theorem 5 deals with $v_p(S(n, kp^m))$, the theorem is restricted to valuations less than or equal to m since the key congruences used in Theorem 5 are in modulo p^m. The congruences obtained in Theorem 9 are in modulo p^n for arbitrary n, usually greater than or equal to m of $S(n, kp^m)$; the next theorem is an application of such congruence.

Theorem 10 *Let p be an odd prime and a, u, k, and m be positive integers such that $p \nmid a$ and $a \equiv 0 \pmod{p-1}$. If $p^{m-1} \leq u = \frac{a}{p-1} < p^m$ and $\frac{u}{p^{m-1}} + k \geq p$, then*

$$v_p(S(kp^m + a, kp^m)) = m. \tag{53}$$

Proof Replace n and a in the first result of Theorem 9 with $m+1$ and $u(p-1)$, respectively; we get

$$\begin{aligned} S(a+kp^m, kp^m) &\equiv \sum_j \binom{p^m+j-1}{j} \\ &\quad \times s(bp^m, bp^m-(u-j)(p-1)) \pmod{p^{m+1}} \end{aligned} \tag{54}$$

where $b = p-k$.
Let $u = \sum_{i=0}^{m-1} u_i p^i$ be the p-adic expansion of u. We know that $s(bp^m, bp^m-(u-j)(p-1))$ is divisible by p^m if $bp^m - (u-j)(p-1) < bp^{m-1}$ or $j < \sum_{i=0}^{m-1} u_i p^i - bp^{m-1} = \alpha$ (*say*). We can also confirm that p divides $\binom{p^m+j-1}{j}$ if $j < \alpha$, unless $j=0$, in which case $s(bp^m, bp^m-(u-j)(p-1)) = 0$ since $bp^m - u(p-1) < 0$. Thus, all the j-th terms with $0 \leq j < \alpha$ are divisible by p^{m+1}, and we obtain

$$\begin{aligned} S(a+kp^m, kp^m) &\equiv \sum_{j=\alpha}^{u} \binom{p^m+j-1}{j} \\ &\quad \times s(bp^m, bp^m-(u-j)(p-1)) \pmod{p^{m+1}}. \end{aligned} \tag{55}$$

From Theorem 1(b), we have

$$s(bp^m, bp^m-(u-j)(p-1)) \equiv \binom{bp^{m-1}}{u-j}(-1)^{u-j} \pmod{p^m}. \tag{56}$$

It follows that the p-adic valuation of each term on the RHS of Equation (55) is $m-v_p(j)+m-1-v_p(u-j)$. If $p\nmid j$ and $j\neq u$, then the valuation is $2m-1-v_p(u-j)$, which is greater than or equal to $m+1$ unless $v_p(u-j)=m-1$. On the other hand, if $p\mid j$, then $p\nmid(u-j)$, and the valuation becomes $2m-1-v_p(j)$, which is greater than or equal to $m+1$ unless $v_p(j)=m-1$. If $v_p(u-j)=m-1$, then $u-j=rp^{m-1}$ for some r, $p\nmid r$. If $v_p(j)=m-1$, then $j=tp^{m-1}$ for some t, $p\nmid t$. The only remaining term whose p-adic valuation is less than $m+1$ is the term with $j=u$. Thus, Equation (55) reduces to

$$\begin{aligned} S(a+kp^m, kp^m) &\equiv \sum_{r=1}^{b}\binom{p^m+u-rp^{m-1}-1}{p^m-1}\varphi_{r(p-1)p^{m-1}} \\ &+\binom{p^m+u-rp^{m-1}-1}{p^m-1}\varphi_{r(p-1)p^{m-1}} \\ &+\sum_{t=u_{m-1}-b+1}^{u_{m-1}}\binom{p^m+tp^{m-1}-1}{p^m-1}\varphi_{(u-tp^{m-1})(p-1)} \pmod{p^{m+1}}, \end{aligned} \tag{57}$$

where $\varphi_q=s(bp^m, bp^m-q)$, and the preceding equation can be written as

$$\begin{aligned} S(a+kp^m, kp^m) &\equiv \sum_{r=0}^{b}\binom{p^m+u-rp^{m-1}-1}{p^m-1}\varphi_{r(p-1)p^{m-1}} \\ &+\sum_{t=u_{m-1}-b+1}^{u_{m-1}}\binom{p^m+tp^{m-1}-1}{p^m-1}\varphi_{(u-tp^{m-1})(p-1)} \pmod{p^{m+1}}. \end{aligned} \tag{58}$$

Now, we have the following congruences

$$\begin{aligned} \binom{p^m+u-rp^{m-1}-1}{p^m-1} &= \frac{p^m}{p^m+u-rp^{m-1}}\binom{p^m+u-rp^{m-1}}{p^m} \\ &\equiv \frac{p^m}{u} \pmod{p^{m+1}}, \end{aligned} \tag{59}$$

$$\varphi_{rp^{m-1}(p-1)} \equiv \binom{bp^{m-1}}{rp^{m-1}}(-1)^r \equiv \binom{b}{r}(-1)^r \pmod{p}, \tag{60}$$

$$\begin{aligned} \binom{p^m+tp^{m-1}-1}{p^m-1} &= \frac{p^m}{p^m+tp^{m-1}}\binom{p^m+tp^{m-1}}{p^m} \\ &\equiv \frac{p}{t} \pmod{p^2}, \end{aligned} \tag{61}$$

and

$$\begin{aligned} \varphi_{(u-tp^{m-1})(p-1)} &\equiv \binom{bp^{m-1}}{u-tp^{m-1}}(-1)^{u-t} \pmod{p^m} \\ &\equiv \frac{bp^{m-1}}{u-tp^{m-1}}\binom{bp^{m-1}-1}{u-tp^{m-1}-1}(-1)^{u-t} \pmod{p^m} \\ &\equiv \frac{bp^{m-1}}{u}\binom{b-1}{u_{m-1}-t}(-1)^{s_p(u)-u_{m-1}-1}(-1)^{u-t} \pmod{p^m} \\ &\equiv \frac{bp^{m-1}}{u}\binom{b-1}{u_{m-1}-t}(-1)^{t-u_{m-1}-1} \pmod{p^m}. \end{aligned} \tag{62}$$

Let

$$X=\sum_{r=0}^{b}\binom{p^m+u-rp^{m-1}-1}{p^m-1}s(bp^m,bp^m-r(p-1)p^{m-1}) \tag{63}$$

and

$$Y=\sum_{t=u_{m-1}-b+1}^{u_{m-1}}\binom{p^m+tp^{m-1}-1}{p^m-1}s(bp^m,bp^m-(u-tp^{m-1})(p-1)). \tag{64}$$

From Equations (59), (60), and (63), we get

$$\begin{aligned}X&\equiv\frac{p^m}{u}\sum_{r=1}^{b}\binom{b}{r}(-1)^r \pmod{p^{m+1}}\\&\equiv 0 \pmod{p^{m+1}}.\end{aligned} \tag{65}$$

From Equations (61), (62), and (64), we get

$$\begin{aligned}Y&\equiv\frac{p^m}{u}\sum_{t=u_{m-1}-b+1}^{u_{m-1}}\frac{b}{t}\binom{b-1}{u_{m-1}-t}(-1)^{t-u_{m-1}-1} \pmod{p^{m+1}}\\&\equiv\frac{p^m}{u}\sum_{t=0}^{b-1}\frac{b}{u_{m-1}-t}\binom{b-1}{t}(-1)^{t-1} \pmod{p^{m+1}}\\&\equiv\frac{p^m}{u}\sum_{t=0}^{b-1}\frac{b-t}{u_{m-1}-t}\binom{b}{t}(-1)^{t-1} \pmod{p^{m+1}}.\end{aligned} \tag{66}$$

Using partial fraction decomposition, we get

$$\sum_{t=0}^{b-1}\frac{b-t}{u_{m-1}-t}\binom{b}{t}(-1)^{b-t-1}=\frac{1}{\binom{u_{m-1}}{b}}. \tag{67}$$

From Equations (58) and (63) – (67), we obtain

$$S(a+kp^m,kp^m)\equiv\frac{(-1)^b p^m}{u\binom{u_{m-1}}{b}} \pmod{p^{m+1}}. \tag{68}$$

Since $p\nmid u\binom{u_{m-1}}{b}$,

$$v_p(S(a+kp^m,kp^m))=m. \tag{69}$$

Hence, the theorem holds. □

The following theorem gives a generalization of Theorem 8 to congruence modulo p^n for any positive integer n greater than m.

Theorem 11 *For an odd prime p and positive integers a, u, and n such that $a\le p^n$, the following two congruences holds;*

$$\begin{aligned}S(u+a,a)\equiv(-1)^u\sum_j&\binom{p^{n-1}+j-1}{j}\\&\times s(p^n-a,\,p^n-a-u+j(p-1)) \pmod{p^n}\end{aligned} \tag{70}$$

and

$$\begin{aligned}s(a,a-u)\equiv\sum_j&(-1)^{u+j}\binom{p^{n-1}}{j}\\&\times S(u-j(p-1)+p^n-a,\,p^n-a) \pmod{p^n}.\end{aligned} \tag{71}$$

Proof The proof is similar to the proof of Theorems 8 and 9. □

Remark 6 It follows from Theorem 11 that if $0 \leq u < p-1$, then the index j has only one possible value, which is $j = 0$. Therefore,

$$S(u+a, a) \equiv (-1)^u s(p^n - a, p^n - a - u) \pmod{p^n} \tag{72}$$

and

$$s(a, a-u) \equiv (-1)^u S(u + p^n - a, p^n - a) \pmod{p^n}. \tag{73}$$

Declarations

Conflicts of interest The authors have no conflicts of interest to declare. All co-authors have seen and agree with the contents of the manuscript and there is no financial interest to report. We certify that the submission is original work and is not under review at any other publication.

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Conferences/Seminars/Workshops

1. Attended and presented a paper entitled with "p-adic valuations of Stirling numbers of second kind" in Multidisciplinary International Seminar on a perspective of Global Research Process: Presented Scenario and Future Challenges organized by Manipur University, Canchipur-795 003, Manipur on 19^{th}-20^{th} January, 2019.

2. Attended the International Conference on Chemistry and Environmental Sustainability (ICCES-2019) organized by Department of Chemistry, Mizoram University, Aizawl-796 004, Mizoram on 19^{th}-22^{nd} February, 2019.

3. Attended the '30 Days Faculty induction Programme' organized by Faculty Development Centre, Mizoram University, Mizoram on 5^{th} March, 2019 -4^{th} April, 2019.

4. Attended the Instructional School for Teachers(IST-2019) on Mathematical Modelling in Continuum Mechanics and Ecology organized by Department of Mathematics and Computer Science, Mizoram University, Aizawl-796 004, Mizoram on 3^{rd}-15^{th} June, 2019.

5. Attended national workshop on Ethics in Research and Preventing Plagiarism (ERPP-2019) organized by Department of Physics, Mizoram University, Aizawl-796004, Mizoram on 3^{rd} October, 2019.

6. Attended the One Day National Seminar on Total Quality Assurance for Higher Educational Institutions organized by College Development Council, Mizoram University, Aizawl-796 004, Mizoram on 25^{th} February, 2020.

7. Attended the webinar on 'Mathematical modelling of Infectious Diseases: Its relevance in time of COVID' and 'Binary recurrence sequences and its arithmetic' organized by the Department of Mathematics and Computer Science, Mizoram University, Aizawl-796 004, Mizoram on 11^{th} June, 2020.

8. Attended and presented a paper entitled with "On the p-adic valuations of Stirling numbers of the second kind" in International Seminar on Recent Advances in Science and Technology organized by NEAST and Mizoram University, Aizawl-796 004, Mizoram on 16^{th}-18^{th} November, 2020.

9. Attended the 'Five Days Faculty Development Programme (Online Mode)' organized by National Institute of Technology Manipur, Langol, Manipur on 15^{th}-19^{th} March, 2021.

10. Attended the International workshop-cum-conference on Mathematics Education (IWCME2021) organized by the Department of Mathematics and Computer Science, Mizoram University, Aizawl-796 004, Mizoram on 15^{th}-20^{th} November, 2021.

11. Attended One week training program on Mathematical mdelling and Computing organized by the Department of Mathematics and Computer Science, Mizoram University, Aizawl-796 004, Mizoram during 26^{th} April - 2^{nd} May, 2022.

12. Attended the International Faculty Development Programme on Mathematical Modelling of Biosystems with Special Focus on Epidemiology held at Mizoram University organized by the Indian Statistical Institute, Kolkata and Mizoram University, Mizoram on 22^{nd}-27^{th} August, 2022.

13. Attended the workshop on Teachers Enrichment Workshop(TEW-2022) on Groups, Rings and Number Theory organized by the Department of Mathematics and Computer Science, Mizoram University, Aizawl-796 004, Mizoram on 12^{th}-17^{th} December, 2022.

14. Attended and presented a paper entitled with "Periodicity and Divisibility of Stirling numbers of the second kind" in the National Conference on Recent Developments in Mathematics and Computer Science (NCRDMCS23)

jointly organized by Mizoram Mathematical Research Association (MMRA) and Department of Mathematics, Govt Champhai College, Mizoram held during 13^{th}- 14^{th} March, 2023.

15. Attended and presented in the workshop on Importance of Mathematics in the other branch of Science organized by Mizoram Science, Technology and innovation Council (MISTIC) and Department of Mathematics, Govt Champhai College, Mizoram on 20^{th}-25^{th} March, 2023.

Particulars of the candidate

Name of Candidate : A Lalchhuangliana

Degree : Doctor of Philosophy

Department : Mathematics and Computer Science

Title of Thesis : P-Adic Valuations of Certain Classes of Stirling Numbers of the Second Kind

Date of Admission : 30.08.2018

Approval of Research Proposal

1. DRC : 15.04.2019
2. BOS : 24.04.2019
3. School Board : 08.05.2019

MZU Registration No. : 1900096

Ph.D. Registration No. and date : MZU/Ph.D./1269 OF 30.08.2018

Extension (If any) : NA

(Prof. M SUNDARARAJAN)

Head

Department of Mathematics and Computer Science

ABSTRACT

P-ADIC VALUATIONS OF CERTAIN CLASSES OF STIRLING NUMBERS OF THE SECOND KIND

AN ABSTRACT SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

A. LALCHHUANGLIANA

MZU REGN. No. : 1900096

Ph.D. REGN. No. : MZU/Ph.D./1269 of 30.08.2018

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

SCHOOL OF PHYSICAL SCIENCES

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P-ADIC VALUATIONS OF CERTAIN CLASSES OF STIRLING NUMBERS OF THE SECOND KIND

BY

A. LALCHHUANGLIANA

Department of Mathematics and Computer Science

Supervisor : Prof. S. Sarat Singh

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ABSTRACT

Sequences of integers and their divisibility properties are interesting topic in number theory. There are many Mathematicians who have been introducing different results, particularly powers of primes dividing integers. Nowadays, the divisibility properties of integers and more general, rational numbers are expressed in terms of p-adic valuations. The sequence of Stirling numbers of the second kind have deep importance in combinatorics and its divisibility properties connect to different areas of mathematics. Motivated by the above reasons, we have taken up the following three objectives in the present thesis:

1. Obtain the p-adic valuations of Stirling numbers of the second kind of the classes $S(p^n, k)$ and $S(p^n, kp)$, $1 \leq k \leq p - 1$.

2. Derive the relationship of p-adic valuations of sequence of integers with Stirling numbers of the second kind.

3. Obtain the relationship between p-adic valuations of Stirling numbers of the second kind and minimum periods of { S(n,k) modulo p^N }.

The thesis consists of six chapters and deals with various approaches to determine the p-adic valuations of certain classes of Stirling numbers of the second kind. The p-adic valuations of these numbers are mainly obtained through congruence relations. Some cases are also tackled through an algebraic and combinatorial approach. The first chapter is General Introduction and it contains basic definitions, divisibility and congruence, p-adic Valuation, Stirling Numbers, Periodicity, applications of Stirling numbers and review of literature.

In Chapter 2, the problem of divisibility of certain classes of Stirling numbers of the second kind is investigated. We derive a new identity of Stirling numbers of the second kind. A combinatorial approach helps to obtain the lower bounds of p-adic valuations of some classes of $S(n, k)$ for an odd prime p. We also extend an existing congruence relation in modulo of a power of an odd prime, which is useful in determining the lower bound of $v_p(S(p^n, kp))$ when k is odd and less than $p - 1$. We obtain the lower bound of $v_p(S(p^2, kp))$ when k is even and its value is greater than the one when k is odd. We also discuss the congruence behaviour of $S(p^n, k)$ and the involvement of p-adic digits of k on the congruence when k is not divisible by p.

In Chapter 3, we study the p-adic valuations of $S(n, k)$ when n is a power of a prime. We find that the results when k is divisible by p (or p^m) are quite different

from the ones where k is not divisible by p. We have proved that $v_p(S(p^2, kp)) \geq 5$ when k is even, which confirms the lower bound of the Conjecture in Chapter 2. Furthermore, we find that the values of $v_p(S(n, kp^m))$ are affected by the parity of n and k. In fact, if n and k are opposite in parity, i.e., $n-k$ is odd, then $v_p(S(n, kp^m)) \geq 2m$ when $(p-1) \nmid (n-k)$ and $v_p(S(n, kp^m)) \geq m$ when $(p-1) \mid (n-k)$. However, if the parity of n and k are the same, i.e., $n-k$ is even, then $v_p(S(n, kp^m)) \geq m$ when $(p-1) \nmid (n-k)$. We further investigate the divisibility of $S(p^n, k)$ when p does not divide k and we have found that the divisibility depends on the sum of the p-adic digits of k.

Chapter 4 focuses on the congruence relation between Stirling numbers of the first and the second kind. Their generating function is the bridge between the two numbers. We present their congruence relations with sums involving binomial coefficients for the two numbers. We also express the first kind in terms of sums involving the second kind modulo a power of a prime and vice versa. The congruence obtained helps to acquire the p-adic valuations of some classes of the two numbers. We even establish a congruence relation between the two numbers in modulo p^n for any positive integer n.

In Chapter 5, the relationship between minimum periods and p-adic valuations of Stirling numbers of the second kind has been studied. We discuss the periodicity, period, and minimum period of the sequence $\{S(n, k) \pmod{p^N}\}_{n\geq 0}$ for some fixed positive integers N and k. We find that the cycle of the sequence sometimes starts even when n is less than k. We present some results about the divisibility of a partial Stirling number, which is effective in evaluating some classes of $S(n, k)$. The periodicity and minimum periods help to determine a class of $S(n, k)$ holding the same p-adic valuation.

Chapter 6 is the summary and conclusions of the thesis.

A list of references is presented at the end.