A STUDY ON CERTAIN CLASSES OF ALMOST CONTACT MANIFOLDS AND SPACETIMES

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CERTIFICATE

This is to certify that the thesis entitled "A Study on Certain Classes of Almost Contact Manifolds and Spacetimes" submitted by Mr. C. Zosangzuala (Registration No: MZU/Ph.D./1711 of 05.11.2020) for the degree of Doctor of Philosophy (Ph.D.) of the Mizoram University, embodies the record of original investigation carried out by him under my supervision. He has been duly registered and the thesis presented is worthy of being considered for the award of the Ph.D. degree. This work has not been submitted for any degree from any other university.

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Month: July **Year: 2024**

DECLARATION

I, C. Zosangzuala, hereby declare that the subject matter of this thesis is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to do the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in any other University/Institute.

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PREFACE

The present thesis entitled **"A Study on Certain Classes of Almost Contact Manifolds and Spacetimes"** is an outcome of the research carried out by me under the joint supervisor, Prof. Jay Prakash Singh, Professor, Department of Mathematics, Central University of South Bihar, Gaya-824236, Bihar, and supervisor, Prof. S. Sarat Singh, Professor, Department of Mathematics and Computer Science, Mizoram University, Aizawl-796004, Mizoram.

This thesis has been divided into six chapters and each chapter is subdivided into smaller sections. The first chapter is the introduction which includes the basic definitions and formulas of differential geometry such as topological manifolds, smooth manifolds, Riemannian manifolds, almost contact metric manifolds, Kenmotsu manifolds, almost Kenmotsu manifolds, hyperbolic Kenmotsu manifolds, almost cosymplectic manifolds, Vaidya spacetime, Submanifolds and Ricci-Yamabe solitons, and review of literature.

The second chapter is the classification of Ricci-Yamabe solitons. This chapter is divided into three sections. In the first section, we examine the isometries of almost Ricci-Yamabe solitons. Firstly, the conditions under which a compact gradient almost Ricci-Yamabe soliton is isometric to Euclidean sphere $S^n(r)$ are obtained. Next, we have studied complete gradient almost Ricci-Yamabe soliton with $\alpha \neq 0$ and non-trivial conformal vector field with non-negative scalar curvature and proved that it is either isometric to Euclidean space E^n or Euclidean sphere S^n. Also, solenoidal and torse-forming vector fields are considered. Then, some non-trivial examples are constructed to verify the results. In the second section, we characterize Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) admitting Ricci-Yamabe solitons. It is shown that an $(LCS)_n$-manifold which admits the Ricci-Yamabe soliton becomes flat when the soliton is steady. Next, we have constructed examples of 3-dimensional and

5-dimensional $(LCS)_n$-manifold satisfying the result. Moreover, we extend our study to $\eta-$Ricci-Yamabe soliton on a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold and we have shown the conditions for the soliton to be shrinking, steady and expanding with ξ being a torse forming vector field. Lastly, the third section investigates almost $*-$Ricci-Yamabe solitons on a Sasakian manifold in which we have proved that the manifold is isometric to the unit sphere $\mathbb{S}^{2n+1}$ if its metric represents a complete almost $*-$Ricci-Yamabe solitons with $\alpha \neq 0$. Certain conditions under which the soliton reduces to $*$-Ricci-Yamabe soliton and when it becomes steady are also obtained.

In the third chapter, we study almost cosymplectic manifolds and its extension. In the first section, we investigate almost cosymplectic manifolds admitting almost Ricci-Yamabe solitons and show the conditions for local isomorphism to Lie group $G_{\sqrt{-\kappa}}$. Non-existence of such solitons on compact (κ, μ)-almost cosymplectic manifolds with $\kappa < 0$ is established. Scalar curvature equations are derived and the findings are validated with a 3-dimensional example. The second section characterizes almost Kenmotsu manifolds admitting conformal Ricci-Yamabe solitons and it is shown that $(\kappa, \mu)'$ manifolds M^{2n+1} are locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ under specific conditions. Conditions for conformal pressure and the potential vector field as an infinitesimal contact transformation are given with an example of a 3-dimensional manifold.

In the fourth chapter, we study the characterization of invariant submanifolds of hyperbolic Kenmotsu manifolds. First, we have proved that an invariant submanifolds of a hyperbolic Kenmotsu manifold is again a hyperbolic Kenmotsu manifold and is minimal. Next, the conditions for the invariant submanifolds to be totally geodesic are obtained. Also, it is shown that a 3-dimensional submanifolds is totally geodesic if and only if it is invariant. Moreover, an invariant submanifold of a hyperbolic Kenmotsu manifold admitting η-Ricci-Bourguinon soliton is examined and verified with an example.

The fifth chapter is geometrical properties of spacetime and it is divided into three main sections. The first section is devoted to the study of Vaidya spacetime under the effect of a conformal Ricci soliton vector field. The study demonstrates the reduction of spacetime to Schwarzschild spacetime and the existence of a conformal gradient Ricci soliton in Vaidya spacetime determining its shrinking, steady or expanding condition. The second section investigates relativistic magneto fluid spacetime stuffing in $f(R)$-gravity. We characterize the spacetime and obtain the expressions for Ricci tensor, scalar curvature and equation of state. We explore the emergence of a black hole and a trapped surface. The study also reveals that gravitational dynamics are influenced by magnetic field strength, permeability, and density affecting total pressure on the spacetime. Finally, the third section explores the dynamics of a string cloud using $f(R)$-gravity theory analyzing its properties. We have found that there is a balance between particle density and string tension. We also use the Ricci soliton metric to determine conditions for its behavior under different vector fields and explore the formation of black holes and trapped surfaces.

In Chapter 6, we present the summary and conclusion followed by a list of references.

Contents

Chapter 1

Introduction

1.1 Topological Manifold

Definition 1.1. *Let M be a topological space. We define M as a topological manifold of dimension n or a topological n-manifold if it meets the following criteria:*

1. *M is a Hausdorff space: for any two distinct points $p, q \in M$, there exist open sets U and V in M such that $p \in U$, $q \in V$, and $U \cap V = \emptyset$.*

2. *M is second-countable: there is a countable collection of open sets that forms a basis for the topology of M.*

3. *M is locally Euclidean of dimension n.*

1.2 Smooth Manifold

Let M denotes a topological n-manifold. A coordinate chart for M is a pair (U, ϕ), where U is an open subset of M and $\phi : U \to \bar{U}$ is a homeomorphism that connects U to an open subset $\bar{U} = \phi(U) \subseteq \mathbb{R}^n$. If (U, ϕ) and (V, ψ) are two charts with $U \cap V \neq \varnothing$, the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is termed as the transition map from ϕ to ψ. The charts (U, ϕ) and (V, ψ) are said to be smoothly compatible if either $U \cap V = \varnothing$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism. An atlas $\mathcal{A}$ consists of charts that cover M. A smooth atlas is defined as any two charts in $\mathcal{A}$ that are smoothly compatible. The smooth atlas $\mathcal{A}$ on M is maximum if it is not properly contained in any larger smooth atlas.

Definition 1.2. *Suppose M is a topological manifold, a smooth or differentiable structure ($\mathbb{C}^{\infty}$-structure) on M is a maximum smooth atlas. A smooth manifold is defined as $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.*

1.3 Riemannian Manifold

The Riemannian metric enables the definition of geometric concepts such as lengths, angles and distances on smooth manifolds. Much like the inner product in a vector space, the Riemannian metric on a manifold is a smoothly varying inner product on each tangent space.

Definition 1.3. *Consider a smooth manifold M, which may or may not have a boundary. A Riemannian metric on M is a smooth, symmetric, covariant 2-tensor field on M that is positive definite at every point. A Riemannian manifold is defined as the pair (M, g), where M is a smooth manifold and g represents the Riemannian metric on M.*

1.4 Connection on Riemannian Manifold

An affine or linear connection on a smooth manifold M is an $\mathbb{R}$-bilinear mapping

$$\nabla : \chi(M) \times \chi(M) \to \chi(M),$$

which satisfies the following conditions:

1. $\nabla_{fX}Y = f\nabla_X Y$,

2. $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$,

for any vector fields $X, Y \in \chi(M)$ and smooth function f. On a Riemannian manifold M of dimension n, the affine connection ∇ is termed as a Levi-Civita connection (Myers, 1935) or Riemannian connection if it meets the following criteria:

1. ∇ is symmetric or torsion-free, i.e., $\nabla_X Y - \nabla_Y X = [X, Y]$, and

2. ∇ is metric-compatible, i.e., $(\nabla_X g)(Y, Z) = 0$, for all $X, Y, Z \in \chi(M)$.

1.5 Almost Contact Metric Manifolds

A $(2n+1)$-dimensional smooth manifold M is called an almost contact metric manifold if it admits a $(1,1)$-tensor field ϕ, a unit vector field ξ (called the Reeb vector field) and a 1-form η such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \cdot \phi = 0, \tag{1.1}$$

which is equivalent to a reduction of the structural group of the tangent bundle to $U(n) \times 1$ (Sasaki, 1960; Sasaki and Hatakeyama, 1961). A Riemannian metric g is said to be an associated (or compatible) metric if it satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.2}$$

for all $X, Y \in \chi(M)$. An almost contact manifold $M^{2n+1}(\phi, \xi, \eta)$ together with a compatible metric g is known as an almost contact metric manifold (Blair, 1976). Chinea and Gonzalez (1990) obtained a complete classification for almost contact metric manifolds through the study of the covariant derivative of the fundamental 2-form. The fundamental 2-form Φ of an almost contact metric manifold (M, ϕ, ξ, η, g) is defined by

$$\Phi(X, Y) = g(X, \phi Y),$$

for all $X, Y \in \chi(M)$, and this form satisfies $\eta \wedge \Phi^n \neq 0$. This means that every almost contact metric manifold is orientable. Moreover, an almost contact metric manifold is said to be a contact metric manifold (Perrone, 2004) if $d\eta = \Phi$. The following formula holds on a contact metric manifold (Blair, 2002):

$$\nabla_X \xi = -\phi X - \phi h X. \tag{1.3}$$

Further, we define two self-adjoint operators h and ℓ by $h = \frac{1}{2}(\mathcal{L}_\xi \phi)$, where $\mathcal{L}_\xi \phi$ denotes the Lie-derivative of ϕ along ξ, and $\ell = R(\cdot, \xi)\xi$. These operators satisfy

$$h\xi = \ell\xi = 0, \quad h\phi + \phi h = 0, \quad \operatorname{Tr} h = \operatorname{Tr}(h\phi) = 0, \tag{1.4}$$

$$\operatorname{Tr} \ell = Ric(\xi, \xi) = 2n - \|h\|^2. \tag{1.5}$$

Here, "Tr" denotes trace. When a unit vector ξ is Killing, i.e., $h = 0$ or $\operatorname{Tr} \ell = 2n$, then the contact metric manifold is called K-contact. On the K-contact manifold, the following condition holds:

$$R(X, \xi)\xi = X - \eta(X)\xi. \tag{1.6}$$

An almost contact structure (ϕ, η, ξ) and almost contact manifold M are said to be normal if the almost complex structure on $M \times \mathbb{R}$ defined by $J(X, f\partial_t) = (\phi X - f\xi, \eta(X)\partial_t)$, where f is a real function on $M \times \mathbb{R}$ and t a coordinate on $\mathbb{R}$, is integrable (Blair, 1976, 2002). The necessary and sufficient conditions for the almost contact structure (ϕ, η, ξ) to be normal is

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where the pair $[\phi, \phi]$ is the Nijenhuis tensor of ϕ defined by

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y],$$

for all $X, Y \in \chi(M)$. A normal almost contact metric manifold is a Sasakian manifold. It is shown that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{1.7}$$

for any $X, Y \in \chi(M)$. A Sasakian manifold is K-contact, but the converse is true only in dimension 3. Olszak (1986) showed that a 3-dimensional almost contact metric manifold M is normal if and only if $\nabla\xi \cdot \phi = \phi \cdot \nabla\xi$ or equivalently

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{1.8}$$

where $2\alpha = \operatorname{div}\xi$, $2\beta = \operatorname{Tr}(\phi\nabla\xi)$, $\operatorname{div}\xi$ is the divergence of ξ defined by $\operatorname{div}\xi = \operatorname{Tr}\{X \to \nabla_X\xi\}$ and $\operatorname{Tr}(\phi\nabla\xi) = \operatorname{Tr}\{X \to \phi\nabla_X\xi\}$. On a 3-dimensional normal almost contact metric manifold, the following relations hold (Olszak, 1986):

$$Ric(X, \xi) = -X\alpha - (\phi X)\beta - \{\xi\alpha + 2(\alpha^2 - \beta^2)\}\eta(X), \tag{1.9}$$

$$\xi\alpha + 2\alpha\beta = 0, \tag{1.10}$$

for any $X \in \chi(M)$.

1.6 (κ, μ)-contact metric Manifold

The (κ, μ)-nullity distribution of an almost contact metric manifold $M(\phi, \xi, \eta, g)$ is a distribution (Blair et al., 1995; Shaikh and Yadav, 2019):

$$\begin{aligned} N_p(\kappa, \mu) = \{Z \in T_pM | R(X, Y)Z =& \kappa(g(Y, Z)X - g(X, Z)Y) \\ &+ \mu(g(Y, Z)hX - g(X, Z)hY)\}, \end{aligned} \tag{1.11}$$

where $\kappa, \mu \in \mathbb{R}$ and $X, Y, Z \in \chi(M)$. If $\mu = 0$, the (κ, μ)-nullity distribution $N(\kappa, \mu)$ is called the κ-nullity distribution $N(\kappa)$ (Koufogiorgos, 1993). An almost contact metric manifold M with $\xi \in N(\kappa, \mu)$ is called a (κ, μ)-contact metric manifold (Singh and Khatri, 2021). A (κ, μ)-contact metric manifold becomes a Sasakian manifold if $\kappa = 1$ and $\mu = 0$. In a (κ, μ)-contact metric manifold, the following relations hold (Papantoniou, 1993; Blair et al., 1995):

$$h^2 = (k-1)\varphi^2, \qquad k \leq 1, \tag{1.12}$$

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{1.13}$$

$$\begin{aligned} Ric(X,Y) = [2(n-1) - n\mu]g(X,Y) + [2(n-1) + \mu]g(hX,Y) \\ +[2(1-n) + n(2k+\mu)]\eta(X)\eta(Y), \end{aligned} \tag{1.14}$$

for any vector fields $X, Y \in \chi(M)$.

1.7 Curvatures on Riemannian Manifold

A conformal transformation is a map that preserves angles. Suppose g and $\bar{g}$ are two metrics on an n-dimensional Riemannian manifold M related by

$$\bar{g}(X,Y) = e^{2\sigma} g(X,Y), \tag{1.15}$$

for all vector fields X, Y on M and some scalar function σ. The angle between any two tangent vectors at a point $p \in M$ remains unchanged under this transformation. In this case, M and $\bar{M}$ are said to be conformally related, and the mapping between them is called a conformal transformation (Obata, 1970). One significant curvature tensor for investigating the intrinsic properties of a Riemannian manifold is the Weyl conformal curvature tensor, introduced by Yano and Kon (1984). This curvature tensor is invariant under conformal transformations. The conformal curvature tensor C of type (1,3) on a $(2n+1)$-dimensional Riemannian manifold (M, g), where $n > 1$, is defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{2n-1}\Big[Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY\Big] + \frac{r}{2n(2n-1)}\Big[g(Y,Z)X - g(X,Z)Y\Big], \end{aligned} \tag{1.16}$$

where r is the scalar curvature of M, Q is the Ricci operator, and Ric is the Ricci tensor.

A harmonic function is defined as a function whose Laplacian is zero. Generally, harmonic functions are not invariant under conformal transformations. To address this, Ishi (1957) identified the condition under which a harmonic function remains invariant by introducing the conharmonic transformation as a subgroup of conformal transformations (1.15) satisfying

$$\sigma^{i}_{,i} + \sigma_{,i}\sigma^{,i} = 0.$$

The tensor H, which remains invariant under conharmonic transformation, is known as the conharmonic curvature tensor. For a Riemannian manifold M of dimension $(2n+1)$, the conharmonic curvature tensor is given by

$$\begin{aligned} H(X,Y)Z &= R(X,Y)Z - \frac{1}{(2n-1)}\Big[Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY\Big], \end{aligned} \tag{1.17}$$

for all vector fields X, Y, Z on M. Also, Yano and Kon (1984) defined the projective curvature tensor as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[g(QY,Z)X - g(QX,Z)Y]. \tag{1.18}$$

The Riemannian curvature tensor is also defined by Ozgur (2003) as

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{1.19}$$

1.8 Kenmotsu Manifold

To investigate manifolds with negative curvature, Bishop and O'Neill (1969) introduced the concept of a warped product as a generalization of the Riemannian product. Tanno (1969) classified connected $(2n+1)$-dimensional almost contact metric manifolds M based on their automorphism groups having the maximum dimension of $(n+1)^2$. For such manifolds, if the sectional curvature of plane sections containing ξ is a constant k, then there are three categories:

1. If $k > 0$, M is a homogeneous Sasakian manifold with constant holomorphic sectional curvature.

2. If $k = 0$, M is the global Riemannian product of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature.

3. If $k < 0$, M is a warped product space $\mathbb{R} \times_f \mathbb{C}^n$.

Kenmotsu (1972) studied the third category and derived its geometrical properties which lead to what is now known as the Kenmotsu structure, and the corresponding manifolds are called Kenmotsu manifolds (Janssens and Vanhecke, 1981). Generally, a Kenmotsu manifold is not Sasakian. A Kenmotsu manifold can be characterized as a normal almost contact metric manifold satisfying $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. K. Kenmotsu demonstrated that such a manifold is locally a warped product $I \times_f N^{2n}$, where I is an open interval with coordinate t, $f = ce^t$ is the warping function for some positive constant c, and N^{2n} is a Kählerian manifold (Kenmotsu, 1972). A Kenmotsu manifold can be further characterized by its Levi-Civita connection ∇ satisfying:

$$\nabla_X \xi = X - \eta(X)\xi, \tag{1.20}$$

$$(\nabla_X \phi)Y = -\eta(Y)\phi X - g(X, \phi Y)\xi, \tag{1.21}$$

for any $X, Y \in \chi(M)$. In a Kenmotsu manifold M, the following properties hold (Kenmotsu, 1972):

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{1.22}$$

$$Q\xi = -2n\xi, \tag{1.23}$$

for any vector fields X, Y on M.

1.9 Almost Kenmotsu Manifolds

Olszak (1989) and Kim and Pak (2005) investigated almost contact metric manifolds where η is closed and $d\Phi = 2\eta \wedge \Phi$, referring to them as almost Kenmotsu manifolds. A normal almost Kenmotsu manifold is a Kenmotsu manifold. In an almost Kenmotsu manifold M, the following relation holds

$$\nabla_X \xi = -\phi^2 X - \phi h X, \tag{1.24}$$

for any vector field X on M. Dileo and Pastore (2009) examined almost Kenmotsu manifolds with (κ, μ)-nullity distribution and $(\kappa, \mu)'$-nullity distribution. Pastore and Saltarelli (2011) later extended this to generalized nullity distributions. An almost Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is termed a generalized (κ, μ)-almost Kenmotsu manifold if ξ belongs to the generalized (κ, μ)-nullity distribution, i.e.,

$$R(X, Y)\xi = \kappa\left[\eta(Y)X - \eta(X)Y\right] + \mu\left[\eta(Y)hX - \eta(X)hY\right], \tag{1.25}$$

for all vector fields X, Y on M, where κ and μ are smooth functions on M. Similarly, an almost Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is termed a generalized $(\kappa, \mu)'$-almost Kenmotsu manifold if ξ belongs to the generalized $(\kappa, \mu)'$-nullity distribution, i.e.,

$$R(X, Y)\xi = \kappa\left[\eta(Y)X - \eta(X)Y\right] + \mu\left[\eta(Y)h'X - \eta(X)h'Y\right], \tag{1.26}$$

for all vector fields X, Y on M, where κ and μ are smooth functions on M. Furthermore, if both κ and μ are constants in (1.26), then M is called a $(\kappa,\mu)'$-almost Kenmotsu manifold (Dileo and Pastore, 2009). For generalized (κ,μ) or $(\kappa,\mu)'$-almost Kenmotsu manifolds with $h \neq 0$ (equivalently, $h' \neq 0$), the following relations hold (Dileo and Pastore, 2009):

$$h'^2 = (\kappa+1)\phi^2, \qquad h^2 = (\kappa+1)\phi^2, \tag{1.27}$$

$$Q\xi = 2n\kappa\xi. \tag{1.28}$$

Almost contact metric manifolds such that η is closed and $d\Phi = 2\eta \wedge \Phi$ are studied by many geometers (Blair et al., 1995; Dileo and Pastore, 2007, 2019; Pastore and Saltarelli, 2011; Wang and Liu, 2016; Khatri and Singh, 2024b) and this type of manifolds are called almost Kenmotsu manifolds. A normal almost Kenmotsu manifold is called a Kenmotsu manifold (Kenmotsu, 1972). Let us denote the distribution orthogonal to ξ by $\mathfrak{D}$ defined by Dey and Majhi (2019) as

$$\mathfrak{D} = Ker(\eta) = Im(\phi),$$

where $\mathfrak{D}$ is an integrable distribution on an almost Kenmotsu manifold as η is closed.

Many geometers studied and characterized almost Kenmotsu manifold admitting solitons and deduced some notion and conditions on the manifold (For details see Basu and Bhattacharya, 2015; Naik et al., 2020; Venkatesha et al., 2020). Let M^{2n+1} be an almost Kenmotsu manifold (in short, akm). We denote by $h = \frac{1}{2}\mathcal{L}_\xi\phi$ and $l = R(\cdot,\xi)\xi$ on M^{2n+1}. The tensor fields l and h are symmetric operators and satisfy the following relations (Pastore and Saltarelli, 2011):

$$h\xi = 0,\ l\xi = 0,\ tr(h) = 0,\ tr(h\phi) = 0,\ h\phi = \phi h = 0, \tag{1.29}$$

$$\nabla_X \xi = X - \eta(X)\xi - \phi h X, \tag{1.30}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{1.31}$$

$$R(X,Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y, \tag{1.32}$$

for any vector fields X, Y. From (1.30), we see that

$$\nabla_\xi \xi = 0. \tag{1.33}$$

We define the $(1,1)$-type symmetric tensor field by $h' = h \circ \phi$, where h' is anti-commuting with ϕ and $h'\xi = 0$. Also, it satisfies the following relations

$$h = 0 \iff h' = 0,\ h'^2 = (\kappa+1)\phi^2(\iff h^2 = (\kappa+1)\phi^2). \tag{1.34}$$

1.10 Hyperbolic Kenmotsu Manifold

A $(2n+1)$-dimensional smooth manifold $\tilde{M}$ is called an almost hyperbolic contact metric manifold (Upadhyay and Dube, 1976) if it admits a timelike vector field ζ, 1-form η and a $(1,1)$-tensor ϕ satisfying (Singh et al., 2024):

$$\phi^2 X = X + \eta(X)\zeta, \tag{1.35}$$

$$\eta(\zeta) = -1,\ \phi(\zeta) = 0, \tag{1.36}$$

$$rank\ \phi = 2n,\ \eta \circ \phi = 0, \tag{1.37}$$

$$g(\phi X, \phi Y) = -g(X,Y) - \eta(X)\eta(Y), \tag{1.38}$$

$$g(\phi X, Y) = -g(X, \phi Y), \tag{1.39}$$

$$g(X, \zeta) = \eta(X), \tag{1.40}$$

for all $X, Y \in \chi(\tilde{M})$. The structure (ϕ, ζ, η, g) is said to be hyperbolic almost contact metric structure (see Dube and Niwas, 1978; Joshi and Dube, 2001; Zulekha et al., 2016; Pankaj and Chaubey, 2021). Therefore, the manifold $\tilde{M}$ is called hyperbolic Kenmotsu manifold if and only if

$$(\tilde{\nabla}_X \phi)Y = g(\phi X, Y)\zeta - \eta(Y)\phi X. \tag{1.41}$$

It is obvious that

$$d\eta = 0, \ \tilde{\nabla}_X \zeta = -X - \eta(X)\zeta \tag{1.42}$$

and

$$(\tilde{\nabla}_X \eta)Y = -g(X, Y) - \eta(X)\eta(Y). \tag{1.43}$$

In hyperbolic Kenmotsu manifold, we have

$$\tilde{R}(X, Y)\zeta = \eta(Y)X - \eta(X)Y, \tag{1.44}$$

$$\tilde{R}(X, \zeta)\zeta = -X - \eta(X)\zeta, \tag{1.45}$$

$$\tilde{R}(\zeta, X)Y = g(X, Y)\zeta - \eta(Y)X, \tag{1.46}$$

$$\tilde{Ric}(X, \zeta) = 2n\eta(X), \tag{1.47}$$

$$\tilde{Q}\zeta = -2n\zeta, \tag{1.48}$$

for all vector fields X, Y, where $\tilde{R}, \tilde{Ric}, \tilde{Q}$ are the Riemann tensor, Ricci tensor and the Ricci operator of $\tilde{M}$ respectively.

1.11 Almost Cosymplectic Manifolds

An almost cosymplectic manifold is a smooth manifold with a 1-form η and a metric g that meets certain compatibility constraints. Let (M, η, g) denotes a smooth manifold M with a non-degenerate 1-form η and a metric g. If the following criteria hold, then (M, η, g) is an almost cosymplectic manifold:

1. **Compatibility condition**:The metric g is compatible with the 1-form η, which means that for all vector fields $X, Y \in \chi(M)$, the condition

$$g(X, Y) = \eta(X)\eta(Y) + g(\phi X, \phi Y), \tag{1.49}$$

holds.

2. **Closedness condition**: The 1-form η is closed, which means that $d\eta = 0$, where d is the exterior derivative.

Almost cosymplectic manifolds are generalization of symplectic manifolds with a relaxed non-degeneracy requirement. In a symplectic manifold, the 1-form must be closed and non-degenerate, but in an almost cosymplectic manifold, just the non-degeneracy requirement is necessary.

The investigation of almost cosymplectic manifolds provides a rich framework for investigating geometric structures and their interactions with curvature features. These manifolds are related to many different fields of mathematics, including Riemannian geometry, symplectic geometry, and mathematical physics. The features and categorization of almost cosymplectic manifolds are important to examine the relationship with the almost Ricci-Yamabe solitons.

For any $X, Y \in \chi(M)$, the fundamental 2-form on M is defined as $\omega(X, Y) = g(\phi X, Y)$. An almost α-cosymplectic manifold is an almost contact metric manifold in which the basic form ω and 1-form η satisfy $d\omega = 0$ and $d\omega = 2\alpha\eta \wedge \omega$ (Shaikh and Yadav, 2019). An α-cosymplectic manifold is a normal nearly α-cosymplectic manifold. When $\alpha = 0$, M is a nearly cosymplectic manifold. Additionally, if $\alpha = 1$, the manifold represents the Kenmotsu manifold.

Suppose that M is a nearly α-cosymplectic manifold. We remember that there is a self-dual operator $h = \frac{1}{2}\mathcal{L}_\xi\phi$. If $\dim M = 3$ and $h = 0$, then M is normal.

We know from Ozturk et al.(2010) that

$$2g((\nabla_X \phi)Y, Z) = 2\alpha g(g(\phi X, Y)\xi - \eta(Y)\phi X, Z) + g(N(Y, Z), \phi X), \tag{1.50}$$

for any vector field X, Y, where N is the Nijenhuis torsion of M. Then the following relations hold (Blair, 1976)

$$\begin{aligned} \text{trace}(h) = 0, \quad h\xi = 0, \quad \varphi h = -h\varphi \\ g(hX, Y) = g(X, hY), \quad \forall X, Y \in TM. \end{aligned} \tag{1.51}$$

Using equation (1.50), a simple calculation provides

$$\nabla_X \xi = -\alpha\phi^2 X - \phi h X, \quad \nabla_\xi \varphi = 0. \tag{1.52}$$

For an almost α-cosymplectic manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$, the following equations were proven in Ozturk et al.(2010):

$$R(X, \xi)\xi - \varphi R(\varphi X, \xi)\xi = 2[\alpha^2\varphi^2 X - h^2 X], \quad \text{trace}(\varphi h) = 0,$$

$$R(X, \xi)\xi = \alpha^2\varphi^2 X + 2\alpha\varphi h X - h^2 X + \varphi(\nabla_\xi h)X,$$

for any vector fields X, Y on M.

The Study of almost cosymplectic manifolds requires an understanding of the (κ, μ)-nullity distribution. It decomposes the tangent space geometrically into subspaces corresponding with distinct eigenvalues of the κ operator. This decomposition results in the categorization of almost cosymplectic manifolds and aids in the comprehension of the geometrical characteristics and curvature requirements. An almost cosymplectic manifold is known as a (κ, μ)-almost cosymplectic manifold when ξ connects to the (κ, μ)-nullity distribution. Furthermore, Cappelletti-Montano (2013) provided the following relationships:

$$Q = 2n\kappa\eta \otimes \xi + \mu h, \tag{1.53}$$

$$h^2 = \kappa\phi^2. \tag{1.54}$$

1.12 Submanifold

Let M and N be smooth manifolds such that $dim(M) \leq dim(N)$. Let $F : M \to N$ be a smooth map and let p be a point in M. We say that F is an immersion at p if the differential map $d_p(F) : T_p(M) \to T_{F(p)}(N)$ is injective and that F is an immersion if it is an immersion at every p in M.

Definition 1.4. *Suppose $(N, \tilde{g})$ is a Riemannian manifold of dimension m, M is a manifold of dimension n and $\iota : M \to N$ is an immersion. If M is given the induced Riemannian metric $g := \iota * \tilde{g}$, then ι is said to be an isometric immersion. If, in addition, ι is injective, so that M is an immersed submanifold of N, then M is said to be a Riemannian submanifold of N.*

In recent decades, the geometry of submanifolds has garnered substantial interest due to its significant applications in both applied mathematics and theoretical physics. Submanifolds constitute a fundamental concept in the realm of differential geometry, serving as essential building blocks for understanding the geometric properties of higher-dimensional spaces. Essentially, a submanifold is a subset of a manifold that retains its own intrinsic manifold structure, inheriting certain geometric properties from the ambient space. Notably, the study of invariant submanifolds offers insights into the properties of non-linear autonomous systems and such submanifolds inherit "almost all geometric properties of the ambient manifold". Another crucial type of submanifold is the totally geodesic submanifold which is distinguished by the fact that geodesics of the ambient manifold remain geodesics within these submanifolds. The concept of geodesics plays a pivotal role in the theory of relativity. The foundational work on the geometry of invariant submanifolds within almost contact manifolds was initiated by Yano and Ishihara (1969).

Consider that M be an immersed submanifold of a Riemannian manifold $\tilde{M}$ with an induced metric g. The tangent and normal subspaces of M in $\tilde{M}$ are denoted by $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, respectively. The induced connections on the tangent bundle, TM and the normal bundle, $T^{\perp}M$ of M are denoted by ∇ and $\nabla^{\perp}$, respectively. The Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \mu(X, Y) \tag{1.55}$$

and

$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{1.56}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where μ and A are second fundamental form and shape operator of M respectively. These are related by:

$$g(A_V X, Y) = g(\mu(X, Y), V). \tag{1.57}$$

The mean curvature H of M is defined as:

$$H = \frac{1}{n}\mathrm{Tr}(\mu),$$

where Tr denotes the trace.

A submanifold M is called minimal if $H \equiv 0$. It is termed as totally geodesic if $\mu(X, Y) = 0$ for any $X, Y \in \Gamma(TM)$. The covariant derivative of the second fundamental form μ is defined by (Atceken, 2021):

$$(\tilde{\nabla}_X \mu)(Y, Z) = \nabla^{\perp}_X \mu(Y, Z) - \mu(\nabla_X Y, Z) - \mu(Y, \nabla_X Z), \tag{1.58}$$

for all $X, Y, Z \in \Gamma(TM)$, where $\tilde{\nabla}$ represents the Vander-Waerden-Bortolotti connection on M. The tensor $\tilde{\nabla}\mu$ is a tensor of type (0, 3) and it is valued in the normal bundle as the third fundamental form. If $\tilde{\nabla}\mu = 0$, then M is said to have a parallel second fundamental form. The Gauss equation for the Riemannian curvature R of the submanifold M is given by

$$\tilde{R}(X,Y)Z = R(X,Y)Z + A_{\mu(X,Z)}Y - A_{\mu(Y,Z)}X, \tag{1.59}$$

for any $X, Y, Z \in \Gamma(TM)$.

1.13 Vaidya Spacetime

The exploration of spacetime geometry and its connections to physical phenomena has been a captivating subject of investigation spanning the domains of both physics and mathematics. Over time, the intricate interplay between geometry and gravity has evolved into a vibrant field of research and give rises to various mathematical frameworks which are aimed to understand the complex interrelationships. With its profound implications in mathematical physics and cosmology, the study of spacetime has experienced significant growth in recent years. Albert Einstein's iconic theory of general relativity (Einstein, 1915) serves as a pivotal bridge between the physical attributes of spacetime and its underlying geometrical structure which is expressed by the field equation

$$Ric - \frac{R}{2}g + \Lambda g = \frac{8\pi G}{c^4}\mathcal{T}, \tag{1.60}$$

where Λ represents the cosmological constant, while c represents the speed of light in a vacuum, G, Ric, R and $\mathcal{T}$ are the gravitational constant, the Ricci tensor, the scalar curvature and the energy-momentum tensor respectively. The best known non-trivial exact solution of the field equation when $\Lambda = 0$ is the Schwarzschild metric (Griffiths and Podolsky, 2009) given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.61}$$

where m is an arbitrary parameter, r is the areal radius. This metric (1.61) specifies the geometry of a non-rotating, non-charged black hole, and it has been researched by numerous physicists and geometers such as Lindquist et al.(1958), Vishveshwara(1970), Vaidya (1999a,1999b), Simpson and Visser(2019) and Hashimoto et al.(2020).

Vaidya (1951) proposed a spacetime metric which characterizes the behaviour of spherically symmetric (i.e., non-rotating) stars or black holes interacting with null dust, either emitting or absorbing it. Consequently, their mass undergoes corresponding decreases or increases which is different from the Schwarzschild spacetime, where the mass remains constant. In reality, astronomical bodies experience mass variations as they absorb or emit radiation rendering the space-time around them time-dependent. The Vaidya metric offers a relatively straightforward yet captivating framework for exploring the attributes of such dynamic space-times, notably the presence of a photon sphere and the formation of a shadow.

Now, in Eddington–Finkelstein coordinates, the Vaidya metric is given by (Vaidya, 1951):

$$ds^2 = -\left(1 - \frac{2m(u)}{r}\right) du^2 - 2drdu + r^2 \left(d\theta^2 + sin^2\theta d\phi^2\right). \tag{1.62}$$

If $m(u)$ =constant in the above equation, then the metric becomes Schwarzschild's metric. In the absence of matter or energy sources, the Schwarzschild metric describes the geometry of spacetime.

The non-vanishing components of the curvature tensor R in (1.62) are given by (Shaikh et al., 2019):

$$\begin{aligned}
&R_{urur} = -\frac{2m}{r^3}, \quad R_{u\theta u\theta} = -\frac{2m^2 + r^2m' - mr}{r^2}, \quad R_{u\theta r\theta} = \frac{m}{r}, \\
&R_{u\phi r\phi} = \frac{msin^2\theta}{r}, \quad R_{u\phi u\phi} = -\frac{(2m^2 + r^2m' - mr)sin^2\theta}{r^2}, \\
&R_{\theta\phi\theta\phi} = 2mr\sin^2\theta.
\end{aligned}$$

Furthermore, the expression for Ricci curvature is provided by:

$$Ric_{uu} = \frac{2m'}{r^2}. \tag{1.63}$$

Here, m' denotes the derivative of m with respect to the coordinate, u.

1.14 $f(R)$-gravity theory

The relationship between spacetime and gravity constitutes a cornerstone of modern theoretical physics (Thorne et al., 2000). Massive objects induce curvature in spacetime and create gravitational fields that controls the trajectories of nearby objects. This curvature alters the path of particles and compelling them to follow geodesics dictated by the underlying spacetime geometry. The recent detection of gravitational waves further confirms the profound link between spacetime and gravity. These waves propagate as ripples in spacetime which carry energy and provide insights into cataclysmic cosmic events.

General Relativity (GR) is an important theoretical framework for comprehending the Universe's large-scale structure (Hawking and Ellis, 1973) yet it fails to explain the cosmic acceleration events. To solve this, researchers investigated theoretical revisions that include the ideas such as inflation and Dark Energy (DE). Einstein's gravitational field equations are altered to better account for observable cosmic dynamics. This often requires an introduction of an additional speculative component known as Dark Matter (Overduin and Wesson, 2004) which aims to reconcile theoretical predictions with observable data. Dark energy is believed to be the driving force behind cosmic expansion, while modified gravity theories propose alterations to the gravitational framework without dark energy. Both approaches aim to reconcile theoretical predictions with observational data but each presents unique challenges.

Beyond the fundamental Einstein-Hilbert action, Nojiri and Odintsov (2005) proposed $f(G)$-gravity, Sotiriou and Faraoni (2010) discussed $f(R)$-gravity and Cai et al. (2016) introduced $f(T)$-gravity theory. These modified gravitational frameworks give novel insights that go beyond Einstein's gravitational theory. Furthermore, these alterations are expected to give effective approximations to the elusive domain of quantum gravity (Parker and Toms, 2009). The addition of a function $f(R)$ allows General Relativity (GR) to be extended into the world of $f(R)$-gravity beyond the Einstein-Hilbert Lagrangian density. The Einstein-Hilbert action for $f(R)$-gravity has the expression

$$\mathcal{H} = \frac{1}{\kappa^2}\int [f(R) + \mathcal{L}_m]\sqrt{-g}d^4x, \tag{1.64}$$

where $f(R)$ represents an arbitrary function of the Ricci scalar R and $\mathcal{L}_m$ denotes the Lagrangian of the scalar field. The integration of higher-order curvature components causes the equations of motion within this framework to exhibit bigger degrees giving a solution to the dilemma encountered by massive neutron stars (Astashenok et al., 2013, 2015, 2017).

1.15 Lie Derivative

For any smooth vector fields X and Y on a manifold M, let θ be the flow of X, we define a vector $(\mathcal{L}_X Y)_p$ at each $p \in M$, called the Lie Derivative of Y with respect to X at p, by

$$(\mathcal{L}_X Y)_p = \left.\frac{d}{dt}\right|_{t=0} (\theta_{-t})_* Y_{\theta_{t(p)}} = \lim_{t\to 0} \frac{(\theta_{-t})_* Y_{\theta_{t(p)}} - Y_p}{t},$$

provided the derivative exists.

1.16 Generalized Quasi Einstein and Some Vector Fields

Definition 1.5 (Barros and Ribeiro, 2012). *A smooth vector field X on a Riemannian manifold is said to be a conformal vector field if there exists a smooth function ψ on M that satisfies*

$$\mathcal{L}_X g = 2\psi g.$$

We say that X is non-trivial if X is not Killing, that is, $\psi \neq 0$.

Definition 1.6 (Chaki, 2001). *A semi-Riemannian manifold M $(n > 3)$ is a generalized quasi-Einstein (GQE) manifold if its Ricci tensor (Ric) does not vanish identically and satisfies the equation:*

$$Ric = ag + b\eta \otimes \eta + c\gamma \otimes \gamma,$$

where $a, b,$ and c are scalars, with the condition that b and c are non-zero and η and γ signify non-zero 1-forms. For every vector field $X, g(X, \xi) = \eta(X)$ and $g(X, \zeta) = \gamma(X)$ hold. The generators of the manifold are the unit vectors ξ and ζ, which are mutually orthogonal and correspond to the 1-forms η and γ. The manifold M reduces to a quasi-Einstein manifold if $c = 0$.

Definition 1.7 (Yano, 1944). *A Lorentzian manifold features a torse-forming vector field (TFV) denoted as ζ. This vector field satisfies*

$$\nabla_X \zeta = \omega X + \gamma(X)\zeta, \tag{1.65}$$

where ω represents a scalar function and γ is a non-zero 1-form, for any $X \in \chi(M)$.

A time-like TFV unit is $\xi = X$ on a n-dimensional Lorentzian manifold M

and has the following form (Yano, 1944):

$$\nabla_X Y = \omega[X + \eta(X)Y], \tag{1.66}$$

where η is a 1-form and $g(X, Y) = \eta(X)$ for all X.

Definition 1.8 (Hinterleitner and Kiosak, 2008)**.** *On a Lorentzian manifold M, a vector field ϕ is termed as $\phi(Ric)$-vector field when it satisfies the following equation*

$$\nabla_X \phi = \mu Ric X, \tag{1.67}$$

where ∇ represents the Levi-Civita connection and μ is a constant. In case of a non-zero μ, the vector field ϕ is specifically identified as a proper $\phi(Ric)$-vector field. Conversely, in the case when $\mu = 0$ in (1.67)*, the vector field ϕ is categorized as covariantly constant.*

1.17 Ricci-Yamabe Solitons

Henri Poincaré, in 1904, wondered if there is a way to recognize a three dimensional sphere while the necessary measurements are operated from inside the shape, which later becomes the famous Poincaré conjecture. Since then, to prove or disprove the conjecture has become the core agenda of many researchers. The study of geometric flows amongst other methods is one of the main attraction for geometers due to its application in mathematical physics and it helps us to understand shapes in three or more dimensional spaces. One such significant flow is the Ricci flow introduced by Hamilton (1988), who used it to prove a three-dimensional sphere theorem (Hamilton, 1982) and which later become the heart of proof of the Poincaré conjecture by Perelman (2002). The Ricci soliton (Ghosh, 2011) on a Riemannian manifold (M, g) are the self-similar solutions to

Ricci flow and is defined by

$$\frac{1}{2}\mathcal{L}_V g + Ric = \lambda g, \tag{1.68}$$

where $\mathcal{L}_V g$ denotes the Lie-derivative of g along potential vector field V, Ric is the Ricci curvature of M^{2n+1} and λ, a real constant. When the vector field V is the gradient of a smooth function f on M^{2n+1}, that is, $V = \nabla f$, then we say that Ricci soliton is a gradient. According to Petersen (2009), a gradient Ricci soliton is rigid if it is a flat $N \times_\Gamma \mathbb{R}^k$, where N is Einstein and represents certain classification. The notion of almost Ricci soliton was introduced by Pigola et al. (2011) by taking λ as a smooth function in the definition of Ricci soliton (1.68).

To tackle the Yamabe problem on manifolds of positive conformal Yamabe invariant, Hamilton (1998) introduced the geometric flow known as Yamabe flow. The Yamabe soliton is a self-similar solution to the Yamabe flow. On a Riemannian manifold (M, g), a Yamabe soliton is given by

$$\frac{1}{2}\mathcal{L}_V g = (\tau - \lambda)g, \tag{1.69}$$

where τ is the scalar curvature of the manifold and λ, a real constant. The Yamabe soliton preserves the conformal class of the metric but the Ricci soliton does not in general. However, in dimension $n = 2$, both the solitons are similar. If λ is a smooth function in (1.69), then it is called almost Yamabe soliton.

Guler and Crasmareanu (2019) introduced a new type of geometric flow which is a scalar combination of Ricci flow and Yamabe flow under the name Ricci-Yamabe map and define the following:

Definition 1.9. *The map $RY^{(\alpha,\beta,g)} : I \to T_2^s(M)$ given by:*

$$RY^{(\alpha,\beta,g)} = \frac{\partial g}{\partial t}(t) + 2\alpha Ric(t) + \beta r(t) g(t),$$

is called the (α, β)*-Ricci-Yamabe map of the Riemannian flow* (M, g)*. If*

$$RY^{(\alpha,\beta,g)} \equiv 0,$$

then $g(.)$ *will be called an* (α, β)*-Ricci-Yamabe flow.*

The Ricci-Yamabe flow can be Riemannian or semi-Riemannian or singular Riemannian flow due to the involvement of scalars α and β. These kind of different choices can be useful in some physical models such as relativity theory. The Ricci-Yamabe soliton emerges as the limit of the solution of Ricci-Yamabe flow.

Definition 1.10. *A Riemannian manifold* $(M^n, g), n > 2$ *is said to admit almost Ricci-Yamabe soliton* $(g, V, \lambda, \alpha, \beta)$ *if there exist smooth function* λ *such that*

$$\mathcal{L}_V g + 2\alpha Ric = (2\lambda - \beta\tau)g, \tag{1.70}$$

where $\alpha, \beta \in \mathbb{R}$*.*

Almost Ricci-Yamabe soliton is of particular interest as it generalizes a large group of well-known solitons such as:

1. Ricci almost soliton $(\alpha = 1, \beta = 0)$.
2. almost Yamabe soliton $(\alpha = 0, \beta = 1)$.
3. Ricci-Bourguignon almost soliton $(\alpha = 1, \beta = -2\rho)$.

Also, if λ is constant, then it includes Ricci soliton, Yamabe soliton and Ricci-Bourguignon soliton among others. If V is a gradient of some smooth function f on M, then the above notion is called gradient almost Ricci-Yamabe soliton and then (1.70) reduces to

$$\nabla^2 f + \alpha Ric = (\lambda - \frac{1}{2}\beta\tau)g, \tag{1.71}$$

where $\nabla^2 f$ is the Hessian of f. The almost Ricci-Yamabe soliton (ARYS) is said to be expanding, shrinking or steady if $\lambda < 0, \lambda > 0$ or $\lambda = 0$ respectively. In particular, if λ is constant, then almost Ricci-Yamabe soliton reduces to Ricci-Yamabe soliton. Extending the notion of Ricci soliton, Cho and Kimura (2009) introduced the η-Ricci soliton which is obtained by perturbing the equation (1.70) with a multiple of a certain $(0,2)$-tensor field $\eta \otimes \eta$. Siddiqi and Akyol (2020) extended to η-Ricci-Yamabe soliton of type (α, β) which is defined by:

$$\mathcal{L}_V g + 2\alpha Ric + (2\lambda - \beta\tau) + 2\omega\eta \otimes \eta = 0. \tag{1.72}$$

1.18 Review of Literature

A Ricci soliton is a self-similar solution to the Ricci flow (Hamilton, 1982). Ricci flow has various applications including Ricci flow gravity (Graf, 2007), non-linear reaction-diffusion systems in biology, chemistry and physics (Ivancevic and Ivancevic, 2011), brain surface conformal parametrization with the Ricci flow (Wang et al., 2012) and economic modeling (Sandhu et al., 2016). Gradient Ricci solitons were introduced and studied by Cao (2006, 2009) and Petersen and Wylie (2009). Wylie (2008) showed that complete shrinking Ricci solitons have a finite fundamental group. Cao and Zhou (2010) studied complete shrinking Ricci solitons. Munteanu and Wang (2017) demonstrated that positively curved shrinking Ricci solitons are compact. Ricci solitons with Jacobi-type vector fields were explored by Deshmukh (2012), while Deshmukh et al. (2020) characterized trivial Ricci solitons. For more details, see Ghosh (2013, 2019, 2020) and Duggal (2017).

Cho and Sharma (2010) initiated the study of Ricci solitons in contact geometry. The concept of almost Ricci solitons was introduced by Pigola et al. (2011). Barros et al. (2021) studied the rigidity of gradient almost Ricci solitons and showed that they are isometric to either the Euclidean space $\mathbb{R}^n$ or the sphere $\mathbb{S}^n$. Various rigidity results were presented by Cao et al. (2011), Barros et al. (2013) and Yang and Zhang (2017). Catino et al. (2016a) also discussed the analytic and geometric properties of generic Ricci solitons. Chu and Wang (2013) provided scalar curvature estimates for gradient Yamabe solitons. Subsequent studies on Yamabe solitons were conducted by Wang (2016b) and Suh and De (2020). Shaikh et al. (2021) gave the characterizations of gradient Yamabe solitons. Chaubey et al.(2022) presented a complete classification of Yamabe solitons on real hypersurfaces in the complex quadric, $Q^m = SO_{m+1}/SO_2SO_m$. Extending the concept of Yamabe solitons, Barbosa and Ribeiro (2013) introduced almost Yamabe solitons. Seko and Maeta (2019) classified almost Yamabe solitons in Euclidean spaces, while Alkhaldi et al. (2021) characterized almost Yamabe solitons with conformal vector fields. Miao and Tam (2009, 2011) studied on the volume functional of a critical metric and compact manifolds.

Guler and Crasmareanu (2019) introduced the Ricci-Yamabe solitons. De et al. (2022) characterized Ricci-Yamabe solitons on a 3-dimensional Riemannian manifold. Sardar and Sarkar (2022) analyzed Ricci-Yamabe solitons on a class of generalized Sasakian space forms. Singh and Khatri (2021) studied perfect fluid spacetime using Ricci-Yamabe solitons with torse forming vector field and deduced its condition for the soliton to be shrinking, expanding and steady. Siddiqi and De (2022) investigated Ricci-Yamabe solitons in relativistic perfect fluid spacetimes and derived the Poisson and Liouville equation. Siddiqi et al. (2022) presented the problem of almost Ricci-Yamabe solitons on static spacetimes using conformal Killing vector field. Zhang et al. (2022) studied perfect fluid spacetimes and introduced the notion of conformal Ricci-Yamabe solitons. Also, Yoldas (2021) studied η-Ricci-Yamabe solitons on Kenmotsu manifolds and proved that the scalar curvature is constant.

Endo (2002) investigated the non-existence of almost cosymplectic manifolds satisfying certain curvature conditions. Goldberg and Yano (1969) derived the integrability condition of almost cosymplectic structures. Olszak (1981) and Olszak and Rosca (1991) studied locally conformal and normal locally conformal almost cosymplectic manifolds. Chen (2020a, 2020b) studied quasi Einstein structure and almost quasi-Yamabe solitons on almost cosymplectic manifolds. Sardar and De (2023) studied almost cosymplectic manifolds with Schouten solitons (Schouten, 1954).

Blair (2002) provided a comprehensive overview of the geometric structures involved in the Sasakian manifolds, while Boyer and Galicki (2007) presented an in-depth exploration of the properties and significance of Sasakian manifolds in both mathematics and theoretical physics. Many geometers studied and characterized almost Kenmotsu manifold such as Basu and Bhattacharya (2015), Naik et al. (2020), Venkatesha et al. (2020), Chaubey et al. (2021), Satarelli (2015), Wang (2017), Wang and Wang (2017), Patra and Ghosh (2018), Patra et al. (2020), Khatri and Singh (2023a, 2023b) and De et al. (2023).

Sasaki (1960) introduced Sasakian manifold. Sasakian manifold attracts geometers and physicists due to its application in complex geometry and string theory(Maldecena, 1999; Friedrich and Ivanov, 2002). Further, developments extend to the study of almost Ricci solitons isometric to spheres(Deshmukh, 2019) and the characterization of generalized Ricci-Yamabe solitons on Sasakian 3-metrics(Dey and Majhi, 2022). Dwivedi and Patra (2022) introduced the notion of almost $*$-Ricci-Bourguinon soliton and obtained its geometric characterization on Sasakian manifold.

The study of the geometry of invariant submanifolds of almost contact manifolds was initiated by Yano and Ishihara (1969). Chern(1968) introduced the minimal submanifolds and Chen (1973) studied the geometry of submanifolds and derived the theorem for minimal submanifolds. There are many applications of invariant submanifolds to spacetimes(Chen, 1993, 1994, 1995a, 1995b, 1996).

Kenmotsu (1969) investigated invariant submanifolds in Sasakian manifold. Kon (1973) studied conditions for an invariant submanifolds of normal contact metric manifolds to be totally geodesic. Milnor (1976) derived the curvature properties of left invariant metrics on Lie groups. Bejanchu and Papaghiuc (1981) introduced the semi-invariant submanifolds of Sasakian manifolds, while Joshi et al.(2001) studied r- almost contact hyperbolic metric manifold. Many geometers have attempted various problems related with invariant submanifolds, i.e., Kon (1973), Endo (1986), Anitha and Bagewadi (2003), Yildiz and Murathan (2009), Vanli and Sari (2014), De and Majhi (2015), Shaikh et al. (2016), Eyasmin and Baishya (2020), Atceken et al. (2020), Atceken (2021), Atceken and Uygun (2021), Blaga and Ozgur (2022), Chaubey et al. (2022) and Khatri et al. (2022).

In the last decade, significant work has been done on η-Ricci solitons and η-Yamabe solitons within the context of Riemannian geometry. Geometric flows have recently been applied to cosmological models, such as perfect fluid spacetimes. Venkatesha and Kumara (2019) analyzed Ricci solitons in perfect fluid spacetimes with torse-forming vector fields. Conformal Ricci solitons in perfect fluid spacetimes have also been examined (Siddiqi and Siddiqui, 2020). Blaga (2020) studied η-Ricci and η-Einstein solitons in perfect fluid spacetimes and derived the Poisson equation from the soliton equation. Praveena et al. (2021) investigated solitons in Kahlerian spacetime manifolds.

Minkowski (1908) introduced the geometric interpretation of spacetime. The spacetime of special theory of relativity is now known as Minkowski spacetime. Spacetime in general relativity (Wienberg, 1972) simply means a four dimensional connected semi-Riemannian manifold (M^4, g) with Lorentz metric g of signature $(-, +, +, +)$. A Lorentzian manifold M of dimension n is an $n-$dimensional semi-Riemannian manifold (O'Neill, 1983) endowed with a Lorentzian metric g of signature $(\underbrace{+, +, \ldots, +}_{n-1\ times}, -)$. Alias et al. (1995) introduced the notion of generalized

Robertson-Walker (GRW) spacetimes. A Lorentzian manifold M of dimension $n \geq 3$ is named GRW-spacetime if it is the warped product $M = -I \times q^2 M^*$, where I is an open interval of real numbers with base $(I, -dt^2)$, warping function q and the fibre (M^*, g^*).

A Lorentzian manifold with Ricci tensor of the form (Hawking and Ellis, 1973)

$$Ric\,(X,Y) = \alpha g\,(X,Y) + \beta A\,(X)\,A\,(Y)\,,$$

is called perfect fluid spacetimes. Here, α, β are scalar fields and U is a unit timelike vector field corresponding to the $1-$form A, i.e., $g\,(U,U) = -1$. De et al. (2002) introduced the notion of Lorentzian Para-Sasakian manifolds with coefficient α which is known as LP-Sasakian manifold with coefficient α. Many researchers from physics and mathematics have been studying different types of spacetime such as Shaikh et al. (2009) on Quasi-Einstein spacetimes, Gutiérrez et al. (2009) on GRW space, Mantica et al. (2015) on perfect fluid spacetime to be GRW spacetime, Yadav et al. (2019) on perfect fluid LP-Sasakian spacetime, De and Sardar (2020) on relativistic properties of LP-Sasakian type spacetime, Blaga (2020) on solitons and geometrical structures in a perfect fluid spacetime, Chattopadhyay et al. (2021) on hyper-generalized Quasi-Einstein spacetime and Siddiqui and De (2022) on relativistic perfect fluid spacetime.

The study of spacetime evolves exponentially, with the approach from both physical and mathematical point of view. Vaidya (1951) proposed a spacetime metric which characterizes the behaviour of spherically symmetric (i.e., non-rotating) stars or black holes interacting with null dust, either emitting or absorbing it. After this, several authors such as Lindquist et al. (1958), Dwivedi and Joshi (1989), Virbhadra (1992), Rudra et al. (2016), Simpson et al. (2019), Shaikh et al. (2019) and Piesnack and Kassner (2022) extended the research on Vaidya spacetime. Vishveshwara (1970) investigated the scattering of gravitational radiation by a Schwarzschild Black-hole. Simpson and Visser (2019) discussed the black bounce in traversable wormhole. Siddiqi and Siddiqui (2020) studied conformal Ricci soliton in a perfect fluid spacetime. For more information on spacetime, blackholes and its relation to cosmology, one can see Boucher (1984), Masood-ul-Alam (1987), Karchar (1992), Schmidt (1993), Philbin (1996), Stuchlik and Hledik (1999), Sahni and Starobinski (2000), Stephani et al. (2003), Griffiths and Podolsky (2009), Limoncu (2010), Qing and Yuan (2013), Mantica et al. (2016), Bronnikov et al. (2016), Hwang et al. (2016), Batool and Hussain (2017), Carroll (2019), Coutinho (2019), Leandro and Solorzano (2019), De et al. (2021), Kumara et al. (2021) and Siddiqi (2022), Cao et al. (2022), Fathi et al. (2022), Siddiqi et al. (2023), Yang et al. (2023).

The perfect fluid spacetime model postulates matter distribution as a perfect fluid without viscosity. Solutions to Einstein's field equations are often used to depict astrophysical scenarios such as star interiors or cosmic environments with perfect fluid behavior (Van Elst and Ellis, 1996; Carot and Sintes, 1997; Zhao et al., 2021). The connection between Ricci solitons and Perelman's Ricci flow (Perelman, 2002) provides new research opportunities for understanding spacetime geometry under systematic flows revealing fundamental principles which regulate the structure of the cosmos and help in cosmological modeling. Many mathematicians and physicists investigate the interplay of solitons, spacetimes and modified gravity such as Sidhoumi and Batat (2017), Khan et al.(2018), Capozziello et al. (2019), Mandal (2021), Chaubey (2021), De et al. (2021, 2022), Guler and Gunal (2022), Ali and Khan (2022), De and De (2022, 2023), Khatri et al. (2023) Suh and Chaubey (2023) and De et al. (2023).

Chapter 2

Classification of Ricci-Yamabe Solitons

This chapter is divided into three main sections. Section 2.1 deals with almost Ricci-Yamabe solitons on Riemannian manifold. In Section 2.2, Ricci-Yamabe solitons on $(LCS)_n$-manifolds are discussed and Section 2.3 is devoted to the characterization of almost $*$-Ricci-Yamabe solitons.

2.1 Isometries on almost Ricci-Yamabe solitons

This section investigates isometries of almost Ricci-Yamabe solitons. We obtain conditions for compact and complete gradient almost Ricci-Yamabe solitons to be isometric to Euclidean space or sphere.

M. Khatri, Z. Chhakchhuak and J.P. Singh (2023). Isometries on almost Ricci–Yamabe solitons, *Arab. J. Math.* **12**, 127–138.

2.1.1 Some rigidity results on almost Ricci-Yamabe solitons

Before proceeding to the main results, we obtain several lemmas on almost Ricci-Yamabe solitons (ARYS) and gradient ARYS which would be used later.

Lemma 2.1. *For a gradient ARYS* $(M^n, g, \nabla f, \lambda)$*, the following formula holds:*

(1) $2\Delta f + (2\alpha + n\beta)\tau = 2n\lambda$.

(2) $\{\alpha + (n-1)\beta\}\nabla_i\tau = 2(m-1)\nabla_i\lambda + 2R_{is}\nabla^s f, \ \ \alpha \neq 0,\ n \geq 3.$

(3) $\alpha(\nabla_j R_{ik} - \nabla_i R_{jk}) = \frac{\alpha}{\alpha+(n-1)\beta}\left[(\nabla_j\lambda)g_{ik} - (\nabla_i\lambda)g_{jk}\right] + \frac{\alpha+(n-3)\beta}{\alpha+(n-1)\beta}R_{ijks}\nabla^s f, \ \ \alpha + (n-1)\beta \neq 0.$

(4) For $\alpha + (n-1)\beta \neq 0$*, we have*

$$\begin{aligned}\frac{1}{2}\nabla(\tau + |\nabla f|^2) &= \frac{n-1}{\alpha+(n-1)\beta}\nabla\lambda + (\lambda - \frac{\beta\tau}{2})\nabla f \\ &+ \frac{1-\alpha^2-(n-1)\alpha\beta}{\alpha+(n-1)\beta}Ric(\nabla f).\end{aligned}$$

Proof. Result (1) is directly obtained by taking trace of the soliton equation.

For result (2), we consider Schur's Lemma ($n > 2$), we have

$$\begin{aligned}&\frac{1}{2}\nabla_i\tau = divRic_i = g^{jk}\nabla_k R_{ij}, \\ \Longrightarrow\ &\frac{\alpha}{2}\nabla_i\tau = g^{jk}\{(\nabla_k\lambda)g_{ij} - \frac{\beta}{2}(\nabla_k\tau)g_{ij}\} - g^{jk}\nabla_k\nabla_i\nabla_j f.\end{aligned}$$

Then, using Ricci identity in the above expression gives

$$(\alpha + \beta)\nabla_i\tau = 2\nabla_i\lambda - 2\nabla_i(\Delta f) - 2R_{is}\nabla^s f.$$

Thus, in regard of result (1) yields

$$[\alpha + (n-1)\beta]\nabla_i\tau = 2(n-1)\nabla_i\lambda + 2R_{is}\nabla^s f.$$

This gives result (2).

In consequence of result (2) and Ricci identity, we obtain

$$R_{jiks}\nabla^s f + \alpha(\nabla_j R_{ik} - \nabla_i R_{jk}) = (\nabla_j \lambda) g_{ik} - (\nabla_i \lambda) g_{jk} + \frac{\beta}{2}[(\nabla_i \tau) g_{jk} - (\nabla_j \tau) g_{ik}].$$

Further, inserting result (2) in the above expression and then simplifying, we obtain result (3). Now, using result (3) and the fundamental equation as a (1,1)-tensor, result (4) follows, which thus completes the proof. □

Petersen and Wylie (2009) obtained the following Bochner formula for Killing and gradient field as:

Lemma 2.2. *Given a vector field X on a Riemannian manifold (M^n, g), we have*

$$div(\mathcal{L}_X g)(X) = \frac{1}{2}\Delta|X|^2 - |\nabla X|^2 + Ric(X, X) + D_X div X.$$

When $X = \nabla f$ is a gradient field and Z is any vector field , we have

$$div(\mathcal{L}_{\nabla f} g)(Z) = 2Ric(Z, \nabla f) + 2D_Z div \nabla f,$$

or, in (1,1)-tensor notation,

$$div \nabla\nabla f = Ric(\nabla f) + \nabla \Delta f.$$

Taking an inner product of result (2) in Lemma 2.1 by arbitrary vector field Z gives

$$[\alpha + (n-1)\beta] g(\nabla\tau, Z) = 2(n-1) g(\nabla\lambda, Z) + 2Ric(\nabla f, Z). \tag{2.1}$$

In particular,

$$[\alpha + (n-1)\beta] g(\nabla\tau, \nabla f) = 2(n-1) g(\nabla\lambda, \nabla f) + 2Ric(\nabla f, \nabla f). \tag{2.2}$$

and

$$[\alpha + (n-1)\beta] |\nabla\tau|^2 = 2(n-1) g(\nabla\lambda, \nabla\tau) + 2Ric(\nabla f, \nabla\tau). \tag{2.3}$$

Lemma 2.3. *For an ARYS* (M^n, g, X, λ) $(n \geq 3)$ *with* $\alpha \neq 0$*, we have*

$$\frac{2\alpha+n\beta}{2}\Delta|X|^2 - (2\alpha+n\beta)|\nabla X|^2 + \beta(2\alpha+n\beta)g(\nabla\tau, X)$$
$$+ (2\alpha+n\beta)Ric(X,X) + 2[(n-2)\alpha - n\beta]g(\nabla\lambda, X) + n\beta D_X divX = 0.$$

Proof. Taking divergence of ARYS equation yields

$$div(\mathcal{L}_X g)(X) + 2(\alpha+\beta)divRic(X) = 2D_X\lambda. \tag{2.4}$$

We have from (1.70), $2divX + (2\alpha+n\beta)\tau = 2n\lambda$, which gives

$$2D_X divX + (2\alpha+n\beta)D_X R = 2nD_X\lambda. \tag{2.5}$$

Making use of Schur's Lemma, Lemma 2.2, (2.4) and (2.5), we get the required results. This completes the proof. □

Moreover, from (1.70) we have

$$\frac{1}{2}\mathcal{L}_X g(X,X) + \alpha Ric(X,X) = (\lambda - \frac{\beta\tau}{2})|X|^2.$$

In consequence of this in Lemma 2.3, we get

$$\frac{2\alpha+n\beta}{2}(\Delta - \frac{D_X}{\alpha})|X|^2 = (2\alpha+n\beta)|\nabla X|^2 - \beta(2\alpha+n\beta)g(\nabla\tau, X)$$
$$- \frac{2\alpha+n\beta}{\alpha}(\lambda - \frac{\beta\tau}{2})|X|^2 + 2[n\beta - (n-2)\alpha]g(\nabla\lambda, X) - n\beta D_X divX. \tag{2.6}$$

Corollary 2.1. *For a gradient ARYS* $(M^n, g, \nabla f, \lambda)$ $(n \geq 3)$ *with* $\alpha \neq 0$*, we have*

$$\frac{2\alpha+n\beta}{2}\Delta|\nabla f|^2 = (2\alpha+n\beta)|Hess\ f|^2 - \beta(2\alpha+n\beta)g(\nabla\tau, \nabla f)$$
$$- (2\alpha+n\beta)Ric(\nabla f, \nabla f) + 2[n\beta - (n-2)\alpha]g(\nabla\lambda, \nabla f) - n\beta D_{\nabla f}div(\nabla f).$$

Theorem 2.1. *Let* (M^n, g, X, λ) $(n \geq 3)$ *be a compact ARYS. If* $\alpha \neq \{0, -\frac{n\beta}{2}\}$

and

$$\int_M \Big\{Ric(X,X) + \beta g(\nabla\tau,X) + \frac{n\beta}{2\alpha+n\beta}\nabla_X divX + \frac{2[(n-2)\alpha-n\beta]}{2\alpha+n\beta} g(\nabla\lambda,X)\Big\} dv_g \leq 0,$$

then X is Killing and M^n is RYS.

Proof. Since M^n is compact, taking integration of Lemma 2.3 gives

$$\int_M |\nabla X|^2 dv_g = \int_M \Big\{Ric(X,X) + \beta g(\nabla\tau,X) + \frac{n\beta}{2\alpha+n\beta}\nabla_X divX + \frac{2[(n-2)\alpha-n\beta]}{2\alpha+n\beta} g(\nabla\lambda,X)\Big\} dv_g. \tag{2.7}$$

In view of our hypothesis

$$\int_M \Big\{Ric(X,X) + \beta g(\nabla\tau,X) + \frac{n\beta}{2\alpha+n\beta}\nabla_X divX + \frac{2[(n-2)\alpha-n\beta]}{2\alpha+n\beta} g(\nabla\lambda,X)\Big\} dv_g \leq 0,$$

and (2.7), we get $\nabla X = 0$ which implies $\mathcal{L}_X g = 0$, i.e., X is Killing vector field. In this case, ARYS will be simply RYS since M^n will be Einstein manifold, which implies that λ is constant. This completes the proof. □

Corollary 2.2. *Let (M^n, g, X, λ) $(n \geq 3)$ be a compact RYS. If $\alpha \neq \{0, -\frac{n\beta}{2}\}$ and*

$$\int_M \left[Ric(X,X) + \beta g(\nabla\tau,X) + \frac{n\beta}{2\alpha+n\beta}\nabla_X divX\right] dv_g \leq 0,$$

then X is Killing.

In particular, for $\alpha = 1$ and $\beta = -2\rho$ in Theorem 2.1, we recover Theorem 1.6 of (Dwivedi, 2021). Moreover, Theorem 3 in (Barros and Ribeiro, 2012) for compact Ricci soliton is obtained for $\alpha = 1, \beta = 0$.

The next theorem generalizes Theorem 3.5 of (Dwivedi, 2021) which is obtained for compact gradient Ricci-Bourguignon almost soliton, which is the case for $\alpha = 1$ and $\beta = -2\rho$.

Theorem 2.2. *Let* $(M^n, g, \nabla f, \lambda)$ $(n \geq 3)$ *be a compact ARYS with* $\alpha \neq 0$ *and* $\alpha + (n-1)\beta \neq 0$. *Then we have*

(1) $\int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 dv_g = \frac{\alpha(n-2)}{2n} \int_M g(\nabla\tau, \nabla f) dv_g$.

(2) $\int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 dv_g = \frac{\alpha(n-2)}{2n[\alpha+(n-1)\beta]} \int_M [2(n-1)g(\nabla\lambda, \nabla f) + 2Ric(\nabla f, \nabla f)] dv_g$.

Proof. From the gradient ARYS, from (1.71) we have

$$(Hess\ f)(\nabla f) + \alpha Ric(\nabla f) = (\lambda - \frac{\beta\tau}{2})\nabla f. \tag{2.8}$$

Combining second argument of Lemma 2.1 and (2.8), then taking divergence of the obtained expression yields

$$\begin{aligned} \alpha[\alpha + (n-1)\beta]\Delta\tau = {} & 2\alpha(n-1)\Delta\lambda + (2\lambda - \beta\tau)\Delta f - \Delta|\nabla f|^2 \\ & + 2g(\nabla\lambda, \nabla f) - \beta g(\nabla\tau, \nabla f). \end{aligned} \tag{2.9}$$

Now, using commuting covariant derivative and Ricci identity, we have

$$\begin{aligned} \nabla_i\nabla_i(g(\nabla_j f, \nabla_j f)) &= 2\nabla_i(g(\nabla_i\nabla_j f, \nabla_j f)), \\ &= 2g(\nabla_i\nabla_i\nabla_j f, \nabla_j f) + 2|\nabla^2 f|^2, \\ &= 2g(\nabla_i\nabla_i\nabla_j f + R_{iijs}\nabla^s f, \nabla_j f) + 2|\nabla^2 f|^2, \\ &= 2g(\nabla(\Delta f), \nabla f) + 2Ric(\nabla f, \nabla f) + 2|\nabla^2 f|^2. \end{aligned}$$

Making use of the above expression in (2.9), we get

$$\begin{aligned} & \alpha\{\alpha + (n-1)\beta\}\Delta\tau + 2g(\nabla\Delta f, \nabla f) + 2Ric(\nabla f, \nabla f) + 2|\nabla^2 f|^2 \\ & = 2\alpha(n-1)\Delta\lambda + (2\lambda - \beta\tau)\Delta f + 2g(\nabla\lambda, \nabla f) - \beta g(\nabla\tau, \nabla f). \end{aligned} \tag{2.10}$$

Combining first argument of Lemma 2.1, (2.2) and (2.10), we obtain

$$\alpha\{\alpha+(n-1)\beta\}\Delta\tau - \alpha g(\nabla\tau, \nabla f) + 2|\nabla^2 f|^2 = 2\alpha(n-1)\Delta\lambda + (2\lambda - \beta\tau)\Delta f. \tag{2.11}$$

Making use of the fact that $|\nabla^2 f - \frac{\Delta f}{n} g|^2 = |\nabla^2 f|^2 - \frac{(\Delta f)^2}{n}$ in (2.11) gives

$$\alpha\{\alpha + (n-1)\beta\}\Delta\tau + 2|\nabla^2 f - \frac{\Delta f}{n} g|^2 - \alpha g(\nabla\tau, \nabla f) = 2\alpha(n-1)\Delta\lambda + \frac{2\alpha}{n}\tau\Delta f. \tag{2.12}$$

By hypothesis, since M^n is compact, we get

$$\int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 dv_g = \frac{\alpha}{2}\int_M g(\nabla\tau, \nabla f) dv_g + \frac{\alpha}{n}\int_M R\Delta f dv_g. \tag{2.13}$$

Also, we know that $\int_M R\Delta f dv_g = -\int_M g(\nabla\tau, \nabla f) dv_g$, then (2.13) becomes

$$\int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 dv_g = \frac{\alpha(n-2)}{2n}\int_M g(\nabla\tau, \nabla f) dv_g. \tag{2.14}$$

Combining (2.2) in (2.14) proves the second part provided $\alpha + (n-1)\beta \neq 0$. This completes the proof. □

Now, for a gradient ARYS $(M^n, g, \nabla f, \lambda)$, from (1.71) and Lemma 2.1 we can write

$$\begin{aligned}\alpha(Ric - \frac{\tau}{n} g) &= (\lambda - \frac{\beta\tau}{2})g - \nabla^2 f - \frac{\alpha\tau}{n} g,\\ &= \lambda g - \frac{(2\alpha + n\beta)\tau}{2n} g - \nabla^2 f,\\ &= \frac{\Delta f}{n} g - \nabla^2 f.\end{aligned}$$

Now, using the foregoing equation in (2.14) yields

$$\int_M |Ric - \frac{\tau}{n} g|^2 dv_g = \frac{\alpha(n-2)}{2n|\alpha|^2}\int_M g(\nabla\tau, \nabla f) dv_g. \tag{2.15}$$

Corollary 2.3. *Let $(M^n, g, \nabla f, \lambda)$ $(n \geq 3)$ be a gradient ARYS with $\alpha \neq 0$. Then we have*

(1) $\{\alpha + (n-1)\beta\}\Delta\tau + 2\alpha|Ric - \frac{\tau}{n} g|^2 - g(\nabla\tau, \nabla f) = 2(n-1)\Delta\lambda + \frac{2}{n}\tau\Delta f$.

(2) If M^n is compact, then $\int_M |Ric - \frac{\tau}{n} g|^2 dv_g = \frac{(n-2)}{2n\alpha}\int_M g(\nabla\tau, \nabla f) dv_g$.

With regard to Theorem 2.2, Corollary 2.3 and Tashiro's result (Tashiro, 1965) which states that a compact Riemannian manifold (M^n, g) is conformally equivalent to $S^n(r)$ provided there exists a non-trivial function $f : M^n \to \mathbb{R}$ such that $\nabla^2 f = \frac{\Delta f}{n} g$. We obtain the following result which is a generalization of Corollary 1 of (Barros and Ribeiro, 2012) and Corollary 1.10 of (Dwivedi, 2021).

Corollary 2.4. *A non-trivial compact gradient ARYS* $(M^n, g, \nabla f, \lambda)$ $(n \geq 3)$ *with* $\alpha \neq \{0, (1-n)\beta\}$ *is isometric to a Euclidean sphere* $S^n(r)$ *if one of the following conditions hold:*
(1) M^n *has constant scalar curvature.*
(2) M^n *is a homogeneous manifold.*
(3) $\int_M [2(n-1)g(\nabla\lambda, \nabla f) + 2Ric(\nabla f, \nabla f)]dv_g \geq 0$ *and* $0 < \alpha < (1-n)\beta$ *or* $0 > \alpha > (1-n)\beta$.
(4) $\int_M [2(n-1)g(\nabla\lambda, \nabla f) + 2Ric(\nabla f, \nabla f)]dv_g \leq 0$ *with non-negative constants* α *and* β.

Hodge-de Rham decomposition theorem states that we may decompose the vector field X over a compact oriented Riemannian manifold as a sum of the gradient of a function h and a divergence free vector field Y, i.e.,

$$X = \nabla h + Y, \tag{2.16}$$

where $div\ Y = 0$.
Taking divergence of (2.16) gives $div\ X = \Delta h$. From the fundamental equation, we have $2div\ X + (2\alpha + n\beta)\tau = 2n\lambda$. Therefore, combining both equations result in the following:

$$2\Delta h + (2\alpha + n\beta)\tau = 2n\lambda. \tag{2.17}$$

On the other hand, if $(M^n, g, \nabla f, \lambda)$ is also a compact gradient ARYS, then from result (1) of Lemma 2.1, we have

$$2\Delta f + (2\alpha + n\beta)\tau = 2n\lambda. \tag{2.18}$$

Comparing (2.17) and (2.18), we get $\Delta(h-f) = 0$. Now, by using Hopf's theorem, we see that $f = h + c$, where c is a constant. Hence, we can state the following:

Theorem 2.3. *Let (M^n, g, X, λ) be a compact ARYS. If M^n is also gradient ARYS with potential f, then upto a constant, it agrees with the Hodge-de Rham potential h.*

2.1.2 Almost Ricci-Yamabe solitons with certain conditions on the potential vector field

In this subsection, we consider ARYS whose potential vector field satisfies certain conditions such as conformal, solenoidal and torse-forming vector fields. Conformal vector field under almost Ricci soliton and almost Ricci-Bourguignon solitons were considered by authors in (Barros and Ribeiro, 2012; Blaga and Tastan, 2021) and obtained interesting results. Now, we state and prove the following lemma.

Lemma 2.4. *Let $(n \geq 3)$ be ARYS with $\alpha \neq 0$. If X is a conformal vector field with potential function ψ, then τ and $\lambda - \psi$ are constants.*

Proof. Since X is a conformal vector field, we have $\mathcal{L}_X g = 2\psi g$. Making use of this in the soliton equation (1.70) yields

$$\alpha Ric = (\lambda - \frac{\beta\tau}{2} - \psi)g. \tag{2.19}$$

which further gives

$$(2\alpha + n\beta)\tau = 2n(\lambda - \psi), \tag{2.20}$$

and

$$\alpha \; divRic = \nabla(\lambda - \frac{\beta\tau}{2} - \psi). \tag{2.21}$$

Making use of Schur's Lemma in (2.21) and inserting it in the covariant derivative of (2.20) results in $(n-2)\alpha\nabla\tau = 0$. As $\alpha \neq 0$, then τ is constant, which implies then from (2.20) that $\lambda - \psi$ is also constant. This completes the proof. □

Theorem 2.4. *Let (M^n, g, X, λ) $(n \geq 3)$ be a compact ARYS with $\alpha \neq 0$. If X is a non-trivial conformal vector field, then M^n is isometric to Euclidean sphere $S^n(r)$.*

Proof. In regard of Lemma 2.4, we know that τ and $\lambda - \psi$ are constants. Moreover, using Lemma 2.3 of (Yano, 1970), we conclude that $\tau \neq 0$, otherwise $\psi = 0$, a contradiction as $\psi \neq 0$.

Taking Lie derivative of (2.19) and using the fact that τ and $\lambda - \psi$ are constants give

$$\alpha \mathcal{L}_X Ric = (\lambda - \frac{\beta\tau}{2} - \psi)\mathcal{L}_X g = (\lambda - \frac{\beta\tau}{2} - \psi)\psi g.$$

Now, applying Theorem 4.2 of (Yano, 1970) to conclude that M^n is isometric to Euclidean sphere $S^n(r)$. This completes the proof. □

Now, we look at gradient ARYS admitting conformal vector field on which we state and prove the following:

Theorem 2.5. *Let $(M^n, g, \nabla f, \lambda)$ $(n \geq 3)$ be a complete gradient ARYS with $\alpha \neq 0$. If ∇f is a non-trivial conformal vector field with non-negative scalar curvature, then either*

(1) M^n is isometric to a Euclidean space E^n.

or

(2) M^n is isometric to a Euclidean sphere S^n. Moreover, ψ is a first eigenfunction of Laplacian and $\lambda = \frac{2\alpha + n\beta}{2n}\tau - \frac{\lambda_1}{n} f + k$, where k is a constant.

Proof. Since ∇f is a non-trivial conformal vector field, we have $\mathcal{L}_{\nabla f} g = 2\psi g$, $\psi \neq 0$. Now, in consequence of argument (1) of Lemma 2.1, we get $\psi = \frac{\Delta f}{n} \neq 0$. Moreover, from Lemma 2.4, we know that τ and $\lambda - \psi$ are constants. Suppose $\tau = 0$, then this implies that M^n is Ricci flat and by using Tashiro's theorem (Tashiro, 1965) in the fundamental equation, we conclude that M^n is isometric to a Euclidean space E^n. On the other hand, suppose $\tau \neq 0$. Then, making use of Lemma 2.1 in $\psi = \frac{\Delta f}{n}$ gives $\lambda = \psi + (\frac{2\alpha+n\beta}{2n})\tau$. As a consequence, (2.19) becomes $Ric = \frac{\tau}{n}g$ for $\alpha \neq 0$. Therefore, by involving a theorem by Nagano and Yano (Nagano and Yano, 1959), we can conclude that M^n is isometric to a Euclidean sphere S^n. Furthermore, taking into account of the fact that $Ric = \frac{\tau}{n}g$, we can use Lichnerowicz's theorem (Lichnerowicz, 1958), the first eigenvalue of the Laplacian of M^n is $\lambda_1 = \frac{\tau}{n-1}$. Now, we make use of well known formula by Obata and Yano (Obata and Yano, 1970), which gives

$$\Delta\psi + \frac{\tau}{n-1}\psi = 0. \tag{2.22}$$

In view of (2.22), one can easily obtain $\Delta\psi = -\lambda_1\psi$, that is, ψ is a first eigenfunction of the Laplacian. Also, we get $\Delta(\Delta f + \lambda_1 f) = 0$. Then, by Hopf theorem, we obtain $\Delta f + \lambda_1 f = c$, where c is a constant. Combining the last expression with Lemma 2.1 give us the required expression for λ. This completes the proof. □

In the paper of Blaga and Tastan (Blaga and Tastan, 2021), the authors considered almost Ricci-Bourguignon soliton and almost η-Ricci-Bourguignon soliton with solenoidal and torse-forming vector field and obtained several rigidity results. Following similar methods, we examine ARYS (M^n, g, ξ, λ) with solenoidal and torse-forming vector fields.

Let ξ be a solenoidal vector field. Then, by taking trace of the ARYS equation (1.70), we get

$$\tau = \frac{2}{2\alpha + n\beta}(\lambda n - div(\xi)), \tag{2.23}$$

provided $\alpha \neq -\frac{n\beta}{2}$. If $\alpha = -\frac{n\beta}{2}$, then $\lambda = \frac{div(\xi)}{n}$. For $\alpha \neq \{0, -\frac{n\beta}{2}\}$, the soliton equation can be written as

$$\frac{1}{2}\mathcal{L}_\xi g + \alpha Ric = \frac{\beta div(\xi) + 2\alpha\lambda}{2\alpha + n\beta}g. \tag{2.24}$$

Taking an inner product with Ric in (2.24) gives

$$\langle \mathcal{L}_\xi g, Ric\rangle = -2\alpha|Ric|^2 + \frac{4}{(2\alpha+n\beta)^2}[(n\beta -2\alpha)div(\xi)\lambda - \beta(div(\xi))^2 + 2\alpha n\lambda^2]. \tag{2.25}$$

Again, taking an inner product with $\mathcal{L}_\xi g$ in (2.24) and considering $|\mathcal{L}_\xi g|^2 = 4|\nabla\xi|^2$, we have

$$\langle \mathcal{L}_\xi g, Ric\rangle = -\frac{2}{\alpha}|\nabla\xi|^2 + \frac{2}{\alpha(2\alpha+n\beta)}[\beta(div(\xi))^2 + 2\alpha\lambda div(\xi)]. \tag{2.26}$$

Comparing (2.25) and (2.26), we get

$$|Ric|^2 = \frac{1}{\alpha^2}|\nabla\xi|^2 + \frac{1}{\alpha^2(2\alpha+n\beta)^2}[4\alpha^2 n\lambda^2 - 8\alpha^2\lambda div(\xi) - (4\alpha+n\beta)\beta(div(\xi))^2].$$

which leads to the following:

Proposition 2.1. *For an ARYS (M^n, g, ξ, λ) with $\alpha \neq \{0, -\frac{n\beta}{2}\}$ and a solenoidal vector field ξ, we have*

$$|Ric|^2 \geq \frac{1}{\alpha^2}|\nabla\xi|^2.$$

Now, let ξ be a gradient vector field. Making use of Bochner formula (Blaga, 2017), we have

$$Ric(\xi,\xi) = \frac{1}{2}\Delta(|\xi|^2) - |\nabla\xi|^2 - \xi(div(\xi)). \tag{2.27}$$

Using (2.23) in the soliton equation (1.70), we get

$$\frac{1}{2}\mathcal{L}_\xi g + \alpha Ric = \left[\lambda - \frac{\beta}{2\alpha+n\beta}(\lambda n - div(\xi))\right] g. \tag{2.28}$$

From (2.28), we have

$$\alpha Ric(\xi,\xi) = -\frac{1}{2}\xi(|\xi|^2) + \left[\lambda - \frac{\beta}{2\alpha+n\beta}(\lambda n - div(\xi))\right]|\xi|^2. \tag{2.29}$$

Comparing (2.24) and (2.29), we can state the following:

Theorem 2.6. *A gradient ARYS (M^n, g, ξ, λ) with $\alpha \neq \{0, -\frac{n\beta}{2}\}$ has the function*

λ expressed in terms of ξ as

$$\lambda = \frac{2\alpha + n\beta}{4\alpha|\xi|^2}[\alpha\Delta(|\xi|^2) - 2\alpha|\nabla\xi|^2 + \xi(|\xi|^2) - 2\alpha\xi div(\xi)] - \frac{\beta}{2\alpha}div(\xi).$$

In particular, for $\alpha = 1$ and $\beta = -2\rho$, where $\rho \in \mathbb{R}$ and $\rho \neq \frac{1}{n}$, we recover Theorem 2.2 of (Blaga and Tastan, 2021).

If $\xi = \nabla f$ with f a smooth function on M^n and $\alpha \neq \{0, -\frac{n\beta}{2}\}$, the soliton equation becomes

$$Hess\ f + \alpha Ric = (\lambda - \frac{\beta\tau}{2})g. \tag{2.30}$$

and (2.23) becomes

$$\tau = \frac{2}{2\alpha + n\beta}(\lambda n - \Delta f). \tag{2.31}$$

Differentiating the above expression gives

$$d(\Delta f) = nd\lambda - \frac{2\alpha + n\beta}{2}d\tau, \tag{2.32}$$
$$\implies \nabla(\Delta f) = n\nabla\lambda - \frac{2\alpha + n\beta}{2}\nabla\tau.$$

Taking divergence of (2.30) and using Schur's Lemma, we get

$$div(Hess\ f) = d\lambda - \frac{\alpha + \beta}{2}d\tau. \tag{2.33}$$

Also, from (Blaga, 2017), we have

$$div(Hess\ f) = d(\Delta f) + i_{Q(\nabla f)}g, \tag{2.34}$$

where i denotes the interior product and Q is the Ricci operator.

Comparing (2.33) and (2.34) yields

$$d(\Delta f) = d\lambda - \frac{\alpha + \beta}{2}d\tau - i_{Q(\nabla f)}g. \tag{2.35}$$

From (2.32) and (2.35), we have

$$(n-1)d\lambda = \frac{\alpha + (n-1)\beta}{2}d\tau - Q(\nabla f). \tag{2.36}$$

Therefore we can state the following:

Proposition 2.2. *For a gradient ARYS on M^n with $\alpha \neq \{0, -\frac{n\beta}{2}\}$, we have*

$$grad(\lambda) = \frac{\alpha + (n-1)\beta}{2(n-1)} grad(\tau) - \frac{1}{n-1} Q(grad\ f).$$

Moreover, if $grad\ f \in Ker(Q)$, then

$$grad(\lambda) = \frac{\alpha + (n-1)\beta}{2(n-1)} grad(\tau). \tag{2.37}$$

In the gradient case, we have $\xi = \nabla f$, if $\alpha \neq \{0, -\frac{n\beta}{2}\}$, then from (2.31), we get

$$\lambda = \frac{2\alpha + n\beta}{2n}\tau + \frac{\Delta f}{n}. \tag{2.38}$$

Then, (2.30) becomes

$$Hess\ f + \alpha Ric = \frac{\alpha\tau + \Delta f}{n} g. \tag{2.39}$$

Taking inner product with Ric and $Hess\ f$ respectively in (2.39) yields

$$\alpha |Ric|^2 = \frac{\alpha\tau + \Delta f}{n}\tau - \langle Hess\ f, Ric\rangle, \tag{2.40}$$

and

$$\frac{1}{\alpha}|Hess\ f|^2 = \frac{\alpha\tau + \Delta f}{\alpha n}\Delta f - \langle Ric, Hess\ f\rangle. \tag{2.41}$$

On comparing (2.40) and (2.41), we get

$$\alpha|Ric|^2 - \frac{1}{\alpha}|Hess\ f|^2 = \frac{\alpha^2\tau^2 - (\Delta f)^2}{\alpha n},$$

which leads to the following:

Theorem 2.7. *For a gradient ARYS $(M^n, g, \nabla f, \lambda)$ on M^n with $\alpha \neq \{0, -\frac{n\beta}{2}\}$, we have*

$$\frac{1}{\alpha^2}|Hess\ f|^2 - \frac{(\Delta f)^2}{\alpha^2 n} \leq |Ric|^2 \leq \frac{1}{\alpha^2}|Hess\ f|^2 + \frac{\tau^2}{n}.$$

Again, let us consider a torse forming vector field ξ, then, $\nabla\xi = \gamma I + \psi \otimes \xi$, where γ is a smooth function, ψ is a 1-form and I is the identity endomorphism on the space of vector fields. Then, we have

$$div(\xi) = n\gamma + \psi(\xi),$$

$$\mathcal{L}_\xi g = 2\gamma g + \psi \otimes \theta + \theta \otimes \psi,$$

where θ is the dual 1-form of ξ. From (1.70), we get for $\alpha \neq \{0, -\frac{n\beta}{2}\}$ that

$$Ric = \frac{\beta\psi(\xi) - 2\alpha(\gamma - \lambda)}{\alpha(2\alpha + n\beta)} g - \frac{1}{2\alpha}(\psi \otimes \theta + \theta \otimes \psi). \tag{2.42}$$

Thus,

$$Q = \frac{\beta\psi(\xi) - 2\alpha(\gamma - \lambda)}{\alpha(2\alpha + n\beta)} I - \frac{1}{2\alpha}(\psi \otimes \xi + \theta \otimes \zeta),$$

which implies

$$\tau = \frac{n\beta\psi(\xi) - 2\alpha n(\gamma - \lambda)}{\alpha(2\alpha + n\beta)},$$

where ζ is the dual vector field of ψ.

Computing the Riemann curvature for $\nabla\xi = \gamma I + \psi \otimes \xi$, we get

$$R(X,Y)\xi = (d\gamma - \gamma\psi)(X)Y - (d\gamma - \gamma\psi)(Y)X + [(\nabla_X\psi)Y - (\nabla_Y\psi)X]\xi,$$

for any $X, Y \in \chi(M^n)$. If ψ is a Codazzi tensor field, i.e., $(\nabla_X\psi)Y = (\nabla_Y\psi)X$, then

$$Ric(\xi,\xi) = (1-n)[\xi(\gamma) - \gamma\psi(\xi)]. \tag{2.43}$$

Also, from (2.42), we have

$$Ric(\xi,\xi) = \frac{|\xi|^2}{\alpha(2\alpha + n\beta)}[2\alpha(\lambda - \gamma) - \{2\alpha + (n-1)\beta\}\psi(\xi)]. \tag{2.44}$$

Then, comparing (2.43) and (2.44) yields

Proposition 2.3. *Let (M^n, g, ξ, λ) defines an ARYS with $\alpha \neq \{0, -\frac{n\beta}{2}\}$ such that ξ is a torse forming vector field and ψ is a Codazzi tensor field, then*

$$\lambda = \gamma + \frac{2\alpha + n\beta}{2|\xi|^2}(1-n)\xi(\gamma) + \frac{1}{\alpha|\xi|^2}[\{2\alpha + (n-1)\beta\}|\xi|^2 + \alpha(n-1)(2\alpha + n\beta)\gamma]\psi(\xi).$$

Let us verify the obtained results by assuming non-trivial examples constructed in (Blaga and Tastan, 2021).

Example 2.1. *On the 3-dimensional manifold* $M = \{(x, y, z) \in \mathbb{R}^3,\ z > 0\}$, *where* (x, y, z) *are the standard coordinates in* $\mathbb{R}^3$ *with the Riemannian metric*

$$g := \frac{1}{z^2}(dx^2 + dy^2 + dz^2).$$

Then $\big(g, \xi = \frac{\partial}{\partial z}, \lambda = \frac{3\beta}{2\alpha z} - \frac{2\alpha+3\beta}{2\alpha}(2 + \frac{1}{z})\big)$ *defines a gradient ARYS.*
Precisely, $\xi = \nabla f$ *for* $f(x, y, z) = -\frac{1}{z}$ *where* $|\xi|^2 = \frac{1}{z^2}$, $\xi(|\xi|^2) = -\frac{2}{z^3}$, $\Delta(|\xi|^2) = \frac{8}{z^2}$, $|\nabla\xi|^2 = \frac{3}{z^2}$, $div(\xi) = -\frac{3}{z}$, $\xi(div(\xi)) = \frac{3}{z^2}$. *Therefore,* $\lambda = \frac{3\beta}{2\alpha z} - \frac{2\alpha+3\beta}{2\alpha}(2 + \frac{1}{z})$ *is obtained from Theorem 2.6.*

Example 2.2. *Let* $M = \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$. *Consider the Riemannian metric*

$$g := \exp(2z)(dx^2 + dy^2) + dz^2.$$

Then, $(g, \xi = \exp(z)\frac{\partial}{\partial z}, \lambda = \frac{2\alpha+3\beta}{2\alpha}(\exp(z) - 2\alpha) - \frac{3\beta}{2\alpha}\exp(z))$ *defines a gradient ARYS with* $\xi = \nabla f$, *where* $f(x, y, z) = \exp(z)$. *On the other hand, one can check that* $|\xi|^2 = \exp(2z)$, $\xi(|\xi|^2) = 2\exp(3z)$, $\Delta(|\xi|^2) = 8\exp(2z)$, $|\nabla\xi|^2 = 3\exp(2z)$, $div(\xi) = 3\exp(z)$, $\xi(div(\xi)) = 3\exp(2z)$, *therefore,* $\lambda = \frac{2\alpha+3\beta}{2\alpha}(\exp(z) - 2\alpha) - \frac{3\beta}{2\alpha}\exp(z)$ *is immediately obtained from Theorem 2.6.*

2.2 Certain results of Ricci-Yamabe solitons on $(LCS)_n$-manifolds

Siddiqi and Akyol (2020) constructed the geometrical bearing on Riemannian submersions in terms of $\eta-$Ricci-Yamabe soliton with the potential field and presented the classification of any fiber of Riemannian submersion is a $\eta-$Ricci-Yamabe soliton, $\eta-$Ricci soliton and $\eta-$Yamabe soliton.

J.P. Singh and Z. Chhakchhuak (2022). Certain results of Ricci-Yamabe solitons on $(LCS)_n$-manifolds, *Facta Univ. Math. Inform.* **37(4)**, 797–812.

2.2.1 Preliminaries

Let an $n-$dimensional Lorentzian manifold M admits the characteristic vector field ξ. Then, we have $g(\xi, \xi) = -1$. Since ξ is a unit concircular vector field, we have a non-zero $1-$form η such that for $g(X, \xi) = \eta(X)$, the following equation holds for all vector fields X and Y on M:

$$(\nabla_X \eta)(Y) = \gamma\{g(X, Y) + \eta(X)\eta(Y)\}, \gamma \neq 0, \tag{2.45}$$

where γ is a scalar function on M which satisfies

$$\nabla_X \gamma = X(\gamma) = d\gamma(X) = \rho\eta(X), \tag{2.46}$$

for $\rho \in \mathbb{C}^\infty(M)$, where ∇ is the Levi-Civita connection of g. Let us take a symmetric $(1, 1)$ tensor field ϕ denoted by

$$\phi(X) = X(\gamma) = \frac{1}{\gamma}\nabla_X \xi, \tag{2.47}$$

called the structure tensor of the manifold. Thus, the Lorentzian manifold M equipped with ξ, η and ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n-$manifold) (Shaikh, 2003). In fact, if we take $\gamma = 1$, then we obtain the LP-Sasakian structure of Matsumoto (Matsumoto, 1989). The following relations hold in an $(LCS)_n-$manifold $(n > 2)$ for any X, Y, Z on M (Shaikh 2003; Roy et al., 2020):

$$\phi X = X + \eta(X)\xi, \tag{2.48}$$

$$\phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.49}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad \text{and} \quad g(\phi X, Y) = g(X, \phi Y), \tag{2.50}$$

$$(\nabla_X \phi)Y = \gamma[g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \tag{2.51}$$

$$\eta(\nabla_X \xi) = 0, \quad \nabla_\xi \xi = 0, \tag{2.52}$$

$$R(X,Y)Z = (\gamma^2 - \rho)[g(Y,Z)X - g(X,Z)Y], \tag{2.53}$$

$$Ric(X,Y) = (\gamma^2 - \rho)(n-1)g(X,Y), \tag{2.54}$$

$$\tau = n(n-1)(\gamma^2 - \rho), \tag{2.55}$$

$$\nabla \eta = \gamma(g + \eta \otimes \eta), \quad \nabla_\xi \eta = 0, \tag{2.56}$$

$$\mathcal{L}_\xi \phi = 0, \quad \mathcal{L}_\xi \eta = 0, \quad \mathcal{L}_\xi g = 2\nabla \eta = 2\gamma(g + \eta \otimes \eta), \tag{2.57}$$

where R is the Riemannian curvature tensor, Ric is the Ricci tensor, τ is the scalar curvature and ∇ is the Levi-Civita connection associated with g.

Shaikh (2009) studied a conformally flat $(LCS)_n-$manifold and shown that a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold is $\eta-$Einstein and its Ricci tensor is given by

$$Ric(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.58}$$

where $a = \frac{\tau}{n-1} - (\gamma^2 - \rho)$ and $b = n(\gamma^2 - \rho) - \frac{\tau}{n-1}$. Then, from (2.58), we have

$$Ric(X,\xi) = (a-b)\eta(X), \tag{2.59}$$

and

$$Ric(\xi,\xi) = -(a-b). \tag{2.60}$$

Definition 2.1 (Hui and Chakraborty, 2016). *A vector field ξ is called torse forming if it satisfies*

$$\nabla_X \xi = fX + \nu(X)\xi, \tag{2.61}$$

for a smooth function $f \in \mathbb{C}^\infty(M)$ and ν is an $1-$form, for all vector field X on M.

Remark 2.1. *Let ξ be a torse forming vector field on an $(LCS)_n$-manifold. We know that in an $(LCS)_n$-manifold, $\nabla_\xi \xi = 0$. Taking $X = \xi$ in (2.61), we get $(f-1)\xi = 0$. Since $\xi \neq 0$, then $f = 1$.*

2.2.2 Ricci-Yamabe Soliton (RYS) on $(LCS)_n$ manifolds

Now, we assume that ξ is the Reeb vector field of the Lorentzian concircular structure.

Consider the Ricci-Yamabe soliton (RYS) on an $n-$dimensional $(LCS)_n$ manifold as

$$\mathcal{L}_\xi g + 2\alpha Ric = (2\lambda - \beta\tau)g. \tag{2.62}$$

From (2.57), we have

$$2\gamma(g(X,Y) + \eta(X)\eta(Y)) + 2\alpha Ric(X,Y) = (2\lambda - \beta\tau)g(X,Y), \tag{2.63}$$

for all vector fields X, Y on M. This implies

$$(2\lambda - \beta\tau - 2\gamma)g(X,Y) - 2\alpha g(QX,Y) - 2\gamma\eta(X)\eta(Y) = 0. \tag{2.64}$$

Setting $Y = \xi$ in the above equation and using (2.49), we get

$$g((2\lambda - \beta\tau)X - 2\alpha QX, \xi) = 0. \tag{2.65}$$

Then, we have

$$QX = \frac{2\lambda - \beta\tau}{2\alpha}X, \alpha \neq 0. \tag{2.66}$$

Contracting the foregoing equation, we obtain

$$\tau = \frac{2\lambda n}{2\alpha + n\beta}, 2\alpha + n\beta \neq 0. \tag{2.67}$$

Since, α, β, λ are constants, τ is also constant. Now, using (2.54), (2.55) and (2.67), we have

$$Ric(X,Y) = \frac{2\lambda}{2\alpha + n\beta}g(X,Y), 2\alpha + n\beta \neq 0. \tag{2.68}$$

Also, using (2.67) and (2.68) in (2.62), we get

$$\mathcal{L}_\xi g = 0. \tag{2.69}$$

Thus, we can state the following theorem:

Theorem 2.8. *An $(LCS)_n-$manifold admitting a Ricci-Yamabe soliton has constant scalar curvature and the manifold becomes Einstein manifold provided that $\alpha \neq \{0, -\frac{n\beta}{2}\}$. Moreover, ξ is the killing vector field.*

Let us assume that the Ricci tensor, Ric of an $(LCS)_n$ manifold is $\eta-$recurrent, we have

$$\nabla Ric = \eta \otimes Ric, \tag{2.70}$$

which implies

$$(\nabla_X Ric)(Y, Z) = \eta(X) Ric(Y, Z), \tag{2.71}$$

for all vector fields X, Y, Z on M. From (2.68), we get $\nabla Ric = 0$. Now, from (2.71), we have

$$\eta(X) Ric(Y, Z) = 0. \tag{2.72}$$

Since $\eta(X) \neq 0$, we obtain $Ric(Y, Z) = 0$. Thus, from the expression of Ric in (2.68), we have $\lambda = 0$. Also, from (2.67), we get $\tau = 0$, which then implies from (2.55) that $(\gamma^2 - \rho) = 0$ provided $n > 1$. Again, in view of (2.53), we have $R(X, Y)Z = 0$ for all vector fields X, Y, Z on M. This results to the following:

Proposition 2.4. *If the Ricci tensor Ric of an $(LCS)_n$ $(n > 1)$ manifold admitting a RYS is $\eta-$recurrent, then the soliton is steady and the manifold becomes flat.*

Let us consider a symmetric $(0, 2)$ tensor field h such that

$$h = \frac{1}{2\alpha}\mathcal{L}_\xi g - \frac{2\lambda - \beta\tau}{2\alpha} g, \alpha \neq 0. \tag{2.73}$$

This implies $\nabla h = 0$. Then,

$$\begin{aligned} h(\xi, \xi) &= \mathcal{L}_\xi g(\xi, \xi) - \frac{2\lambda - \beta\tau}{2\alpha} g(\xi, \xi) \\ &= \frac{2\lambda - \beta\tau}{2\alpha}. \end{aligned} \tag{2.74}$$

As $\nabla h = 0$, then using (2.74) and the results obtained in Chandra et al. (2015), (2.73) becomes

$$\mathcal{L}_\xi g(X,Y) + 2\alpha Ric(X,Y) = (2\lambda - \beta\tau)g(X,Y), \tag{2.75}$$

for all vector fields X, Y on M. This leads to the following theorem:

Theorem 2.9. *Let $(M, g, \xi, \eta, \phi, \gamma)$ be an $(LCS)_n-$manifold such that a symmetric $(0,2)$ tensor field h given by $h = \frac{1}{2\alpha}\mathcal{L}_\xi g - \frac{2\lambda-\beta\tau}{2\alpha} g$ with $\alpha \neq 0$ and $\nabla h = 0$. Then, (g,ξ) yields a Ricci-Yamabe soliton on M.*

Let us define a Ricci-Yamabe soliton (RYS) on an n-dimensional $(LCS)_n$-manifold M as

$$\mathcal{L}_V g + 2\alpha Ric = (2\lambda - \beta\tau)g. \tag{2.76}$$

Let $V = t\xi$, where t is a function on M. Then,

$$\mathcal{L}_{t\xi} g(X,Y) + 2\alpha Ric(X,Y) = (2\lambda - \beta\tau)g(X,Y), \tag{2.77}$$

for any vector fields X, Y on M. Applying the property of Lie derivative and Levi-Civita connection, we have

$$tg(\nabla_X\xi, Y) + (Xt)\eta(Y) + tg(\nabla_Y\xi, X) + (Yt)\eta(X) + 2\alpha g(QX,Y) = (2\lambda - \beta\tau)g(X,Y). \tag{2.78}$$

Taking $Y = \xi$ in the above equation and using (2.52), we get

$$-Xt + \left(\xi t + \frac{4\alpha\lambda}{2\alpha + n\beta} - 2\lambda + \beta\tau\right)\eta(X) = 0, 2\alpha + n\beta \neq 0. \tag{2.79}$$

Taking $X = \xi$ in the foregoing equation, we obtain

$$\xi t = \frac{2\lambda - \beta\tau}{2} - \frac{2\alpha\lambda}{2\alpha + n\beta}. \tag{2.80}$$

Using (2.80), (2.79) becomes

$$Xt = -\frac{2n\lambda\beta - \beta\tau(2\alpha + n\beta)}{2(2\alpha + n\beta)}\eta(X). \tag{2.81}$$

Applying exterior differentiation in (2.81), we get

$$\frac{2n\lambda\beta - \beta\tau(2\alpha + n\beta)}{2(2\alpha + n\beta)} d\eta = 0. \tag{2.82}$$

We know that in an $n-$dimensional $(LCS)_n-$manifold, we have

$$(d\eta)(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]), \tag{2.83}$$

which implies

$$(d\eta)(X, Y) = g(Y, \nabla_X \xi) - g(X, \nabla_Y \xi). \tag{2.84}$$

Using (2.47) and (2.50) in (2.84), we get

$$(d\eta)(X, Y) = 0. \tag{2.85}$$

Hence, the $1-$form η is closed. Then, using the above equation, (2.82) implies either

$$\tau = \frac{2\lambda n}{2\alpha + n\beta} \quad \text{or} \quad \tau \neq \frac{2\lambda n}{2\alpha + n\beta}. \tag{2.86}$$

Now, if $\tau \neq \frac{2\lambda n}{2\alpha + n\beta}$, we have

$$\mathcal{L}_V g + 2\alpha Ric = (2\lambda - \beta\tau) g. \tag{2.87}$$

Replacing the expression of Ric from (2.68) in (2.87), we get

$$\mathcal{L}_V g = \left(2\lambda - \frac{4\alpha\lambda}{2\alpha + n\beta} - \beta\tau\right) g, \tag{2.88}$$

which implies that V is a conformal Killing vector field. Again, if $\tau = \frac{2\lambda n}{2\alpha + n\beta}$, then from (2.81), we get

$$Xt = 0, \tag{2.89}$$

which implies that t is constant. Therefore, we can state the following theorem:

Theorem 2.10. *If a vector field V on an $(LCS)_n-$manifold admitting a Ricci-Yamabe soliton is pointwise collinear with ξ, then either V is a conformal Killing vector field, or V is a constant multiple of ξ, provided that $\tau \neq \frac{2\lambda n}{2\alpha + n\beta}$ and $2\alpha +$*

$n\beta \neq 0$.

Remark 2.2. *The above Theorem 2.10 is a generalization of Theorem 3.8 in (Roy et al., 2020), where they obtained the condition for V to be a conformal Killing vector field is $\tau \neq \lambda$. It is easy to see that for $\alpha = 0$ and $\beta = 2$, Theorem 3.8 in (Roy et al., 2020) can be obtained from Theorem 2.10.*

As a consequence of Theorem 2.10, substituting $\tau = \frac{2\lambda n}{2\alpha+n\beta}, \quad 2\alpha + n\beta \neq 0$ in (2.87), we get that

$$\mathcal{L}_V g = 0, \tag{2.90}$$

implying V is a Killing vector field. Then, we have:

Corollary 2.5. *If a vector field V on an $(LCS)_n-$manifold admitting a Ricci-Yamabe soliton is pointwise collinear with ξ and $\tau = \frac{2\lambda n}{2\alpha+n\beta}$ with $2\alpha + n\beta \neq 0$, then V becomes a Killing vector field.*

Setting $Z = \xi$ in (1.18), we get

$$P(X,Y)\xi = R(X,Y)\xi - \frac{1}{(n-1)}[Ric(Y,\xi)X - Ric(X,\xi)Y]. \tag{2.91}$$

Taking $Z = \xi$ in (2.53), then using the result and (2.68) in the above equation, we get

$$P(X,Y)\xi = \left((\gamma^2 - \rho) - \frac{2\lambda}{(n-1)(2\alpha + n\beta)}\right)[\eta(Y)X - \eta(X)Y]. \tag{2.92}$$

Using (2.55) and (2.67) in (2.92), we obtain

$$P(X,Y)\xi = 0, \tag{2.93}$$

which results to the following:

Proposition 2.5. *An $(LCS)_n-$manifold admitting a Ricci-Yamabe soliton is $\xi-$projectively flat.*

Again, taking $Z = \xi$ in (1.17), we get

$$H(X,Y)\xi = R(X,Y)\xi - \frac{1}{(n-2)} \quad [g(Y,\xi)QX - g(X,\xi)QY \qquad (2.94)$$
$$+Ric(Y,\xi)X - Ric(X,\xi)Y].$$

Putting $Z = \xi$ in (2.53), then using the result and (2.68) in (2.94), we obtain

$$H(X,Y)\xi = \left((\gamma^2 - \rho) - \frac{4\lambda}{(n-2)(2\alpha + n\beta)}\right)[\eta(Y)X - \eta(X)Y]. \qquad (2.95)$$

Using (2.55) and (2.67) in the above equation, we get

$$H(X,Y)\xi = -\frac{2n\lambda}{(n-1)(n-2)(2\alpha + n\beta)}[\eta(Y)X - \eta(X)Y]. \qquad (2.96)$$

This implies that $H(X,Y)\xi = 0$ if and only if $\lambda = 0$. Hence, we can state the following:

Proposition 2.6. *An $(LCS)_n-$manifold admitting a Ricci-Yamabe soliton is $\xi-$conharmonically flat if and only if the soliton is steady.*

Now, we know that

$$R(\xi,X)\cdot Ric = Ric(R(\xi,X)Y,Z) + Ric(Y,R(\xi,X)Z), \qquad (2.97)$$

for all vector fields X, Y, Z on M. Interchanging Y and Z, then putting $Z = \xi$ in (2.53), then using the result and (2.68) in (2.97), the above equation becomes

$$R(\xi,X)\cdot Ric = \tfrac{2\lambda}{2\alpha+n\beta}(\gamma^2 - \rho)[g(X,Y)\eta(Z) - g(X,Z)\eta(Y) \qquad (2.98)$$
$$+g(X,Z)\eta(Y) - g(Y,X)\eta(Z)],$$

which implies that $R(\xi,X)\cdot Ric = 0$. This leads to the following theorem:

Theorem 2.11. *If an $(LCS)_n-$manifold admits a Ricci-Yamabe soliton, then the manifold is $\xi-$semi symmetric.*

Let us assume that $Ric(\xi, X) \cdot R = 0$, which implies

$$Ric(X, R(Y,Z)W)\xi - Ric(\xi, R(Y,Z)W)X + Ric(X,Y)R(\xi,Z)W \quad (2.99)$$
$$-Ric(\xi,Y)R(X,Z)W + Ric(X,Z)R(Y,\xi)W - Ric(\xi,Z)R(Y,X)W$$
$$+Ric(X,W)R(Y,Z)\xi - Ric(\xi,W)R(Y,Z)X = 0,$$

for any vector fields X, Y, Z, W on M. Taking the inner product with ξ, (2.99) becomes

$$-Ric(X, R(Y,Z)W) - Ric(\xi, R(Y,Z)W)\eta(X) + Ric(X,Y)\eta(R(\xi,Z)W) \quad (2.100)$$
$$-Ric(\xi,Y)\eta(R(X,Z)W) + Ric(X,Z)\eta(R(Y,\xi)W) - Ric(\xi,Z)\eta(R(Y,X)W)$$
$$+Ric(X,W)\eta(R(Y,Z)\xi) - Ric(\xi,W)\eta(R(Y,Z)X) = 0.$$

Taking $Z = W = \xi$ in (2.147) and replacing the expression of Ric from (2.68), we get

$$\tfrac{2\lambda}{2\alpha+n\beta}[-g(X,R(Y,\xi)\xi) - \eta(R(Y,\xi)\xi)\eta(X) + g(X,Y)\eta(R(\xi,\xi)\xi) \quad (2.101)$$
$$-\eta(Y)\eta(R(X,\xi)\xi) + \eta(X)\eta(R(Y,\xi)\xi) + \eta(R(Y,X)\xi) + \eta(X)\eta(R(Y,\xi)\xi)$$
$$+\eta(R(Y,\xi)X)] = 0.$$

In view of (2.53) and on simplification, the above equation becomes

$$\frac{4\lambda}{2\alpha+n\beta}(\gamma^2 - \rho)[g(X,Y) + \eta(X)\eta(Y)] = 0. \quad (2.102)$$

Using (2.50), we get

$$\frac{4\lambda}{2\alpha+n\beta}(\gamma^2 - \rho)g(\phi X, \phi Y) = 0, \quad (2.103)$$

for all vector fields X, Y on M, which implies that

$$\frac{4\lambda}{2\alpha+n\beta}(\gamma^2 - \rho) = 0. \quad (2.104)$$

Using (2.55) and (2.67) in (2.104), we get

$$\frac{8\lambda^2}{(n-1)(2\alpha+n\beta)^2}=0. \tag{2.105}$$

This implies that $\lambda=0$, then using (2.67), $\tau=0$. From (2.55), $\tau=0$ implies $(\gamma^2-\rho)=0$ provided $n>1$. Again, in view of (2.53), we have $R(X,Y)Z=0$ for all vector fields X,Y,Z on M. Hence, we can state the following theorem:

Theorem 2.12. *If an $(LCS)_n$ $(n>1)$ manifold admitting a Ricci-Yamabe soliton satisfies $Ric(\xi,X)\cdot R=0$, then the manifold becomes flat and the soliton is steady.*

2.2.3 Examples of $(LCS)_3$ and $(LCS)_5-$ manifold satisfying RYS

In this subsection, we construct examples for the 3-dimensional and 5-dimensional $(LCS)_n$-manifold in which we verify our results.

Example 2.3. *Consider the $3-$dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3, z\neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$.*

Let E_1,E_2,E_3 be a linearly independent system of vector fields on M given by

$$E_1=y\frac{\partial}{\partial x},\quad E_2=y\frac{\partial}{\partial y},\quad E_3=y\frac{\partial}{\partial z}.$$

Let g be the Lorentzian metric defined by

$$g(E_1,E_1)=g(E_2,E_2)=1, g(E_3,E_3)-1,$$
$$g(E_i,E_j)=0\quad \forall i\neq j; i,j=1,2,3.$$

Let η be the $1-$form defined by $\eta(Z)=g(Z,e_2)$ and ϕ be the $(1,1)-$tensor field defined by

$$\phi E_1=E_1,\quad \phi E_2=0,\quad \phi E_3=E_3.$$

Then, using the linearity of ϕ and g, we have

$$\eta(E_2) = -1, \quad \phi^2(Z) = Z + \eta(Z)E_2,$$
$$\text{and} \quad g(\phi Z, \phi V) = g(Z, V) + \eta(Z)\eta(V),$$

for all $Z, V \in \chi(M)$. Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then, we have

$$[E_1, E_2] = -E_1, \quad [E_2, E_3] = E_3, \quad [E_1, E_3] = 0.$$

Using Koszuls formula and taking $\xi = E_2$, we can easily calculate

$$\nabla_{E_1} E_3 = 0, \quad \nabla_{E_2} E_3 = 0, \quad \nabla_{E_3} E_3 = \tfrac{1}{2}(1 + 2E_2),$$
$$\nabla_{E_1} E_1 = \tfrac{1}{2}(1 + 2E_2), \quad \nabla_{E_2} E_1 = 0, \quad \nabla_{E_3} E_1 = 0,$$
$$\nabla_{E_1} E_2 = -E_1, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_3} E_2 = -E_3.$$

Hence, in this case, the data $(g, \xi, \eta, \phi, \gamma)$ is an $(LCS)_3-$structure on M, where $\gamma = -1$. Also, as $\gamma = -1$, then $\rho = 0$ and consequently $(M, g, \xi, \eta, \phi, \gamma)$ is an $(LCS)_3-$manifold.

Now, from (2.55), we have $\tau = 6$. Let us consider g defines a RYS on M. Putting the value of τ and γ in (2.63), we have

$$\left(2\lambda - 6\beta + 2 - \frac{4\alpha\lambda}{2\alpha + 3\beta}\right) g(X, Y) + 2\eta(X)\eta(Y) = 0.$$

Taking $X = Y = \xi$, we get $\lambda = 2\alpha + 3\beta$ which satisfies (2.67). Again, from (2.68), we have $Ric(X, Y) = 2g(X, Y)$ and therefore, this example verifies Theorem 2.8 in 3-dimension.

Example 2.4. *Consider the* $5-$*dimensional manifold* $M = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_5 \neq 0\}$*, where* $(x_1, x_2, x_3, x_4, x_5)$ *are the standard coordinates in* $\mathbb{R}^5$*.*

Let E_1, E_2, E_3, E_4, E_5 be a linearly independent global frame on M given by

$$E_1 = x_5\left(\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right), \quad E_2 = x_5\frac{\partial}{\partial x_2}, \quad E_3 = x_5\left(\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\right),$$
$$E_4 = x_5\frac{\partial}{\partial x_4}, \quad E_5 = (x_5)^4\frac{\partial}{\partial x_5}.$$

Let us define ϕ, ξ, η, g by

$$\phi E_1 = E_1, \quad \phi E_2 = E_2, \quad \phi E_3 = E_3, \quad \phi E_4 = E_4, \quad \phi E_5 = 0, \quad \xi = E_5,$$
$$\eta(X) = g(X, E_5) \quad \text{for any} \quad X \in \chi(M), \quad g(E_i, E_i) = 1 \quad \forall i = 1,2,3,4,$$
$$\text{and} \quad g(E_5, E_5) = -1, \quad g(E_i, E_j) = 0, \quad \forall i \neq j, i, j = 1,2,3,4,5.$$

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g, then we have

$$[E_1, E_5] = -(x_5)^3 E_1, \quad [E_2, E_5] = -(x_5)^3 E_2, \quad [E_3, E_5] = -(x_5)^3 E_3,$$
$$[E_4, E_5] = -(x_5)^3 E_4, \quad [E_1, E_2] = -(x_5) E_2,$$

and all other remaining $[E_i, E_j]$ vanishes. Taking $\xi = E_5$ and using Koszul formula for the Lorentzian metric g, we obtain

$$\nabla_{E_1} E_5 = -(x_5)^3 E_1, \quad \nabla_{E_2} E_5 = -(x_5)^3 E_2, \quad \nabla_{E_3} E_5 = -(x_5)^3 E_3,$$
$$\nabla_{E_4} E_5 = -(x_5)^3 E_4, \quad \nabla_{E_2} E_1 = (x_5) E_2,$$
$$\nabla_{E_1} E_1 = \nabla_{E_3} E_3 = \nabla_{E_4} E_4 = -\tfrac{1}{2}(1 - 2(x_5)^3 E_5),$$
$$\nabla_{E_2} E_2 = -\tfrac{1}{2}(1 - 2(x_5)^3 E_5 + 2(x_5) E_1).$$

Hence, $(g, \phi, \xi, \eta, \gamma)$ is an $(LCS)_5-$structure on M. Consequently, $M^5(g, \xi, \eta, \phi, \gamma)$ is an $(LCS)_5-$manifold with $\gamma = -(x_5)^3 \neq 0$, where $\rho = 3(x_5)^6$.
Now, from (2.55), we get $\tau = -40(x_5)^6$. Assume g defines a RYS on M and from (2.63), we obtain

$$\left(2\lambda + 40(x_5)^6\beta + 2(x_5)^3 - \frac{4\alpha\lambda}{2\alpha + 5\beta}\right) g(X, Y) + 2(x_5)^3\eta(X)\eta(Y) = 0.$$

Setting $X = Y = \xi$, we get $\lambda = -4(x_5)^6(2\alpha + 5\beta)$ which satisfies (2.67). Again, from (2.68), we have $Ric(X,Y) = -8(x_5)^6 g(X,Y)$ and thus confirms Theorem 2.8 in 5-dimension.

2.2.4 Conformal Ricci-Yamabe soliton on $(LCS)_n$-manifolds

Here, we obtain the expression for the scalar λ on an $(LCS)_n$-manifold admitting a conformal Ricci-Yamabe soliton, where the notion of the soliton was introduced by Zhang et al. (2022) while studying a perfect fluid spacetime. The soliton is given by

$$\mathcal{L}_V g + 2\alpha Ric = \left[2\lambda - \beta\tau - \left(p + \frac{2}{n}\right)\right] g, \tag{2.106}$$

where p is a conformal pressure.

Taking $V = \xi$ in (2.106), we get

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2\alpha Ric(X,Y) = \left[2\lambda - \beta\tau - \left(p + \frac{2}{n}\right)\right] g(X,Y), \tag{2.107}$$

for all X, Y. Now, using (2.47), (2.48) and (2.50) in the above equation and on simplification, we obtain

$$Ric(X,Y) = \frac{1}{\alpha}\left[\lambda - \frac{\beta\tau}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \gamma\right] g(X,Y) - \frac{\gamma}{\alpha}\eta(X)\eta(Y), \alpha \neq 0. \tag{2.108}$$

Now, contracting (2.108), we have

$$\lambda = \frac{p}{2} + \frac{(2\alpha + n\beta)}{2n}\tau + \frac{(n-1)\gamma + 1}{n}. \tag{2.109}$$

Thus, we can state the following:

Theorem 2.13. *If an $(LCS)_n$-manifold admits a conformal Ricci-Yamabe soliton, then the manifold becomes η-Einstein and the scalar λ is given by $\lambda = \frac{p}{2} + \frac{(2\alpha+n\beta)}{2n}\tau + \frac{(n-1)\gamma+1}{n}$ provided $\alpha \neq 0$.*

2.2.5 $\eta-$Ricci-Yamabe Soliton ($\eta-$RYS) on a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold

Here, we study a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold which admits $\eta-$Ricci-Yamabe soliton.

Lemma 2.5. *If a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold admits a $\eta-RYS$, then*

$$\alpha(a-b) + \lambda - \frac{\beta\tau}{2} - \mu = 0.$$

Proof. From (1.72), we have

$$g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) + 2\alpha Ric(X,Y) + (2\lambda - \beta\tau)g(X,Y) \qquad (2.110)$$
$$+2\mu\eta(X)\eta(Y) = 0.$$

Substituting (2.58) in the foregoing equation, we get

$$g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) \qquad (2.111)$$
$$+2\left[\left(\lambda - \tfrac{\beta\tau}{2} + a\alpha\right)g(X,Y) + (b\alpha + \mu)\eta(X)\eta(Y)\right] = 0.$$

Putting $X = Y = \xi$ in (2.111), we get

$$g(\nabla_\xi\xi, \xi) = \alpha(a-b) + \lambda - \frac{\beta\tau}{2} - \mu. \qquad (2.112)$$

Using (2.52), $g(\nabla_\xi\xi, \xi) = 0$. Then,

$$\alpha(a-b) + \lambda - \frac{\beta\tau}{2} - \mu = 0. \qquad (2.113)$$

Hence, we get the result. □

Remark 2.3. *For a particular case such that $\alpha = 1$ and $\beta = 0$ in the above Lemma 2.5, the relation becomes $a - b + \lambda - \mu = 0$. This result is obtained by Hui and Chakraborty (2016).*

Theorem 2.14. *If ξ is a torse forming vector field on a conformally flat $(LCS)_n-$manifold admitting $\eta-$RYS, then $\lambda = \frac{\beta\tau}{2} - a\alpha - 1$, η is closed and*

$$b = a - (n-1) \quad and \quad \mu = \alpha(n-1) + \lambda - \frac{\beta\tau}{2}.$$

Proof. Let ξ be a torse forming vector field on a conformally flat $(LCS)_n-$manifold which admits $\eta-$RYS. Then, taking inner product with ξ in (2.61), we get

$$f\eta(X) = \nu(X). \tag{2.114}$$

In view of the above relation, (2.61) becomes

$$\nabla_X \xi = f[X + \eta(X)\xi]. \tag{2.115}$$

Using (2.115) in (2.111) and in view of Lemma 2.5, we get

$$\left(f + \lambda - \frac{\beta\tau}{2} + a\alpha\right)[g(X,Y) + \eta(X)\eta(Y)] = 0, \tag{2.116}$$

for all vector fields X and Y and hence it follows that

$$f = -\left(\lambda - \frac{\beta\tau}{2} + a\alpha\right). \tag{2.117}$$

Using the fact that $f = 1$ from Remark 2.2 in (2.117), we get

$$\lambda = \frac{\beta\tau}{2} - a\alpha - 1. \tag{2.118}$$

Now, using (2.117) in (2.115), we get

$$\nabla_X \xi = -\left(\lambda - \frac{\beta\tau}{2} + a\alpha\right)[X + \eta(X)\xi], \tag{2.119}$$

which means that $\nabla_X \xi$ is collinear to $\phi^2 X$ for all X and hence we get $d\eta = 0$, i.e., η is closed. From (1.19), we know that

$$R(X,Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi. \tag{2.120}$$

In view of (2.119), (2.120) yields

$$R(X,Y)\xi = \left(\lambda - \frac{\beta\tau}{2} + a\alpha\right)^2 [\eta(Y)X - \eta(X)Y]. \tag{2.121}$$

Again, in view of (2.53), (2.54), (2.118) and (2.121), we get

$$Ric(X,\xi) = (n-1)\eta(X). \tag{2.122}$$

Comparing (2.122) with (2.59), we obtain

$$b = a - (n-1), \tag{2.123}$$

$$\text{and} \quad \mu = \alpha(n-1) + \lambda - \frac{\beta\tau}{2}. \tag{2.124}$$

Hence, we get the theorem. □

The following result follows immediately from Theorem 2.14

Corollary 2.6. *If ξ is a torse forming vector field on a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold admitting $\eta-RYS$, then the soliton is expanding, steady and shrinking according as $\beta\tau < 2(a\alpha+1)$, $\beta\tau = 2(a\alpha+1)$ and $\beta\tau > 2(a\alpha+1)$ respectively.*

Again, as a consequence of Theorem 2.14, in particular, if $\mu = 0$, then we obtain $\lambda = \alpha(b-a) + \frac{\beta\tau}{2}$ which results in the following corollary:

Corollary 2.7. *If ξ is a torse forming vector field on a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold admitting Ricci-Yamabe soliton with $\alpha \neq 0$, then the soliton is shrinking, steady and expanding according as $\beta\tau > 2\alpha(a-b)$, $\beta\tau = 2\alpha(a-b)$, and $\beta\tau < 2\alpha(a-b)$ respectively.*

2.3 Characterization of almost $*$-Ricci-Yamabe solitons

Recently, Dwivedi and Patra (2022) introduced the notion of almost $*$-Ricci-Bourguinon soliton and studied its geometric characterization on Sasakian manifold. One of the most interesting geometric property for a soliton which represents a metric of a manifold is its isometry, may it be spheres or hyperbolic space. Obata (1962) has shown that "In order for a complete Riemannian manifold of dimension $n \geq 2$ to admit a non-constant function ϕ with $\nabla_X d\phi = -c^2\phi X$ for any vector X, it is necessary and sufficient that the manifold be isometric with a sphere $\mathbb{S}(c)$ of radius $1/c$ in the $(n+1)$-Euclidean space. " Deshmukh (2019) obtained certain conditions and bounds for an almost Ricci soliton to be isometric to spheres. Many other geometers also obtained conditions under which the almost Ricci-Bourguinon soliton, almost $*$-Ricci-Bourguinon soliton, almost Ricci-Yamabe soliton are isometric to spheres (see for further details: Ghosh and Patra, 2018; Dwivedi, 2021; Dwivedi and Patra, 2022). Inspired from these mentioned works, we pondered if we assume a complete Sasakian manifold and define its metric by gradient almost $*$-Ricci-Yamabe soliton and almost $*$-Ricci-Yamabe soliton, would it still be isometric to the unit sphere? And if it does, then what are the conditions it need to satisfy?

To answer the questions that we asked, we introduce the notion of almost $*-$Ricci-Yamabe soliton as

$$\mathcal{L}_U g + 2\alpha Ric^* = (2\lambda - \beta\tau^*)g, \tag{2.125}$$

where λ is a smooth function, $\alpha, \beta \in \mathbb{R}$, Ric^* is the Ricci curvature tensor and τ^*, the $*-$scalar curvature. If $U = \nabla f$, where ∇ denotes the gradient in (2.125),

Z. Chhakchhuak and J.P. Singh (2024). Characterization of almost $*$-Ricci-Yamabe solitons isometric to a unit sphere, *Novi Sad J. Math.* https://doi.org/10.30755/NSJOM.15574 then (2.125) reduces to gradient almost $*-$Ricci-Yamabe soliton as

$$\nabla^2 f + \alpha Ric^* = (\lambda - \frac{\beta\tau^*}{2})g, \tag{2.126}$$

where $\nabla^2 f = Hess\ f$ is the Hessian of a smooth function f.

2.3.1 Preliminaries

In this subsection, we give some basic results which will be useful for proving our results.

A contact manifold is a $(2n+1)$ dimensional which admits a contact 1-form η satisfying $\eta \wedge (d\eta)^n \neq 0$. Therefore, for a contact structure, there exists a characteristic vector field ξ satisfying $d\eta(\xi, \cdot) = 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact sub-bundle defined by $\eta = 0$ by $\mathfrak{D}$ gives a $(1,1)-$tensor field ϕ and a Riemannian metric g such that the following relations hold:

$$\phi^2 X = -X + \eta(X)\xi,\ \eta = g(X_1, \xi), \tag{2.127}$$

$$d\eta(X,Y) = g(X, \phi Y Y. \tag{2.128}$$

This structure is called a contact metric structure and the manifold equipped with such structure is called a contact metric manifold M of dimension $(2n+1)$ with an associated metric g. We know that from the foregoing equations

$$\eta \circ \phi = 0,\ \phi(\xi) = 0,\ \text{and } rank(\phi) = 2n. \tag{2.129}$$

The Riemannian curvature tensor R of g is given by the formula

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \quad \text{for all } X, Y, Z \in \chi(M), \tag{2.130}$$

where ∇ and $\chi(M)$ are the Levi-Civita connection of g and the Lie algebra of vector fields on the manifold respectively. The Ricci operator Q is a $(1,1)-$tensor field defined by

$$g(QX, Y) = Ric(X, Y), \quad X, Y \in \chi(M),$$

and the scalar curvature g and the gradient of the scalar curvature τ are respectively the smooth function defined by $\tau = tr\ Q$ and

$$\frac{1}{2}g(X, \nabla\tau) = (div\ Q)(X),\ X \in \chi(M). \tag{2.131}$$

We call such manifold a Sasakian manifold if any of the following three equivalent conditions hold (Blair, 2002; Boyer and Galicki, 2007):

1. The metric cone $(C(M), \bar{g}) = (\mathbb{R}_+ \times M, dr^2 \oplus \tau^2 g)$ is Kähler.

2. The Riemann curvature tensor R of g satisfies the identity

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,\ X, Y \in \chi(M). \tag{2.132}$$

3. The structural tensor field ϕ satisfies the identity

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,\ X, Y \in \chi(M). \tag{2.133}$$

We know that M is a $K-$contact manifold if ξ is Killing. Also, A Sasakian manifold is a $K-$ contact manifold but a $K-$contact manifold is not Sasakian for $dim \neq 3$. The following relations are valid on a Sasakian manifold (Blair, 2002):

$$\nabla_X \xi = -\phi X,\ \nabla_\xi \xi = 0,\ QX = 2nX,\ X \in \chi(M). \tag{2.134}$$

Further, setting $X = \xi$ on the last term of the above equation and then taking covariant derivative along $X \in \chi(M)$, we get

$$(\nabla_X Q)\xi = Q\phi X - 2n\phi X. \tag{2.135}$$

A contact metric manifold is said to be $\eta-$Einstein if

$$Ric(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),\ X, Y \in \chi(M), \tag{2.136}$$

where a, b are smooth functions on M. However, by Okumura (Okumura, 1962), if the manifold is a $K-$contact manifold with $dim > 3$, then a, b becomes constants.

From the geometric point of view, preserving the structure upon transformation is important. Therefore, such preservation of a $K-$contact and Sasakian structures can be obtained by a $D-$homothetic deformation which is described as

$$\bar{\eta} = \nu\eta,\ \bar{\xi} = \frac{1}{\nu}\xi,\ \bar{\phi} = \phi,\ \bar{g} = \nu g + \nu(\nu-1)\eta\otimes\eta, \tag{2.137}$$

where $\nu \in \mathbb{R}_+$. Now, we recall the following for later use:

Definition 2.2 (Ghosh and Patra, 2018)**.** *A K-contact η-Einstein manifold with $a = -2$ is D-homothetically fixed.*

Following Tanno (Tanno, 1963), we define an infinitesimal contact transformation as

Definition 2.3. *A potential vector field U is infinitesimal contact transformation on an almost contact metric manifold if $\mathcal{L}_U\eta = \psi\eta$ for some function ψ. In particular, if $\mathcal{L}_U\eta = 0$, then U is said to be strict infinitesimal contact transformation. Moreover, U is called an infinitesimal automorphism if it leaves all the structure tensors invariant.*

Lemma 2.6 (Ghosh and Sharma, 2021)**.** *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Sasakian manifold and $\{E_i\}_{1\leq i\leq 2n+1}$ be a local orthonormal frame on M. Then, for $Y \in \chi(M)$, we have*

$$\sum_i g((\nabla_{\phi Y}Q)\phi E_i, E_i) = 0,\ \sum_i g((\nabla_{\phi E_i}Q)\phi Y, E_i) = \frac{1}{2}X(\tau).$$

Lemma 2.7 (Dwivedi and Patra, 2022)**.** *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Sasakian manifold and f be a smooth function on M. If $\{E_i\}_{1\leq i\leq 2n+1}$ is a local orthonormal* *frame on* M*, then for* $Y \in \chi(M)$*, we have*

$$\sum_i g(Y, \nabla_{\phi E_i} \nabla f) g(\xi, E_i) = 0,$$

$$\sum_i g(\xi, \nabla_{E_i} \nabla f) g(\phi Y, E_i) = g(\phi Y, \nabla_\xi \nabla f),$$

$$\sum_i g(\phi Y, \nabla_{E_i} \nabla f) g(\xi, E_i) = g(\xi, \nabla_{\phi Y} \nabla f),$$

$$\sum_i g(\xi, R(E_i, Y) \nabla f) g(\xi, E_i) = Y(f) - \eta(\nabla f) \eta(Y).$$

Also, we recall the expression of $*-$Ricci tensor on a Sasakian manifold by the lemma:

Lemma 2.8 (Ghosh and Patra, 2018)**.** *The expression of* $*-$*Ricci tensor* Ric^* *on a Sasakian manifold* $M^{2n+1}(\phi, \xi, \eta, g)$ *is*

$$Ric^*(X, Y) = Ric(X, Y) - (2n-1) g(X, Y) - \eta(X)\eta(Y), \;\; X, Y \in \chi(M). \quad (2.138)$$

As a direct consequence of the above lemma, we have the following:

Corollary 2.8. *The* $*$*-scalar curvature* τ^* *on a Sasakian manifold* M^{2n+1} (ϕ, ξ, η, g) *is given by* $\tau^* = \tau - 4n^2$.

2.3.2 Main Results

In this subsection, we will prove our main results. All the vector fields we considered here are on the Sasakian manifold M.

First of all, let us make use of (2.138) in (2.125), we get

$$\begin{aligned}(\mathcal{L}_U g)(X, Y) + 2\alpha Ric(X, Y) =& (2\lambda - \beta\tau^* + 2\alpha(2n-1)) g(X, Y) \\ &+ 2\alpha\eta(X)\eta(Y). \end{aligned} \quad (2.139)$$

Taking Lie derivative of $R(X, \xi)\xi = X - \eta(X)\xi$ along U gives

$$(\mathcal{L}_U R)(X, \xi)\xi + R(X, \xi)\mathcal{L}_U \xi + g(X, \mathcal{L}_U \xi)\xi + (\mathcal{L}_U g)(X, \xi)\xi + \eta(\mathcal{L}_U \xi) X = 0. \quad (2.140)$$

Using the fact that $Q\xi = 2n\xi$ and (2.139), we get

$$(\mathcal{L}_U g)(X, \xi) = (2\lambda - \beta\tau^*)\eta(X). \tag{2.141}$$

Again, taking Lie derivative of $\eta(X) = g(X, \xi)$ and $g(\xi, \xi) = 1$ along U respectively results in

$$(\mathcal{L}_U \eta)X - g(X, \mathcal{L}_U \xi) = (2\lambda - \beta\tau^*)\eta(X), \tag{2.142}$$

and

$$\eta(\mathcal{L}_U \xi) = -\frac{1}{2}(\mathcal{L}_U g)(\xi, \xi). \tag{2.143}$$

From (2.141), we get

$$\eta(\mathcal{L}_U \xi) = -\left(\lambda - \frac{\beta\tau^*}{2}\right). \tag{2.144}$$

Again, from (2.142), we obtain

$$(\mathcal{L}_U \eta)\xi = \left(\lambda - \frac{\beta\tau^*}{2}\right). \tag{2.145}$$

Making use of these foregoing equations in (2.140) results in the following lemma.

Lemma 2.9. *If a Sasakian metric g represents an almost $*-RYS$ with $\alpha \neq 0$, then the relation*

$$(\mathcal{L}_U R)(X, \xi)\xi = (2\Lambda - \beta\tau^*)(X - \eta(X)\xi),\ X \in \chi(M),$$

holds.

Now, using Lemma 2.8, we can write (2.126) as

$$\nabla_X \nabla f + \alpha Q X = \left(\lambda - \frac{\beta\tau^*}{2} + \alpha(2n-1)\right) X + \alpha\eta(X)\xi. \tag{2.146}$$

Applying covariant derivative in

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.147}$$

to obtain the curvature tensor expression

$$\begin{aligned}R(X,Z)\nabla f &= \nabla_X\nabla_Z\nabla f - \nabla_Z\nabla_X\nabla f - \nabla_{[X,Z]}\nabla f\\&= X\left(\lambda - \frac{\beta\tau^*}{2}\right)Z - \alpha\eta(Z)\phi X + 2\alpha g(X,\phi Z)\xi\\&- Z\left(\lambda - \frac{\beta\tau^*}{2}\right)X - \alpha(\nabla_X Q)Z + \alpha(\nabla_Z Q)X + \alpha\eta(X)\phi Z.\end{aligned}$$

Taking inner product with respect to the vector field Y in the above equation, we obtain

$$\begin{aligned}g(R(X,Z)\nabla f, Y) = X\left(\lambda - \frac{\beta\tau^*}{2}\right)g(Y,Z) - \alpha\eta(Z)g(Y,\phi X)\\+ 2\alpha g(X,\phi Z)\eta(Y) - Z\left(\lambda - \frac{\beta\tau^*}{2}\right)g(X,Y)\\- \alpha g((\nabla_X Q)Z, Y) + \alpha g((\nabla_Z Q)X, Y)\\+ \alpha\eta(X)g(Y,\phi Z).\end{aligned} \tag{2.148}$$

Since Q and ϕ commute on a Sasakian manifold, we have (see Lemma 2.1 of (Ghosh and Patra, 2018))

$$\nabla_\xi Q = Q\phi - \phi Q \text{ and } \nabla_\xi Q = 0. \tag{2.149}$$

Also, utilizing the symmetric and anti-symmetric properties of Q and ϕ respectively and thus setting $Y = Z = \xi$ in (2.148) and using (2.135), (2.149) and the fact that $\phi(\xi) = 0$, we obtain

$$X\left(\lambda - \frac{\beta\tau^*}{2} + f\right) = \xi\left(\lambda - \frac{\beta\tau^*}{2} + f\right)\eta(X). \tag{2.150}$$

Putting $\zeta = \lambda - \frac{\beta\tau^*}{2} + f$, (2.150) becomes

$$X(\zeta) = \xi(\zeta)\eta(X). \tag{2.151}$$

Now, taking covariant derivative of (2.151) along $Y \in \chi(M)$ and using (2.134) gives

$$g(\nabla_Y \nabla \zeta, X) = Y(\xi(\zeta))\eta(X) + \xi(\zeta)g(\phi X, Y). \tag{2.152}$$

Ustilizing (2.152) in the symmetric property of $Hess_\zeta$, we get

$$2\xi(\zeta)g(\phi X, Y) = X(\xi(\zeta))\eta(Y) - Y(\xi(\zeta))\eta(X).$$

Hence, we choose $X, Y \perp \xi$ to get $\xi(\zeta) = 0$ on the manifold as $d\eta$ is non-zero on the manifold. This further implies that $\nabla\zeta = 0$ on M. Thus, we conclude that $\zeta = \lambda - \frac{\beta\tau^*}{2} + f$ is constant on M. On the contrary, replacing X by ξ in (2.148) and using (2.149) and (2.149) results in

$$\begin{aligned} g(R(\xi, Z)\nabla f, Y) &= g(Q\phi Z, Y) - (2n-1)g(\phi Z, Y) \\ &\quad + \xi\left(\lambda - \frac{\beta\tau^*}{2}\right)g(Y, Z) - Z\left(\lambda - \frac{\beta\tau^*}{2}\right)\eta(Y). \end{aligned} \tag{2.153}$$

Also, the relation

$$R(\xi, Z)\nabla f = Z(f)\xi - \xi(f)Z, \tag{2.154}$$

follows directly from (2.132). Substituting the above equation in (2.153), we obtain

$$g(Q\phi Z, Y) - (2n-1)g(\phi Z, Y) = Z(\zeta)\eta(Y) - \xi(\zeta)g(Y, Z). \tag{2.155}$$

Since ζ is constant, (2.155) yields $Q\phi Z = (2n-1)\phi Z$. Then, setting $Z = \phi Z$ and using (2.127), (2.134), we obtain

$$Ric(Y, Z) = (2n-1)g(Y, Z) + \eta(Y)\eta(Z), \ Y, Z \in \chi(M). \tag{2.156}$$

Hence, M becomes $\eta-$Einstein manifold. Further, suppose that M is complete, then (2.156) shows that M is compact and positive Sasakian. Again, plugging in (2.156) into (2.146) yields

$$\nabla_X \nabla f = \left(\lambda - \frac{\beta\tau^*}{2}\right)X, \ X \in \chi(M). \tag{2.157}$$

As $\zeta = \left(\lambda - \frac{\beta\tau^*}{2} + f\right)$ is a constant, the foregoing expression can be written as

$$\nabla_X D\gamma = -\gamma X, \tag{2.158}$$

where $\gamma = \lambda - \frac{\beta\tau^*}{2}$ and λ is a non-constant smooth function on the manifold. Therefore, by invoking Obata's theorem (Obata, 1962) to our results, we conclude with the following theorem.

Theorem 2.15. *A complete Sasakian manifold admitting a gradient almost $*$-Ricci-Yamabe soliton as its metric is $*$-Ricci flat, compact positive-Sasakian and isometric to the unit sphere $\mathbb{S}^{2n+1}$ provided $\alpha \neq 0$.*

By looking at the above theorem, a natural question arise as to whether the soliton would still behave as such without considering the gradient vector field of a smooth function f or not. Before giving answer to this logical assumption, let us deduce some propositions which we will use later in the proofs.

Proposition 2.7. *For a Sasakian metric g admitting almost $*-RYS$ with $\alpha \neq 0$, the following formula holds:*

$$(\mathcal{L}_U\nabla)(X,\xi) = 2\alpha(2n-1)\phi X - 2\alpha\phi QX + X\left(\lambda - \frac{\beta\tau^*}{2}\right)\xi$$
$$+\xi\left(\lambda - \frac{\beta\tau^*}{2}\right)X - \eta(X)\nabla\left(\lambda - \frac{\beta\tau^*}{2}\right).$$

Proof. Taking covariant derivative of (2.139) along an arbitrary $Z \in \chi(M)$ and using (2.134), we get

$$(\nabla_Z\mathcal{L}_Ug)(X,Y) + 2\alpha(\nabla_Z Ric)(X,Y) = Z(2\lambda - \beta\tau^*)g(X,Y)$$
$$- 2\alpha[\eta(X)g(Y,\phi Z) + \eta(Y)g(X,\phi Z)]. \tag{2.159}$$

Now, we recall the formula given by Yano (1970)

$$(\mathcal{L}_U\nabla_Z g - \nabla_Z\mathcal{L}_U g - \nabla_{[U,Z]}g)(X,Y) = -g((\mathcal{L}_U\nabla)(Z,X),Y)$$
$$- g((\mathcal{L}_U\nabla)(Z,Y),X), \tag{2.160}$$

for all $X, Y, Z \in \chi(M)$. Since g is parallel, inserting (2.159) into (2.160), we obtain

$$\begin{aligned} g((\mathcal{L}_U\nabla)(Z,X),Y) + g((\mathcal{L}_U\nabla)(Z,Y),X) + 2\alpha(\nabla_Z Ric)(X,Y) \\ = 2Z\left(\lambda - \frac{\beta\tau^*}{2}\right) g(X,Y) - 2\alpha[\eta(X)g(Y,\phi Z) \\ + \eta(Y)g(X,\phi Z)]. \end{aligned} \tag{2.161}$$

Interchanging cyclically X, Y, Z in the foregoing equation and using the symmetry $(\mathcal{L}_U\nabla)(X,Y) = (\mathcal{L}_U\nabla)(Y,X)$ results in

$$\begin{aligned} g((\mathcal{L}_U\nabla)(X,Y),Z) &= \alpha[(\nabla_Z Ric)(X,Y) - (\nabla_X Ric)(Y,Z) \\ &- (\nabla_Y Ric)(Z,X)] + X\left(\lambda - \frac{\beta\tau^*}{2}\right) g(Y,Z) + Y\left(\lambda - \frac{\beta\tau^*}{2}\right) g(X,Z) \\ &- Z\left(\lambda - \frac{\beta\tau^*}{2}\right) g(X,Y) - 2\alpha[\eta(X)g(\phi Y,Z) + \eta(Y)g(\phi X,Z)]. \end{aligned} \tag{2.162}$$

Finally, setting $Y = \xi$ in (2.162) and utilizing (2.135) and (2.149), (2.149) completes the proof. □

Proposition 2.8. *For a Sasakian metric g admitting almost $*-RYS$, the formula*

$$\begin{aligned} g\left(\xi, \nabla_X\nabla\left(\lambda - \frac{\beta\tau}{2}\right)\right)\xi - \eta(X)\nabla_\xi\nabla\left(\lambda - \frac{\beta\tau}{2}\right) &= 4\alpha QX \\ &- 2\left(\lambda - \frac{\beta\tau^*}{2} + 2\alpha(2n-1)\right)X \\ &+ 2\left(\lambda - \frac{\beta\tau^*}{2} - 2\alpha\right)\eta(X)\xi + 2(\phi X)\left(\lambda - \frac{\beta\tau}{2}\right)\xi \\ &- \nabla_X\nabla\left(\lambda - \frac{\beta\tau}{2}\right), \end{aligned}$$

is valid provided ξ leaves λ invariant and $\alpha \neq 0$.

Proof. We know that ξ is Killing on a Sasakian manifold, this implies $\mathcal{L}_\xi Ric = 0$, from which we obtain $\xi(\tau) = 0$. Also, from Corollary 2.8, we have $\tau^* = \tau - 4n^2$. This further implies $\xi(\tau^*) = 0$ and $X\left(\lambda - \frac{\beta\tau^*}{2}\right) = X\left(\lambda - \frac{\beta\tau}{2}\right)$ due to the fact that $\xi(\lambda) = 0$. Thus, Proposition 2.7 reduces to

$$(\mathcal{L}_U \nabla)(X, \xi) = 2\alpha(2n-1)\phi X - 2\alpha\phi Q X + X\left(\lambda - \frac{\beta\tau}{2}\right)\xi - \eta(X)\nabla\left(\lambda - \frac{\beta\tau}{2}\right), \tag{2.163}$$

where $X \in \chi(M)$. Setting $X = \xi$ in (2.163) and using (2.156) results in

$$(\mathcal{L}_U \nabla)(\xi, \xi) = -\nabla\left(\lambda - \frac{\beta\tau}{2}\right). \tag{2.164}$$

Taking the covariant derivative along X in (2.164) and using (2.134) yields

$$(\nabla_X \mathcal{L}_U \nabla)(\xi, \xi) = 4\alpha Q X - 4\alpha(2n-1)X - 4\alpha\eta(X)\xi + 2(\phi X)\left(\lambda - \frac{\beta\tau}{2}\right)\xi - \nabla_X\nabla\left(\lambda - \frac{\beta\tau}{2}\right). \tag{2.165}$$

On the other hand, differentiating (2.163) along ξ gives

$$(\nabla_\xi \mathcal{L}_U \nabla)(X, \xi) = g\left(X, \nabla_\xi\nabla\left(\lambda - \frac{\beta\tau}{2}\right)\right)\xi - \eta(X)\nabla_\xi\nabla\left(\lambda - \frac{\beta\tau}{2}\right), \tag{2.166}$$

where we used $\nabla_\xi Q = \nabla_\xi \xi = \nabla_\xi \phi = 0$. Now, from the commutation formula by Yano (Yano, 1970):

$$(\mathcal{L}_U R)(X, Y)Z = (\nabla_X \mathcal{L}_U \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_U \nabla)(X, Z). \tag{2.167}$$

Finally, setting $Y = Z = \xi$ in the above expression and applying Lemma 2.9, (2.165) and (2.166) thus completes the proof. □

Proposition 2.9. *For a Sasakian metric g representing almost $*-$RYS, the formula*

$$Ric\left(Y, \nabla\left(\lambda - \frac{\beta\tau}{2}\right)\right) = (4n-1)Y\left(\lambda - \frac{\beta\tau}{2}\right) + 4\alpha g\left(\phi Y, \nabla_\xi\nabla\left(\lambda - \frac{\beta\tau}{2}\right)\right) + \eta(Y)div\left(\nabla_\xi\nabla\left(\lambda - \frac{\beta\tau}{2}\right)\right) - g\left(\xi, \nabla_Y\nabla_\xi\nabla\left(\lambda - \frac{\beta\tau}{2}\right)\right) - 2\alpha Y(\tau),$$

holds true provided $\xi(\lambda) = 0$ *and* $\alpha \neq 0$.

Proof. By hypothesis, $\xi(\lambda) = 0$ and since $\xi(\tau) = 0$, this implies that $\xi\left(\lambda - \frac{\beta\tau}{2}\right) = 0$. Therefore,

$$(\phi X)\left(\lambda - \frac{\beta\tau}{2}\right) = g\left(\xi, \nabla_X \nabla\left(\lambda - \frac{\beta\tau}{2}\right)\right).$$

Thus, Proposition 2.8 reduces to

$$\begin{aligned}\nabla_X \nabla\left(\lambda - \frac{\beta\tau}{2}\right) &= 4\alpha QX - 2\left(\lambda - \frac{\beta\tau^*}{2} + 2\alpha(2n-1)\right)X \\ &+ 2\left(\lambda - \frac{\beta\tau^*}{2} - 2\alpha\right)\eta(X)\xi + g\left(\xi, \nabla_X \nabla\left(\lambda - \frac{\beta\tau}{2}\right)\right)\xi \\ &+ \eta(X)\nabla_\xi \nabla\left(\lambda - \frac{\beta\tau}{2}\right). \end{aligned} \tag{2.168}$$

Taking covariant derivative of (2.168) along Y, we get

$$\begin{aligned}\nabla_Y \nabla_X \nabla\left(\lambda - \frac{\beta\tau}{2}\right) &= g\left(\xi, \nabla_Y \nabla_X \nabla\left(\lambda - \frac{\beta\tau}{2}\right)\right)\xi \\ &- g\left(\phi Y, \nabla_X \nabla\left(\lambda - \frac{\beta\tau}{2}\right)\right)\xi \\ &- g\left(\xi, \nabla_X \nabla\left(\lambda - \frac{\beta\tau}{2}\right)\right)\phi Y \\ &+ \eta(X)\nabla_Y \nabla_\xi \nabla\left(\lambda - \frac{\beta\tau}{2}\right) - g(X, \phi Y)\nabla_\xi \nabla\left(\lambda - \frac{\beta\tau}{2}\right) \\ &+ 4\alpha(\nabla_Y Q)X - 2Y\left(\lambda - \frac{\beta\tau^*}{2}\right)X \\ &2Y\left(\lambda - \frac{\beta\tau^*}{2}\right)\eta(X)\xi - 2\left(\lambda - \frac{\beta\tau^*}{2} - 2\alpha\right)g(X, \phi Y)\xi \\ &- 2\left(\lambda - \frac{\beta\tau^*}{2} - 2\alpha\right)\eta(X)\phi Y. \end{aligned}$$

Utilizing the symmetry of $Hess$, anti-symmetric property of ϕ, (2.168) and the foregoing equation in (2.130), we get

$$\begin{aligned}R(X, Y)\nabla\left(\lambda - \frac{\beta\tau}{2}\right) &= \nabla_X \nabla_Y \nabla\left(\lambda - \frac{\beta\tau}{2}\right) - \nabla_Y \nabla_X \nabla\left(\lambda - \frac{\beta\tau}{2}\right) \\ &- \nabla_{[X,Y]} \nabla\left(\lambda - \frac{\beta\tau}{2}\right). \end{aligned}$$

Now, contracting the obtained results over X after a few steps of calculation of the foregoing expression and making use of the fact that $Tr\phi = 0 = \phi(\xi)$ Lemma 2.7, the required result is obtained. □

Now, we recall the commutation formula (Yano, 1970)

$$\mathcal{L}_Y \mathcal{L}_X g - \mathcal{L}_X \mathcal{L}_Y g = \mathcal{L}_{[Y,X]} g, \tag{2.169}$$

where $X, Y \in \chi(M)$. Since ξ is Killing, we have $\mathcal{L}_\xi g = \mathcal{L}_\xi Ric = 0$. Utilizing (2.139) in (2.169), we get

$$\mathcal{L}_{[U,\xi]} g = -2\xi(\lambda) g \tag{2.170}$$

where we used $\xi(\tau^*) = 0$ on M. Thus, the vector field $[U, \xi]$ is conformal and thus results in the following two cases:

Case I: $[U, \xi]$ is homothetic.

Case II: $[U, \xi]$ is non-homothetic.

Proceeding the calculation as in Theorem 1.4 of (Dwivedi and Patra, 2022), we can conclude that Case I cannot happen as it results in a contradiction of λ being a constant regardless of the fact that it is non-constant. Thus, invoking Okumura's theorem in Case II yields that the manifold is isometric to the unit sphere $\mathbb{S}^{2n+1}$. Hence, we can state the following theorem.

Theorem 2.16. *A complete Sasakian manifold admitting almost $*-RYS$ of dim $>$ 3 with $\lambda \neq$ constant is isometric to the unit sphere $\mathbb{S}^{2n+1}$ provided $\alpha \neq 0$.*

Remark 2.4. *For a particular value of $\alpha = 1, \beta = -2\rho$, the above theorem reduces to Theorem 1.4 of (Dwivedi and Patra, 2021).*

Suppose that the vector field U is parallel to the Reeb vector field ξ, this means that $U = \sigma\xi$, where σ is some smooth function on M. Then from (2.134), we get

$$(\mathcal{L}_U g)(X, Y) = X(\sigma)\eta(Y) + Y(\sigma)\eta(X). \tag{2.171}$$

Using the antisymmetry of ϕ, (2.139) implies

$$X(\sigma)\eta(Y) + Y(\sigma)\eta(X) + 2\alpha Ric(X,Y) = [2\lambda - \beta\tau^* + 2\alpha(2n-1)]g(X,Y) + 2\alpha\eta(X)\eta(Y). \tag{2.172}$$

Putting $X = Y = \xi$ in (2.172) and using (2.134), we get $\xi(\sigma) = \lambda - \frac{\beta\tau^*}{2}$. Similarly, setting $Y = \xi$ in (2.172) yields

$$X(\sigma) = \xi(\sigma)\eta(X) \tag{2.173}$$

Taking the covariant differentiation of (2.173) along $Y \in \chi(M)$ and using (2.134), we get

$$g(\nabla_Y \nabla\sigma, X) = Y(\xi(\sigma))\eta(X) + \xi(\sigma)g(\phi X, Y).$$

Using the symmetric property of $Hess_\sigma$, it follows that

$$X(\xi(\sigma))\eta(Y) - Y(\xi(\sigma))\eta(X) = 2\xi(\sigma)g(\phi X, Y),$$

which further gives

$$\xi(\sigma)d\eta(X,Y) = 0, \ \forall \ X, Y \perp \xi.$$

Now, since $d\eta$ is non-zero, we get $\xi(\sigma) = 0$ and hence $\nabla\sigma = 0$. This implies σ is constant and thus U is Killing. Further, $2\lambda - \beta\tau^* = 0$ which leads to the fact that

$$Ric(X,Y) = (2n-1)g(X,Y) + \eta(X)\eta(Y).$$

Hence, M is $*-$Ricci flat and $*-$scalar curvature $\tau^* = 0$. Moreover, $\tau = 4n^2$ and $\lambda = 0$. Therefore, we can conclude the results with the following theorem.

Theorem 2.17. *If U is parallel to the characteristic vector field ξ on a Sasakian manifold M admitting almost $*-RYS$ with $\alpha \neq 0$, then U is Killing and $*-$Ricci flat with constant scalar curvature $4n^2$. Moreover, the soliton is steady for any σ.*

Now, let us consider that U is an infinitesimal contact transformation on M.

Thus, setting $Y = \xi$ in (2.139), we get

$$(\mathcal{L}_U g)(X, \xi) = (2\lambda - \beta\tau^*)\eta(X). \tag{2.174}$$

Then, from $\eta(\xi) = 1$, we obtain by taking Lie derivative along U

$$(\mathcal{L}_U \eta)(\xi) = -\eta(\mathcal{L}_U \xi) = \lambda - \frac{\beta\tau^*}{2},$$

and from Definition 2.3, $\psi = \lambda - \frac{\beta\tau^*}{2}$. In view of this and the Lie derivative of $\eta(X) = g(X, \xi)$ along U gives

$$g(\mathcal{L}_U \xi, X) = -\left(\lambda - \frac{\beta\tau^*}{2}\right)\eta(X). \tag{2.175}$$

Taking exterior derivative of $\mathcal{L}_U \eta = \psi\eta$ yields

$$\begin{aligned}(\mathcal{L}_U d\eta)(X, Y) &= d(\mathcal{L}_U \eta)(X, Y) \\ &= \frac{1}{2}[X(\psi)\eta(Y) - Y(\psi)\eta(X)] + \psi d\eta(X, Y).\end{aligned} \tag{2.176}$$

Now, Lie derivative of (2.128) along U and using Definition 2.3, (2.139) and (2.176) yields

$$\begin{aligned}2(\mathcal{L}_U \phi)(X) + 2[2\lambda - \beta\tau^* + 2\alpha(2n-1)]\phi X &= 4\phi Q X - X(\psi)\xi \\ &\quad + \eta(X)\nabla\psi.\end{aligned} \tag{2.177}$$

Utilizing $\phi\xi = 0$, we have $(\mathcal{L}_U \phi)\xi = 0$ and setting $X = \xi$ in (2.177) results in

$$\nabla\psi = \xi(\psi)\xi.$$

Hence, ψ is constant on M. Thus, utilizing $\psi = \lambda - \frac{\beta\tau^*}{2}$ and (2.127) in (2.177), we obtain

$$(\mathcal{L}_U\phi)(\phi X) = \phi(\mathcal{L}_U\phi)(X) \\ = -2QX + \left(\lambda - \frac{\beta\tau^*}{2} + 2\alpha(2n-1)\right)X \\ - \left(\lambda - \frac{\beta\tau^*}{2} - 2\alpha\right)\eta(X)\xi, \tag{2.178}$$

where we used $Q\phi = \phi Q$. Furthermore, taking Lie derivative of (2.127) yields

$$(\mathcal{L}_U\phi)(\phi X) + \phi(\mathcal{L}_U\phi)(X) = (\mathcal{L}_U\eta)(X)\xi + \eta(X)\mathcal{L}_U\xi. \tag{2.179}$$

Combining Definition 2.3, (2.134), (2.178) and (2.179), we get

$$2\alpha Ric = \left[\lambda - \frac{\beta\tau^*}{2} + 2\alpha(2n-1)\right]g - \left(\lambda - \frac{\beta\tau^*}{2} - 2\alpha\right)\eta\otimes\eta. \tag{2.180}$$

Utilizing (2.180) in (2.177) yields $\mathcal{L}_U\phi = 0$ which implies that U leaves ϕ invariant. Taking Lie derivative of the possible volume form $\omega = \eta\wedge(d\eta)^n \neq 0$ along U yields $\mathcal{L}_U\omega = (n+1)\psi\omega$. Invoking the result $\mathcal{L}_U\omega = (divU)\omega$ implies $divU = (n+1)\psi$ and then integrating it over a compact M where we applied divergence theorem to get $\psi = 0$ and thus $\lambda = \frac{\beta\tau^*}{2}$. Hence, (2.180) becomes

$$Ric = (2n-1)g + \eta\otimes\eta,$$

which gives $\tau = 4n^2$. Thus, M is $*-$Ricci flat and $\tau^* = 0$. Further, V is Killing and $\lambda = 0$. Moreover, from (2.175) and (2.134), $U(\eta) = U(\xi) = 0$ which leads to the following theorem.

Theorem 2.18. *If U is an infinitesimal contact transformation on a Sasakian manifold M admitting almost $*-RYS$ with $\alpha \neq 0$, then M is $*-$Ricci flat and $\tau = 4n^2$. Moreover, U becomes an infinitesimal automorphism and the soliton is steady for any values of β.*

Again, recall the formula

$$\nabla_Y \nabla_X U - \nabla_{\nabla_Y X} U + R(U,Y)X = (\mathcal{L}_U \nabla)(Y,X). \tag{2.181}$$

Setting $X = Y = \xi$ in the foregoing equation and utilizing Proposition 2.7, we obtain

$$\nabla_\xi \nabla_\xi U + R(U,\xi)\xi = \xi(2\lambda - \beta\tau^*)\xi - \nabla\left(\lambda - \frac{\beta\tau^*}{2}\right). \tag{2.182}$$

Suppose that U is a Jacobi field such that

$$\nabla_\xi \nabla_\xi U + R(U,\xi)\xi = 0.$$

Let $\gamma = \lambda - \frac{\beta\tau^*}{2}$. Utilizing the above equation into (2.182) yields $2\xi(\gamma)\xi = \nabla\gamma$. Also,

$$X(\xi(\gamma))\eta(Y) - \xi(\gamma)g(\phi X, Y) = \frac{1}{2}g(\nabla_X \nabla\gamma, Y).$$

Making use of the symmetric and anti-symmetric properties of $Hess_\gamma$ and ϕ respectively, it follows that

$$\xi(\gamma)d\eta(X,Y) = 0, \ \forall \ X, Y \perp \xi.$$

This implies that $\xi(\gamma) = 0$ and consequently, $\nabla\gamma = 0$ and hence $\gamma = \lambda - \frac{\beta\tau^*}{2}$ is constant. Thus, we can state the following result.

Theorem 2.19. *If U is a Jacobi field along trajectories of ξ on a Sasakian manifold M admitting almost $*-RYS$, then the soliton reduces to $*-RYS$ provided $\alpha \neq 0$.*

Let us now see an example of a Sasakian manifold satisfying gradient almost $*-$RYS.

Example 2.5. *From Example 3.1 of (Ghosh and Patra, 2018), we see that*

$*-$Ricci tensor on a Sasakian manifold satisfies the equation

$$Ric^*(X,Y) = [(n+1)c - (n-1)]g(X,Y), \forall\ X, Y \perp \xi, \tag{2.183}$$

where c is a constant ϕ sectional curvature. Again, using Example 4.1 of (Ghosh and Patra, 2018), we define a vector field U on the unit sphere $\mathbb{S}^{2n+1}$ such that $U = -D\gamma + \omega\xi$, where D is the gradient operator on the sphere and ω is constant. It follows that U is conformal from Obata's theorem and (2.134). *On applying a $D-$homothetic deformation to the unit sphere, we obtain a Sasakian structure on $\mathbb{S}^{2n+1}$ with constant $c = \frac{4}{a} - 3$. Now, choosing $a = \frac{2(n+1)}{2n+1}$, we also observe that from* (2.183) *the $*-$Ricci tensor vanishes and thus the Ricci tensor satisfies* (2.156)*. Hence, this example satisfies Theorem 2.15.*

2.4 Conclusion

We have explored the properties and isometries of almost Ricci-Yamabe solitons (ARYS) in Section 2.1. We have established several key results that advance the understanding of these solitons in the context of Riemannian geometry. Firstly, we derived the conditions under which a compact gradient almost Ricci-Yamabe soliton is isometric to a Euclidean sphere $S^n(r)$. This involved demonstrating the potential function f of a compact gradient almost Ricci-Yamabe soliton coincides with the Hodge-de Rham potential h. This result is significant as it ties the geometric structure of the soliton to a well-known and well-studied geometric form known as the Euclidean sphere, thereby providing a concrete example of these abstract structures. Secondly, we examined complete gradient almost Ricci-Yamabe solitons with non-zero α and a non-trivial conformal vector field. We showed that these solitons under the condition of non-negative scalar curvature must be isometric to either Euclidean space E^n or a Euclidean sphere S^n. This result not only extends the rigidity results known for Ricci solitons but also illustrates the restrictive nature of almost Ricci-Yamabe solitons under these conditions enhancing our understanding of their geometric properties. Additionally, we considered ARYS with solenoidal and torse-forming vector fields providing a comprehensive analysis of their structure. Through various lemmas and theorems, we demonstrated the rigidity of these solitons proving that they admit few deformations under the given conditions. This rigidity is an essential characteristic as it implies stability and uniqueness of the geometric structures described by these solitons. We have provided explicit examples to verify the theoretical results obtained. These examples serve to illustrate the applicability of the theoretical findings and provide a concrete foundation for further research. By constructing non-trivial examples, we not only validated our theoretical work but also opened up new avenues for exploring the practical implications of almost Ricci-Yamabe solitons in various geometric contexts.

The work carried out in Section 2.2 is an extension of the work done on $(LCS)_n$-manifolds by Roy et al. (2020). We generalized their results and obtained more general value for the scalar curvature tensor on an $(LCS)_n$-manifold admitting the Ricci-Yamabe soliton and shown that it is constant. The prominence of this result is that it holds for a larger group of solitons. We have also verified our result by constructing a 3-dimensional and 5-dimensional $(LCS)_n$-manifold. The expression for the scalar λ when the manifold admits a conformal Ricci-Yamabe soliton is also obtained. Moreover, the conditions under which a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold admitting a torse forming vector field ξ is expanding, steady and shrinking η-Ricci-Yamabe soliton is obtained. Also, we give the expression for λ in a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold which admits a torse forming η-Ricci-Yamabe soliton. Further, it is shown that the results obtained in (Roy et al., 2020) are particular results.

Throughout the work done in Section 2.3, we introduced and investigated almost $*-$Ricci-Yamabe solitons on a Sasakian manifold M. Following the method used by Dwivedi and Patra (2022) and extending their results, we give analytic answer to the question raised in the beginning of the section and hence we proved that if a complete Sasakian manifold admits almost $*$-Ricci-Yamabe soliton and gradient almost $*$-Ricci-Yamabe soliton as its metric, then it is isometric to the unit sphere $\mathbb{S}^{2n+1}$ under the condition that α is non-zero. Furthermore, we obtained certain conditions for the soliton to become steady. Also, we found that if the potential vector field U is assumed to be an infinitesimal contact transformation, it becomes an infinitesimal automorphism. Lastly, we used the example constructed in (Ghosh and Patra, 2018) to verify our results. However, we have studied the solitons only on a Sasakian manifold and found the results, further work of the almost $*$-Ricci-Yamabe solitons on Riemannian manifold can be carried out and is highly suggested.

Chapter 3

Characterization of Almost Cosymplectic Manifolds

This chapter comprises of two main sections. Section 3.1 deals with almost cosymplectic manifolds with almost Ricci-Yamabe solitons as its metric. We take a conformal Ricci-Yamabe solitons on almost Kenmotsu manifolds in section 3.2.

3.1 Investigations on almost cosymplectic manifolds admitting almost Ricci-Yamabe solitons

The categorization of almost cosymplectic manifolds admitting almost Ricci-Yamabe solitons is the focus of this section. Almost cosymplectic structures are a natural generalisation of cosymplectic structures in which the symplectic form might be degenerate. Because of their importance in both mathematical and scientific contexts such as the study of generalised complex geometry and supersymmetric field theories, there has been a great deal of interest in these structures lately. The main aim of this investigation is to look into the existence and characteristics of almost Ricci-Yamabe solitons on almost cosymplectic manifolds. We hope to gain a comprehensive understanding of the underlying geometric structures and uncover novel insights into the rich interplay between curvature and soliton dynamics by investigating the interplay between the geometric properties of almost cosymplectic structures and the solitonic behaviour induced by almost Ricci-Yamabe solitons.

One of the important notion in the study of mathematical structures is the concept of isomorphism because it allows us to identify and compare various things based on their underlying qualities. Understanding the isomorphism between almost cosymplectic manifolds and particular classes of Lie groups give vital insights into the nature of these geometric objects and develops linkages to group-theoretic issues in the world of almost cosymplectic manifolds. The classification of almost cosymplectic manifolds admitting almost Ricci-Yamabe solitons not only adds to the theory of solitons and geometric flows but it also has consequences in other fields of mathematics and theoretical physics.

Now, we derive from equation (1.53) that the scalar curvature $\tau = 2n\kappa$ and $Q\xi = 2n\kappa\xi$ using equation (1.51). We can clearly see from equation (1.54), that $\kappa \leq 0$ and $\kappa = 0$ if and only if M is a cosymplectic manifold. As a result, we will always assume $\kappa < 0$ throughout the following discussions. Let us recall the following lemma:

Lemma 3.1 (Ozturk et al., 2010)**.** *On a (κ, μ)-almost cosymplectic manifold with $\kappa < 0$, we have*

$$\begin{aligned}(\nabla_X h)Y - (\nabla_Y h)X =& \kappa[\eta(Y)\phi X - \eta(X)\phi Y - 2g(X, \phi Y)\xi] \\ &+ \mu[\eta(X)h'Y - \eta(Y)h'X], \end{aligned} \tag{3.1}$$

and

$$(\nabla_X h')Y - (\nabla_Y h')X = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \quad (3.2)$$

Also, if $\mu = 0$, we have the following theorem:

Theorem 3.1 (Dacko, 2000)**.** *For some $\kappa < 0$, an almost cosymplectic $(\kappa, 0)$-manifold is locally isomorphic to a Lie group G_q equipped with the almost cosymplectic structure, where $q = \sqrt{-\kappa}$.*

3.1.1 Main Results

In this subsection, we investigate the behaviour of an almost cosymplectic manifolds while considering the manifold to be compact and belonging to the (κ, μ)-nullity distributions. Here, we take an almost Ricci-Yamabe solitons as the metric of the manifold and we obtain the following results.

Theorem 3.2. *Given that $b \neq 0$, no almost Ricci-Yamabe solitons exist on compact (κ, μ)-almost cosymplectic manifolds with $\kappa < 0$.*

Proof. For any $X, Y \in \chi(M)$, we can write equation (1.70) as

$$\mathcal{L}_U g(X, Y) + 2a\ Ric(X, Y) - (2\lambda - b\ \tau)g(X, Y) = 0, \quad (3.3)$$

which implies

$$g(\nabla_X U, Y) + g(X, \nabla_Y U) + 2a\ Ric(X, Y) + (2\lambda - b\ \tau)g(X, Y) = 0. \quad (3.4)$$

Setting $U = \xi$ in the above equation and employing equations (1.52), (1.53) gives

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2a\ Ric(X, Y) + (2\Lambda - b\ \tau)g(X, Y) = 0,$$
$$-2g(\phi hX, Y) + 2a[\mu g(hX, Y) + 2n\kappa\eta(X)\eta(Y)] + (2\lambda - b\ \tau)g(X, Y) = 0. \quad (3.5)$$

Putting $X = Y = \xi$ in equation (3.5), we get

$$\tau = \frac{4na\kappa + 2\lambda}{b}, \; b \neq 0. \tag{3.6}$$

Contracting X and Y in equation (3.5) results in

$$div\ U + 2na\kappa + 2na\kappa + (2n+1)\lambda - \frac{(2n+1)b}{2}\tau = 0. \tag{3.7}$$

Utilizing equation (3.6) yields

$$div\ U - 4n^2 a\kappa = 0. \tag{3.8}$$

Integrating equation (3.8) and using divergence theorem, we obtain

$$\int_M 4n^2 a\kappa \; dM = 0, \tag{3.9}$$

where the manifold M's volume form is expressed as dM. Since $\kappa < 0$, equation (3.9) does not hold. This completes the proof. □

Theorem 3.3. *If a (κ, μ)-almost cosymplectic manifold admits an ARYS with a potential vector field such that $U = d\xi$, it becomes a RYS and the soliton shrinks.*

Proof. Consider that the potential vector field U is pointwise collinear with ξ, then $U = d\xi$, where d is a smooth function. Then equation (3.4) implies that

$$(XXd)\eta(Y) + (Yd)\eta(X) - 2dg(\phi hX, Y) + 2a\ Ric(X,Y) + (2\lambda - b\ \tau)g(X,Y) = 0. \tag{3.10}$$

Replacing X by ϕX and Y by ϕY in equation (3.10) results in

$$2dg(\phi^2 h\phi X, Y) + 2a\mu g(\phi h\phi X, Y) + (2\lambda - b\ \tau)g(\phi^2 X, Y) = 0, \tag{3.11}$$

which further implies

$$2d\phi^2 h\phi X + 2a\mu\phi^2 hX + (2\lambda - b\ \tau)\phi^2 X = 0. \tag{3.12}$$

Then, using equation (1.1) and taking inner product with Y, we get

$$2dg(h\phi X, Y) - 2a\mu g(hX, Y) - (2\lambda - b\tau)g(X, Y) = 0. \tag{3.13}$$

Contracting the foregoing expression and using equation (1.51), we obtain

$$\lambda = nb\kappa, \tag{3.14}$$

which is constant. Moreover, since $\kappa < 0$, $\lambda < 0$, completing the proof. □

Theorem 3.4. *A Lie group G_q equipped with the almost cosymplectic structure where $q = \sqrt{-\kappa}$ is locally isomorphic to a (κ, μ)-almost cosymplectic manifold admitting a gradient RYS with $a \neq 0$. Otherwise, the manifold does not admit a gradient RYS.*

Proof. From the soliton equation, we have

$$\nabla^2 f + a\ Ric + \left(\lambda - \frac{b\,\tau}{2}\right) g = 0. \tag{3.15}$$

Then,

$$\nabla_X Df + a\ QX + \left(\lambda - \frac{b\,\tau}{2}\right) X = 0, \tag{3.16}$$

where D is the gradient operator. Taking covariant differentiation of the equation (3.16) along Y implies

$$\nabla_Y \nabla_X Df + a\ \nabla_Y QX + \left(\lambda - \frac{b\,\tau}{2}\right) \nabla_Y X - \frac{b}{2} Y(\tau)X = 0. \tag{3.17}$$

Now, swapping X and Y in the equation (3.17), we have

$$\nabla_X \nabla_Y Df + a\ \nabla_X QY + \left(\lambda - \frac{b\,\tau}{2}\right) \nabla_X Y - \frac{b}{2} X(\tau)Y = 0. \tag{3.18}$$

From equation (3.16), we obtain

$$\nabla_{[X,Y]} Df + a\ Q[X, Y] + \left(\lambda - \frac{b\,\tau}{2}\right) [X, Y] = 0. \tag{3.19}$$

Utilizing the last three equations, we get

$$R(X,Y)Df = \frac{b}{2}\{(Y(\tau))X - (X(\tau))Y\} - a\{(\nabla_X Q)Y - (\nabla_Y Q)X\}. \tag{3.20}$$

Differentiating equation (1.53) along Y results in

$$(\nabla_Y Q)X = \mu(\nabla_Y h)X - 2n\kappa\{g(\phi hX, Y)\xi + \eta(X)\phi hY\}. \tag{3.21}$$

Utilizing equation (3.21) in equation (3.20), we get

$$\begin{aligned} R(X,Y)Df =& \frac{b}{2}\{(Y(\tau))X - (X(\tau))Y\} - a\mu\{(\nabla_X h)Y - (\nabla_Y h)X\} \\ &+ 2na\kappa\{\eta(Y)\phi hX - \eta(X)\phi hY\}. \end{aligned} \tag{3.22}$$

Invoking Lemma 3.1 in the above expression yields

$$\begin{aligned} R(X,Y)Df =& \frac{b}{2}\{(Y(\tau))X - (X(\tau))Y\} - a\kappa\mu\{\eta(Y)\phi X - \eta(X)\phi Y \\ &- 2g(X, \phi Y)\xi\} - a\mu^2\{\eta(X)h'Y - \eta(Y)h'X\} \\ &+ 2na\kappa\{\eta(Y)\phi hX - \eta(X)\phi hY\}. \end{aligned} \tag{3.23}$$

Contracting equation (3.23) along X, we obtain

$$Ric(Y, Df) = 0. \tag{3.24}$$

Again, from equation (1.53), we have

$$Ric(X,Y) = \mu g(hX, Y) + 2n\kappa\eta(X)\eta(Y). \tag{3.25}$$

Replacing X by Df in equation (3.25) and using equation (3.24) results in

$$\mu g(Df, hY) + 2n\kappa(\xi f)\eta(Y) = 0. \tag{3.26}$$

Substituting $Y = \xi$ in the above equation yields

$$2n\kappa(\xi f) = 0. \tag{3.27}$$

Since $\kappa < 0$, equation (3.27) implies

$$\xi f = 0. \tag{3.28}$$

Utilizing the foregoing equation in equation (3.26), we get

$$\mu g(Df, hY) = 0, \tag{3.29}$$

which implies that either $\mu = 0$ or $\mu \neq 0$.

Case I: If $\mu = 0$, the manifold becomes $N(\kappa)$-almost cosymplectic manifold and Theorem 3.1 follows.

Case II: If $\mu = 0$, then equation (3.29) implies that $g(Df, hY) = 0$. Hence, $g(Df, h^2Y) = 0$. Using $h^2 = \kappa\phi^2$, we have

$$\kappa[Yf - (\xi f)\eta(Y)] = 0$$

Also, since $\kappa < 0$, $Yf = 0$ implying that f is constant. Hence, using equation (3.17), we have

$$Ric(X, Y) = \frac{1}{a}\left(\lambda - \frac{b\,\tau}{2}\right) g(X, Y),\ a \neq 0,$$

resulting in Einstein manifold, which is a contradiction to equation (3.25). This completes the proof. □

Substituting the values $a = 1$ and $b = -2\rho$ in equation (3.17), we get

$$\nabla_X Df + QX = (\lambda + \rho\tau)X,$$

which is a gradient Ricci-Bourguinon soliton (GRBS) and equation (3.26) becomes

$$Ric(X, Y) = (\lambda + \rho\tau)g(X, Y),$$

which contradict equation (3.25). Hence the corollary follows

Corollary 3.1. *A (κ, μ)-almost cosymplectic manifolds does not admit a GRBS.*

Theorem 3.5. *If a compact α-almost cosymplectic manifolds admits an ARYS, then the manifold becomes almost cosymplectic manifold.*

Proof. Setting $U = \xi$ in equation (3.4) and using (1.52), we get

$$2\alpha[g(X,Y) - \eta(X)\eta(Y)] - 2g(\phi hX, Y) + 2a\ Ric(X,Y) + (2\lambda - b\ \tau)g(X,Y) = 0.$$

Contracting X and Y from the above expression yields

$$(2\lambda - b\ \tau)(2n+1) + 2a\ \tau = -4n\alpha. \tag{3.30}$$

Again, contracting X and Y from the soliton equation (1.70), we obtain

$$div\ U + a\ \tau + \left(\lambda - \frac{b\ \tau}{2}\right)(2n+1) = 0. \tag{3.31}$$

Utilizing equation (3.30), equation (3.31) becomes

$$div\ U = 2n\alpha. \tag{3.32}$$

Integrating the foregoing equation and invoking the divergence theorem yields

$$\int_M 2n\alpha\ dM = 0. \tag{3.33}$$

Hence, $\alpha = 0$ implying that M is an almost cosymplectic manifold. □

Theorem 3.6. *If an ARYS's soliton vector field is pointwise collinear with ξ on a compact α-almost cosymplectic manifold, then $\tau = \frac{2\lambda}{(2n+1)b-2a}$ provided that $a \neq \{0, \frac{(2n+1)b}{2}\}$.*

Proof. Putting $U = d\xi$ in equation (3.4), we get

$$(Xd)\eta(Y) + (Yd)\eta(X) + 2d\alpha[g(X,Y) - \eta(X)\eta(Y)] - 2dg(\phi hX, Y) + 2a\ Ric(X,Y) + (2\lambda - b\ \tau)g(X,Y). \tag{3.34}$$

Setting $Y = \phi Y$ and $X = \xi$ in equation (3.34), we obtain

$$g(Dd, \phi Y) = 0, \tag{3.35}$$

which further implies that $Yd = (\xi d)\eta(Y)$. Differentiating $U = d\xi$ along X and then taking inner product with Y gives

$$g(\nabla_X U, Y) = (Xd)\eta(Y) + d\alpha[g(X,Y) - \eta(X)\eta(Y)] - dg(\phi hX, Y). \tag{3.36}$$

Contracting the preceding equation gives

$$div\ U = (\xi d) + 2nd\alpha. \tag{3.37}$$

Integrating equation (3.37) and invoking the divergence theorem yields

$$\int_M [(\xi d) + 2nd\alpha]\ dM = 0, \tag{3.38}$$

which implies that $\xi d = -2nd\alpha$. Again, contracting equation (3.34) and using the value of ξd, we obtain

$$\tau = \frac{2\lambda}{(2n+1)b - 2a},\ a \neq \{0, \frac{(2n+1)b}{2}\}. \tag{3.39}$$

Hence, the proof. □

Theorem 3.7. *If a 3-dimensional cosymplectic manifold admits GRYS, then it is either flat or the scalar curvature is constant, assuming that $a \neq b$.*

Proof. Setting $X = Df$ in the following equation

$$Ric(X,Y) = \frac{\tau}{2}[g(X,Y) - \eta(X)\eta(Y)], \tag{3.40}$$

and using equation (3.24), we get

$$\tau[Yf - (\xi f)\eta(Y)] = 0, \tag{3.41}$$

which gives either $\tau = 0$ or $\tau \neq 0$. If $\tau = 0$, then $Ric(X,Y) = 0$ which further implies that $R(X,Y)Z = 0$ resulting in a flat manifold.

Now, suppose that $\tau \neq 0$, then

$$Df = (\xi f)\xi. \tag{3.42}$$

Then, $\nabla_X Df = [X(\xi f)]\xi$. So, from equation (3.16) we have

$$a\ Ric(X,Y) + [X(\xi f)]\eta(Y) + \left(\lambda - \frac{b\ \tau}{2}\right) g(X,Y) = 0. \tag{3.43}$$

Putting $Y = \xi$ in the foregoing expression, we obtain

$$X(\xi f) = -\left(\lambda - \frac{b\ \tau}{2}\right) \eta(X), \tag{3.44}$$

Substituting the preceding equation into equation (3.43) yields

$$\tau = \frac{2\lambda}{b-a},\ a \neq b. \tag{3.45}$$

This completes the proof. □

3.1.2 Example on 3-dimensional manifold

Consider a 3-dimensional manifold $M^3 = \{(x,y,z) \in \mathbb{R}^3\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Let us choose E_1, E_2 and E_3 to be the three vector fields in $\mathbb{R}^3$ that satisfy

$$[E_1,E_2] = E_2,\ [E_1,E_3] = E_3 \text{ and } [E_i,E_j] = 0\ \forall\ i,j = 2,3.$$

and let g be the Riemannian metric such that

$$g(E_i,E_j) = \begin{cases} 1,\ i = j, \\ 0,\ i \neq j,\ \forall\ i,j = 1,2,3. \end{cases}$$

Here, we take E_1 as the Reeb vector field. Suppose ϕ is the $(1,1)-$tensor field and η be the $1-$form defined such that

$$\phi(E_1) = 0, \ \phi(E_2) = E_3, \ \phi(E_3) = 2E_2,$$
$$\eta(V) = g(V, E_1), \text{ for any } V \in T(M^3).$$

By the linearity of ϕ and g, we have

$$\eta(E_1) = 1, \ \phi^2(V) = -V + \eta(V)E_1, \ g(\phi V, \phi U) = g(V, U) - \eta(V)\eta(U),$$

for any V, U. The following relations are obtained directly by using Koszul's formula:

$$\nabla_{E_1}E_1 = 0, \ \nabla_{E_1}E_2 = 0, \ \nabla_{E_1}E_3 = 0,$$
$$\nabla_{E_2}E_1 = -E_2, \ \nabla_{E_2}E_2 = E_1 - \frac{1}{2}, \ \nabla_{E_2}E_3 = 0,$$
$$\nabla_{E_3}E_1 = -E_3, \ \nabla_{E_3}E_2 = 0, \ \nabla_{E_3}E_3 = E_1 - \frac{1}{2}.$$

From the above relations, we obtain the components of the curvature tensor R as follows:

$$R(E_1, E_2)E_3 = 0, \ R(E_2, E_3)E_3 = -E_2, \ R(E_1, E_3)E_1 = E_3,$$
$$R(E_1, E_3)E_3 = -E_1 + \frac{1}{2}, \ R(E_1, E_2)E_2 = 0, \ R(E_2, E_3)E_1 = 0,$$
$$R(E_2, E_3)E_2 = E_3, \ R(E_1, E_3)E_2 = 0, \ R(E_1, E_2)E_1 = -E_1 + \frac{1}{2}.$$

From these expressions, we conclude that

$$R(X, Y)\xi = -[\eta(Y)X - \eta(X)Y],$$

for any X, Y. Therefore M^3 is a $N(-1)$-almost cosymplectic manifold. The Ricci tensor are obtained as below

$$Ric(E_1, E_1) = -2, \quad Ric(E_2, E_2) = 0, \quad Ric(E_3, E_3) = 0.$$

Setting $f = z^2$ and $a = 1, \lambda = -b \in \mathbb{R}$, equation (3.16) is met. As a result, M specifies a gradient RYS. Furthermore, because the manifold is $N(-1)$-almost cosymplectic, Theorem 3.4 is validated.

3.2 On almost Kenmotsu manifolds admitting conformal Ricci-Yamabe solitons

Wang (2016a, 2016c) proved that a $(\kappa, \mu)' - akm$ admitting Ricci soliton is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. Venkatesha and Kumara (2019) proved the same result for a gradient ρ-Einstein soliton. Also, the result is extended by the authors in (Dey and Majhi, 2019) for Ricci-Yamabe soliton and for conformal Ricci soliton. So, a logical question arises as:

Will the above result also holds true if a $(2n+1)$ *dimensional* $(\kappa, \mu)' - akm$ *admits a conformal Ricci-Yamabe soliton or a conformal gradient Ricci-Yamabe soliton?*

In this section, we shall try to give a reasonable answer to this question.

On an almost Kenmotsu manifold, the authors in (Dileo and Pastore, 2009) defined for any $q \in M$ and $\kappa, \mu \in \mathbb{R}$ the $(\kappa, \mu)'-$nullity distribution as

$$\begin{aligned} N_q(\kappa, \mu)' = \{Z \in T_q(M) : R(X, Y)Z = \kappa[g(Y, Z)X - g(X, Z)Y] \\ + \mu[g(Y, Z)h'X - g(X, Z)h'Y]\}. \end{aligned} \tag{3.46}$$

This is called generalized nullity distribution when κ, μ are smooth functions.

Let $X \in \mathfrak{D}$ be the eigenvector of h' analogous to the eigenvalue γ. Then, from (1.34) it is obvious that $\gamma^2 = -(\kappa + 1)$, a constant. Therefore $\kappa \leq -1$ and $\gamma = \pm\sqrt{-\kappa - 1}$. We denote $[\gamma]'$ and $[-\gamma]'$, the corresponding eigenspaces akin to the non-zero eigenvalues γ and $-\gamma$ of h' respectively. It has been proved in (Dileo and Pastore, 2019) that in a $(\kappa, \mu)' - akm$ M^{2n+1} with $h' \neq 0$, $\kappa < -1$, $\mu = -2$

J.P. Singh and Z. Chhakchhuak (2024). On almost Kenmotsu manifolds admitting conformal Ricci-Yamabe solitons, *Bol. Soc. Paran. Math.* (accepted) and $Spec(h') = \{0, \gamma, -\gamma\}$ with 0 as a simple eigenvalue and $\gamma = \sqrt{-\kappa - 1}$. Also,

$$(\nabla_X h')Y = -g(h'X + h'^2 X, Y)\xi - \eta(Y)(h'X + h'^2 X). \tag{3.47}$$

Further, from Wang and Liu (2016), we have

$$QX = -2nX + 2n(\kappa + 1)\eta(X)\xi - 2nh'X. \tag{3.48}$$

From (3.48), the scalar curvature of M^{2n+1} is $2n(\kappa - 2n)$. From (3.46), we have

$$R(X,Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{3.49}$$

where $\kappa, \mu \in \mathbb{R}$. Again, from above equation, we get

$$R(\xi, X)Y = \kappa[g(X,Y)\xi - \eta(Y)X] + \mu[g(h'X, Y)\xi - \eta(Y)h'X]. \tag{3.50}$$

Contracting X in (3.48) yields

$$Ric(Y, \xi) = 2n\kappa\eta(Y). \tag{3.51}$$

Using (1.30) and (1.33), we have

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y) + g(h'X, Y). \tag{3.52}$$

Dai et al.(2019) defined an infinitesimal contact transformation on M as

Definition 3.1. *A potential vector field V is infinitesimal contact transformation on an almost contact metric manifold if $\mathcal{L}_V \eta = \zeta\eta$ for some function ζ. In particular, if $\mathcal{L}_V \eta = 0$, then V is said to be strict infinitesimal contact transformation.*

3.2.1 Conformal Ricci-Yamabe Soliton (CRYS) on $(\kappa, \mu)'$-almost Kenmotsu manifolds

In this subsection, we study $(\kappa, \mu)' - akm$ and generalized $(\kappa, \mu)' - akm$ admitting a conformal Ricci-Yamabe soliton.

A Conformal Ricci-Yamabe soliton (CRYS) on an almost Kenmotsu manifolds with dimension $(2n+1)$ can be defined as the data $(g, V, \lambda, \alpha, \beta)$ satisfying

$$\mathcal{L}_V g + 2\alpha Ric = \left[2\lambda - \beta\tau - \left(p + \frac{2}{2n+1}\right)\right] g, \tag{3.53}$$

where Ric, $\mathcal{L}_V$ are the Ricci tensor and Lie derivative along a vector field V respectively.

Lemma 3.2 (Dey and Majhi, 2019)**.** *In a $(\kappa, \mu)' - akm$ M^{2n+1} with $h' \neq 0$, we have*

$$(\nabla_Z Ric)(X, Y) - (\nabla_X Ric)(Y, Z) - (\nabla_Y Ric)(X, Z) = -4n(\kappa + 2)g(h'X, Y)\eta(Z).$$

Lemma 3.3 (Dey and Majhi, 2019)**.** *In a $(\kappa, \mu)' - akm$ M^{2n+1}, $(\mathcal{L}_X h')Y = 0$ for any $X, Y \in [\gamma]'$ or $X, Y \in [-\gamma]'$, where $Spec(h') = \{0, \gamma, -\gamma\}$.*

Theorem 3.8. *A conformal Ricci-Yamabe soliton (CRYS) with $\alpha, \beta > 0$ on a $(2n+1)-$ dimensional $(\kappa, \mu)' - akm$ is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$, provided that $2\lambda - \beta\tau \neq 4\alpha n\kappa - (p + \frac{2}{2n+1})$ or the $CRYS$ is (i) expanding, (ii) steady or (iii) shrinking, according to whether the conformal pressure p is*
(1) $p < 2n\beta(2n - \kappa) - 4\alpha n\kappa - \frac{2}{2n+1}$,
(2) $p = 2n\beta(2n - \kappa) - 4\alpha n\kappa - \frac{2}{2n+1}$,
(3) $p > \frac{2n\beta(2n+1)^2 + 4\alpha n(2n+1) - 2}{2n+1}$.

Proof. From the soliton equation (3.53), we have

$$(\mathcal{L}_V g)(X, Y) + 2\alpha Ric(X, Y) = \left[2\lambda - \beta\tau - \left(p + \frac{2}{2n+1}\right)\right] g(X, Y). \tag{3.54}$$

Taking covariant derivative of (3.54) along any vector field Z yields

$$(\nabla_Z \mathcal{L}_V g)(X, Y) = -2\alpha(\nabla_Z Ric)(X, Y). \tag{3.55}$$

Using the result of Yano (Yano, 1970), we have

$$(\mathcal{L}_V \nabla_X g - \mathcal{L}_X \nabla_V g - \nabla_{[V,X]} g)(Y, Z) = -g((\mathcal{L}_V \nabla)(X, Y), Z) - g((\mathcal{L}_V \nabla)(X, Z), Y).$$

Since g is parallel with respect to the Levi-civita connection ∇, then the above relation becomes

$$(\nabla_X \mathcal{L}_V g)(Y, Z) = g((\mathcal{L}_V \nabla)(X, Y), Z) + g((\mathcal{L}_V \nabla)(X, Z), Y). \tag{3.56}$$

Since $\mathcal{L}_V \nabla$ is symmetric, then from (3.56), we obtain

$$g((\mathcal{L}_V \nabla)(X, Y), Z) = \frac{1}{2}(\nabla_X \mathcal{L}_V g)(Y, Z) + \frac{1}{2}(\nabla_Y \mathcal{L}_V g)(X, Z) - \frac{1}{2}(\nabla_Z \mathcal{L}_V g)(X, Y). \tag{3.57}$$

Utilizing (3.55) in (3.57), we get

$$g((\mathcal{L}_V \nabla)(X, Y), Z) = \alpha[(\nabla_Z Ric)(X, Y) - (\nabla_X Ric)(Y, Z) - (\nabla_Y Ric)(X, Z). \tag{3.58}$$

In view of Lemma 3.2, (3.58) can be written as

$$g((\mathcal{L}_V \nabla)(X, Y), Z) = -4n\alpha(\kappa + 2)g(h'X, Y)\eta(Z),$$

which implies

$$(\mathcal{L}_V \nabla)(X, Y) = -4n\alpha(\kappa + 2)g(h'X, Y)\xi. \tag{3.59}$$

Taking $Y = \xi$ in (3.59), we get

$$(\mathcal{L}_V \nabla)(X, \xi) = 0.$$

From the above expression, we can have $\nabla_Y (\mathcal{L}_V \nabla)(X, \xi) = 0$, which results in

$$(\nabla_Y \mathcal{L}_V \nabla)(X, \xi) + (\mathcal{L}_V \nabla)(\nabla_Y X, \xi) + (\mathcal{L}_V \nabla)(X, \nabla_Y \xi) = 0. \tag{3.60}$$

Using $(\mathcal{L}_V \nabla)(X, \xi) = 0$, (3.58) and (1.30) in (3.60) we get

$$(\nabla_Y \mathcal{L}_V \nabla)(X, \xi) = 4n\alpha(\kappa + 2)(g(h'X, Y) + g(h'^2 X, Y))\xi. \tag{3.61}$$

It is known that (Yano, 1970)

$$(\mathcal{L}_V R)(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z),$$

Taking $Y = Z = \xi$ in (3.61) and utilizing the result in the foregoing equation, we have

$$(\mathcal{L}_V R)(X,\xi)\xi = 0. \tag{3.62}$$

Now, setting $Y = \xi$ in (3.54) and in view of (3.71), we obtain

$$(\mathcal{L}_V g)(X,\xi) = \left[2\lambda - \beta\tau - 4n\alpha\kappa - \left(p + \frac{2}{2n+1}\right)\right]\eta(X). \tag{3.63}$$

Taking Lie differentiation of $g(X,\xi) = \eta(X)$ along V and using (3.63) yields

$$(\mathcal{L}_V \eta)X - g(X,\mathcal{L}_V\xi) = \left[2\lambda - \beta\tau - 4n\alpha\kappa - \left(p + \frac{2}{2n+1}\right)\right]\eta(X). \tag{3.64}$$

Putting $X = \xi$ in the foregoing expression, we get

$$\eta(\mathcal{L}_V\xi) = \left[2\lambda - \beta\tau - 4n\alpha\kappa - \left(p + \frac{2}{2n+1}\right)\right]. \tag{3.65}$$

From (3.49) we have

$$R(X,\xi)\xi = \kappa(X - \eta(X)\xi) - 2h'X. \tag{3.66}$$

Utilizing (3.64)-(3.66) and (3.49)-(3.50), we get

$$\begin{aligned}(\mathcal{L}_V R)(X,\xi)\xi = \kappa&\left[2\lambda - \beta\tau - 4n\alpha\kappa - \left(p + \frac{2}{2n+1}\right)\right](X - \eta(X)\xi) - 2(\mathcal{L}_V h')X\\ &- 2\left[2\lambda - \beta\tau - 4n\alpha\kappa - \left(p + \frac{2}{2n+1}\right)\right]h'X - 2n\eta(X)h'(\mathcal{L}_V\xi)\\ &- 2g(h'X,\mathcal{L}_V\xi)\xi.\end{aligned} \tag{3.67}$$

Equating (3.62) and (3.67) and then taking an inner product with Y, we obtain

$$\begin{aligned}\kappa&\left[2\lambda - \beta\tau - 4n\alpha\kappa - \left(p + \frac{2}{2n+1}\right)\right][g(X,Y) - \eta(X)\eta(Y)] - 2g((\mathcal{L}_V h')X,Y)\\ &- 2\left[2\lambda - \beta\tau - 4n\alpha\kappa - \left(p + \frac{2}{2n+1}\right)\right]g(h'X,Y) - 2n\eta(X)g(h'(\mathcal{L}_V\xi),Y)\\ &- 2g(h'X,\mathcal{L}_V\xi)\eta(Y) = 0.\end{aligned} \tag{3.68}$$

Replacing X by ϕX in the foregoing expression, we obtain

$$\kappa\left[2\lambda-\beta\tau-4n\alpha\kappa-\left(p+\frac{2}{2n+1}\right)\right]g(\phi X,Y)-2g((\mathcal{L}_V h')\phi X,Y)$$
$$-2\left[2\lambda-\beta\tau-4n\alpha\kappa-\left(p+\frac{2}{2n+1}\right)\right]g(h'\phi X,Y)=0. \quad (3.69)$$

Letting $X\in[-\gamma]'$ and $V\in[\gamma]'$, then $\phi X\in[\gamma]'$. Thus (3.69) yields

$$(\kappa-2\gamma)\left[2\lambda-\beta\tau-4n\alpha\kappa-\left(p+\frac{2}{2n+1}\right)\right]g(\phi X,Y)$$
$$-2g((\mathcal{L}_V h')\phi X,Y)=0. \quad (3.70)$$

Since $V,\phi X\in[\gamma]'$, using Lemma 3.3 we have $(\mathcal{L}_V h')\phi X=0$. Therefore (3.70) becomes

$$(\kappa-2\gamma)\left[2\lambda-\beta\tau-4n\alpha\kappa-\left(p+\frac{2}{2n+1}\right)\right]g(\phi X,Y)=0,$$

which implies either $\kappa=2\gamma$ or $2\lambda=\beta\tau+4\alpha n\kappa+\left(p+\frac{2}{2n+1}\right)$.

Case I: If $\kappa=2\gamma$, then from $\gamma^2=-(\kappa+1)$, we get $\gamma=-1$ and thus $\kappa=-2$. Therefore, from Proposition 4.2 of (Dileo and Patore, 2009), we have

$$R(X_{1_\gamma},X_{2_\gamma})X_{3_\gamma}=0,$$

and

$$R(X_{1_\gamma},X_{2_\gamma})X_{3_\gamma}=-4[g(X_{2_{-\gamma}},X_{3_{-\gamma}})X_{1_{-\gamma}}-g(X_{1_{-\gamma}},X_{3_{-\gamma}})X_{2_{-\gamma}}],$$

for any $X_{1_\gamma},X_{2_\gamma},X_{3_\gamma}\in[\gamma]'$ and $X_{1_{-\gamma}},X_{2_{-\gamma}},X_{3_{-\gamma}}\in[-\gamma]'$. Also, $\mu=-2$, it follows from Proposition 4.3 of (Dileo and Pastore, 2009) that $\kappa(X,\xi)=-4$ for any $X\in[-\gamma]'$ and $\kappa(X,\xi)=0$ for any $X\in[\gamma]'$. Again, we see that $\kappa(X,Y)=-4$ for any $X,Y\in[-\gamma]'$ and $\kappa(X,Y)=0$ for any $X,Y\in[\gamma]'$. As is shown in (Dileo and Pastore, 2009), the distribution $[\xi]\oplus[\gamma]'$ is integrable with totally geodesic leaves and the distribution $[-\gamma]'$ is integrable with totally umbilical leaves by $\mathbb{H}=-(1-\gamma)\xi$, where $\mathbb{H}$ is the mean curvature tensor field for the leaves of

$[-\gamma]'$ immersed in M^{2n+1}. Here, $\gamma = -1$, then both the orthogonal distributions $[\xi] \oplus [\gamma]'$ and $[-\gamma]'$ are integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Case II: Let $2\lambda = \beta\tau + 4n\alpha\kappa + (p + \frac{2}{2n+1})$. Since $\tau = 2n(\kappa - 2n)$ in an *akm* of dimension $2n+1$, we get

$$2\lambda = 2\beta n(\kappa - 2n) + 4\alpha n\kappa + \left(p + \frac{2}{2n+1}\right).$$

Now the CRYS is expanding, steady or shrinking depending on whether $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ respectively. Therefore the CRYS is expanding when

$$p < -4\alpha n\kappa - 2\beta n(\kappa - 2n) - \frac{2}{2n+1},$$

steady when

$$p = 2\beta n(2n - \kappa) - 4\alpha n\kappa - \frac{2}{2n+1},$$

and shrinking when

$$p > 2\beta n(2n - \kappa) - 4\alpha n\kappa - \frac{2}{2n+1},$$

the last expression is obtained by taking $\kappa = -1$ which completes the proof. □

Theorem 3.9. *If $(g, \xi, \lambda, \alpha, \beta)$ is a CRYS in a generalized $(\kappa, \mu)' - akm$ M^{2n+1}, then the manifold is $\eta-$Einstein provided $2n\alpha - 1 \neq 0$ and the expression for λ is given by*

$$\lambda = \frac{p}{2} + \frac{1}{2n+1} + (2\alpha + \beta)n\kappa - 2n^2\beta. \tag{3.71}$$

Proof. From the soliton equation (3.53) we have

$$(\mathcal{L}_\xi g)(X,Y) + 2\alpha Ric(X,Y) = \left[2\lambda - \beta\tau - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y). \tag{3.72}$$

Using (1.30), we get

$$(\mathcal{L}_\xi g)(X,Y) = 2g(X,Y) - 2\eta(X)\eta(Y) - 2g(\phi hX, Y). \tag{3.73}$$

Utilizing (3.73) in (3.72), we obtain

$$2g(X,Y) - 2\eta(X)\eta(Y) - 2g(\phi hX,Y) + 2\alpha Ric(X,Y) = \left[2\lambda - \beta\tau - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y). \tag{3.74}$$

From (3.48), we get

$$g(\phi hX,Y) = \frac{1}{2n} Ric(X,Y) + g(X,Y) - (\kappa+1)\eta(X)\eta(Y). \tag{3.75}$$

Now, substituting (3.75) in (3.74), we get

$$Ric(X,Y) = \frac{n[2\lambda - 2n\beta(\kappa - 2n) - (p + \frac{2}{2n+1})]}{2n\alpha - 1} g(X,Y) - \frac{2n\kappa}{2n\alpha - 1}\eta(X)\eta(Y), \tag{3.76}$$

which implies that the manifold is $\eta-$Einstein. Now taking $X = Y = \xi$ in (3.76) yields (3.71). This completes the proof. □

3.2.2 Conformal Gradient Ricci-Yamabe Soliton on $(\kappa,\mu)'$-almost Kenmotsu manifolds

From (3.53), we get the conformal gradient Ricci-Yamabe soliton equation by considering the vector field V to be a gradient of some smooth function ζ on the manifold as

$$\nabla\nabla\zeta + \alpha Ric = \left[2\lambda - \beta\tau - \left(p + \frac{2}{2n+1}\right)\right] g. \tag{3.77}$$

Lemma 3.4. *If $(g, D\zeta, \lambda, \alpha, \beta)$ is a conformal gradient Ricci-Yamabe soliton (CGRYS) with $\alpha \neq 0$ on a $(\kappa,\mu)' - akm$ M^{2n+1}, then following relation:*

$$R(X,Y)D\zeta = \alpha[2n(\kappa+2)(\eta(X)h'Y - \eta(Y)h'X)],$$

holds. Here, ζ is a smooth function such that $V = D\zeta$, where D is the gradient operator.

Proof. From (3.77), we have

$$\nabla_X D\zeta = \left[\lambda - \frac{\beta\tau}{2} - \left(\frac{p}{2} + \frac{1}{2n+1}\right)\right] X - \alpha QX. \tag{3.78}$$

Covariant derivative of the above relation along Y yields

$$\nabla_Y \nabla_X D\zeta = \left[\lambda - \frac{\beta\tau}{2} - \left(\frac{p}{2} + \frac{1}{2n+1}\right)\right] \nabla_Y X - \alpha \nabla_Y QX. \tag{3.79}$$

Interchanging X and Y in the above equation yields

$$\nabla_X \nabla_Y D\zeta = [\lambda - \frac{\beta\tau}{2} - (\frac{p}{2} + \frac{1}{2n+1})]\nabla_X Y - \alpha \nabla_X QY. \tag{3.80}$$

Again, from (3.78), we get

$$\nabla_{[X,Y]} D\zeta = \left[\lambda - \frac{\beta\tau}{2} - \left(\frac{p}{2} + \frac{1}{2n+1}\right)\right] (\nabla_X Y - \nabla_Y X) - \alpha Q(\nabla_X Y - \nabla_Y X). \tag{3.81}$$

Utilizing (3.79)-(3.81) in the equation

$$R(X,Y)D\zeta = \nabla_X \nabla_Y D\zeta - \nabla_Y \nabla_X D\zeta - \nabla_{[X,Y]} D\zeta,$$

results in

$$R(X,Y)D\zeta = \alpha[(\nabla_Y Q)X - (\nabla_X Q)Y]. \tag{3.82}$$

Now, utilizing (1.30), (3.47), (3.48) and (3.52) we obtain

$$\begin{aligned}(\nabla_Y Q)X &= \nabla_Y X - Q(\nabla_Y X)\\ &= 2n(\kappa+1)[g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y)]\xi\\ &+ 2n(\kappa+1)\eta(X)(Y - \eta(Y)\xi - \phi hY) + 2ng(h'Y + h'^2Y, X)\xi\\ &+ 2n\eta(X)(h'Y + h'^2Y). \end{aligned} \tag{3.83}$$

Interchanging X and Y in (3.83) yields the expression for $(\nabla_X Q)Y$. Then, on simplification and using (1.34), (3.82) becomes

$$R(X,Y)D\zeta = \alpha[2n(\kappa+2)(\eta(X)h'Y - \eta(Y)h'X)],$$

which completes the proof. $\square$

Theorem 3.10. *A* $(\kappa,\mu)'-akm$ M^{2n+1} *with* $h'\neq 0$ *admitting a conformal gradient Ricci-Yamabe soliton with* $\alpha\neq 0$ *is locally isometric to* $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$ *provided* V *is not pointwise collinear with the Reeb vector field, otherwise the manifold does not admit a conformal gradient Ricci-Yamabe soliton.*

Proof. Setting $X=\xi$ in Lemma 3.4 and then taking inner product with X yields

$$g(R(\xi,Y)D\zeta,X)=\alpha 2n(\kappa+2)g(h'Y,X). \tag{3.84}$$

Again from (3.50), we have

$$\begin{aligned} g(R(\xi,Y)D\zeta,X) &= -g(R(\xi,Y)X,D\zeta) \\ &= -\kappa g(X,Y)(\xi\zeta)+\kappa\eta(X)(Y\zeta) \\ &\quad + 2g(h'X,Y)(\xi\zeta)-2\eta(X)((h'Y)\zeta). \end{aligned} \tag{3.85}$$

Equating (3.84) and (3.85) we get

$$\begin{aligned} -\kappa g(X,Y)(\xi\zeta)+\kappa\eta(X)(Y\zeta)+2g(h'X,Y)(\xi\zeta) \\ -2\eta(X)((h'Y)\zeta)=2n(\kappa+2)\alpha g(h'Y,X). \end{aligned}$$

Antisymmetrizing the above relation results in

$$\kappa\eta(X)(Y\zeta)-\kappa\eta(Y)(X\zeta)-2\eta(X)((h'Y)\zeta)+2\eta(Y)((h'X)\zeta)=0. \tag{3.86}$$

Putting $X=\xi$ in (3.86) yields

$$\kappa(Y\zeta)-\kappa(\xi\zeta)\eta(Y)-2((h'Y)\zeta)=0,$$

which implies

$$\kappa[(D\zeta)-(\xi\zeta)\xi]-2h'(D\zeta)=0. \tag{3.87}$$

Operating on h' in the above equation and using (1.34), we get

$$h'(D\zeta) = -\frac{2(\kappa+1)}{\kappa}[d\zeta - (\xi\zeta)\xi]. \tag{3.88}$$

Utilizing (3.88) in (3.87) yields

$$(\kappa+2)^2[D\zeta - (\xi\zeta)\xi] = 0,$$

which implies either $\kappa = -2$ or $D\zeta = (\xi\zeta)\xi$. Let us consider these two cases in the following:

Case I: For $\kappa = -2$, then by the same argument made in Case I of Theorem 3.8, the manifold M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Case II: For $V = D\zeta = (\xi\zeta)\xi$, then V is pointwise collinear with the Reeb vector field ξ. Then, Differentiating $D\zeta = (\xi\zeta)\xi$ covariantly along X and using (1.30), we obtain

$$\nabla_X D\zeta = (X(\xi\zeta))\xi + (\xi\zeta)(X - \eta(X)\xi - \phi hX). \tag{3.89}$$

Equating (3.78) and (3.89), we obtain

$$\alpha QX = \left(\left[\lambda - \frac{\beta\tau}{2} - \left(\frac{p}{2} + \frac{1}{2n+1}\right)\right] - (\xi\zeta)\right) X + ((\xi\zeta)\eta(X) - X(\xi\zeta))\xi + (\xi\zeta)\phi hX. \tag{3.90}$$

Comparing (3.48) and the above equation, we get

$$\left[\lambda - \frac{\beta\tau}{2} - \left(\frac{p}{2} + \frac{1}{2n+1}\right)\right] - \xi\zeta = -2n\alpha, \tag{3.91}$$

$$(\xi\zeta)\eta(X) - X(\xi\zeta) = 2n(\kappa+1)\eta(X), \tag{3.92}$$

$$(\xi\zeta)\phi h = -2nh'. \tag{3.93}$$

Utilizing (3.93) in (3.91) and (3.92) we get

$$\lambda = \frac{\beta\tau}{2} + \left(\frac{p}{2} + \frac{1}{2n+1}\right) + 4n(1-\alpha), \tag{3.94}$$

and

$$2n\eta(X) = 2n(\kappa + 1)\eta(X), \tag{3.95}$$

for any vector field X which implies $\kappa = 0$ which is a contradiction as $\kappa \leq -1$. This completes the proof. □

Moreover, if $V = D\zeta = (\xi\zeta)\xi = d\xi$, where $d = \xi\zeta$ is a smooth function on the manifold M^{2n+1}. By (1.30), we have

$$(\mathcal{L}_{d\xi}g)(X,Y) = (Xd)\eta(Y) + (Yd)\eta(X) + 2d[g(X,Y) - \eta(X)\eta(Y) - g(\phi hX, Y)]. \tag{3.96}$$

Utilizing (3.96) in (3.54), we obtain

$$\begin{aligned}(Xd)\eta(Y) + (Yd)\eta(X) + 2d[g(X,Y) - \eta(X)\eta(Y) \\ - g(\phi hX, Y)] = \left[2\lambda - \beta\tau - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) - 2\alpha Ric(X,Y).\end{aligned} \tag{3.97}$$

Setting $X = Y = \xi$ in the foregoing relation and utilizing (3.71) yields

$$2(\xi d) = \left[2\lambda - \beta\tau - \left(p + \frac{2}{2n+1}\right)\right] - 4n\alpha\kappa. \tag{3.98}$$

Again, considering the orthonormal basis of the tangent space $\{E_j\}$ at each point of the manifold and setting $X = Y = E_j$ in (3.97) and then summing over j results as

$$2(\xi d) = \left[2\lambda - \beta\tau - \left(p + \frac{2}{2n+1}\right)\right](2n+1) - 2\alpha\tau - 4nd. \tag{3.99}$$

Since $\alpha, \beta, \lambda, \tau, p$ are all constants, so, from (3.98) and (3.99) d is also a constant. Therefore, (3.98) results in

$$\left[2\lambda - \beta\tau - \left(p + \frac{2}{2n+1}\right)\right] = 4n\alpha\kappa. \tag{3.100}$$

Further, since d is constant, we get $\mathcal{L}_V\xi = 0$. Utilizing (3.100) in (3.63) yields $(\mathcal{L}_V\eta)X = 0$ for any vector field X. Now, substituting $\mathcal{L}_V\xi = 0$ and (3.100) in

(3.68) results in $(\mathcal{L}_V h')X = 0$ for any vector field X, which implies that V leaves h' invariant.

Thus, we can state the following:

Corollary 3.2. *On a $(\kappa,\mu)' - akm$ M^{2n+1} with $\kappa \neq -2$ admitting a conformal gradient RYS, then V is a constant multiple of ξ which further implies the potential vector field V is a strict infinitesimal contact transformation and leaves h' invariant.*

3.2.3 Example of a 3-dimensional $(\kappa,\mu)' - akm$ satisfying CRYS

Consider a 3-dimensional manifold $M^3 = \{(x,y,z) \in \mathbb{R}^3\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Let us take E_1, E_2 and E_3 to be the three vector fields in $\mathbb{R}^3$ which satisfies

$$[E_1, E_2] = E_2,\ [E_1, E_3] = E_3,\ \text{and } [E_i, E_j] = 0,\ \forall\ i,j = 2,3,$$

and let g be the Riemannian metric such that

$$g(E_i, E_j) = \begin{cases} 1,\ i = j, \\ 0,\ i \neq j,\ \forall\ i,j = 1,2,3. \end{cases}$$

Here, we take E_1 as the Reeb vector field. Suppose ϕ is the $(1,1)-$tensor field and η be the $1-$form defined such that

$$\phi(E_1) = 0,\ \phi(E_2) = E_3,\ \phi(E_3) = 2E_2,$$
$$\eta(U) = g(U, E_1),\ \text{for any } U \in T(M^3).$$

Moreover, let $h'E_1 = 0,\ h'E_2 = E_2,\ h'E_3 = E_3$. By the linearity of ϕ and g, we have

$$\eta(E_1) = 1,\ \phi^2(U) = -U + \eta(U)E_1,\ g(\phi U, \phi V) = g(U,V) - \eta(U)\eta(V).$$

The following relations are obtained directly by using Koszul's formula:

$$\begin{aligned}
&\nabla_{E_1}E_1 = 0,\ \nabla_{E_1}E_2 = 0,\ \nabla_{E_1}E_3 = 0,\\
&\nabla_{E_2}E_1 = -E_2,\ \nabla_{E_2}E_2 = E_1 - \frac{1}{2},\ \nabla_{E_2}E_3 = 0,\\
&\nabla_{E_3}E_1 = -E_3,\ \nabla_{E_3}E_2 = 0,\ \nabla_{E_3}E_3 = E_1 - \frac{1}{2}.
\end{aligned}$$

From the above relations, we get that

$$\nabla_X E_1 = -\phi^2 X + h'X,$$

for any $X \in T(M^3)$. Thus, the structure (ϕ, E_1, η, g) is an almost contact metric structure so that M^3 is an almost Kenmotsu manifold (akm) of dimension 3. Utilizing the above results, we can calculate the components of the curvature tensor R as follows:

$$\begin{aligned}
R(E_1,E_2)E_1 &= R(E_3,E_2)E_3 = 4E_2,\\
R(E_1,E_2)E_2 &= R(E_1,E_3)E_3 = -4E_1,\\
R(E_1,E_3)E_1 &= R(E_2,E_3)E_2 = 4E_3,\\
R(E_2,E_1)E_1 &= R(E_2,E_3)E_3 = -4E_2,\\
R(E_3,E_1)E_1 &= R(E_3,E_2)E_2 = -4E_3,\\
R(E_2,E_2)E_2 &= R(E_3,E_1)E_3 = 4E_1.
\end{aligned}$$

In view of the above results obtained for the curvature tensor R, we observe that the Reeb vector field E_1 belongs to the $(\kappa,\mu)'-$nullity distribution where $\kappa = -2$ and $\mu = -2$. Thus, from the formula $\gamma^2 = -(\kappa+1)$, we get $\gamma = \pm 1$. Considering $\gamma = -1$, thus, by the same argument made in Case I of Theorem 3.8, we conclude that M^3 is locally isometric to $\mathbb{H}^2(-4) \times \mathbb{R}$.

Using the curvature tensor formula, we have

$$R(X,Y)Z = -4[g(Y,Z)X - g(X,Z)Y].$$

From the foregoing equation, we obtain

$$Ric(Y,Z) = -8g(Y,Z),$$

which implies $\tau = -24$. Now, we can easily see that

$$(\mathcal{L}_{E_1}g)(E_1,E_1) = 0, \ (\mathcal{L}_{E_1}g)(E_2,E_2) = (\mathcal{L}_{E_1}g)(E_3,E_3) = -2.$$

Consider $V = E_1$ and then tracing the soliton equation (3.53), we get

$$\lambda = \frac{p}{2} + \frac{1}{3} - 4(2\alpha + 3\beta) - \frac{2}{3}.$$

Hence, $(g, E_1, \lambda, \alpha, \beta)$ is a CRYS on M^3. Therefore, Theorem 3.8 is verified.

3.3 Conclusion

In Section 3.1, we have shown that almost Ricci-Yamabe solitons are significantly limited in their existence on compact (κ, μ)-almost cosymplectic manifolds with $\kappa < 0$. This discovery underlines a restriction on the occurrence of such solitons inside this particular framework. Furthermore, the transition from an almost Ricci-Yamabe soliton (ARYS) to a more refined Ricci-Yamabe soliton (RYS) occurs on a (κ, μ)-almost cosymplectic manifold. This transition signifies an increase in geometric regularity and structure as well as the appearance of a shrinking soliton. The shrinking behaviour of the soliton suggests a contraction and convergence of geometric characteristics as the manifold evolves, demonstrating the soliton's stability and coherence. Our main finding is that a (κ, μ)-almost cosymplectic manifold with a gradient RYS (GRYS) and $a \neq 0$ is either locally isomorphic to a Lie group $G_{\sqrt{-\kappa}}$ with an almost cosymplectic structure or does not accept a GRYS. This result reveals a deep relationship between the manifold's geometric qualities and its local isomorphism to a certain Lie group, enhancing our knowledge of the manifold's structural aspects. We have also observed that a (κ, μ)-almost cosymplectic manifold does not take gradient Ricci-Bourguinon solitons (GRBS). Furthermore, we see that allowing an ARYS on a compact α-almost cosymplectic manifold changes it into an almost cosymplectic manifold. This observation emphasises the transformation and refinement of the geometric structure of the manifold in the presence of an ARYS.

We also found a link between the scalar curvature (τ) and the parameters λ, a, b and n in our study of an ARYS vector field being pointwise collinear with ξ on a compact α-almost cosymplectic manifold. In particular, we obtain that $\tau = \frac{2\lambda}{(2n+1)b-2a}$, subject to the constraint $\alpha \neq \{0, \frac{(2n+1)\beta}{2}\}$. This connection provides vital insights into the interplay between geometric quantities and soliton characteristics allowing for a better understanding of the curvature dynamics of the manifold. We discovered that a 3-dimensional cosymplectic manifolds that permits a GRYS with $a \neq b$ is either flat or has a constant scalar curvature. This conclusion is useful in understanding the geometric configurations and curvature aspects of 3-dimensional cosymplectic manifolds. Moreover, we construct an example of a 3-dimensional manifold admitting a GRYS validating our results.

In Section 3.2, we have conducted a thorough investigation into the properties and structures of almost Kenmotsu manifolds that admits conformal Ricci-Yamabe solitons (CRYS). By extending the existing results on Ricci solitons and Ricci-Yamabe solitons to the more generalized setting of CRYS on $(\kappa, \mu)'$-almost Kenmotsu manifolds, we demonstrated that a $(2n+1)$-dimensional $(\kappa, \mu)'$-almost Kenmotsu manifold admitting a CRYS is locally isometric to $H^{n+1}(-4) \times \mathbb{R}^n$ provided that $2\lambda - \beta\tau \neq 4\alpha n\kappa - \left(p + \frac{2}{2n+1}\right)$. This result is significant as it reveals a specific geometric structure that these manifolds possess when they admit such solitons, establishing a notable connection to hyperbolic spaces and Euclidean spaces. Additionally, it was proven that a generalized $(\kappa, \mu)'$-almost Kenmotsu manifold admitting a CRYS is an η-Einstein manifold. For $(\kappa, \mu)'$-almost Kenmotsu manifolds admitting a conformal gradient Ricci-Yamabe soliton, it was shown that the potential vector field is a strict infinitesimal contact transformation. This result indicates that the vector field associated with the soliton preserves the contact structure of the manifold, thereby maintaining the integrity of the underlying almost Kenmotsu structure. Furthermore, an example of a 3-dimensional $(\kappa, \mu)'$-almost Kenmotsu manifold is constructed to illustrate and verify the theoretical findings. The results not only extend the theory of Ricci-Yamabe solitons to a broader class of manifolds but also provide a deeper understanding of the interaction between soliton equations and the geometric structures of almost Kenmotsu manifolds. The local isometry to $H^{n+1}(-4) \times \mathbb{R}^n$ opens up new avenues for exploring the curvature and topology of these manifolds in relation to well-known geometric spaces. Moreover, the η-Einstein condition and the strict infinitesimal contact transformation property offer new insights into the curvature conditions and the preservation of geometric structures under the influence of CRYS. These findings could have further implications in the study of geometric flows and their limiting behaviors in various geometric contexts.

Chapter 4

On Invariant submanifolds

This chapter deals with the invariant submanifolds of hyperbolic Kenmotsu manifolds and its characterization when η-Ricci-Bourguignon soliton is admitted as its metric.

4.1 Invariant submanifolds of hyperbolic Kenmotsu Manifolds

Definition 4.1. *A submanifold M of a hyperbolic Kenmotsu manifold $\tilde{M}$ is termed invariant if the structure vector field ζ is tangent to M at every point and ϕX is tangent to M for any vector field X tangent to M at every point, i.e., $\phi(TM) \subset TM$ at every point of M.*

It is easy to see that for invariant submanifolds of a hyperbolic Kenmotsu manifolds, we have

$$\mu(X, \zeta) = 0, \tag{4.1}$$

for any $X \in \Gamma(TM)$.

Proposition 4.1. *Suppose M is an invariant submanifold of a hyperbolic Kenmotsu manifold $\tilde{M}$. Then the following equations are satisfied:*

$$\nabla_X \zeta = -X - \eta(X)\zeta = -\phi^2 X, \tag{4.2}$$

$$\phi\mu(X,Y) = \mu(\phi X, Y) = \mu(X, \phi Y), \tag{4.3}$$

$$(\nabla_X \phi)Y = g(\phi X, Y)\zeta - \eta(Y)\phi X, \tag{4.4}$$

$$R(X,Y)\zeta = \eta(Y)X - \eta(X)Y, \tag{4.5}$$

where ∇, μ, and R represent the induced Levi-Civita connection, shape operator, and Riemannian curvature tensor of M, respectively.

Proof. By employing (1.41), (1.42), (1.44), (1.55), (1.59), and (4.1), we can directly derive the required results. □

Therefore, in view of the above proposition, we can state the following lemma:

Lemma 4.1. *A submanifold M that is invariant under a hyperbolic Kenmotsu manifold $\tilde{M}$ is itself a hyperbolic Kenmotsu manifold and is a minimal submanifold.*

Proof. The first part is a direct consequence of Proposition 4.1.

For the second part, let us consider an orthonormal basis $E_1, ..., E_{2n+1}$ of M such that $E_{n+x} = \phi E_x (x = 1, ..., n)$ and $E_{2n+1} = \zeta$. Then, according to (4.1) and (4.4), we obtain

$$\mu(\phi E_i, \phi E_i) = \phi^2 \mu(E_i, E_i) = -\mu(E_i, E_i).$$

Thus, we have

$$Tr(\mu) = \sum_{i=1}^{2n+1} (\mu(E_i, E_i) + \mu(\phi E_i, \phi E_i)) + \mu(\zeta, \zeta) = 0.$$

This completes the proof. □

Now $\tilde{R} \cdot \mu$ is given by

$$(\tilde{R}(X,Y) \cdot \mu)(Z,U) = R^{\perp}(X,Y)\mu(Z,U) - \mu(R(X,Y)Z,U) \\ -\mu(Z,R(X,Y)U), \qquad (4.6)$$

for all $X, Y, Z, U \in \Gamma(TM)$, where

$$R^{\perp}(X,Y) = [\nabla^{\perp}_X, \nabla^{\perp}_Y] - \nabla^{\perp}_{[X,Y]}.$$

If $\tilde{R} \cdot \mu = 0$, then the submanifold is said to be semiparallel. Arslan et al. (1990) defined and studied submanifolds satisfying the condition $\tilde{R}(X,Y) \cdot \tilde{\nabla}\mu = 0$ for all $X, Y \in \Gamma(TM)$ and called it as 2-semiparallel. We can write

$$(\tilde{R}(X,Y) \cdot \tilde{\nabla}\mu)(Z,U,V) = R^{\perp}(X,Y)(\tilde{\nabla}\mu)(Z,U,V) \\ - (\tilde{\nabla}\mu)(R(X,Y)Z,U,V) \\ - (\tilde{\nabla}\mu)(Z,R(X,Y)U,V) \\ - (\tilde{\nabla}\mu)(Z,U,R(X,Y)V), \qquad (4.7)$$

for all $X, Y, Z, U, V \in \Gamma(TM)$ and $(\tilde{\nabla}\mu)(Z,U,V) = (\tilde{\nabla}_Z\mu)(U,V)$.

For a $(0,k)$-type tensor field J, $k \geq 1$ and a $(0,2)$-type tensor field G on a Riemannian manifold M. $D(G,J)$-tensor field is defined by Atceken and Uygun (2021) which is as follows:

$$D(G,J)(J_1, J_2, ...J_k; X, Y) = -J((X \wedge_G Y)J_1, J_2, ...J_k) \\ .... - J(J_1, J_2, ...J_{k-1}, (X \wedge_G Y)J_k), \qquad (4.8)$$

for all $J_1, J_2, ...J_k, X, Y \in \Gamma(TM)$, where

$$(X \wedge_G Y)Z = G(Y,Z)X - G(X,Z)Y.$$

Definition 4.2 (Atceken et al., 2020)**.** *Suppose M is a submanifold of a Rie-*

mannian manifold $(\tilde{M}, g)$. *If there exist functions* L_1, L_2, L_3, *and* L_4 *on* $\tilde{M}$ *such that*

$$\tilde{R} \cdot \mu = L_1 D(g, \mu), \tag{4.9}$$

$$\tilde{R} \cdot \tilde{\nabla}\mu = L_2 D(g, \tilde{\nabla}\mu), \tag{4.10}$$

$$\tilde{R} \cdot \mu = L_3 D(Ric, \mu), \tag{4.11}$$

$$\tilde{R} \cdot \tilde{\nabla}\mu = L_4 D(Ric, \tilde{\nabla}\mu), \tag{4.12}$$

then M *is, respectively, pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel, and 2-Ricci-generalized pseudoparallel submanifold. In fact, if* $L_1 = 0$ *or* $L_3 = 0$ *(resp.,* $L_2 = 0$ *or* $L_4 = 0$*), then* M *is called semiparallel (resp., 2-semiparallel).*

Following this, we establish certain characteristic arguments for totally geodesic submanifolds of hyperbolic Kenmotsu manifolds.

Theorem 4.1. *Consider* M *as an invariant submanifold of a hyperbolic Kenmotsu manifold* $\tilde{M}$. *Then* M *is totally geodesic if and only if its second fundamental form is parallel.*

Proof. Suppose $\tilde{\nabla}\mu = 0$, then from (1.58) we have

$$(\tilde{\nabla}_X \mu)(Y, Z) = \nabla^{\perp}_X \mu(Y, Z) - \mu(\nabla_X Y, Z) - \mu(Y, \nabla_X Z) = 0. \tag{4.13}$$

Replacing Z by ζ in (4.13) and using (1.35), (4.1) and (4.2) gives

$$\mu(X, Y) = 0,$$

which completes the proof. □

Theorem 4.2. *Suppose M is an invariant submanifold of a hyperbolic Kenmotsu manifold $\tilde{M}$. If we assume that M is a pseudoparallel submanifold of hyperbolic Kenmotsu manifold, then it is either totally geodesic or the function L_1 satisfies $L_1 = -1$.*

Proof. Consider that M is a pseudoparallel submanifold of a hyperbolic Kenmotsu manifold, then from (4.9) we have

$$(\tilde{R}(X,Y)\cdot\mu)(Z,U) = L_1 D(g,\mu)(Z,U;X,Y), \tag{4.14}$$

for all $X, Y, Z, U \in \Gamma(TM)$. Utilizing (4.6) and (4.8) in (4.14) yields

$$\begin{aligned} R^{\perp}(X,Y)\mu(Z,U) - \mu(R(X,Y)Z,U) - \mu(Z,R(X,Y)U) \\ = -L_1\{\mu((X\wedge_g Y)Z,U) + \mu(Z,(X\wedge_g Y)U)\}. \end{aligned} \tag{4.15}$$

Setting $X = Z = \zeta$ and using (4.1) in (4.15) we get

$$(L_1+1)\mu(Y,U) = 0,$$

for all $U, Y \in \Gamma(TM)$ which completes the proof. □

Corollary 4.1. *Consider M as an invariant submanifold of a hyperbolic Kenmotsu manifold $\tilde{M}$. Then M is totally geodesic if and only if it is semiparallel.*

Theorem 4.3. *Suppose M is an invariant submanifold of a hyperbolic Kenmotsu manifold $\tilde{M}$. If M is a 2-pseudoparallel submanifold of a hyperbolic Kenmotsu manifold, then it is either totally geodesic or the function L_2 satisfies $L_2 = -3$.*

Proof. Suppose M is 2-pseudoparallel, then from (4.10) we have

$$(\tilde{R}(X,Y)\cdot\tilde{\nabla}\mu)(U,V,Z) = L_2 D(g,\tilde{\nabla}\mu)(U,V,Z;X,Y), \tag{4.16}$$

for all $X, Y, Z, U, V \in \Gamma(TM)$. Making use of (4.6) and (4.8) in (4.16) results in

$$\begin{aligned}R^{\perp}(X,Y)(\tilde{\nabla}_U\mu)(V,Z) - (\tilde{\nabla}_{R(X,Y)U}\mu)(V,Z) - (\tilde{\nabla}_U\mu)(R(X,Y)V,Z) \\ - (\tilde{\nabla}_U\mu)(V,R(X,Y)Z) = -L_2\{(\tilde{\nabla}_{(X\wedge_g Y)U}\mu)(V,Z) \\ + (\tilde{\nabla}_U\mu)((X\wedge_g Y)V,Z) + (\tilde{\nabla}_U\mu)(V,(X\wedge_g Y)Z)\}.\end{aligned} \tag{4.17}$$

Putting $X = V = \zeta$ in (4.17) yields

$$\begin{aligned}R^{\perp}(\zeta,Y)(\tilde{\nabla}_U\mu)(\zeta,Z) - (\tilde{\nabla}_{R(\zeta,Y)U}\mu)(\zeta,Z) - (\tilde{\nabla}_U\mu)(R(\zeta,Y)\zeta,Z) \\ - (\tilde{\nabla}_U\mu)(\zeta,R(\zeta,Y)Z) = -L_2\{(\tilde{\nabla}_{(\zeta\wedge_g Y)U}\mu)(\zeta,Z) \\ + (\tilde{\nabla}_U\mu)((\zeta\wedge_g Y)\zeta,Z) + (\tilde{\nabla}_U\mu)(\zeta,(\zeta\wedge_g Y)Z)\}.\end{aligned} \tag{4.18}$$

Solving the individual terms of (4.18) and making use of (1.35), (1.58), (4.1), (4.2) and (4.5), yields the following:

$$\begin{aligned}(\tilde{\nabla}_{(\zeta\wedge_g Y)U}\mu)(\zeta,Z) &= \mu(\phi^2(\zeta\wedge_g Y)U,Z) \\ &= -\eta(U)\mu(Y,Z).\end{aligned} \tag{4.19}$$

$$\begin{aligned}(\tilde{\nabla}_U\mu)((\zeta\wedge_g Y)\zeta,Z) &= (\tilde{\nabla}_U\mu)(\eta(Y)\zeta - Y,Z) \\ &= -\mu(\nabla_U\zeta,Z)\eta(Y) - (\tilde{\nabla}_U\mu)(Y,Z) \\ &= \mu(\phi^2U,Z)\eta(Y) - (\tilde{\nabla}_U\mu)(Y,Z) \\ &= \mu(U,Z)\eta(Y) - (\tilde{\nabla}_U\mu)(Y,Z).\end{aligned} \tag{4.20}$$

$$\begin{aligned}(\tilde{\nabla}_U\mu)(\zeta,(\zeta\wedge_g Y)Z) &= (\tilde{\nabla}_U\mu)(\zeta,-\eta(Z)Y) \\ &= -\mu(U,Y)\eta(Z).\end{aligned} \tag{4.21}$$

$$\begin{aligned}R^{\perp}(\zeta,Y)(\tilde{\nabla}_U\mu)(\zeta,Z) &= -R^{\perp}(\zeta,Y\mu(\nabla_U\zeta,Z) \\ &= R^{\perp}(\zeta,Y)\mu(U,Z).\end{aligned} \tag{4.22}$$

$$
\begin{aligned}
(\tilde{\nabla}_{R(\zeta,Y)U}\mu)(\zeta,Z) &= -\mu(\nabla_{R(\zeta,Y)U}\zeta,Z) \\
&= \mu(\phi^2 R(\zeta,Y)U,Z) \\
&= \mu(Y,Z)\eta(U). \qquad (4.23) \\
(\tilde{\nabla}_U\mu)(R(\zeta,Y)\zeta,Z) &= (\tilde{\nabla}_U\mu)(Y+\eta(Y)\zeta,Z) \\
&= \mu(\nabla_U\zeta,Z)\eta(Y)+(\tilde{\nabla}_U\mu)(Y,Z) \\
&= -\mu(U,Z)\eta(Y)+(\tilde{\nabla}_U\mu)(Y,Z). \qquad (4.24) \\
(\tilde{\nabla}_U\mu)(\zeta,R(\zeta,Y)Z) &= -\mu(\nabla_U\zeta,R(\zeta,Y)Z) \\
&= \mu(\phi^2 U,R(\zeta,Y)Z) \\
&= -\mu(U,Y)\eta(Z). \qquad (4.25)
\end{aligned}
$$

Combining (4.18)-(4.25) and then replacing Z by ζ in the forgoing equation leads to

$$(3+L_2)\mu(Y,U)=0,$$

for all $Y,U \in \Gamma(TM)$. This completes the proof. □

Corollary 4.2. *Suppose M is an invariant submanifold of a hyperbolic Kenmotsu manifold. Then M is 2-semiparallel if and only if it is totally geodesic.*

Theorem 4.4. *Consider M to be an invariant submanifold of a hyperbolic Kenmotsu manifold $\tilde{M}$. If M is a Ricci-generalized pseudoparallel submanifold of a hyperbolic Kenmotsu manifold, then M is either totally geodesic or the function L_3 satisfies $L_3=-\frac{1}{2n}$.*

Proof. Suppose that M is Ricci-generalized pseudoparallel submanifold of a hyperbolic Kenmotsu manifold, then from (4.11) becomes

$$(\tilde{R}(X,Y)\cdot\mu)(Z,U)=L_3 D(Ric,\mu)(Z,U;X,Y), \qquad (4.26)$$

for all $X, Y, Z, U \in \Gamma(TM)$. Making use of (4.6) and (4.8) in (4.26) gives

$$R^{\perp}(X,Y)\mu(Z,U) - \mu(R(X,Y)Z,U) - \mu(Z,R(X,Y)U)$$
$$= -L_3\{\mu((X \wedge_{Ric} Y)Z,U) + \mu(Z,(X \wedge_{Ric} Y)U)\}. \quad (4.27)$$

Setting $X = U = \zeta$ in (4.27) and then using (4.1) and (4.5) we obtain

$$(1 + 2nL_3)\mu(Z,Y) = 0,$$

for all vector fields Z, Y. This completes the proof. □

Theorem 4.5. *Suppose that M is an invariant submanifold of a hyperbolic Kenmotsu manifold $\tilde{M}$. If M is a 2-generalized Ricci pseudoparallel submanifold of a hyperbolic Kenmotsu manifold, then M is either totally geodesic or the function L_4 satisfies $L_4 = -\frac{1}{2n}$.*

Proof. By hypothesis, from (4.12) we have

$$(\tilde{R}(X,Y) \cdot \tilde{\nabla}\mu)(U,V,Z) = L_4 D(Ric, \tilde{\nabla}\mu)(U,V,Z;X,Y), \quad (4.28)$$

for all vector fields X, Y, Z, U, V. Making use of (4.6) and (4.8) in (4.28) gives

$$R^{\perp}(X,Y)(\tilde{\nabla}_U\mu)(V,Z) - (\tilde{\nabla}_{R(X,Y)U}\mu)(V,Z) - (\tilde{\nabla}_U\mu)(R(X,Y)V,Z)$$
$$-(\tilde{\nabla}_U\mu)(V,R(X,Y)Z) = -L_4\{(\tilde{\nabla}_{(X\wedge_{Ric}Y)U}\mu)(V,Z)$$
$$+(\tilde{\nabla}_U\mu)((X \wedge_{Ric} Y)V,Z) + (\tilde{\nabla}_U\mu)(V,(X \wedge_{Ric} Y)Z)\}. \quad (4.29)$$

Replacing $X = V = \zeta$ in (4.29) gives

$$R^{\perp}(\zeta,Y)(\tilde{\nabla}_U\mu)(\zeta,Z) - (\tilde{\nabla}_{R(\zeta,Y)U}\mu)(\zeta,Z) - (\tilde{\nabla}_U\mu)(R(\zeta,Y)\zeta,Z)$$
$$-(\tilde{\nabla}_U\mu)(\zeta,R(\zeta,Y)Z) = -L_4\{(\tilde{\nabla}_{(\zeta\wedge_{Ric}Y)U}\mu)(\zeta,Z)$$
$$+(\tilde{\nabla}_U\mu)((\zeta \wedge_{Ric} Y)\zeta,Z) + (\tilde{\nabla}_U\mu)(\zeta,(\zeta \wedge_{Ric} Y)Z)\}. \quad (4.30)$$

Computing each terms separately gives

$$
\begin{aligned}
(\tilde{\nabla}_{(\zeta\wedge_{Ric}Y)U}\mu)(\zeta, Z) &= -\mu(\nabla_{(\zeta\wedge_{Ric}Y)U}\zeta, Z) \\
&= \mu(\phi^2(\zeta\wedge_{Ric} Y)U, Z) \\
&= -Ric(\zeta, U)\mu(Y, Z). \qquad (4.31)
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{\nabla}_U\mu)((\zeta\wedge_{Ric} Y)\zeta, Z) &= (\tilde{\nabla}_U\mu)(Ric(Y,\zeta)\zeta - Ric(\zeta,\zeta)Y, Z) \\
&= Ric(Y,\zeta)(\tilde{\nabla}_U\mu)(\zeta, Z) - Ric(\zeta,\zeta)(\tilde{\nabla}_U\mu)(Y, Z). \qquad (4.32)
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{\nabla}_U\mu)(\zeta, (\zeta\wedge_{Ric} Y)Z) &= (\tilde{\nabla}_U\mu)(\zeta, Ric(Y,Z)\zeta - Ric(\zeta, Z)Y) \\
&= -Ric(\zeta, Z)(\tilde{\nabla}_U\mu)(\zeta, Y). \qquad (4.33)
\end{aligned}
$$

Utilizing (4.22)-(4.25) and (4.31)-(4.33) in (4.30) then setting $Z = \zeta$ results in

$$(1 + 2nL_4)\mu(Y, U) = 0,$$

for all $Y, U \in \Gamma(TM)$. This completes the proof. □

4.1.1 3-dimensional invariant submanifold of hyperbolic Kenmotsu manifold

Lemma 4.2. *Consider an invariant submanifold M of a hyperbolic Kenmotsu manifold $\tilde{M}$. Then there exists the distributions D and $D^\perp$ such that*

$$TM = D \oplus D^\perp \oplus <\zeta>, \quad \phi(D) \subset D^\perp and \quad \phi(D^\perp) \subset D.$$

Proof. The result is due to Lemma 4.1 of Chaubey et al. (2022) and Proposition 6.1 of Shaikh et al. (2016). □

Theorem 4.6. *A 3-dimensional submanifold M of a hyperbolic Kenmotsu manifold $\tilde{M}$ is invariant if and only if it is totally geodesic.*

Proof. Suppose that a 3-dimensional submanifold M of a hyperbolic Kenmotsu manifold $\tilde{M}$ is invariant, then from (4.3) for $X_1, Y_1 \in D$ we have

$$\phi\mu(X_1, Y_1) = \mu(\phi X_1, Y_1) = \mu(X_1, \phi Y_1). \tag{4.34}$$

Operating (4.34) by ϕ and using (1.35) gives

$$\phi\mu(X_1, \phi Y_1) = \phi^2\mu(X_1, Y_1) = \mu(X_1, Y_1) + \eta(\mu(X_1, Y_1))\zeta. \tag{4.35}$$

Since $\mu(X_1, Y_1) \subset T^{\perp}M$, $\mu(X_1, Y_1)$ is orthogonal to $\zeta \in TM$. In consequence, from (4.34) and (4.35) we get

$$\mu(\phi X_1, \phi Y_1) = \mu(X_2, Y_2) = -\mu(X_1, Y_1), \tag{4.36}$$

where $X_2 = \phi X_1, Y_2 = \phi Y_1 \in D^{\perp}$. Now for any $X_1, Y_1 \in D$ and $X_2, Y_2 \in D^{\perp}$ we see that

$$\mu(X_1 + X_2 + \zeta, Y_1) = \mu(X_1, Y_1) + \mu(X_2, Y_1) + \mu(\zeta, Y_1),$$
$$\mu(X_1 + X_2 + \zeta, Y_2) = \mu(X_1, Y_2) + \mu(X_2, Y_2) + \mu(\zeta, Y_2),$$
$$\mu(X_1 + X_2 + \zeta, \zeta) = \mu(X_1, \zeta) + \mu(X_2, \zeta) + \mu(\zeta, \zeta).$$

In view of above equations and (4.22), we can write

$$\mu(X_1 + X_2 + \zeta, Y_1 + Y_2 + \zeta) = \mu(X_2, Y_1) + \mu(X_1, Y_2). \tag{4.37}$$

Taking $U, V \in TM$ as $U = X_1 + X_2 + \zeta$ and $V = Y_1 + Y_2 + \zeta$, (4.37) becomes

$$\mu(U, V) = \mu(X_2, Y_1) + \mu(X_1, Y_2).$$

Solving the above equation with ϕ and utilizing (4.34) and (4.36), we obtain

$$\phi\mu(U, V) = \mu(X_2, \phi Y_1) + \mu(X_1, \phi Y_2) = 0.$$

Again operating by ϕ gives $\mu(U, V) = 0$, for any vector fields U, V. Therefore, M is totally geodesic.

Conversely, suppose M is totally geodesic, then

$$\mu(X,Y) = 0, \quad \forall\ X,Y \in TM.$$

We have to show that $\phi X \notin T^{\perp}M$. To demonstrate this, suppose for contradiction that the vector field ϕX has a component, say LX, along the normal vector field of M. Clearly, $A_{LX}Y \in TM$. Let $Z = A_{LX}Y \neq 0$. Then,

$$g(Z,Z) = g(A_{LX}Y,Z) = g(\mu(Y,Z),LX) = 0.$$

Since Z is a non-null and non-zero vector field in TM, it follows that $g(Z,Z) \neq 0$, which is a contradiction. Therefore, $\phi X \in TM$ and thus M is invariant. $\square$

4.1.2 η-Ricci-Bourguignon solitons on invariant submanifolds of hyperbolic Kenmotsu manifolds

Let M be an invariant submanifold of a hyperbolic Kenmotsu manifold. Consider the equation

$$\frac{1}{2}(\mathcal{L}_{\zeta}g)(X,Y) + Ric(X,Y) + (\lambda + \rho\tau)g(X,Y) + \omega\eta(X)\eta(Y) = 0, \qquad (4.38)$$

for any $X, Y \in \Gamma(TM)$, where $\mathcal{L}_{\zeta}g$ denote the Lie-derivative of g with respect to ζ, Ric is the Ricci tensor of g, τ is the scalar curvature and λ, ρ and ω are real constants. The data $(g, \zeta, \lambda, \omega)$ satisfying (4.38) is referred to as an η-Ricci-Bourguignon soliton on M (for more detail, see Chaubey et al., 2022; Dey and Roy, 2022; Dogru, 2023). Specifically, if $\omega = 0$, it is known as a Ricci-Bourguignon soliton (Khatri and Singh, 2024a) and it is classified as growing, steady, or decreasing depending on whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively.

Utilizing (4.2), we have

$$(\mathcal{L}_\zeta g)(X,Y) = g(\nabla_X \zeta, Y) + g(X, \nabla_Y \zeta)$$
$$= -2\{g(X,Y) + \eta(X)\eta(Y)\}.$$

In view of the last equation in (4.38) gives

$$Ric(X,Y) = (1 - \lambda - \rho\tau)g(X,Y) + (1 - \omega)\eta(X)\eta(Y), \tag{4.39}$$

for any vector fields X, Y on M. Thus we conclude with the following:

Theorem 4.7. *Let an invariant submanifold M of a hyperbolic Kenmotsu manifold $\tilde{M}$ admit an η-Ricci-Bourguignon soliton, then M is η-Einstein.*

Specifically, for $\omega = 0$, we have

Corollary 4.3. *Consider an invariant submanifold M of hyperbolic Kenmotsu manifold $\tilde{M}$ admitting a Ricci-Bourguignon soliton, then M is η-Einstein.*

Setting $X = Y = \zeta$ in (4.39) and making use of (4.5), we obtain

$$\lambda = \omega - 2n - \rho\tau.$$

Theorem 4.8. *If an invariant submanifold M of a hyperbolic Kenmotsu manifold $\tilde{M}$ admits an η-Ricci-Bourguignon soliton, then $\lambda = \omega - 2n - \rho\tau$.*

Corollary 4.4. *Suppose that an invariant submanifold M of a hyperbolic Kenmotsu manifold $\tilde{M}$ admit a Ricci-Bourguignon soliton as its metric. Then M is η-Einstein and the soliton is shrinking.*

Assume that invariant submanifold M of a hyperbolic Kenmotsu manifold $\bar{M}$ admits an η-Ricci-Bourguignon soliton as its metric. Let the unit timelike vector field ζ of M is the gradient of some smooth function ψ, that is, $\zeta = grad\ \psi$. Then equation (4.38) becomes

$$\frac{1}{2}\{g(\nabla_X grad\ \psi, Y) + g(X, \nabla_Y grad\ \psi)\} + Ric(X,Y) + (\lambda + \rho\tau)g(X,Y) + \omega\eta(X)\eta(Y) = 0.$$

Contracting the foregoing expression over X and Y yields

$$\nabla^2\psi = \Omega, \tag{4.40}$$

where $\Omega = -[(2n+1)\lambda + \{(2n+1)\rho+1\}\tau + \omega]$, ∇^2 denotes the Laplacian operator of g and τ represents the scalar curvature of M.

A smooth function ψ on a Riemannian manifold M is said to indulge Poisson's equation if (4.40) holds true for some smooth function Ω on M. In particular, if $\Omega = 0$, then Poisson's equation simplifies to the Laplace equation, and ψ is called harmonic.

In view of these facts, we conclude the following theorems:

Theorem 4.9. *Consider an invariant submanifold M of a hyperbolic Kenmotsu manifold $\tilde{M}$ that admits an η-Ricci-Bourguignon soliton. If the unit timelike vector field of M is the gradient of a smooth function ψ, then ψ satisfies Poisson's equation (4.40).*

Theorem 4.10. *Consider an invariant submanifold M of a hyperbolic Kenmotsu manifold $\tilde{M}$ that admits an η-Ricci soliton. If the unit timelike vector field of M is the gradient of a smooth function ψ, then ψ satisfies the Laplace equation if and only if $\lambda = -\frac{\{(2n+1)\rho+1\}\tau+\omega}{2n+1}$.*

Now, let us consider a concircular vector field ξ and thus it satisfies

$$\nabla_X\xi = \psi X, \tag{4.41}$$

for any $X \in TM$ and ψ is a smooth function. For an invariant submanifold, from

Lemma 4.2, we have

$$\xi = \xi^T + \xi^\perp, \tag{4.42}$$

where $\xi \in TM$, $\xi^T \in D$ and $\xi^\perp \in D^\perp$. Since ξ is a concircular vector field, we can write

$$\psi X = \tilde{\nabla}_X \xi^T + \tilde{\nabla}_X \xi^\perp, \tag{4.43}$$

for any $X \in D$. Now, from (1.55) and (1.56), we have

$$\psi X = \nabla_X \xi^T + \mu(X, \xi^T) - A_{\xi^\perp} X + \nabla^\perp_X \xi^\perp. \tag{4.44}$$

Comparing the tangential and normal components of the above expression, we get

$$\mu(X, \xi^T) = -\nabla^\perp_X \xi^\perp, \ \nabla_X \xi^T = \psi X - A_{\xi^\perp} X. \tag{4.45}$$

Next, we prove the following:

Theorem 4.11. *Suppose M is an invariant submanifold of a hyperbolic Kenmotsu manifold that admits an η-Ricci-Bourguignon soliton with a concircular vector field ξ. Then, the Ricci tensor on the invariant distribution is expressed as*

$$Ric(X, Y) = -[(\psi + \lambda + \rho\tau) g(X, Y) - g(\mu(X, Y), \xi^\perp) + \omega\eta(X)\eta(Y)].$$

Proof. We know that the Lie derivative has the form

$$(\mathcal{L}_{\xi^T} g)(X, Y) = g(\nabla_X \xi^T, Y) + g(X, \nabla_Y \xi^T). \tag{4.46}$$

Utilizing (4.45) yields

$$(\mathcal{L}_{\xi^T} g)(X, Y) = 2\psi g(X, Y) - 2g(\mu(X, Y), \xi^\perp). \tag{4.47}$$

Then, substituting the above expression in (4.38) completes the proof. □

Also, we we assume the invariant distribution to be D-geodesic, then $g(\mu(X, Y), \xi^\perp)$ vanishes and hence, we state:

Corollary 4.5. *Suppose that an invariant submanifold M of a hyperbolic Kenmotsu manifold admits an η-Ricci-Bourguignon soliton with a concircular vector field ξ. If the invariant distribution is D-geodesic, then the invariant distribution is η-Einstein.*

Furthermore, if we assume $\psi = 1$, then we get a concurrent vector field ξ. As a consequence, we get the following results:

Theorem 4.12. *Consider an invariant submanifold M of a hyperbolic Kenmotsu manifold admitting an η-Ricci-Bourguignon soliton as its metric with a concurrent vector field ξ. Then, the Ricci tensor on the invariant distribution is given by the following relation:*

$$Ric(X,Y) = -[(1+\lambda+\rho\tau)g(X,Y) - g(\mu(X,Y),\xi^{\perp}) + \omega\eta(X)\eta(Y)].$$

Corollary 4.6. *Assume that M is an invariant submanifold of a hyperbolic Kenmotsu manifold that admits an η-Ricci-Bourguignon soliton with a concurrent vector field ξ. If the invariant distribution is D-geodesic, then the invariant distribution is η-Einstein.*

Remark 4.1. *In view of the results obtained in this section, we can extract the results from every theorems and corollaries for η-Ricci soliton, η-Schouten soliton and η-Einstein soliton by putting different values for ρ as $\frac{1}{2(n-1)}$ and $\frac{1}{2}$ respectively.*

4.1.3 Examples of an invariant submanifold of a hyperbolic Kenmotsu manifolds

Example 4.1. *Consider a 5-dimensional manifold $\tilde{M} = \{(x,y,z,v,t) \in \mathbb{R}^5 : t \neq 0\}$ where (x,y,z,v,t) are the standard coordinates in $\mathbb{R}^5$.*

Now let $\{E_1, E_2, E_3, E_4, E_5\}$ be a linearly independent global frame on $\tilde{M}$. Let g be a Riemannian metric on $\tilde{M}$ defined as

$$g(E_i, E_j) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Set $E_5 = \zeta$, then it is obvious that $\eta(E_5) = -1$ and $\eta(E_i) = 0$ for $i = 1, 2, 3, 4$. Also, we define (1,1)-tensor ϕ as

$$\phi(E_1) = -E_2, \quad \phi(E_2) = -E_1, \quad \phi(E_3) = E_4, \quad \phi(E_4) = E_3, \quad \phi(E_5) = 0.$$

As an immediate result of the above equations we can see that $\phi^2 X = X + \eta(X)\zeta$ and $g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y)$, for any vector fields X, Y on $\tilde{M}$. Clearly, the structures we defined satisfies the condition of an almost hyperbolic contact metric manifold. Let $\tilde{\nabla}$ be the Levi-Civita connection with respect to the metric g. Then we define the Lie bracket for our vector fields $\{E_1, E_2, E_3, E_4, E_5\}$ as follows:

$$[E_i, E_j] = \begin{cases} -E_i, & \text{if} \quad i = 1, 2, 3, 4; j = 5, \\ 0, & \text{otherwise.} \end{cases} \tag{4.48}$$

Using Koszul formula and (4.48), we obtain the following

$$\tilde{\nabla}_{E_1} E_1 = E_5, \tilde{\nabla}_{E_1} E_2 = 0, \tilde{\nabla}_{E_1} E_3 = 0, \tilde{\nabla}_{E_1} E_4 = 0, \tilde{\nabla}_{E_1} E_5 = -E_1,$$
$$\tilde{\nabla}_{E_2} E_2 = E_5, \tilde{\nabla}_{E_2} E_3 = 0, \tilde{\nabla}_{E_2} E_4 = 0, \tilde{\nabla}_{E_2} E_5 = -E_2, \tilde{\nabla}_{E_3} E_3 = E_5,$$
$$\tilde{\nabla}_{E_3} E_4 = 0, \tilde{\nabla}_{E_3} E_5 = -E_3, \tilde{\nabla}_{E_4} E_5 = -E_4, \tilde{\nabla}_{E_4} E_4 = E_5, \tilde{\nabla}_{E_5} E_5 = 0.$$

We can easily see from the above relations that the manifold satisfies $\tilde{\nabla}_X \zeta = -\phi^2 X$, for $\zeta = E_5$. Hence, $\tilde{M}$ is a hyperbolic Kenmotsu manifold.

Let M be a subset of $\tilde{M}$ and consider the isometric immersion $\pi : M \to \tilde{M}$ defined by $\pi(x, z, t) = (x, 0, z, 0, t)$. Clearly, $M = \{(x, z, t) \in \mathbb{R}^3 : (x, z, t) \neq 0\}$

is a 3-dimensional submanifold of $\tilde{M}$, where the triplet (x, z, t) are standard coordinates in $\mathbb{R}^3$. We choose the vector fields $\{E_1, E_3, E_5\}$ such that their Lie bracket is defined as:

$$[E_i, E_j] = \begin{cases} -E_i, & \text{if } \ i = 1, 3, \ \text{ and } \ j = 5, \\ 0, & \text{otherwise.} \end{cases}$$

We define g_1 such that $\{E_1, E_3, E_5\}$ is an orthonormal basis of M as follows:

$$g_1(E_i, E_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Set $\zeta = E_5$. Then define 1-form η_1 and (1,1)-tensor field ϕ_1 as $\eta_1(\cdot) = g_1(\cdot, E_5)$ and $\phi_1(E_1) = E_3, \phi_1(E_3) = E_1, \phi_1(E_5) = 0$.

From the above equations, it is obvious that

$$\eta_1(E_5) = -1, \phi_1^2(X) = X + \eta_1(X)E_5,$$
$$g_1(\phi_1 X, \phi_1 Y) = -g_1(X, Y) - \eta_1(X)\eta_1(Y),$$

for vector fields X, Y on M. Clearly, $M(\eta_1, g_1, E_5, \phi_1)$ is an invariant submanifold of $\tilde{M}$. Let ∇ be the Levi-Civita connection induced by the metric g_1, then we have the following:

$$\nabla_{E_1} E_1 = E_5, \nabla_{E_1} E_3 = 0, \nabla_{E_1} E_5 = -E_1,$$
$$\nabla_{E_3} E_3 = E_5, \nabla_{E_3} E_5 = -E_3, \nabla_{E_5} E_5 = 0.$$

One can see that $M(g_1, \eta_1, \phi_1, E_5)$ forms a 3-dimensional hyperbolic Kenmotsu manifold with $\zeta = E_5$. Thus, Lemma 4.1 is verified.

Let μ be the second fundamental form, then making use of (1.55) and the foregoing equations, we obtaine

$$\mu(X, Y) = 0,$$

for any vector field X, Y on M. Thus, M is a totally geodesic submanifold of $\tilde{M}$. Hence, Theorem 4.6 and Corollary 4.1, 4.2 are verified.

Example 4.2. *Let $\mathbb{R}^n$ be an n-dimensional real number space. Define $M^5 = \{(x, y, z, v, u) : x_i \in \mathbb{R}, i = 1, 2, ..., 5 \;\; and \;\; z \neq 0\}$.*

Let $\{E_1, E_2, E_3, E_4, E_5\}$ be a set of linearly independent vector fields of M^5 with their Lie bracket defined by

$$[E_i, E_j] = \begin{cases} -E_i, & \text{if} \quad i = 1, 2, 4, 5; j = 3, \\ 0, & \text{otherwise.} \end{cases}$$

Let g be the associated metric of M^5 which is define as

$$g(E_i, E_j) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, let us define a (1,1)-tensor field ϕ of M^5 as

$$\phi(E_1) = E_2, \quad \phi(E_2) = E_1, \quad \phi(E_3) = 0, \quad \phi(E_4) = -E_5, \quad \phi(E_5) = -E_4.$$

We know that g and ϕ are linear, thus we can easily see that the following expressions $\phi^2 E_i = E_i + \eta(E_i)E_3, g(E_i, E_3) = \eta(E_i)$ and $g(\phi E_i, \phi E_j) = -g(E_i, E_j) - \eta(E_i)\eta(E_j)$ holds for $i, j = 1, 2, 3, 4, 5$ and $\zeta = E_3$. Thus, $M^5(g, \phi, \eta, \zeta = E_3)$ is an almost hyperbolic contact metric manifold.

Let ∇ denote the Levi-Civita connection, then by Koszul's formula and above expressions, we obtain the following:

$$\nabla_{E_1}E_1 = E_3, \nabla_{E_1}E_2 = 0, \nabla_{E_1}E_3 = -E_1, \nabla_{E_1}E_4 = 0,$$
$$\nabla_{E_1}E_5 = 0, \nabla_{E_2}E_2 = E_3, \nabla_{E_2}E_3 = -E_2, \nabla_{E_2}E_4 = 0,$$
$$\nabla_{E_2}E_5 = 0, \nabla_{E_3}E_3 = 0, \nabla_{E_3}E_4 = E_4, \nabla_{E_3}E_5 = E_5,$$
$$\nabla_{E_4}E_4 = E_3, \nabla_{E_4}E_5 = 0, \nabla_{E_5}E_5 = E_3.$$

Clearly, from the foregoing equations, we can clearly see that $M^5(g, \phi, \zeta, \eta)$ is a hyperbolic Kenmotsu manifold for $\zeta = E_3$.

Let M^3 be a subset of M^5. Now consider an isometric immersion $\pi : M^3 \to M^5$ define as $\pi(x, y, z) = (x, y, z, 0, 0)$ where (x, y, z) is the standard coordinates in $\mathbb{R}^3$. Clearly, $M^3 = \{(x, y, z) \in \mathbb{R}^3 \text{ and } (x, y, z) \neq 0\}$ is a submanifold of M^5. Let $\{E_1, E_2, E_3\}$ be the basis of M^3 whose Lie bracket is defined as

$$[E_i, E_j] = \begin{cases} -E_i, & \text{if } \quad i = 1, 2; j = 3, \\ 0, & \text{otherwise.} \end{cases}$$

Let us define the associate metric g_1 of M^3 as

$$g_1(E_i, E_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Also, 1-from η_1 and (1,1)-tensor field ϕ_1 are define as follows:

$$\phi_1(E_1) = -E_2, \quad \phi_1(E_2) = -E_1, \quad \phi_1(E_3) = 0 \quad \text{and} \quad \eta_1(\cdot) = g_1(\cdot, E_3)$$

Utilizing the above equations, it is obvious that

$$\eta_1(E_3) = -1, \phi_1^2(X) = X + \eta_1(X)E_3,$$
$$g_1(\phi_1 X, \phi_1 Y) = -g_1(X, Y) - \eta_1(X)\eta_1(Y),$$

for any vector fields X, Y on M^3. Clearly, $M^3(\eta_1, g_1, E_3, \phi_1)$ is an invariant submanifold of M^5. Let ∇ be the Levi-Civita connection induced by the metric g_1, then the following results from Koszul formula:

$$\nabla_{E_1}E_1 = E_3, \nabla_{E_1}E_2 = 0, \nabla_{E_1}E_3 = -E_1,$$

$$\nabla_{E_2}E_2 = E_3, \nabla_{E_2}E_3 = -E_2, \nabla_{E_3}E_3 = 0.$$

It is obvious that $M^3(g_1, \phi_1, \eta_1, E_3)$ is also a hyperbolic Kenmotsu manifold. Thus, Lemma 4.1 is verified.

Let μ be the second fundamental form. Making use of (1.55) and the foregoing expressions yields

$$\mu(X, Y) = 0,$$

for any vector field X, Y on M^3. This shows that the 3-dimensional invariant submanifold of a hyperbolic Kenmotsu manifold is totally geodesic. Hence, Theorem 4.6 is verified.

4.2 Conclusion

This study investigates the geometric properties of invariant submanifolds within hyperbolic Kenmotsu manifolds, revealing significant results that enhance our understanding of these complex structures. The findings are crucial for theoretical research and practical applications in fields like mathematics and physics. The study reveals that the structure vector field ζ is tangent to these submanifolds, and the tensor field ϕ preserves tangency, confirming that these submanifolds retain the geometric characteristics of the ambient manifold. Additionally, we demonstrate that invariant submanifolds within hyperbolic Kenmotsu manifolds are minimal, meaning their mean curvature vector vanishes which indicate stability. This condition is essential for understanding geometric flows supported by these submanifolds, contributing to stability analysis and differential geometry applications. The study also identifies the condition under which invariant submanifolds are totally geodesic implying that geodesics in the ambient manifold remain within the submanifold which is a property crucial for applications in relativity theory and spacetime structures.

Further, we explore the curvature properties of pseudoparallel and 2- pseudoparallel submanifolds providing new insights into their geometric behavior. The results show that semiparallel submanifolds are totally geodesic while 2-pseudoparallel submanifolds are either totally geodesic or satisfy a specific functional condition. This extends our understanding of how curvature influences submanifolds' geometric behavior offering a framework for further exploration of curvature conditions in complex geometric settings. The findings have significant implications for differential geometry particularly in the study of spacetime and general relativity. The study also offers valuable tools for researchers investigating the stability and geometric flows of submanifolds and the conditions for pseudoparallelism and 2-pseudoparallelism provide new avenues for exploring the interplay between curvature and geometry potentially leading to advancements in theoretical research and practical applications in fields like material science and cosmology.

Chapter 5

Geometrical Properties of Spacetime

This chapter consists of three sections. The first section deals with Vaidya spacetime and the existence of conformal Ricci soliton in it. Section 5.2 investigates relativistic magneto fluid spacetime in the settings of $f(R)$-gravity theory and the last section is devoted to the study of string cloud spacetime stuffed in $f(R)$-gravity.

5.1 Conformal Ricci Solitons on Vaidya Spacetime

Basu and Bhattacharyya (2015) introduced the notion of conformal Ricci soliton in Kenmotsu manifolds as a limiting solution to the conformal Ricci flow introduced by Fischer (2004), which is given as below:

$$\mathcal{L}_V g + 2Ric = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g,$$

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where p is the conformal pressure and $\lambda \in \mathbb{R}$. We say that the soliton is shrinking, steady or expanding according as $\lambda > 0, \lambda = 0$ or $\lambda < 0$ respectively. Again, if $V = grad\ f$, then the above equation becomes conformal gradient Ricci soliton. Also, Siddiqi and Siddiqui (2020) studied conformal Ricci soliton in a perfect fluid spacetime. For further information about Ricci solitons and conformal Ricci solitons, see for instance (Catino et al., 2016b; Ganguly et al., 2021; Chen et al., 2022; Li et al., 2022; Li and Ganguly, 2023; Li et al., 2023; Khatri et al., 2023; Dey, 2023).

In our investigation, we employ Vaidya spacetime as a framework to elucidate the characteristics of conformal Ricci solitons within the context of gravitational collapse. Vaidya spacetime vividly portrays the dynamic collapse of a null fluid under the influence of gravity. This spacetime model captures fundamental astrophysical phenomena, including the formation of black holes and the emission of gravitational waves.

We have investigated the existence of a conformal Ricci soliton vector field on Vaidya spacetime as a result of the above findings.

5.1.1 Conformal Ricci Soliton vector field in Vaidya spacetime

The existence of a conformal Ricci soliton with its vector field V in Vaidya spacetime is investigated in this subsection.

A conformal Ricci soliton vector field (CRSVF) is a vector field V that meets the following condition:

$$\frac{1}{2}\mathcal{L}_V g + Ric = \left[\lambda - \left(\frac{p}{2} + \frac{1}{n}\right)\right] g, \tag{5.1}$$

where $\mathcal{L}_V g$ denote the Lie derivative of metric g along the vector field V, p is the conformal pressure and λ is a constant. Also, n is the dimension of the manifold that the soliton lies and in our case, we shall consider for $n = 4$ and thus (5.1) becomes

$$\frac{1}{2}\mathcal{L}_V g + Ric = \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right] g. \tag{5.2}$$

Let $V = A\partial_u + B\partial_r + C\partial_\theta + D\partial_\phi$, where A, B, C and D are smooth functions of u, r, θ and ϕ. Utilizing (1.62), we obtain the following:

$$\begin{aligned}
(\mathcal{L}_V g)(\partial_u, \partial_u) &= 2\left(\frac{2m-r}{r}\partial_u A - \partial_u B\right), \\
(\mathcal{L}_V g)(\partial_u, \partial_r) &= -\partial_u A + \left(\frac{2m-r}{r}\right)\partial_r A - \partial_r B, \\
(\mathcal{L}_V g)(\partial_u, \partial_\theta) &= r^2\partial_u C + \left(\frac{2m-r}{r}\right)\partial_\theta A - \partial_\theta B, \\
(\mathcal{L}_V g)(\partial_u, \partial_\phi) &= r^2\sin^2\theta\partial_u D + \left(\frac{2m-r}{r}\right)\partial_\phi X_1 - \partial_\phi B, \\
(\mathcal{L}_V g)(\partial_r, \partial_r) &= -2\partial_r A, \\
(\mathcal{L}_V g)(\partial_r, \partial_\theta) &= r^2\partial_r C - \partial_\theta A, \\
(\mathcal{L}_V g)(\partial_r, \partial_\phi) &= r^2\sin^2\theta\partial_r D - \partial_\phi A, \\
(\mathcal{L}_V g)(\partial_\theta, \partial_\theta) &= 2r^2\partial_\theta C, \\
(\mathcal{L}_V g)(\partial_\theta, \partial_\phi) &= r^2\sin^2\theta\partial_\theta D + r^2\partial_\phi C, \\
(\mathcal{L}_V g)(\partial_\phi, \partial_\phi) &= 2r^2\sin^2\theta\partial_\phi D.
\end{aligned} \tag{5.3}$$

Combining (1.63), (5.2) and (5.3), we get the following set of differential equations

$$\begin{cases} \left(\frac{2m-r}{r}\right)\partial_u A - \partial_u B + \frac{2m'}{r^2} = \left[\lambda - \left(\frac{p}{2}+\frac{1}{4}\right)\right]\left(\frac{2m-r}{r}\right), \\ \left(\frac{2m-r}{r}\right)\partial_r A - \partial_u A - \partial_r B = -\left[2\lambda - \left(p+\frac{1}{2}\right)\right], \\ r^2\partial_u C + \left(\frac{2m-r}{r}\right)\partial_\theta A - \partial_\theta B = 0, \\ r^2\sin^2\theta\partial_u D + \left(\frac{2m-r}{r}\right)\partial_\phi A - \partial_\phi B = 0, \\ \partial_r A = 0, \\ r^2\partial_r C - \partial_\theta A = 0, \\ r^2\sin^2\theta\partial_r D - \partial_\phi A = 0, \\ \partial_\theta C = \left[\lambda - \left(\frac{p}{2}+\frac{1}{4}\right)\right], \\ \sin^2\theta\partial_\theta D + \partial_\phi C = 0, \\ \partial_\phi D = \left[\lambda - \left(\frac{p}{2}+\frac{1}{4}\right)\right]. \end{cases} \tag{5.4}$$

From the fifth equation in (5.4), we get $A = A(u, \theta, \phi)$. Solving the eighth equation in (5.4) gives

$$C = \left[\lambda - \left(\frac{p}{2}+\frac{1}{4}\right)\right]\theta + \mathcal{E}(u, r, \phi), \tag{5.5}$$

where $\mathcal{E}$ is a smooth function. Similarly, solving the tenth equation in (5.4), we obtain

$$D = \left[\lambda - \left(\frac{p}{2}+\frac{1}{4}\right)\right]\phi + \mathcal{F}(u, r, \theta), \tag{5.6}$$

where $\mathcal{F}$ is a smooth function. Since A is independent of r, differentiating the sixth equation of (5.4) with respect to r yields

$$r\partial_r^2 C + 2\partial_r C = 0. \tag{5.7}$$

Utilizing (5.5) in (5.7) results in

$$\mathcal{E}(u,r,\phi)=\frac{1}{r}\bar{\mathcal{E}}(u,\phi)+\bar{\bar{\mathcal{E}}}(u,\phi),$$

where $\bar{\mathcal{E}}$ and $\bar{\bar{\mathcal{E}}}$ are smooth functions. Making use of the foregoing equation with (5.5) yields

$$C=\left[\lambda-\left(\frac{p}{2}+\frac{1}{4}\right)\right]\theta+\frac{1}{r}\bar{\mathcal{E}}(u,\phi)+\bar{\bar{\mathcal{E}}}(u,\phi). \tag{5.8}$$

Similarly, differentiating the seventh equation in (5.4) with respect to r and using (5.6), we get

$$\mathcal{F}(u,r,\theta)=\frac{1}{r}\bar{\mathcal{F}}(u,\theta)+\bar{\bar{\mathcal{F}}}(u,\theta), \tag{5.9}$$

where $\bar{\mathcal{F}}$ and $\bar{\bar{\mathcal{F}}}$ are smooth functions. Combining (5.6) and (5.9) yields

$$D=\left[\lambda-\left(\frac{p}{2}+\frac{1}{4}\right)\right]\phi+\frac{1}{r}\bar{\mathcal{F}}(u,\theta)+\bar{\bar{\mathcal{F}}}(u,\theta). \tag{5.10}$$

Utilizing (5.8) and (5.10) in the ninth equation of (5.4), we get the following expressions:

$$\begin{cases}\partial_\phi\bar{\mathcal{E}}(u,\phi)+\sin^2\theta\partial_\theta\bar{\mathcal{F}}(u,\theta)=0,\\ \partial_\phi\bar{\bar{\mathcal{E}}}(u,\phi)+\sin^2\theta\partial_\theta\bar{\bar{\mathcal{F}}}(u,\theta)=0.\end{cases} \tag{5.11}$$

Solving (5.11), we obtain

$$\begin{cases}\bar{\mathcal{E}}(u,\phi)=-\sin^2\theta\partial_\theta\bar{\mathcal{F}}(u,\theta)\phi+\mathcal{G}(u),\\ \bar{\bar{\mathcal{E}}}(u,\phi)=-\sin^2\theta\partial_\theta\bar{\bar{\mathcal{F}}}(u,\theta)\phi+\bar{\mathcal{G}}(u),\end{cases} \tag{5.12}$$

where $\mathcal{G}$ and $\bar{\mathcal{G}}$ are smooth functions. Combining (5.12) and (5.8) gives

$$C=\left[\lambda-\left(\frac{p}{2}+\frac{1}{4}\right)\right]\theta-\frac{\sin^2\theta}{r}\partial_\theta\bar{\mathcal{F}}(u,\theta)\phi+\frac{1}{r}\mathcal{G}(u)-\sin^2\theta\partial_\theta\bar{\bar{\mathcal{F}}}(u,\theta)\phi+\bar{\mathcal{G}}(u). \tag{5.13}$$

Inserting (5.10) in the seventh equation of (5.4), we get

$$A = -\sin^2\theta \bar{\mathcal{F}}(u,\theta)\phi + \mathcal{H}(u,\theta), \tag{5.14}$$

where $\mathcal{H}$ is a smooth function. Inserting (5.13) and (5.14) in the sixth equation of (5.4), we get

$$2\sin\theta\left(\sin\theta\partial_\theta\bar{\mathcal{F}}(u,\theta) + \cos\theta\bar{\mathcal{F}}(u,\theta)\right)\phi - \mathcal{G}(u) - \partial_\theta\mathcal{H}(u,\theta) = 0.$$

Since the last expression holds for all the values of ϕ, we get

$$\begin{cases} \sin\theta\partial_\theta\bar{\mathcal{F}}(u,\theta) + \cos\theta\bar{\mathcal{F}}(u,\theta) = 0, \\ \partial_\theta\mathcal{H}(u,\theta) + \mathcal{G}(u) = 0. \end{cases} \tag{5.15}$$

Solving (5.15) gives

$$\begin{cases} \mathcal{H}(u,\theta) = -\mathcal{G}(u)\theta + \mathcal{I}(u), \\ \bar{\mathcal{F}}(u,\theta) = \bar{\mathcal{I}}(u)\csc\theta, \end{cases} \tag{5.16}$$

where $\mathcal{I}$ and $\bar{\mathcal{I}}$ are smooth functions. Utilizing (5.16) in (5.14), (5.13) and (5.10) results in

$$\begin{cases} A = -\sin\theta\bar{\mathcal{I}}(u)\phi + \mathcal{I}(u) - \mathcal{G}(u)\theta, \\ C = \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right]\theta + \frac{\cos\theta}{r}\bar{\mathcal{I}}(u) + \frac{1}{r}\mathcal{G}(u) - \sin^2\theta\partial_\theta\bar{\bar{\mathcal{F}}}(u,\theta)\phi + \bar{\mathcal{G}}(u), \\ D = \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right]\phi + \frac{\csc\theta}{r}\bar{\mathcal{I}}(u) + \bar{\bar{\mathcal{F}}}(u,\theta). \end{cases} \tag{5.17}$$

Inserting the value of A from (5.17) in the second equation of (5.4) and integrating, we obtain the following:

$$B = \left(2\lambda - \left(p + \frac{1}{2}\right) + \sin\theta\bar{\mathcal{I}}'(u)\phi - \mathcal{I}'(u) + \mathcal{G}'(u)\theta\right)r + \mathcal{J}(u,\theta,\phi), \tag{5.18}$$

where $\mathcal{J}$ is a smooth function. Making use of (5.17) and (5.18) in the fourth equation of (5.4), we get

$$\sin^2\theta\partial_u\bar{\bar{\bar{\mathcal{F}}}}(u,\theta)r^3 + \big(\sin\theta\bar{\mathcal{I}}(u) - \partial_\phi\mathcal{J}(u,\theta,\phi)\big)\,r - 2m\sin\theta\bar{\mathcal{I}}(u) = 0,$$

which holds for all values of r. Since $m \neq 0$, we get $\bar{\mathcal{I}}(u) = 0$ and

$$r^2\sin^2\theta\partial_u\bar{\bar{\bar{\mathcal{F}}}}(u,\theta) - \partial_\phi\mathcal{J}(u,\theta,\phi) = 0.$$

Solving the last equation gives

$$\mathcal{J}(u,\theta,\phi) = r^2\sin^2\theta\partial_u\bar{\bar{\bar{\mathcal{F}}}}(u,\theta)\phi + \bar{\mathcal{J}}(u,\theta), \tag{5.19}$$

where $\bar{\mathcal{J}}$ is a smooth function. Making use of the above results in (5.18) and (5.17), we get

$$\begin{cases} A &= \mathcal{I}(u) - \mathcal{G}(u)\theta, \\ B &= \big(2\lambda - \big(p+\frac{1}{2}\big) - \mathcal{I}'(u) + \mathcal{G}'(u)\theta\big)\,r + r^2\sin^2\theta\partial_u\bar{\bar{\bar{\mathcal{F}}}}(u,\theta)\phi \\ & \quad +\bar{\mathcal{J}}(u,\theta), \\ C &= \big[\lambda - \big(\frac{p}{2}+\frac{1}{4}\big)\big]\,\theta + \frac{1}{r}\mathcal{G}(u) - \sin^2\theta\partial_\theta\bar{\bar{\mathcal{J}}}(u,\theta)\phi + \bar{\mathcal{G}}(u), \\ D &= \big[\lambda - \big(\frac{p}{2}+\frac{1}{4}\big)\big]\,\phi + \bar{\bar{\mathcal{J}}}(u,\theta). \end{cases} \tag{5.20}$$

Inserting (5.20) in the third equation of (5.4), we obtain

$$\begin{aligned} &\Big(-2r^2\sin^2\theta\partial_u\partial_\theta\bar{\bar{\bar{\mathcal{F}}}}(u,\theta) - r^2\sin 2\theta\partial_u\bar{\bar{\bar{\mathcal{F}}}}(u,\theta)\Big)\,\phi \\ &+\partial_\theta\bar{\mathcal{J}}(u,\theta) + r^2\bar{\mathcal{G}}'(u) - \left(\frac{2m-r}{r}\right)\mathcal{G}(u) = 0, \end{aligned}$$

which holds for all values of ϕ. Thus we get

$$\begin{cases} \partial_\theta\bar{\mathcal{J}}(u,\theta) + r^2\bar{\mathcal{G}}'(u) - \big(\frac{2m-r}{r}\big)\,\mathcal{G}(u) = 0, \\ \sin\theta\partial_\theta\partial_u\bar{\bar{\bar{\mathcal{F}}}}(u,\theta) + \cos\theta\partial_u\bar{\bar{\bar{\mathcal{F}}}}(u,\theta) = 0. \end{cases}$$

Solving the last two equations, we obtain the following relations:

$$\begin{cases} \bar{\mathcal{J}}(u,\theta) = \left(\left(\frac{2m-r}{r}\right)\mathcal{G}(u) - r^2\bar{\mathcal{G}}'(u)\right)\theta + \tilde{\mathcal{J}}(u), \\ \bar{\bar{\mathcal{F}}}(u,\theta) = \csc\theta\int\mathcal{K}(u)du + a, \end{cases} \tag{5.21}$$

where $\tilde{\mathcal{J}}$ and $\mathcal{K}$ are smooth functions and $a \in \mathbb{R}$. As a consequence of (5.21) in (5.20), we get

$$\begin{cases} A &= \mathcal{I}(u) - \mathcal{G}(u)\theta, \\ B &= \left(2\lambda - \left(p + \frac{1}{2}\right) - \mathcal{I}'(u) + \mathcal{G}'(u)\theta\right)r + r^2\sin^2\theta\csc\theta\mathcal{K}(u)\phi \\ &\quad + \left(\left(\frac{2m-r}{r}\right)\mathcal{G}(u) - r^2\bar{\mathcal{G}}'(u)\right)\theta + \tilde{\mathcal{J}}(u), \\ C &= \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right]\theta + \frac{1}{r}\mathcal{G}(u) + \cos\theta\int\mathcal{K}(u)du\phi + \bar{\mathcal{G}}(u), \\ D &= \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right]\phi + \csc\theta\int\mathcal{K}(u) + a. \end{cases} \tag{5.22}$$

Making use of (5.22) in the first equation of (5.4), we obtain

$$\left(\frac{2m-r}{r}\right)(\mathcal{I}'(u) - 2\mathcal{G}'(u)\theta) - (\mathcal{G}''(u)\theta - \mathcal{I}''(u))r - r^2\sin\theta\mathcal{K}'(u)\phi + r^2\bar{\mathcal{G}}''(u)\theta$$
$$-\tilde{\mathcal{J}}'(u) + \frac{2m'}{r^2} = \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right]\left(\frac{2m-r}{r}\right),$$

which implies

$$\begin{cases} r^2\sin\theta\mathcal{K}'(u) = 0, \\ r^2\bar{\mathcal{G}}''(u) - r\mathcal{G}''(u) - 2\left(\frac{2m-r}{r}\right)\mathcal{G}'(u) = 0, \\ \left(\frac{2m-r}{r}\right)\mathcal{I}'(u) + r\mathcal{I}''(u) - \tilde{\mathcal{J}}'(u) + \frac{2m'}{r^2} = \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right]\left(\frac{2m-r}{r}\right). \end{cases} \tag{5.23}$$

Solving the first equation of (5.23), we obtain $\mathcal{K}(u) = \bar{a} \in \mathbb{R}$. Differentiating the second equation three times with respect to r gives $\mathcal{G}(u) = b \in \mathbb{R}$. As a consequence, the second equation of (5.23) becomes $\bar{\mathcal{G}}''(u) = 0$ which implies $\bar{\mathcal{G}}(u) = a'u + b'$, where $a', b' \in \mathbb{R}$.

Differentiating the third equation four times with respect to r yields

$$m\mathcal{I}'(u) + \frac{5m'}{r} = \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right] m. \tag{5.24}$$

Differentiating (5.24) by r yields $m' = 0$, implying that m is constant. As a result, spacetime is transformed into Schwarzschild spacetime. As a consequence, (5.24) becomes $\mathcal{I}'(u) = \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right]$ which implies $\mathcal{I}(u) = \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right] u + d$, where $d \in \mathbb{R}$.

Using the value of $\mathcal{I}$ and $m' = 0$ in the third equation of (5.24), we obtain $\tilde{\mathcal{J}}(u) = k \in \mathbb{R}$. As a consequence of the above values, the components of vector field V are obtained as follows:

$$\begin{cases} A = \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right] u - b\theta + d, \\ B = \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right] r + \left(\frac{2mb}{r} - a'r^2 - b\right) \theta - \bar{a}r^2 \sin\theta\phi + k, \\ C = \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right] \theta + \frac{b}{r} + (\bar{a}\cos\theta)u\phi + a'u + b', \\ D = \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right] \phi + u\bar{a}\csc\theta + a. \end{cases} \tag{5.25}$$

Hence, we can state the following theorem:

Theorem 5.1. *If a Vaidya spacetime with its metric given by* (2.49) *admits a conformal Ricci soliton with its vector field given by* $V = A\partial_u + B\partial_r + C\partial_\theta + D\partial_\phi$, *then the spacetime reduces to a Schwarzschild spacetime with the vector field* V *given by* (5.25).

From (5.24), we have seen that the spacetime mass-energy function becomes constant, i.e., $m'(u) = 0$. If the Vaidya metric's mass-energy function is constant, then the mass of the collapsing or expanding matter is conserved over time. Therefore, there is no matter radiating either inwards or outwards and the mass remains fixed within the evolving spacetime which is stationary. The mass-energy function in the above scenario indicates the entire mass-energy of the core object, which is dispersed symmetrically spherically. As a result, the resultant metric, the Schwarzschild metric represents a static spherically symmetric spacetime.

Remark 5.1. *The above Theorem 5.1 implies that when the spacetime meets the criteria for a conformal Ricci soliton with the provided vector field V, it acts like a Schwarzschild spacetime. The discovery that a Vaidya spacetime, which was originally distinguished by its dynamic nature and reliance on the collapsing or expanding mass, can transform into a Schwarzschild spacetime under the conditions of admitting a conformal Ricci soliton is a remarkable demonstration of the interconnectedness of various gravitational solutions. Schwarzschild spacetime, known as the template for static spherically symmetric black holes, has long been regarded as a foundational concept in our knowledge of gravity. This result not only establishes an intriguing relationship between dynamic and static solutions, but also demonstrates the potency of conformal symmetries in controlling spacetime behavior. It emphasizes the conformal Ricci soliton's adaptability as a geometric term, offering fresh light on the underlying mathematical structure of these spacetimes. This finding opens up new areas of investigation, having implications for both the theoretical study of black holes and a larger knowledge of the links between alternative solutions to Einstein's field equations.*

5.1.2 Conformal Gradient Ricci Soliton in Vaidya Spacetime

Let us consider an arbitrary gradient vector field $V = grad\ f$ on the Vaidya metric (1.62), where f is the potential function. Then, using (1.62), we obtain

$$grad\ f = -(\partial_r f)\partial_u - \left(\partial_u f + \left(\frac{2m-r}{r}\right)\partial_r f\right)\partial_r + \frac{1}{r^2}(\partial_\theta f)\partial_\theta + \frac{1}{r^2\sin^2\theta}(\partial_\phi f)\partial_\phi. \quad (5.26)$$

As previously stated, a Vaidya metric (1.62) admits a conformal Ricci soliton with its vector field V given by (5.25). As a consequence, by comparing Equations

(5.25) and (5.26), we obtain the following expressions:

$$\begin{cases} A = -\partial_r f, \\ B = -\left(\partial_u f + \left(\frac{2m-r}{r}\right)\partial_r f\right), \\ C = \frac{1}{r^2}\partial_\theta f, \\ D = \frac{1}{r^2\sin^2\theta}\partial_\phi f. \end{cases} \tag{5.27}$$

where the values for A, B, C and D are given by (5.25) Solving the first equation of (5.27) gives

$$f = -ur\left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right] + br\theta - dr + \tilde{f}(u,\theta,\phi), \tag{5.28}$$

where $\tilde{f}$ is a smooth function. We obtain the following via inserting (5.28) into the third equation of (5.27) and solving the resultant equation:

$$\tilde{f}(u,\theta,\phi) = \frac{r^2\theta^2}{2}\left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right] + \bar{a}ur^2\phi\sin\theta + (a'u + b')r^2\theta + \tilde{\tilde{f}}(u,\phi), \tag{5.29}$$

where $\tilde{\tilde{f}}$ is a smooth function. Making use of (5.29) and (5.28) in the fourth expression of (5.27), we get

$$\tilde{\tilde{f}}(u,\phi) = \frac{r^2\phi^2\sin^2\theta}{2}\left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right] + r^2 a\phi\sin^2\theta + \bar{f}(u), \tag{5.30}$$

where $\bar{f}$ is a smooth function. Utilizing (5.30) and (5.29) in (5.28) yields

$$\begin{aligned} f =& \frac{r}{2}\left(r\theta^2 + r\phi^2\sin^2\theta - 2u\right)\left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right] + br\theta - dr + \bar{a}ur^2\phi\sin\theta \\ &+ (a'u + b')r^2\theta + r^2 a\phi\sin^2\theta + \bar{f}(u). \end{aligned} \tag{5.31}$$

Using (5.31) in the second equation of (5.27) results in the following:

$$\bar{f}(u) = \left(1 - \frac{2m}{r}\right)\left\{\left(r\theta^2 + r\phi\sin^2\theta - \frac{u}{2}\right)\left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right]u\right\}$$
$$+ \left(1 - \frac{2m}{r}\right)\left\{-du + \bar{a}u^2 r\phi\sin\theta + r\theta(a'u + 2b')u + r^2 au\phi\sin^2\theta\right\}$$
$$+ a'r^2\theta u - ku + 2b\theta u - \frac{4mb\theta}{r}u. \tag{5.32}$$

Combining (5.32) and (5.31), we obtain the following:

$$f = \frac{r}{2}\left(r\theta^2 + r\phi^2\sin^2\theta - 2u\right)\left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right] + br\theta - dr + \bar{a}ur^2\phi\sin\theta$$
$$+ (a'u + b')r^2\theta + r^2 a\phi\sin^2\theta$$
$$+ \left(1 - \frac{2m}{r}\right)\left\{\left(r\theta^2 + r\phi\sin^2\theta - \frac{u}{2}\right)\left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right]u\right\}$$
$$+ \left(1 - \frac{2m}{r}\right)\left\{-du + \bar{a}u^2 r\phi\sin\theta + r\theta(a'u + 2b')u + 2aru\phi\sin^2\theta\right\}$$
$$+ a'r^2\theta u - ku + 2b\theta u - \frac{4mb\theta}{r}u. \tag{5.33}$$

Equation (5.30) is now differentiated three times with respect to θ, and the result is then differentiated twice with respect to ϕ to get the expression for λ as follows:

$$\lambda = \frac{p}{2} + \frac{1}{4}, \tag{5.34}$$

which implies that $a = 0$. Again, differentiating (5.32) and (5.29) two times with respect to r and utilizing (5.34) and the values of a, we get $d = b = 0$ implying $a' = 0$ and $\bar{a} = b' = 0$. Thus, (5.33) reduces to

$$f = -ku. \tag{5.35}$$

Hence, we conclude with the theorems below:

Theorem 5.2. *If a Vaidya metric provided by equation* (1.62) *admits a conformal gradient Ricci soliton, then the potential function f must satisfy equation* (5.35).

Theorem 5.3. *A conformal gradient Ricci soliton on Vaidya spacetime is shrink-*

ing, steady or expanding according as

$$p > -\frac{1}{2},\ p = -\frac{1}{2}\ or\ p < -\frac{1}{2}$$

respectively.

5.2 Investigations on Relativistic Magneto Fluid Spacetime stuffing in $f(R)$-gravity and Ricci Solitons

Siddiqi and De (2021) published their work on "Relativistic Magneto Fluid Spacetime" where the matter content includes magnetism. They characterize the curvature properties of the spacetime and validate to Maxwell equation of magnetism for the magnetic field intensity, H. Also, Siddiqi et al. (2023) studied the solitonic aspects of the spacetime. Here, we study the spacetime in the settings of $f(R)$-gravity.

5.2.1 Magneto Fluid Spacetime in $f(R)$-gravity

A Magneto Fluid Spacetime is a spacetime where the matter content includes magnetic properties such as magnetic strength, intensity, permeability, density, flux and pressure.

In Magneto Fluid spacetime, the magnetic energy momentum tensor $\mathcal{T}$ is of the form (Siddiqi and De, 2021; Siddiqi et al., 2023)

$$\mathcal{T} = pg + (p+\rho)\eta \otimes \eta + \mu \left\{ \left(\eta \otimes \eta + \frac{1}{2}g \right) H - \gamma \otimes \gamma \right\}. \tag{5.36}$$

In this particular framework, the variable ρ is employed to denote the density of the magneto-fluid, while p is used to represent the pressure. The symbol μ is assigned to denote the magnetic permeability, γ stands for the magnetic flux, and

H is employed to indicate the strength of the magnetic field. It is noteworthy that the functions $\eta(X)$ and $g(Y,\zeta)$ are non-zero 1-forms, where $\eta(X) = g(X,\xi)$ and $g(Y,\zeta) = \gamma(Y)$. Moreover, the vector ξ is taken as the unit timelike vector field with the condition that $g(\xi,\xi) = -1$ and ζ is the spacelike magnetic flux vector field, ensuring that $g(\zeta,\zeta) = 1$. Importantly, these vectors are mutually orthogonal, as denoted by $g(\xi,\zeta) = 0$.

Now, the Einstein-Hilbert action for $f(R)$-gravity has the expression

$$\mathcal{H} = \frac{1}{\kappa^2}\int [f(R) + \mathcal{L}_m]\sqrt{-g}d^4x, \tag{5.37}$$

where $f(R)$ represents an arbitrary function of the Ricci scalar R and $\mathcal{L}_m$ denotes the Lagrangian of the scalar field. The tensor $\mathcal{T}$ of the matter is expressed as

$$\mathcal{T}_{\alpha\beta} = \frac{-2\delta(\sqrt{-g})\mathcal{L}_m}{\sqrt{-g}\delta g^{\alpha\beta}}. \tag{5.38}$$

Taking variation of (5.37) with respect to $g_{\alpha\beta}$ yields

$$f_R(R)Ric_{\alpha\beta} - \frac{1}{2}f(R)g_{\alpha\beta} + (g_{\alpha\beta}\nabla_\theta\nabla^\theta - \nabla_\alpha\nabla_\beta)f_R(R) = \kappa^2\mathcal{T}_{\alpha\beta}, \tag{5.39}$$

where Ric is the Ricci tensor, $f_R(R) = \frac{\partial f(R)}{\partial R}$ and $f_R(R) \neq 0$. Taking constant Ricci scalar, the above equation gives

$$Ric_{\alpha\beta} - \frac{R}{2}g_{\alpha\beta} = \frac{\kappa^2}{f_R(R)}\mathcal{T}^{eff}_{\alpha\beta},$$

where

$$\mathcal{T}^{eff}_{\alpha\beta} = \mathcal{T}_{\alpha\beta} + \frac{f(R) - Rf_R(R)}{2\kappa^2}g_{\alpha\beta}.$$

Thus,

$$Ric_{\alpha\beta} - \frac{R}{2}g_{\alpha\beta} = \frac{\kappa^2}{f_R(R)}\mathcal{T}_{\alpha\beta} + \frac{f(R) - Rf_R(R)}{2f_R(R)}g_{\alpha\beta}. \tag{5.40}$$

In view of (5.36), (5.40) can be written as

$$Ric_{\alpha\beta} = \left[\frac{2\kappa^2\left(p+\frac{\mu H}{2}\right)+f(R)}{2f_R(R)}\right]g_{\alpha\beta} + \frac{\kappa^2(p+\rho+\mu H)}{f_R(R)}\eta_\alpha\eta_\beta - \frac{\kappa^2\mu}{f_R(R)}\gamma_\alpha\gamma_\beta. \tag{5.41}$$

The foregoing equation can be written in index free notation for any vector fields X, Y as

$$Ric(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + c\gamma(X)\gamma(Y), \tag{5.42}$$

where $a = \frac{2\kappa^2\left(p+\frac{\mu H}{2}\right)+f(R)}{2f_R(R)}$, $b = \frac{\kappa^2(p+\rho+\mu H)}{f_R(R)}$ and $c = -\frac{\kappa^2\mu}{f_R(R)}$.

Contracting (5.41), we obtain

$$R = \left[\frac{4\kappa^2\left(p+\frac{\mu H}{2}\right)+2f(R)}{f_R(R)}\right] - \frac{\kappa^2(p+\rho+\mu(H+1))}{f_R(R)}. \tag{5.43}$$

Now, utilizing the expressions for the Ricci tensor, Ricci scalar and the magnetic-energy momentum tensor of the Magneto-Fluid Spacetime in $f(R)$-gravity, the Einstein equation (1.60) becomes

$$\left[\Lambda - \kappa^2\frac{p+\rho+\mu(2H+1)+f(R)}{2f_R(R)} - \kappa^2\left(p+\frac{\mu H}{2}\right)\right]g_{\alpha\beta}$$
$$= \kappa^2\left(1-\frac{1}{f_R(R)}\right)(p+\rho+\mu H)\eta_\alpha\eta_\beta + \kappa^2\mu\left(\frac{1}{f_R(R)}-1\right)\gamma_\alpha\gamma_\beta.$$

However, throughout our study, we will assume that $\Lambda = 0$. Now, we state the following results:

Theorem 5.4. *The Ricci tensor in the context of a Magneto Fluid Spacetime within the framework of $f(R)$-gravity theory with constant Ricci scalar takes the following expression:*

$$Ric_{\alpha\beta} = \left[\frac{2\kappa^2\left(p+\frac{\mu H}{2}\right)+f(R)}{2f_R(R)}\right]g_{\alpha\beta} + \frac{\kappa^2(p+\rho+\mu H)}{f_R(R)}\eta_\alpha\eta_\beta - \frac{\kappa^2\mu}{f_R(R)}\gamma_\alpha\gamma_\beta.$$

Corollary 5.1. *The Ricci scalar in the context of a Magneto Fluid Spacetime* *within the framework of $f(R)$-gravity theory takes the following expression:*

$$R = \left[\frac{4\kappa^2\left(p + \frac{\mu H}{2}\right) + 2f(R)}{f_R(R)}\right] - \frac{\kappa^2(p + \rho + \mu(H+1))}{f_R(R)}.$$

In view of (5.42) and Definition 1.6, we conclude:

Theorem 5.5. *The Magneto Fluid Spacetime stuffing in $f(R)$-gravity is a generalized quasi-Einstein spacetime.*

Now, from Corollary 5.1, we obtain

$$p = \frac{1}{3}\left[\frac{Rf_R(R)}{\kappa^2} + \rho + \mu(1-H) - \frac{2f(R)}{\kappa^2}\right]. \tag{5.44}$$

Thus, we conclude with the following:

Corollary 5.2. *In the context of $f(R)$-gravity, when a Magneto Fluid Spacetime is incorporated, the equation of state (EoS) is expressed as* (5.44)*.*

Remark 5.2. *Presently, under the assumption of a radiation-type matter source, we have $EoS = w = \frac{1}{3}$ where $w = \frac{p}{\rho}$, then* (5.44) *gives*

$$H = 1 + \frac{Rf_R(R) - 2f(R)}{\kappa^2\mu}, \tag{5.45}$$

which gives the expression for the magnetic strength of the considered spacetime in the context of $f(R)$-gravity theory. Next, if we, again assume that the matter source is of phantom barrier, then we have $p = -\rho$ which gives

$$p = \frac{1}{4}\left[\frac{Rf_R(R)}{\kappa^2} - \frac{2f(R)}{\kappa^2} + \mu(1-H)\right], \tag{5.46}$$

and

$$\rho = \frac{1}{4}\left[\frac{2f(R)}{\kappa^2} - \frac{Rf_R(R)}{\kappa^2} + \mu(H-1)\right]. \tag{5.47}$$

Also, from (5.42)*, we have*

$$\begin{cases} Ric(X,\xi) = (a+b)\eta(X), \\ Ric(X,\zeta) = (a+c)\gamma(X), \end{cases} \tag{5.48}$$

where a, b, c *are given by* (5.42).

Throughout the study, we shall assume that the $f(R)$-gravity theory considered in our study has a constant Ricci scalar.

5.2.2 Magneto Fluid spacetime in the setting of $f(R)$-gravity and Ricci soliton

We know that the Ricci soliton equation is given by (1.68). Now, for any vector fields X, Y, Z, we have

$$Ric(Y,Z) = -\lambda g(Y,Z) - \frac{1}{2}\mathcal{L}_X g(Y,Z) \tag{5.49}$$

$$= -\lambda g(Y,Z) - \frac{1}{2}[g(\nabla_Y X, Z) + g(Y, \nabla_Z X)]. \tag{5.50}$$

Now, setting $X = \xi$ in (5.50), we have

$$Ric(Y,Z) = -\lambda g(Y,Z) - \frac{1}{2}[g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi)].$$

Contracting the above expression, we obtain

$$R = -4\lambda - div\ \xi. \tag{5.51}$$

Now, comparing (5.51) with (5.43), we get

$$4a - b + c = -4\lambda - div\ \xi. \tag{5.52}$$

Taking $Y = Z = \xi$ in (5.42) and (5.49) yields

$$Ric(\xi,\xi) = -a - b + c,$$

$$Ric(\xi,\xi) = \lambda.$$

Then,

$$c - a - b = \lambda. \tag{5.53}$$

From (5.52) and (5.53), we obtain

$$a = -\lambda - \frac{div\ \xi}{5}, \tag{5.54}$$

which implies that $b - c = \frac{div\ \xi}{5}$. Next, if ξ is killing, then $a = -\lambda$ or $\lambda = -a$ and $b - c = 0$ as $b, c \neq 0$. Then $R = -4\lambda$ which implies that

$$\lambda = -\frac{2\kappa^2\left(p + \frac{\mu H}{2}\right) + f(R)}{2f_R(R)}. \tag{5.55}$$

Hence, we conclude with the following theorem:

Theorem 5.6. *If a Magneto Fluid Spacetime, embedded within the framework of $f(R)$-gravity, possesses a Ricci soliton with a Killing unit timelike vector field ξ, then the soliton exhibits shrinking, steady, or expanding behaviour based on*

$$p < -\frac{1}{2}\left(\frac{f(R)}{\kappa^2} + \mu H\right),\ p = -\frac{1}{2}\left(\frac{f(R)}{\kappa^2} + \mu H\right),\ \text{or}\ p > -\frac{1}{2}\left(\frac{f(R)}{\kappa^2} + \mu H\right),$$

respectively.

Now, setting $X = \zeta$ and using (1.66), we get

$$Ric(Y, Z) = -(\lambda + \omega)g(Y, Z) - \omega\eta(Y)\eta(Z). \tag{5.56}$$

Now, using (5.42) and (5.56), we obtain

$$\left[\frac{2\kappa^2\left(p + \frac{\mu H}{2}\right) + f(R)}{2f_R(R)}\right] g(Y, Z) + \frac{\kappa^2(\mu H + p + \rho)}{f_R(R)}\eta(Y)\eta(Z) - \frac{\kappa^2\mu}{f_R(R)}\gamma(Y)\gamma(Z)$$
$$= -(\lambda + \omega)g(Y, Z) - \omega\eta(Y)\eta(Z). \tag{5.57}$$

Putting $Y = Z = \xi$ in the above equation, we get

$$\lambda = -\kappa^2\left\{3\mu H + 4p + 2\rho\right\} - \frac{f(R)}{2f_R(R)} - 2\omega. \tag{5.58}$$

Thus, we state the following:

Theorem 5.7. *In the scenario where a Magneto Fluid Spacetime within the* *framework of $f(R)$-gravity possesses a Ricci soliton featuring a unit timelike torse-forming vector field ξ with the condition $f_R(R) \neq 0$, the soliton is*

1. *shrinking when* $\frac{f(R)}{2f_R(R)} < -2\omega - \kappa^2 \{3\mu H + 4p + 2\rho\}$,
2. *steady when* $\frac{f(R)}{2f_R(R)} = -2\omega - \kappa^2 \{3\mu H + 4p + 2\rho\}$,
3. *expanding when* $\frac{f(R)}{2f_R(R)} > -2\omega - \kappa^2 \{3\mu H + 4p + 2\rho\}$.

Again, from (5.48), we have

$$\begin{aligned} Ric(\xi, \xi) &= -a - b \\ &= -\kappa^2 \{3\mu H + 4p + 2\rho\} - \frac{f(R)}{2f_R(R)}. \end{aligned} \tag{5.59}$$

So, if $R(\xi, \xi) > 0$, that is

$$\frac{f(R)}{2f_R(R)} < -\kappa^2 \{3\mu H + 4p + 2\rho\},$$

then, the considered spacetime obeys the TCC (Timelike Convergence Condition), which further implies that the spacetime satisfies cosmic SEC (Strong Energy Condition). Utilizing this fact in (5.58) with $\omega < 0$ and $f_R(R) < 0$, we can state:

Theorem 5.8. *If a Magneto Fluid Spacetime within the framework of $f(R)$-gravity contains a Ricci soliton characterized by a unit timelike torse-forming vector field ξ and adhering to the Timelike Convergence Condition (TCC), the soliton shrinks if the scalar function ω and $f_R(R)$ are less than zero.*

We know that the TCC implies NCC (Null Convergence Condition) according to Hawking and Ellis (1973). Thus, combining with the above Theorem 5.8, we have the following:

Corollary 5.3. *In the context of $f(R)$-gravity, if a Magneto Fluid Spacetime accommodates a shrinking soliton characterized by a unit timelike torse-forming vector field ξ, then the spacetime satisfies the Null Convergence Condition (NCC).*

Vilenkin and Wall (2014) extends Penrose Singularity Theorem (Hawking and Ellis, 1973) in which they assume a spacetime M that obeys NCC, has a non-compact, connected Cauchy surface and contains some black holes. They also assume that there is no naked singularities on or outside the horizon and furthermore there is a trapped surface outside the black hole horizon. Then the trapped surface must be completely surrounded by the event horizon.

Now, if we consider the case of having a Magneto Fluid Spacetime embedded in $f(R)$-gravity admitting a Ricci soliton with torse forming vector field, ξ with the scalar function $\omega < 0$ and $f_R(R) < 0$ (i.e., the soliton is shrinking and obeys the NCC), then we assume that the spacetime in $f(R)$-gravity has a non-compact, connected Cauchy surface and contains some black holes with the existence of a trapped surface T and that all the singularities lie in the interior of the event horizon. Hence, the trapped surface must be completely surrounded by the event horizon. Furthermore, either the event horizon extends all the way to past infinity(initial singularity) or there exist black holes whose horizons contain multiple connected components.

5.2.3 Ricci Soliton on Magneto Fluid Spacetime in $f(R)$-gravity along $\phi(Ric)$-vector field

From the soliton equation (1.68), we have for any vector fields X, Y, Z,

$$\mathcal{L}_X g(Y,Z) + 2(a+\lambda)g(Y,Z) + 2b\eta(Y)\eta(Z) + 2c\gamma(Y)\gamma(Z) = 0. \tag{5.60}$$

By the definition of Lie derivative and (1.67), we obtain

$$(\mathcal{L}_\phi g)(Y,Z) = 2\mu Ric(Y,Z), \tag{5.61}$$

for any Y, Z. Then, setting $X = \phi$ in (5.60) and then utilizing (5.61) results in

$$Ric(Y, Z) = -\frac{1}{\mu}[(a + \lambda)g(Y, Z) + b\eta(Y)\eta(Z) + c\gamma(Y)\gamma(Z)]. \tag{5.62}$$

which yields the following result:

Theorem 5.9. *In the context of $f(R)$-gravity, if a Magneto Fluid Spacetime has a Ricci soliton with a vector field ϕ that fits the conditions of being a proper $\phi(Ric)$-vector field, the spacetime is classified as a generalized quasi-Einstein spacetime.*

Setting $Y = Z = \xi$ in (5.62), we get

$$\lambda = -(1 + \mu)(a + b). \tag{5.63}$$

Theorem 5.10. *Consider a Magneto Fluid Spacetime embedded within $f(R)$-gravity, allowing for a Ricci soliton with a suitable $\xi(Ric)$-timelike velocity vector field ξ. In this context, the spacetime is characterized as shrinking, steady, or expanding based on whether*

1. *$\mu < -1, \mu = -1,$ or $\mu > -1$ respectively, provided $a \neq -b$, or*

2. *$a < b, a = b,$ or $a > b$ respectively, provided $\mu \neq -1$.*

Corollary 5.4. *Consider a Magneto Fluid Spacetime embedded within the framework of $f(R)$-gravity. Suppose the spacetime accommodates a Ricci soliton featuring a covariantly constant $\xi(Ric)$-timelike velocity vector field denoted as ξ. In this context, the spacetime exhibits a shrinking, steady, or expanding behaviour depending on whether $a < b$, $a = b$, or $a > b$ respectively.*

Again, contracting (5.62), we obtain

Theorem 5.11. *In the context of $f(R)$-gravity, when a Magneto Fluid Spacetime accommodates a Ricci soliton featuring an appropriate $\phi(Ric)$-vector field denoted* *as ϕ, the expression for the scalar curvature is provided by:*

$$R = \frac{1}{\mu}[b - c - 4(a + \lambda)],$$

where a, b, c are defined by (5.42)*.*

5.2.4 Modified Poisson and Liouville equations with the harmonic feature of Ricci soliton on a Magneto Fluid Spacetime in $f(R)$-gravity

In this subsection, we will be looking for the modified Poisson and Liouville equation of a Ricci soliton in the framework of a Magneto Fluid Spacetime embedded within $f(R)$-gravity theory with its harmonic character.

Now, if we take $\xi = grad\ h$ where h is a smooth function, then from (5.54) we can state the following:

Theorem 5.12. *Consider a Magneto Fluid Spacetime in the setting of $f(R)$-gravity admitting a Ricci soliton. If the velocity vector field ξ associated with this Ricci soliton is of the gradient type, then the function h solves the modified Poisson equation specific to $f(R)$-gravity as*

$$\nabla^2 h = 5\left[\lambda + \frac{2\kappa^2\left(p + \frac{\mu H}{2}\right) + f(R)}{2f_R(R)}\right].$$

Moreover, considering a smooth function h on the manifold M and a vector field ξ, a direct computation yields the expression:

$$\mathrm{div}(h\xi) = \xi(dh) + h\ \mathrm{div}\ \xi.$$

If h belongs to the smooth functions space $C^\infty(M)$ and acts as the last multiplier of ξ concerning the metric g and this implies that $\mathrm{div}(h\xi) = 0$. The associated equation,

$$\xi(d\ln\xi) = -\mathrm{div}(\xi),$$

is referred as the Liouville equation of ξ in the context of the metric g. Considering the above two equations along with (5.54), the following implications arise:

Theorem 5.13. *Consider a Magneto Fluid Spacetime in the setting of $f(R)$-gravity which allow the inclusion of a Ricci soliton. In the event that the velocity vector field ξ linked to this Ricci soliton is characterized as being of the gradient type, then in the context of $f(R)$-gravity, the modified Liouville equation is*

$$\xi(d\ln\ h) = -5\left[\lambda + \frac{2\kappa^2\left(p + \frac{\mu H}{2}\right) + f(R)}{2f_R(R)}\right].$$

Next, we recall that if $\nabla^2 h = 0$, a function h is said to be harmonic. As a result of Theorem 5.12, we get the following results:

Corollary 5.5. *Consider a Magneto Fluid Spacetime in the context of $f(R)$-gravity, which accommodates a Ricci soliton. The velocity vector field ξ associated with this Ricci soliton is classified as gradient. Furthermore, if the function h exhibits harmonic behaviour on the spacetime, then the spacetime exhibits either decreasing, stable, or expanding behaviour, depending on*

$$p < -\frac{\mu H}{2} - \frac{f(R)}{2\kappa^2},\ p = -\frac{\mu H}{2} - \frac{f(R)}{2\kappa^2},\ \text{or}\ p > -\frac{\mu H}{2} - \frac{f(R)}{2\kappa^2}$$

respectively.

Corollary 5.6. *Consider a Magneto Fluid Spacetime with $f(R)$-gravity and a Ricci soliton. This Ricci soliton is coupled with a gradient velocity vector field ξ. Furthermore, we see that, in the case, when the function f shows harmonic behaviour,*

$$p = -\frac{1}{2\kappa^2}[2\lambda f_R(R) + f(R)] - \frac{\mu H}{2}.$$

5.2.5 Magneto Fluid Spacetime in the framework of $f(R)$-gravity and Gradient Ricci soliton

This subsection deals with gradient Ricci soliton as a metric for a Magneto Fluid Spacetime in the framework of $f(R)$-gravity.

Consider X to be the gradient of a smooth function h, with D as the gradient operator. In this case, (1.68) can be expressed as

$$\nabla_Y Dh + QY + \lambda Y = 0. \tag{5.64}$$

Utilizing the relationship

$$R(Y,X)Dh = \nabla_Y \nabla_X Dh - \nabla_X \nabla_Y Dh - \nabla_{[Y,X]} Dh, \tag{5.65}$$

equation (5.64) transforms into

$$R(Y,X)Dh = (\nabla_Y Q)X - (\nabla_X Q)Y. \tag{5.66}$$

Differentiating (5.64) covariantly along X results in

$$\nabla_X \nabla_Y Dh = -[(\nabla_X Q)Y - Q(\nabla_X Y)] - \lambda \nabla_X Y. \tag{5.67}$$

Swapping X and Y in the above equation results in

$$\nabla_Y \nabla_X Dh = -[(\nabla_Y Q)X - Q(\nabla_Y X)] - \lambda \nabla_Y X. \tag{5.68}$$

Now, (5.42) can be expressed as

$$QY = aY + b\eta(Y)\xi + c\gamma(Y)\zeta, \tag{5.69}$$

for all $Y \in \chi(M)$, where M is the Magneto Fluid Spacetime stuffing in $f(R)$-gravity. Again, differentiating (5.69) covariantly along Y, we obtain

$$(\nabla_Y Q)(X) = Y(a)X + b(\nabla_Y \eta)(X)\xi + b\eta(X)\nabla_Y \xi + c(\nabla_Y \gamma)(X)\zeta + c\gamma(X)\nabla_Y \zeta. \tag{5.70}$$

In view of (5.66) and (5.70), we get

$$\begin{aligned} R(Y,X)Dh = {} & Y(a)X - X(a)Y + b[(\nabla_Y \eta)(X)\xi + \eta(X)\nabla_Y \xi - (\nabla_X \eta)(Y)\xi \\ & - \eta(Y)\nabla_X \xi] + c[(\nabla_Y \gamma)(X)\zeta + \gamma(X)\nabla_Y \zeta - (\nabla_X \gamma)(Y)\zeta \\ & - \gamma(Y)\nabla_X \zeta]. \end{aligned} \tag{5.71}$$

Next, contracting the foregoing equation yields

$$Ric(X, Dh) = -3X(a) + b[(\nabla_\xi \eta)(X) - (\nabla_X \eta)(\xi) + \eta(X) div\xi]. \tag{5.72}$$

Also, from (5.42), we have

$$Ric(X, Dh) = aX(h) + b\eta(X)\xi(h) + c\gamma(X)\zeta(h). \tag{5.73}$$

Setting $X = \xi$ in (5.72) and (5.73) and then comparing the obtained results, we get

$$(b - a)\xi(h) = 3\xi(a) + b\ div\xi. \tag{5.74}$$

Let us assume now that ξ is Killing, i.e., $\mathcal{L}_\xi g = 0$, and consider an invariant a acting on the vector field ξ, implying that $\xi(a) = 0$. Therefore, this implies that $div\xi = 0$. Hence, we have from (5.74), either $a = b$ or $\xi(h) = 0$. Now, we have the following two cases:

Case I: Let us first assume that $\xi(h) \neq 0$ and $a = b$. Then, from (5.42), we obtain

$$f(R) = 2\kappa^2 \mu H + 2\kappa^2 \rho, \tag{5.75}$$

which gives the expression for the function $f(R)$ in terms of magnetic strength, permeability and density of the magnetic fluid, provided $f_R(R) \neq 0$.

Case II: Again, let us assume that $a \neq b$ and $\xi(h) = 0$. Then, taking covariant derivative of $g(\xi, Dh) = 0$ along Y with (5.42) and (5.64) gives

$$g(\nabla_Y \xi, Dh) = -[\lambda + (a - b)]\eta(Y). \tag{5.76}$$

We know that ξ is Killing, hence, we have $g(\nabla_Y \xi, X) + g(Y, \nabla_X \xi) = 0$. Next, by substituting $X = \xi$, we get $g(Y, \nabla_\xi \xi) = 0$ since $g(\nabla_Y \xi, \xi) = 0$. Therefore, setting $Y = \xi$ in (5.76) results in

$$\lambda = 2\kappa^2 \mu H + 2\kappa^2 \rho - f(R). \tag{5.77}$$

With the above two cases, we conclude that:

Theorem 5.14. *Suppose a Magneto Fluid Spacetime within $f(R)$-gravity accommodates a gradient Ricci soliton. In the case where the velocity vector field ξ is a Killing vector, and a scalar quantity a remains invariant along ξ, subject to the condition $f_R(R) \neq 0$, then the expression for the function $f(R)$ is determined by* (5.75)*, or alternatively, the soliton exhibits shrinking, steady, or expanding behavior based on*

$$f(R) < 2\kappa^2[\mu H + \rho], \;\; f(R) = 2\kappa^2[\mu H + \rho], \;\; or \; f(R) > 2\kappa^2[\mu H + \rho]$$

respectively.

Remark 5.3. *Looking at the expression of $f(R)$ in* (5.75) *and considering our assumption that the Ricci scalar being constant, we see that $f(R)$ depends solely on the magnetic field intensity or the magnetic field strength, magnetic permeability and the density of the magnetic fluid. However, these three terms would affect the gravitational dynamics of the spacetime when considering gradient Ricci soliton as the metric of the spacetime with the components of the Ricci tensor a* *and b being equal. So, depending on the curvature of the spacetime, the magnetic flux within the spacetime would vary dynamically which could lead to interesting phenomena where gravitational and electromagnetic properties influence one another. Moreover, as it can be seen from Case II, we can say that our argument is valid for the gradient Ricci soliton admitted into the spacetime considered in the framework of $f(R)$-gravity theory.*

5.3 Ricci Solitons and String Cloud Spacetime in $f(R)$-gravity

Letelier (1979), for the first time, introduced clouds of strings. Clouds of strings represent intriguing theoretical constructs within the framework of general relativity, offering novel insights into the gravitational dynamics of cosmic structures. In this theoretical framework, fundamental strings, envisaged as one-dimensional extended objects, aggregate to form macroscopic ensembles known as string clouds. Unlike point particles, strings possess finite size and exhibit rich dynamical behavior, including oscillations, winding modes, and interactions mediated by their tension. Within general relativity, the presence of string clouds induces curvature in spacetime, manifesting as gravitational fields that influence the motion of surrounding matter and radiation. String clouds have been presented as a possible explanation for a variety of astronomical events, including the development of primordial black holes, the generation of cosmic strings in the early universe, and the seeding of structure in the cosmic web. Furthermore, their research connects with other fields of theoretical physics, including string theory, cosmology, and high-energy physics, providing a holistic approach to understanding the underlying nature of spacetime and gravity. While observable evidence for

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the presence of string clouds is still lacking, their theoretical relevance highlights their continuous research as promising paths for investigating the cosmos on both macroscopic and microscopic levels.

Inspired by the aforementioned investigations, we undertake an exploration of the String Cloud Spacetime within the settings of $f(R)$-gravity theory.

5.3.1 String Cloud Spacetime in $f(R)$-gravity

In the context of String Cloud Spacetime within $f(R)$ gravity, we're dealing with a theoretical framework where spacetime is described using string theory and the gravitational dynamics are governed by a modified theory of gravity known as $f(R)$ gravity. This framework aims to incorporate both quantum mechanics (through string theory) and gravitational interactions (through $f(R)$ gravity) into a unified theory. In our study of String Cloud Spacetime, we shall consider its dimension to be 4.

The string cloud energy momentum tensor $\mathcal{T}$ is of the form (Tye, 2008):

$$\mathcal{T} = \rho\eta \otimes \eta - \theta\gamma \otimes \gamma, \tag{5.78}$$

where the variable ρ is employed to denote the associated particle density of the cloud fluid and θ is the string tension. Taking into account the rest energy density of particles, ϵ_0, we have

$$\rho = \theta + \epsilon_0. \tag{5.79}$$

It is essential to note that the functions $\eta(X)$ and $g(Y, \zeta)$ represent non-zero 1-forms, where $\eta(X) = g(X, \xi)$ and $g(Y, \zeta) = \gamma(Y)$. Additionally, the vector ξ is designated as the unit timelike vector field, satisfying the condition $g(\xi, \xi) = -1$, while ζ serves as the unit spacelike vector field, ensuring $g(\zeta, \zeta) = 1$. Crucially, these vectors are mutually orthogonal, as indicated by $g(\xi, \zeta) = 0$.

Let us recall the famous Einstein field equation (EFE's) given in (1.60). For our study, without loss of generality, let us take $G, c = 1$ such that (1.60) becomes

$$Ric + \Lambda g - \frac{R}{2} g = 8\pi \mathcal{T}. \tag{5.80}$$

In index free notation and utilizing (5.78), the above equation becomes

$$Ric(X,Y) + \left(\Lambda - \frac{R}{2}\right) g(X,Y) = 8\pi[\rho\eta(X)\eta(Y) - \theta\gamma(X)\gamma(Y)]. \tag{5.81}$$

Contracting the foregoing equation, we obtain

$$R = 4\Lambda + 8\pi(\rho + \theta). \tag{5.82}$$

Now, in the context of $f(R)$-gravity, the Einstein-Hilbert action for $f(R)$-gravity has the expression

$$\mathcal{H} = \frac{1}{8\pi} \int [f(R) + \mathcal{L}_m]\sqrt{-g} d^4x, \tag{5.83}$$

where $f(R)$ represents an arbitrary function of the Ricci scalar R and $\mathcal{L}_m$ denotes the Lagrangian of the scalar field. The tensor $\mathcal{T}$ characterizing the matter is

$$\mathcal{T}_{\alpha\beta} = \frac{-2\delta(\sqrt{-g})\mathcal{L}_m}{\sqrt{-g}\delta g^{\alpha\beta}}. \tag{5.84}$$

Now, taking variation of (5.83) with respect to $g_{\alpha\beta}$ yields

$$f_R(R) Ric_{\alpha\beta} - \frac{1}{2} f(R) g_{\alpha\beta} + (g_{\alpha\beta}\nabla_c\nabla^c - \nabla_a\nabla_b) f_R(R) = 8\pi\mathcal{T}_{\alpha\beta}, \tag{5.85}$$

where Ric is the Ricci tensor and $f_R(R) = \frac{\partial f(R)}{\partial R}$. We assume that $f(R) \neq 0, f_R(R) \neq 0$ and that $f_R(R) < \infty$. Taking constant Ricci scalar, the above equation gives

$$Ric_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} = \frac{8\pi}{f_R(R)} \mathcal{T}^{eff}_{\alpha\beta},$$

where

$$\mathcal{T}^{eff}_{\alpha\beta} = \mathcal{T}_{\alpha\beta} + \frac{f(R) - Rf_R(R)}{16\pi} g_{\alpha\beta}.$$

Thus,

$$Ric_{\alpha\beta} - \frac{R}{2}g_{\alpha\beta} = \frac{8\pi}{f_R(R)}\mathcal{T}_{\alpha\beta} + \frac{f(R) - Rf_R(R)}{2f_R(R)}g_{\alpha\beta}. \tag{5.86}$$

In view of (5.78), (5.86) can be written as

$$Ric_{\alpha\beta} = \frac{f(R)}{2f_R(R)}g_{\alpha\beta} + \frac{8\pi\rho}{f_R(R)}\eta_\alpha\eta_\beta - \frac{8\pi\theta}{f_R(R)}\gamma_\alpha\gamma_\beta. \tag{5.87}$$

The foregoing equation can be written in index free notation for any vector fields X, Y as

$$Ric(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) + c\gamma(X)\gamma(Y), \tag{5.88}$$

where $a = \frac{f(R)}{2f_R(R)}$, $b = \frac{8\pi\rho}{f_R(R)}$ and $c = -\frac{8\pi\theta}{f_R(R)}$.

Contracting (5.87), we obtain

$$R = \frac{2f(R)}{f_R(R)} - \frac{8\pi\rho}{f_R(R)} - \frac{8\pi\theta}{f_R(R)}. \tag{5.89}$$

Now, we can state the following:

Theorem 5.15. *The Ricci tensor in the context of a String Cloud Spacetime within the framework of $f(R)$-gravity theory takes the following expression:*

$$Ric_{\alpha\beta} = \frac{f(R)}{2f_R(R)}g_{\alpha\beta} + \frac{8\pi\rho}{f_R(R)}\eta_\alpha\eta_\beta - \frac{8\pi\theta}{f_R(R)}\gamma_\alpha\gamma_\beta.$$

Corollary 5.7. *The scalar curvature in the context of a String Cloud Spacetime within the framework of $f(R)$-gravity theory takes the following expression:*

$$R = \frac{2f(R)}{f_R(R)} - \frac{8\pi\rho}{f_R(R)} - \frac{8\pi\theta}{f_R(R)}.$$

In view of (5.88) and Definition 1.6, we conclude:

Theorem 5.16. *The String Cloud Spacetime stuffing in $f(R)$-gravity is a generalized quasi-Einstein spacetime.*

Also, from Corollary 5.7, we have

Theorem 5.17. *Within the domain of $f(R)$-gravity, the equation of state (EoS) for a String Cloud Spacetime is expressed as follows:*

$$\rho + \theta = \frac{2f(R) - Rf_R(R)}{8\pi}.$$

Also, from (5.88), we have

$$\begin{cases} Ric(X, \xi) = (a + b)\eta(X), \\ Ric(X, \zeta) = (a + c)\gamma(X), \end{cases} \tag{5.90}$$

where a, b, c are given by (5.88). Now, from (5.79) and Corollary 5.7, we can state:

Theorem 5.18. *If a String Cloud Spacetime in $f(R)$-gravity obeys the relation given by* (5.79)*, then the particle density ρ of the cloud fluid is given by*

$$\frac{1}{16\pi}(2f(R) - Rf_R(R)) + \frac{\epsilon_0}{2}$$

and the string tension is

$$\frac{1}{16\pi}(2f(R) - Rf_R(R)) - \frac{\epsilon_0}{2}.$$

Combining the value of ρ and θ from the above theorem results in the following:

Corollary 5.8. *If a String Cloud Spacetime adhering to* (5.79) *within $f(R)$-gravity fulfills the condition $\frac{\rho}{\theta} = -1$, then ρ is directly proportional to θ.*

Remark 5.4. *In the context of String Cloud Spacetime within $f(R)$ gravity, the equation $\rho = \theta + \epsilon_0$ describes the total energy density, where ρ is the particle density, θ is the string tension, and ϵ_0 is the rest energy density of the particles. When $\frac{\rho}{\theta} = -1$, it signifies a special condition known as the quintessence era. This condition implies that the total energy density due to particles exactly balances out the string tension. In other words, ρ is proportional to θ, with the proportionality* *factor being a negative one. This equilibrium between the energy stored in the strings and the energy associated with particles has significant implications for the stability and dynamics of the string cloud spacetime, particularly within the framework of $f(R)$ gravity. It represents a delicate balance that could influence the curvature of spacetime and the gravitational interactions within the string cloud.*

Corollary 5.9. *If a String Cloud Spacetime embedded within $f(R)$-gravity adheres to* (5.79) *with a constant R, and satisfies the condition $\frac{\rho}{\theta} = -1$, then the strings within the spacetime are considered massive, thus characterizing the spacetime as a massive String Cloud Spacetime.*

We also know that the energy density σ and the particle density ρ are related to the specific energy e and volume of the fluid V as (Wienberg, 1972; Jackiw et al., 2004):

$$\sigma = \rho e \quad \text{and} \quad \rho = \frac{1}{V}. \tag{5.91}$$

Therefore, in light of the above equation, we conclude that:

Theorem 5.19. *In a String Cloud Spacetime under $f(R)$-gravity with constant R and $\rho = \theta + \epsilon_0$, the specific energy e and volume of the cloud fluid V are given by*

$$\frac{16\pi\sigma}{2f(R) - Rf_R(R) + 8\pi\epsilon_0}$$

and

$$\frac{16\pi}{2f(R) - Rf_R(R) + 8\pi\epsilon_0},$$

respectively.

5.3.2 String Cloud Spacetime and Ricci soliton embedded in $f(R)$-gravity

We know that the Ricci soliton equation is given by (1.68). Now, from the soliton equation, we have for any vector fields Y and Z

$$Ric(Y,Z) = -\lambda g(Y,Z) - \frac{1}{2}\mathcal{L}_X g(Y,Z) \tag{5.92}$$

$$= -\lambda g(Y,Z) - \frac{1}{2}[g(\nabla_Y X, Z) + g(Y, \nabla_Z X)]. \tag{5.93}$$

Now, setting $X = \xi$ in (5.93), we have

$$Ric(Y,Z) = -\lambda g(Y,Z) - \frac{1}{2}[g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi)].$$

Contracting the above expression, we obtain

$$R = -4\lambda - div\ \xi. \tag{5.94}$$

Now, comparing (5.94) with (5.89), we get

$$4a - b + c = -4\lambda - div\ \xi. \tag{5.95}$$

Taking $Y = Z = \xi$ in (5.88) and (5.92) yields

$$Ric(\xi,\xi) = -a - b + c,$$

$$Ric(\xi,\xi) = \lambda.$$

Then,

$$c - a - b = \lambda. \tag{5.96}$$

From (5.95) and (5.96), we obtain

$$a = -\lambda - \frac{div\ \xi}{5}, \tag{5.97}$$

which implies that $b - c = \frac{div\ \xi}{5}$. Next, if ξ is killing, then $a = -\lambda$ or $\lambda = -a$ and $b - c = 0$ as $b, c \neq 0$. Then $R = -4\lambda$ which implies that

$$\lambda = -\frac{f(R)}{2f_R(R)}. \tag{5.98}$$

Hence, we conclude with the following theorem:

Theorem 5.20. *In the context of $f(R)$-gravity, if a String Cloud Spacetime accommodates a Ricci soliton with a killing unit timelike vector field ξ, then the Ricci soliton displays either shrinking, or expanding behavior according as*

$$f(R) < 0, \ \ or\ f(R) > 0,$$

respectively.

A direct consequences of (5.98) and $f(R) \neq 0, f_R(R) < \infty$, is as follows:

Corollary 5.10. *In the case when a String Cloud Spacetime is placed inside the framework of $f(R)$-gravity and employs a Ricci soliton defined by a killing unit timelike vector field ξ, a stable Ricci soliton does not exist.*

Next, setting $X = \zeta$ and using (1.66), we get

$$Ric(Y, Z) = -(\lambda + \omega)g(Y, Z) - \omega\eta(Y)\eta(Z). \tag{5.99}$$

Now, using (5.88) and (5.99), we obtain

$$\begin{aligned} \frac{f(R)}{2f_R(R)}g(Y, Z) + \frac{8\pi\rho}{f_R(R)}\eta(Y)\eta(Z) - \frac{8\pi\lambda}{f_R(R)}\gamma(Y)\gamma(Z) \\ = -(\lambda + \omega)g(Y, Z) - \omega\eta(Y)\eta(Z). \end{aligned} \tag{5.100}$$

Putting $Y = Z = \xi$ in the above equation, we get

$$\lambda = -\omega\left(1 + \frac{8\pi}{f_R(R)}\right) - \frac{8\pi}{f_R(R)}\left(\rho + \frac{f(R)}{16\pi}\right). \tag{5.101}$$

Thus, we state the following:

Theorem 5.21. *In the context where a String Cloud Spacetime is governed by $f(R)$-gravity and harbors a Ricci soliton characterized by a unit timelike torse-forming vector field ξ, and the condition $f_R(R) \neq 0$ holds, the soliton is*

1. *shrinking when $f(R) < -2\omega(8\pi + f_R(R)) - 16\pi\rho$,*
2. *steady when $f(R) = -2\omega(8\pi + f_R(R)) - 16\pi\rho$,*
3. *expanding when $f(R) > -2\omega(8\pi + f_R(R)) - 16\pi\rho$.*

Again, from (5.90), we have

$$\begin{aligned} Ric(\xi, \xi) &= -a - b \\ &= -\frac{f(R)}{2f_R(R)} - \frac{8\pi\rho}{f_R(R)}. \end{aligned} \tag{5.102}$$

So, if $R(\xi, \xi) > 0$, that is

$$f(R) < -16\pi\rho,$$

then the spacetime adheres to the Timelike Convergence Condition (TCC) which indicates its compliance with the Strong Energy Condition (SEC). Utilizing this fact in (5.101) with $\omega > 0$, the following result follows:

Theorem 5.22. *If a String Cloud Spacetime within the framework of $f(R)$-gravity contains a Ricci soliton characterized by a unit timelike torse-forming vector field ξ and adhering to the Timelike Convergence Condition (TCC), the soliton shrinks if the scalar function ω is greater than zero.*

We know that the TCC implies NCC (Null Convergence Condition) according to Hawking and Ellis (1973). Thus, combining with the above Theorem 5.22, we have the following:

Corollary 5.11. *Within the settings of $f(R)$-gravity, if a String Cloud Spacetime accommodates a shrinking Ricci soliton characterized by a unit timelike torse-forming vector field ξ, then the spacetime satisfies the Null Convergence Condition (NCC).*

In light of the theorems derived by Vilenkin and Wall (2014) which states that if a spacetime M obeys the NCC, then M has a non-compact Cauchy surface and contains some black holes, we state the following theorems:

Theorem 5.23. *If a String Cloud Spacetime governed by $f(R)$-gravity allows for a shrinking Ricci soliton characterized by a torse-forming vector field ξ and adheres to the Null Convergence Condition (NCC), then a non-compact Cauchy surface manifests within the spacetime.*

Theorem 5.24. *If a String Cloud Spacetime accommodating a shrinking Ricci soliton characterized by a torse-forming vector field ξ and adhering to the Null Convergence Condition (NCC) which is considered in the context of $f(R)$-gravity theory, then it implies the presence of black holes within this spacetime. Furthermore, the spacetime has a trapped surface lying outside of these black holes.*

5.3.3 Ricci Soliton on String Cloud Spacetime in $f(R)$-gravity along $\phi(Ric)$-vector field

The $\phi(Ric)$ vector field plays a significant role in the study of spacetime dynamics within the framework of $f(R)$ gravity. This vector field is defined based on the Ricci curvature of the spacetime geometry, denoted as Ric, and is characterized by its ability to capture essential geometric properties. This section deals with the study of the spacetime along $\phi(Ric)$ vector field.

From the soliton equation (1.68), we have

$$\mathcal{L}_X g(Y,Z) + 2(a+\lambda)g(Y,Z) + 2b\eta(Y)\eta(Z) + 2c\gamma(Y)\gamma(Z) = 0. \tag{5.103}$$

By the definition of Lie derivative and (1.67), we obtain

$$(\mathcal{L}_\phi g)(Y,Z) = 2\mu Ric(Y,Z), \tag{5.104}$$

for any Y, Z. Then, setting $X = \phi$ in (5.103) and then utilizing (5.104) results in

$$Ric(Y, Z) = -\frac{1}{\mu}[(a + \lambda)g(Y, Z) + b\eta(Y)\eta(Z) + c\gamma(Y)\gamma(Z)], \tag{5.105}$$

which yields the following result:

Theorem 5.25. *Under the framework of $f(R)$-gravity, when a String Cloud Spacetime possesses a Ricci soliton characterized by a vector field ϕ meeting the criteria of a proper $\phi(Ric)$-vector field, the spacetime is designated as a generalized quasi-Einstein spacetime.*

Next, setting $Y = Z = \xi$ in (5.105) yields

$$\lambda = -(1 + \mu)(a + b). \tag{5.106}$$

Theorem 5.26. *Consider a String Cloud Spacetime embedded within $f(R)$-gravity which allow a Ricci soliton with a suitable $\xi(Ric)$-timelike velocity vector field ξ. Then the spacetime is characterized as shrinking, steady or expanding based on whether*

1. *$\mu < -1, \mu = -1,$ or $\mu > -1$ respectively, provided $\rho \neq -\frac{f(R)}{16\pi}$,*

2. *$\rho < \frac{f(R)}{16\pi}, \rho = \frac{f(R)}{16\pi},$ or $\rho > \frac{f(R)}{16\pi}$ respectively, provided $\mu \neq -1$.*

Corollary 5.12. *Consider a String Cloud Spacetime embedded within the framework of $f(R)$-gravity. Suppose the spacetime accommodates a Ricci soliton featuring a covariantly constant $\xi(Ric)$-timelike velocity vector field denoted as ξ. Then the spacetime exhibits a shrinking, steady or expanding behaviour depending on whether $\rho < \frac{f(R)}{16\pi}, \rho = \frac{f(R)}{16\pi}$ or $\rho > \frac{f(R)}{16\pi}$ respectively.*

Again, contracting (5.105), we obtain

Theorem 5.27. *Within the framework of $f(R)$-gravity, when a String Cloud Spacetime accommodates a Ricci soliton featuring an appropriate $\phi(Ric)$-vector* *field denoted as ϕ, the expression for the scalar curvature is provided by:*

$$R = \frac{1}{\mu}\left[\frac{8\pi}{f_R(R)}\{\rho + \theta - 2f(R)\} + \lambda\right],$$

and such that the equation of state is given as

$$\rho + \theta = \frac{f_R(R)}{8\pi}\left[\mu R + \frac{16\pi f(R)}{f_R(R)} - \lambda\right].$$

5.3.4 Modified Equations in String Cloud Spacetime within $f(R)$ gravity

In this subsection, we aim to derive the modified Poisson and Liouville equations describing the behavior of a Ricci soliton within the context of a String Cloud Spacetime governed by $f(R)$-gravity theory using its harmonic nature. This investigation is crucial for understanding the dynamical properties of the spacetime geometry and its interaction with gravitational effects.

Now, if we take $\xi = grad\ h$ where h is a smooth function, then from (5.97) we can state the following:

Theorem 5.28. *Consider a String Cloud Spacetime in the setting of $f(R)$-gravity admitting a Ricci soliton. If the velocity vector field ξ associated with this Ricci soliton is of the gradient type, then the function h solves the modified Poisson equation specific to $f(R)$-gravity as*

$$\nabla^2 h = 5\left[\lambda + \frac{f(R)}{2f_R(R)}\right].$$

Moreover, considering a smooth function h on the manifold M and a vector field ξ, a direct computation yields the expression:

$$\text{div}(h\xi) = \xi(dh) + h\ \text{div}\ \xi.$$

If h belongs to the smooth functions space, $C^\infty(M)$ and acts as the last multiplier of ξ concerning the metric g, then $\text{div}(h\xi) = 0$. The associated equation,

$$\xi(d \ln \xi) = -\text{div}(\xi)$$

is referred to as the Liouville equation of ξ in the context of the metric g. Considering these equations along with (5.97), the following implications arise:

Theorem 5.29. *Consider a String Cloud Spacetime in the setting of $f(R)$-gravity which allows the inclusion of a Ricci soliton. In the event that the velocity vector field ξ linked to this Ricci soliton is characterized as being of the gradient type, then in the context of $f(R)$-gravity, the modified Liouville equation is*

$$\xi(d \ln\ h) = -5 \left[\lambda + \frac{f(R)}{2f_R(R)} \right].$$

Next, we recall that if $\nabla^2 h = 0$, then h is said to be harmonic. As a result of Theorem 5.28, we get the following result:

Theorem 5.30. *Consider a String Cloud Spacetime in the context of $f(R)$-gravity which accommodates a Ricci soliton. The velocity vector field ξ associated with this Ricci soliton is classified as gradient. Furthermore, if the function h exhibits harmonic behaviour on the spacetime, then the spacetime exhibits decreasing and expanding behaviour depending on*

$$f(R) < 0, \;\; \textit{and}\; f(R) > 0$$

respectively. Moreover, the fact that $f(R) \neq 0, f_R(R) < \infty$ implies the absence of steady Ricci soliton in the spacetime.

5.3.5 String Cloud Spacetime in the framework of $f(R)$-gravity and Gradient Ricci soliton

This subsection deals with gradient Ricci soliton as a metric for a String Cloud Spacetime in the framework of $f(R)$-gravity.

Consider X to be the gradient of a smooth function h, with D as the gradient operator. In this case, (1.68) can be expressed as

$$\nabla_Y Dh + QY + \lambda Y = 0. \tag{5.107}$$

Utilizing the relationship

$$R(Y, X)Dh = \nabla_Y \nabla_X Dh - \nabla_X \nabla_Y Dh - \nabla_{[Y,X]} Dh, \tag{5.108}$$

equation (5.107) transforms into

$$R(Y, X)Dh = (\nabla_Y Q)X - (\nabla_X Q)Y. \tag{5.109}$$

Differentiating (5.107) covariantly along X results in

$$\nabla_X \nabla_Y Dh = -[(\nabla_X Q)Y - Q(\nabla_X Y)] - \lambda \nabla_X Y. \tag{5.110}$$

Swapping X and Y in the above equation results in

$$\nabla_Y \nabla_X Dh = -[(\nabla_Y Q)X - Q(\nabla_Y X)] - \lambda \nabla_Y X. \tag{5.111}$$

Now, (5.88) can be expressed as

$$QY = aY + b\eta(Y)\xi + c\gamma(Y)\zeta, \tag{5.112}$$

for all $Y \in \chi(M)$, where M is the String Cloud Spacetime stuffing in $f(R)$-gravity. Again, differentiating (5.112) covariantly along Y, we obtain

$$(\nabla_Y Q)(X) = Y(a)X + b(\nabla_Y \eta)(X)\xi + b\eta(X)\nabla_Y \xi + c(\nabla_Y \gamma)(X)\zeta + c\gamma(X)\nabla_Y \zeta. \tag{5.113}$$

In view of (5.109) and (5.113), we get

$$R(Y,X)Dh = Y(a)X - X(a)Y + b[(\nabla_Y\eta)(X)\xi + \eta(X)\nabla_Y\xi - (\nabla_X\eta)(Y)\xi - \eta(Y)\nabla_X\xi] + c[(\nabla_Y\gamma)(X)\zeta + \gamma(X)\nabla_Y\zeta - (\nabla_X\gamma)(Y)\zeta - \gamma(Y)\nabla_X\zeta]. \tag{5.114}$$

Next, contracting the foregoing equation yields

$$Ric(X,Dh) = -3X(a) + b[(\nabla_\xi\eta)(X) - (\nabla_X\eta)(\xi) + \eta(X)div\xi]. \tag{5.115}$$

Also, from (5.88), we have

$$Ric(X,Dh) = aX(h) + b\eta(X)\xi(h) + c\gamma(X)\zeta(h). \tag{5.116}$$

Setting $X = \xi$ in (5.115) and (5.116) and then comparing the obtained results, we get

$$(b-a)\xi(h) = 3\xi(a) + b\ div\xi. \tag{5.117}$$

Let us assume now that ξ is Killing, i.e., $\mathcal{L}_\xi g = 0$ and consider an invariant, a acting on the vector field ξ which implies that $\xi(a) = 0$. Therefore, this implies that $div\xi = 0$. Hence, we have from (5.117), either $a = b$ or $\xi(h) = 0$. However, if $a = b$, then $f(R) = 16\pi\rho$ which implies that $f(R)$ is a constant and $f_R(R) = 0$, which is a contradiction. Therefore, $\xi(h) = 0$. Thus, taking covariant derivative of $g(\xi, Dh) = 0$ along Y with (5.88) and (5.107) gives

$$g(\nabla_Y\xi, Dh) = -[\lambda + (a-b)]\eta(Y). \tag{5.118}$$

We know that ξ is Killing, hence, we have $g(\nabla_Y\xi, X) + g(Y, \nabla_X\xi) = 0$. Next, by substituting $X = \xi$, we get $g(Y, \nabla_\xi\xi) = 0$ since $g(\nabla_Y\xi, \xi) = 0$. Therefore, setting $Y = \xi$ in (5.118) results in

$$\lambda = \frac{1}{2f_R(R)}[16\pi\rho - f(R)]. \tag{5.119}$$

With the above two cases, we state the following theorem:

Theorem 5.31. *Suppose a String Cloud Spacetime within $f(R)$-gravity accommodates a gradient Ricci soliton. In the case where the velocity vector field ξ is a Killing vector, and a scalar quantity a remains invariant along ξ, the soliton is*

1. *shrinking when $\rho > \frac{f(R)}{16}$,*

2. *steady when $\rho = \frac{f(R)}{16}$,*

3. *expanding when $\rho > \frac{f(R)}{16}$.*

5.4 Conclusion

The study of Vaidya spacetime within the context of conformal gradient Ricci solitons provides a powerful link between geometry and the dynamics of spacetime. The discovery that the potential function f is directly related to the incoming time coordinate u, which is specified by the constant k, stresses the importance of time in determining the conformal structure of these spacetimes.

The results, we have obtained, expand our understanding of the complicated link between geometry and general relativity concepts. It demonstrates the usefulness of mathematical techniques such as conformal Ricci solitons in describing and categorizing the solutions of Einstein's equation. Furthermore, under certain conditions, the connection between Vaidya and Schwarzschild spacetimes emphasizes the unity and beauty of Einstein's gravitational theory.

It is also novel to identify precise formulations that the potential function f must satisfy in order for the Vaidya spacetime to allow a conformal gradient Ricci soliton. A conformal gradient Ricci soliton vector field describes a specific geometric flow through spacetime. The dynamics of the collapsing null fluid and its interaction with the spacetime geometry would be constrained by the potential function f conditions. The presence of a conformal Ricci soliton vector field as well as the accompanying limits on the potential function f, has an influence on the general behaviour and eventual fate of the collapsing null fluid.

From Theorem 5.3, we conclude with the following: the behavior of a conformal gradient Ricci soliton in Vaidya spacetime is intricately related to the value of the Vaidya metric parameter p. The dynamics of spacetime under the influence of the soliton are governed by this parameter.

1. **Shrinking Behavior (When $p > -\frac{1}{2}$):**

   - When p is larger than $-\frac{1}{2}$, the soliton causes the spacetime to compress or shrink.
   - Depending on an understanding of the physical properties of spacetime, this phenomenon may be equivalent to gravitational collapse or spatial point convergence.
   - The presence of p values greater than $-\frac{1}{2}$ implies that the effect of the soliton dominates the expansion induced by the Vaidya metric, resulting in a net shrinkage.

2. **Steady Behavior (When $p = -\frac{1}{2}$):**

   - The soliton has a stabilizing effect on spacetime when p is exactly $-\frac{1}{2}$.
   - At this crucial number, the soliton's actions precisely balance the Vaidya metric's expansion, resulting in a spacetime that remains constant over time.
   - This condition of equilibrium indicates that the soliton's impact is exactly adjusted to counteract any expansion or contraction tendencies.

3. **Expanding Behavior (When $p < -\frac{1}{2}$):**

- When p is smaller than $-\frac{1}{2}$, the soliton causes a growing or expanding behavior in spacetime.
- Depending on the physical environment of the spacetime, this expansion may be equivalent to an expanding cosmos or the divergence of spatial points.
- A value of p less than $-\frac{1}{2}$ implies that the effect of the soliton dominates the growth induced by the Vaidya metric, resulting in a net expansion.

To summarize, the value of the parameter p is a significant determinant of the dynamics of a conformal gradient Ricci soliton in Vaidya spacetime. It determines whether the spacetime contracts (shrinking), remains stable or expands, providing vital insights into the dynamical features of the soliton and its influence on the geometry of the spacetime.

The extensive investigation, we have conducted into the dynamics of Magneto-Fluid Spacetime within the theoretical framework of $f(R)$-gravity, has yielded a number of remarkable insights. These important discoveries, embodied in a set of crucial statements provide light on numerous aspects of the delicate interplay between magneto fluid dynamics and gravitational theories. The findings of our study have made significant advances to our knowledge of Ricci solitons in Magneto-Fluid Spacetime in the settings of $f(R)$-gravity theory. The arguments elaborate on the requirements that determine whether these solitons exhibit decreasing, stable or growing behaviour. These findings, which are based on the complicated interplay of magnetic fields, fluid characteristics, and the modified gravitational framework, add to our understanding of the various behaviours inherent in these complex spacetimes.

A soliton's properties are accurately specified by the interplay of magnetic flux, pressure and gravitational effects. These discoveries add considerably to the larger story of spacetime dynamics and demonstrate the complexities of gravitational systems. Furthermore, our research into the harmonic functions within spacetime in $f(R)$-gravity situations has revealed remarkable discoveries. These claims show how the velocity vector field, harmonic functions, and modified Poisson and Liouville equations are related. These correlations' consequences broaden our understanding of the complicated interaction between geometric characteristics and the changed gravitational field.

The formation of black holes and the trapped surface of a black hole completely surrounded by the event horizons when shrinking Ricci solitons is admitted within Magneto-Fluid Spacetime in the framework of $f(R)$-gravity shed a new light. These significant findings not only improve our theoretical understanding of these systems but also establish the groundwork for future research into the complicated interplay between fluid dynamics and modified gravity theories.

The investigation into string cloud spacetime and Ricci solitons within the realm of $f(R)$ gravity has yielded significant insights into the complex interplay between geometry, matter distribution and gravitational dynamics. The obtained results, encompassing expressions for the Ricci tensor, scalar curvature and equation of state have provided a deeper understanding of the geometric properties of the spacetime, elucidating how it responds to the presence of matter and energy within the framework of $f(R)$ gravity. Furthermore, the identification of String Cloud Spacetime as a generalized quasi-Einstein spacetime underscores its adherence to specific geometric conditions akin to those observed in Einstein's field equations, offering avenues for simplifying the study of its dynamics.

Moreover, the analysis of Ricci solitons has revealed their behavior under $f(R)$ gravity indicating their potential for exhibiting either shrinking or expanding behavior depending on the sign of $f(R)$. This emphasizes the dynamic nature of the spacetime geometry, suggesting that gravitational configurations tend to evolve over time rather than remain static. Additionally, the absence of steady Ricci solitons under certain conditions further highlights the dynamic nature of the spacetime, implying that gravitational interactions within the system are inherently transient and subject to evolution.

Furthermore, the identification of black holes within String Cloud Spacetime featuring shrinking Ricci solitons underscores the gravitational phenomena encapsulated within the framework of $f(R)$ gravity theory. Additionally, the behavior of gradient Ricci solitons offers nuanced insights into the dynamics of the spacetime, providing a comprehensive understanding of its evolution and expansion properties. Moreover, the derived modified Liouville and Poisson equations offer further avenues for investigating the intricate interplay between matter distribution, geometry and gravitational dynamics within the spacetime.

Chapter 6

Summary and Conclusion

In the present thesis, we give classification of almost contact metric manifolds admitting some geometrical structures and also studied their submanifold. Furthermore, we explore the dynamics of different spacetime. The following objectives are taken up in the study:

1. To study the properties of Ricci-Yamabe solitons.
2. To characterize almost cosymplectic manifolds and its extension.
3. To study geometrical properties of spacetimes.
4. To investigate invariant submanifolds of certain classes of almost contact manifolds.

In Chapter 1, we provide a general introduction which includes the basic definitions and formulas of differential geometry such as topological manifolds, smooth manifolds, Riemannian manifolds, almost contact metric manifolds, Kenmotsu manifolds, almost Kenmotsu manifolds, hyperbolic Kenmotsu manifolds, almost cosymplectic manifolds, submanifolds, Vaidya spacetime and Ricci-Yamabe solitons, and review of the literature.

Chapter 2 is divided into three main sections. In the first section, we have investigated the properties and isometries of almost Ricci-Yamabe solitons (ARYS) and established several critical results that enhance the understanding within the realm of Riemannian geometry. Firstly, we derived the necessary conditions under which a compact gradient almost Ricci-Yamabe soliton is isometric to a Euclidean sphere $S^n(r)$. We have observed that the potential function f of a compact gradient almost Ricci-Yamabe soliton aligns with the Hodge-de Rham potential h. This result is significant as it ties the geometric structure of the soliton to the well-known Euclidean sphere, thereby providing a tangible example of these abstract structures. Secondly, we examined complete gradient almost Ricci-Yamabe solitons with non-zero α and a non-trivial conformal vector field. We demonstrated that these solitons, given the condition of non-negative scalar curvature, must be isometric to either Euclidean space E^n or a Euclidean sphere S^n. This finding not only extends known rigidity results for Ricci solitons but also highlights the restrictive nature of almost Ricci-Yamabe solitons under these conditions, enhancing the understanding of their geometric properties.

Additionally, we analyzed ARYS with solenoidal and torse-forming vector fields, providing a comprehensive examination of their structure. Through various lemmas and theorems, we demonstrated the rigidity of these solitons and proved that they admit few deformations under the given conditions. This rigidity is crucial and resulted in the stability and uniqueness of the geometric structures described by these solitons. Furthermore, we provided explicit examples to substantiate the theoretical results. These examples illustrate the applicability of the theoretical findings and provide a concrete foundation for further research. By constructing non-trivial examples, we not only validate our results but also open new avenues for exploring the practical implications of almost Ricci-Yamabe solitons in various geometric contexts. The work in Section 2.2 extends the results of Roy et al. (2020) on $(LCS)_n$-manifolds. We generalized their results and derived a more general value for the scalar curvature tensor an $(LCS)_n$-manifold admitting the Ricci-Yamabe soliton and shown that it is constant. This result is prominent as it applies to a larger group of solitons. We verified our results by constructing 3-dimensional and 5-dimensional $(LCS)_n$-manifolds.

Furthermore, we derived the expression for the scalar, λ when the manifold admits a conformal Ricci-Yamabe soliton. We also identified the conditions under which a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold admitting a torse-forming vector field ξ is an expanding, steady or shrinking η-Ricci-Yamabe soliton. Moreover, we provided the expression for λ in a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold admitting a torse-forming η-Ricci-Yamabe soliton.

In Section 2.3, we investigated almost $*$-Ricci-Yamabe solitons on a Sasakian manifold M.Following and extending the method used by Dwivedi and Patra (2022), we provided an analytic answer to the question posed in the beginning of the section. We proved that if a complete Sasakian manifold admits both almost $*$-Ricci-Yamabe soliton and gradient almost $*$-Ricci-Yamabe soliton as its metric, it is isometric to the unit sphere $\mathbb{S}^{2n+1}$ provided α is non-zero. Additionally, we identified specific conditions under which the soliton becomes steady. We also found that if the potential vector field U is an infinitesimal contact transformation, it becomes an infinitesimal automorphism. Lastly, we validated our results by constructing an example.

Chapter 3 deals with almost cosymplectic manifolds and its extension to almost Kenmotsu manifolds. The chapter comprises of two main sections. In the first section, we found a link between the scalar curvature τ and the parameters (λ, a, b, n) on a compact α-almost cosymplectic manifold. Specifically, we obtained

$$\tau = \frac{2\lambda}{(2n+1)b - 2a} \quad \text{provided} \quad \alpha \neq \left\{0, \frac{(2n+1)\beta}{2}\right\}.$$

This connection provides vital insights into the interplay between geometric quantities and soliton characteristics, allowing for a better understanding of the curvature dynamics of the manifold. We discovered that a manifold permitting a GRYS with $a \neq b$ is either flat or has constant scalar curvature. This result is useful in understanding the geometric configurations and curvature aspects of 3-dimensional cosymplectic manifolds. Lastly, we constructed an example of a 3-dimensional manifold admitting a GRYS to validate the results.

In the second section, we conducted a thorough investigation into the properties and structures of almost Kenmotsu manifolds that admit conformal Ricci-Yamabe solitons (CRYS). By extending the existing results on Ricci solitons and Ricci-Yamabe solitons to the more generalized setting of CRYS on $(\kappa, \mu)'$-almost Kenmotsu manifolds, we provided new insights into the geometric structures that arise in this context. Firstly, it was demonstrated that a $(2n+1)$-dimensional $(\kappa, \mu)'$-almost Kenmotsu manifold admitting a CRYS is locally isometric to $H^{n+1}(-4) \times \mathbb{R}^n$ provided that $2\lambda - \beta\tau \neq 4\alpha n\kappa - \left(p + \frac{2}{2n+1}\right)$. This result is significant as it reveals a specific geometrical structure that these manifolds possess when they admit solitons and established a notable connection to hyperbolic spaces and Euclidean spaces. Additionally, it is proven that a generalized $(\kappa, \mu)'$-almost Kenmotsu manifold admitting a CRYS is an η-Einstein manifold. This finding is important because it links the existence of CRYS to the η-Einstein condition by imposing specific constraints on the Ricci tensor of the manifold.

Understanding the η-Einstein property is crucial for comprehending the curvature properties and the overall geometric behavior of the manifold. We have shown that the potential vector field is a strict infinitesimal contact transformation for $(\kappa, \mu)'$-almost Kenmotsu manifolds admitting a conformal gradient Ricci-Yamabe soliton. This result indicates that the vector field associated with the soliton preserves the contact structure of the manifold, thereby maintaining the integrity of the underlying almost Kenmotsu structure. Furthermore, an example of a 3-dimensional $(\kappa, \mu)'$-almost Kenmotsu manifold was constructed to illustrate and verify the findings. The implications of these results are extensive. The results not only extend the theory of Ricci and Yamabe solitons to a broader class of manifolds but also provide a deeper understanding of the interaction between soliton equations and the geometric structures of almost Kenmotsu manifolds. The local isometry to $H^{n+1}(-4) \times \mathbb{R}^n$ opens up new avenues for exploring the curvature and topology of these manifolds in relation to well-known geometric spaces. Moreover, the η-Einstein condition and the strict infinitesimal contact transformation property offer new insights into the curvature conditions and the preservation of geometric structures under the influence of CRYS. These findings could have further implications in the study of geometric flows and their limiting behaviors in various geometric contexts.

Chapter 4 explored the geometric properties of invariant submanifolds within hyperbolic Kenmotsu manifolds, uncovering several significant results that enhance our understanding of these intricate structures. The implications of our findings are profound, impacting both theoretical research and practical applications in fields such as mathematics and physics. We began by establishing the fundamental properties of invariant submanifolds in hyperbolic Kenmotsu manifolds. Our results show that the structure vector field ζ is tangent to these submanifolds and that the tensor field ϕ preserves tangency. This finding is crucial, as it confirms that these submanifolds inherently retain the geometric characteristics of the ambient manifold, ensuring consistency in geometric behavior across different layers of the manifold.

Moreover, we demonstrated that invariant submanifolds within hyperbolic Kenmotsu manifolds are minimal. This means that their mean curvature vector vanishes which indicate stability. The minimality condition is pivotal for understanding the types of geometric flows that the submanifolds can support and contribute to the stability analysis and applications in differential geometry. One of the key results we identified is the condition under which the invariant submanifolds are totally geodesic. We proved that these submanifolds are totally geodesic if and only if their second fundamental form is parallel. This result is significant as it links the intrinsic geometry of the submanifold with its extrinsic curvature properties. The condition for total geodesy implies that geodesics in the ambient manifold remain geodesics within the submanifold which is a property essential for applications in the theory of relativity and the spacetime structures. We further explored the concepts of pseudoparallel and 2-pseudoparallel submanifolds, providing new insights into the curvature properties of these geometric structures. Our findings show that if an invariant submanifold is semiparallel, it is totally geodesic.

Additionally, for 2-pseudoparallel submanifolds, we established that they are either totally geodesic or satisfy a specific functional condition. These results extend the understanding of how curvature influences the geometric behavior of submanifolds offering a framework for further exploration of curvature conditions in complex geometric settings. The results of our study have far-reaching implications for the field of differential geometry. By establishing the fundamental properties and conditions for invariant submanifolds in hyperbolic Kenmotsu manifolds, we contribute to a deeper understanding of the geometric structures within these manifolds. This has potential applications in theoretical physics, particularly in the spacetime and general relativity, where understanding the geometric properties of submanifolds can provide insights into the nature of the universe and the behavior of gravitational fields. Furthermore, our findings on minimality and total geodesy offer valuable tools for researchers who are investigating the stability and geometric flows of submanifolds. The conditions for pseudoparallelism and 2-pseudoparallelism provide new avenues for exploring the interplay between curvature and geometry which lead to advancements in both theoretical research and practical applications in fields of material science and cosmology.

Chapter 5 is devoted to the study of geometrical structures of spacetime and it consists of three main sections. We start with an investigation on Vaidya spacetime in the first section. The study of Vaidya spacetime within the context of conformal gradient Ricci solitons reveals a profound link between geometry and spacetime dynamics, where the potential function f is directly associated with the incoming time coordinate, u emphasizing the temporal role in determining the conformal structure. This expands our comprehension of the intricate relationship between geometry and general relativity highlighting the utility of mathematical tools like conformal Ricci solitons in categorizing the solutions of Einstein's equation. The connection between Vaidya and Schwarzschild spacetimes underscores the unity of Einstein's theory and the identification of precise formulations for the potential function f elucidates the constraints on the dynamics of collapsing null fluid. The behavior of a conformal gradient Ricci soliton in Vaidya spacetime is intricately linked to the Vaidya metric parameter p, dictating whether spacetime contracts, remains stable or expands, providing crucial insights into the soliton's dynamics.

The second section deals with the relativistic magneto fluid spacetime within $f(R)$-gravity and obtain the expressions for the Ricci tensor, scalar curvature and equation of state. By employing Ricci solitons as metrics, conditions for shrinking, constant or growing behaviors under Killing and torse forming vector fields are established. The emergence of black holes and trapped surfaces outside the black holes are discussed, particularly in the contexts where a shrinking Ricci soliton is admitted imposing constraints on scalar function ω and the first derivative of $f(R)$. Additionally, the influence of magnetic field strength, permeability and fluid density on gravitational dynamics in spacetime admitting a gradient Ricci soliton is highlighted and indicates its impact on total pressure. Furthermore, the study explores the dynamics of string cloud spacetime governed by $f(R)$ gravity in the last section, revealing a balance between particle density ρ and string tension λ during the quintessence era. Modified Poisson and Liouville equations are derived, elucidating the formation of black holes and trapped surfaces in the presence of a shrinking Ricci soliton. Conditions for spacetime contraction, steadiness or expansion under a gradient Ricci soliton are also established based on particle density ρ.

In our study, some key geometrical structures such as Ricci solitons, conformal Ricci solitons, Ricci-Yamabe solitons, conformal Ricci-Yamabe solitons, almost Ricci-Yamabe solitons and almost $*$-Ricci-Yamabe solitons were investigated in the setting of almost contact metric manifolds and various isometric classifications were found. We have also explored the geometrical structure of certain spacetime in the framework of conformal Ricci solitons and $f(R)$-gravity theory and discovered certain conclusions that might be beneficial in theoretical physics, particularly, in the study of general relativity and spacetime. Furthermore, we created additional instances to validate our findings.

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(C. ZOSANGZUALA)

LIST OF RESEARCH PUBLICATIONS/ ACCEPTED/ COMMUNICATIONS

(1) J.P. Singh and **Z. Chhakchhuak** (2022). Certain results of Ricci-Yamabe solitons on $(LCS)_n$-manifolds, *Facta Universitatis, Series Mathematics and Informatics* **37(4)**, 797–812.

(2) M. Khatri, **Z. Chhakchhuak** and J.P. Singh (2023). Isometries on Almost Ricci-Yamabe Solitons, *Arabian Journal of Mathematics* **12(1)**, 127–138.

(3) **Z. Chhakchhuak** and J.P. Singh (2024). Conformal Ricci solitons on Vaidya spacetime, *General Relativity and Gravitation* **15(1)**, 1–15.

(4) **Z. Chhakchhuak** and J.P. Singh (2024). Characterization of almost $*$-Ricci-Yamabe solitons isometric to a unit sphere, *Novi Sad Journal of Mathematics* (Online).

(5) M. Khatri, J.P. Singh, R. Sumlalsanga and **Z. Chhakchhuak** (2024). Classification of conformally harmonic Z-recurrent spacetime in $F(R)$-gravity, *International Journal of Geometric Methods in Modern Physics* (Online).

(6) **Z. Chhakchhuak** and J.P. Singh (2024). Ricci solitons and String cloud spacetime in $f(R)$-gravity, *International Journal of Theoretical Physics* **63**, 185.

(7) J.P. Singh and **Z. Chhakchhuak** (2024). On almost Kenmotsu manifolds admitting conformal Ricci-Yamabe solitons, *Boletim da sociedade paranaense de matematica* (Accepted).

(8) **Z. Chhakchhuak** and J.P. Singh (2024). Investigations on Relativistic Magneto Fluid spacetime stuffing in $f(R)$-gravity and Ricci solitons (communicated).

(9) **Z. Chhakchhuak**, J.P. Singh, and S.S. Singh (2024). Investigations on almost cosymplectic manifolds admitting almost Ricci-Yamabe solitons (communicated).

(10) **Z. Chhakchhuak**, J. P. Singh and S.S. Singh (2024). Characterization of invariant submanifolds of hyperbolic Kenmotsu manifolds (communicated).

CONFERENCES/ SEMINARS/ WORKSHOPS

(1) Attended "NASI TMC Summer School on Differential Geometry" organised by the Department of Mathematics and Statistics, Central University of Punjab, Bathinda during July 05-24, 2021.

(2) Attended "National Workshop On Ancient Indian Mathematics" organized by Central University of Punjab, Bathinda during March 14-16, 2022.

(3) Trainer in the "Mathematics Summer Day Camp 2022" jointly organized by Mizoram Mathematics Society, Mizoram Science, Technology and Innovation Council (MISTIC) and Department of Mathematics, Pachhunga University College, Aizawl-796001, Mizoram (India) held during 28^{th} March, 2022 to 1^{st} April, 2022.

(4) Trainee in the "One Week Training Program on Mathematical Modelling and Computing" organised by Department of Mathematics and Computer Science, Mizoram University, Aizawl-796004, Mizoram (India) held during 26^{th} April, 2022 to 2^{nd} May, 2022.

(5) Attended "International Faculty Development Programme on Mathematical Modelling of Biosystems with Special Focus on Epidemiology," organized by Department of Mathematics and Computer Science, Mizoram University during 22 - 27 August, 2022.

(6) Attended "Workshop on Group Theory and Related Areas (WGTRA)" during November 14, 2022– November 16, 2022 organized by Department of Applied Sciences, Indian Institute of Information Technology, Allahabad, India.

(7) Attended "Teacher's Enrichment Workshop on Groups, Rings and Number Theory (TEW 2022)" organised by Department of Mathematics and Com-

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(8) Presented a paper entitled "Isometries on almost Ricci-Yamabe solitons" at the National Conference on Computational Mathematics organized by NIT Puducherry, Karaikal during December 22–23, 2022.

(9) Trainer in the "Seminar on Higher Secondary School Mathematics Curricula, 2023" jointly organized by Mizoram Mathematics Society, Mizoram Science, Technology and Innovation Council, and Department of Mathematics, Pachhunga University College, Aizawl-796001, Mizoram (India) held during 23^{rd} March, 2023 to 24^{th} March, 2023.

(10) Presented a talk entitled "From Euclidean to Differential Geometry: Unveiling the Mathematical Framework of Modern Physics" on a "National Seminar on Role of Number Theory and Graph Theory in Mathematical Science" organized by Department of Mathematics, Pachhunga University College, Aizawl-796001, during June 12–13, 2023.

(11) Presented a paper entitled "Isometries on almost Ricci-Yamabe solitons" at the 2nd International Conference on Recent Trends in Applied Sciences and Computing Engineering (RTASCE -2023) organized by VIT Bhopal University in association with National Institute of Technology, Warangal during July 7-9, 2023.

(12) Resource person in "One Day Training on Scientific Research Writing Facility and Data Analysis" organized by Department of Mathematics, Lunglei Govt. College, Lunglei-796701, held on January 25, 2024.

(13) Presented a paper entitled "m-quasi Einstein structure and Perfect fluid spacetime in $f(R,T)$-gravity" at the International Conference on Recent

Advances in Mathematical, Physical and Chemical Sciences (ICRAMPC-2024), organized by School of Physical Sciences, Mizoram University, Aizawl-796004, during February 21–23, 2024.

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ABSTRACT

A STUDY ON CERTAIN CLASSES OF ALMOST CONTACT MANIFOLDS AND SPACETIMES

AN ABSTRACT SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

C. ZOSANGZUALA

MZU REGISTRATION NUMBER : 1501450

Ph.D. REGISTRATION NUMBER : MZU/Ph.D./1711 of 05.11.2020

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

SCHOOL OF PHYSICAL SCIENCES

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A STUDY ON CERTAIN CLASSES OF ALMOST CONTACT MANIFOLDS AND SPACETIMES

By

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Submitted

In partial fulfillment of the requirement of the Degree of Doctor of Philosophy in Mathematics of Mizoram University, Aizawl

ABSTRACT

In the present thesis, we have investigated some geometrical structures such as Ricci solitons, conformal Ricci solitons, Ricci-Yamabe solitons, conformal Ricci-Yamabe solitons, almost Ricci-Yamabe solitons and almost $*$-Ricci-Yamabe solitons in the almost contact metric manifolds. We also introduced geometrical structures of certain spacetime in the framework of conformal Ricci solitons and $f(R)$-gravity theory which might be beneficial in theoretical physics, particularly, in the field of general relativity and spacetime. Furthermore, we constructed certain examples to validate the findings.

This thesis aims to deepen the understanding of Riemannian manifolds, almost contact manifolds and their submanifolds and spacetime by addressing the following objectives:

1. To study the properties of Ricci-Yamabe solitons.
2. To characterize almost cosymplectic manifolds and its extension.
3. To study geometrical properties of spacetimes.
4. To investigate invariant submanifolds of certain classes of almost contact manifolds.

In Chapter 1, we provide a general introduction which include basic definitions and formulas of differential geometry such as topological manifolds, smooth manifolds, Riemannian manifolds, almost contact metric manifolds, Kenmotsu manifolds, almost Kenmotsu manifolds, hyperbolic Kenmotsu manifolds, almost cosymplectic manifolds, Ricci-Yamabe solitons, Lorentzian manifolds, Vaidya spacetime, and Submanifolds, as well as a review of the literature.

Chapter 2 consists of three sections. The first section delves to examine the isometries of almost Ricci-Yamabe solitons. Firstly, we consider a compact gradient almost

Ricci-Yamabe soliton. Next, we studied complete gradient almost Ricci-Yamabe soliton with $\alpha \neq 0$ and non-trivial conformal vector field with non-negative scalar curvature and proved that it is either isometric to Euclidean space E^n or Euclidean sphere S^n. Also, solenoidal and torse-forming vector fields are considered. Moreover, some non-trivial examples are constructed to verify the results. The second section characterizes Lorentzian concircular structure manifolds ($(LCS)_n$-manifolds) admitting Ricci-Yamabe solitons and we have shown that they become flat when the soliton is steady. We have constructed 3 and 5-dimensional $(LCS)_n$-manifolds satisfying this property and derived the scalar, λ expression for conformal Ricci-Yamabe solitons. Furthermore, we have discussed η-Ricci-Yamabe solitons on conformally flat $(LCS)_n$ ($n \geq 4$) manifolds and found the conditions for shrinking, steady and expanding solitons when ξ act as a torse forming vector field. We characterized almost $*-$Ricci-Yamabe solitons on a Sasakian manifold in the last section, where we proved that the manifold is isometric to the unit sphere $\mathbb{S}^{2n+1}$ if its metric represents a complete almost $*-$Ricci-Yamabe solitons with $\alpha \neq 0$. Certain conditions under which the soliton reduces to $*$-Ricci-Yamabe soliton and when it becomes steady are also obtained.

In Chapter 3, the first section investigates the behaviour of an almost cosymplectic manifold when an almost Ricci-Yamabe soliton is admitted as its metric. We have shown the condition under which the manifold is locally isomorphic to a Lie group $G_{\sqrt{-\kappa}}$. Next, the non-existence of the almost Ricci-Yamabe solitons on a compact (κ, μ)-almost cosymplectic manifold with $\kappa < 0$ is established. When the soliton vector field is pointwise collinear with ξ, we derived an equation for the scalar curvature of its metric with certain limitations on the metric's parameter a. Then the findings have been validated by constructing an example of a 3-dimensional manifold that defines a gradient Ricci-Yamabe solitons. The second section characterizes almost Kenmotsu manifolds admitting conformal Ricci-Yamabe solitons. Here, we examined $(\kappa, \mu)'$ and generalized $(\kappa, \mu)'$ almost Kenmotsu manifolds and shown that such a manifold M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ under specific conditions. Conditions for conformal pressure when the soliton is expanding, steady or shrinking are obtained. We demonstrated that if the manifold admits a conformal gradient Ricci-Yamabe soliton, the potential vector field is a strict infinitesimal contact transformation. An example of a 3-dimensional manifold is also constructed.

In Chapter 4, we characterized an invariant submanifolds of hyperbolic Kenmotsu manifolds. First, we have proved that an invariant submanifolds of a hyperbolic Kenmotsu manifold is again a hyperbolic Kenmotsu manifold and is minimal. Next, the conditions for the invariant submanifolds to be totally geodesic are obtained. Also, it is shown that a 3-dimensional submanifolds is totally geodesic if and only if it invariant. Moreover, an invariant submanifold of a hyperbolic Kenmotsu manifold admitting η-Ricci-Bourguinon soliton is examined and an example which verify some of the results is constructed.

Chapter 5 is divided into three sections. In the first section, we have examined Vaidya spacetime under a conformal Ricci soliton vector field and observed the reduction of the spacetime to Schwarzschild spacetime and the existence of a conformal gradient Ricci soliton. We then extend our investigation into relativistic magneto fluid spacetime stuffing in $f(R)$-gravity where we have provided the conditions for the emergence of black holes and trapped surfaces. We have also observed that gravitational dynamics are influenced by magnetic field strength, permeability and density, affecting total pressure on spacetime. In the third section, we discussed the dynamics of a string cloud spacetime using $f(R)$-gravity theory and found a balance between particle density and string tension. Moreover, the Ricci soliton metric is used to determine its behavior under different vector fields.

Chapter 6 is devoted for summary and conclusion.

A list of references is given at the end.