

**A STUDY ON CONTACT STRUCTURES, WARPED
PRODUCT AND THEIR APPLICATIONS**

**A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY**

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**A STUDY ON CONTACT STRUCTURES, WARPED PRODUCT
AND THEIR APPLICATIONS**

BY

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**In partial fulfillment of the requirement of the Degree of Doctor of
Philosophy in Mathematics of Mizoram University, Aizawl.**

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CERTIFICATE

This is to certify that the thesis entitled “A study on Contact Structures, Warped Product and their Applications”, submitted by Mr. Robert Sumlalsanga (Registration No. : MZU/Ph.D./1779 of 16.09.2021) for the award of the degree of Doctor of Philosophy (Ph. D.) in Mathematics, is a bonafide record of the original research carried out by him under my supervision. He has been duly registered, and the thesis is worthy of being considered for the award of the Ph. D. degree. This research work has been done under my supervision and has not been submitted for any degree of any other university.

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I, **Robert Sumlalsanga**, hereby declare that the subject matter of this thesis is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to do the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in any other University/Institute.

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Dated:

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PREFACE

The present thesis titled “**A study on contact structures, warped product and their application**” is the result of research conducted by the author under the supervision of Prof. Jay Prakash Singh, Central University of South Bihar, Gaya-824236, INDIA, and Prof. M. Sundararajan, Department of Mathematics and Computer Science, Mizoram University, Aizawl - 796 004, Mizoram, INDIA.

The thesis comprises six chapters. The first chapter is a General Introduction that encompasses fundamental concepts such as definitions, including topological space, topological manifold, smooth manifold, Riemannian manifolds, symmetric manifolds, almost contact metric manifold, Sasakian space forms, Kenmotsu manifold, almost Kenmotsu manifolds, submanifolds, Ricci soliton, $F(R)$ -gravity, and a review of literature is also added.

The second chapter deals with the investigation of almost Kenmotsu manifolds admitting generalized $\mathfrak{h}$ -almost Ricci solitons, where we provide new information on their geometric characteristics. We have shown that such manifolds either conform to Einstein structures or display stable, growing solitons by looking at warped product structures and extending the idea of Ricci solitons. Important findings include the categorization of η -Einstein manifolds, the function of potential vector fields, and the complex connection between curvature and soliton dynamics.

Chapter 3 classifies conformally harmonic Z -recurrent spacetimes within $F(R)$ -gravity theory by contrasting two models: one, exhibiting an exponential functional form and the other, a logarithmic form. The study demonstrates that the positive energy density and the negative Equation of State (EoS) parameter, which categorize both models inside the quintessence phase, facilitate the accelerated expansion of the universe driven by dark energy. The research establishes

significant energy conditions, illustrating that the acceleration is additionally reinforced by the infringement of the Strong Energy Condition (SEC) and the compliance with the Null Energy Condition (NEC), Weak Energy Condition (WEC), and Dominant Energy Condition (DEC). The models align with established cosmological theories and provide a coherent fit with the Λ CDM model under the condition of constant scalar curvature.

In the fourth chapter, we examine the geometry of Ricci-Bourguignon solitons on Euclidean submanifolds, concentrating on cases where the soliton field aligns with the tangential component of the position vector field. We have established essential geometric characteristics and categorizes Ricci-Bourguignon solitons on Euclidean hypersurfaces, illustrating that these hypersurfaces must be either open segments of hyperplanes or hyperspheres, contingent upon the soliton parameter.

In chapter 5, conformal Ricci-Yamabe solitons are studied in detail in the context of warped product manifolds. We investigate the circumstances surrounding the existence of such solitons, paying special attention to how Killing and conformal vector fields interact with the soliton structure. This study reports important findings, such as the categorization of warped product manifolds admitting these solitons and their conversion into Einstein manifolds under certain circumstances involving simultaneous vector fields. The research is strong technically, providing comprehensive proofs and well-founded theorems that advance our knowledge of solitonic structures in geometric flows.

Chapter 6 presents the summary and conclusions of the thesis, followed by a list of references.

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Chapter 1

General Introduction

1.1 Topological Space

A topology on a set S is a collection $\mathcal{T}$ of subsets containing both the empty set $\emptyset$ and the set S , such that $\mathcal{T}$ is closed under arbitrary unions and finite intersections: i.e., if $U_\alpha \in \mathcal{T}$ for all α in an index set A , then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$ and if $U_1, \dots, U_n \in \mathcal{T}$, then $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called *open sets*, and the pair $(S, \mathcal{T})$ is called a *topological space*. A topological space is *second countable* if it has a countable basis. A *neighborhood* of a point p in S is an open set U containing p .

A topological space M is *locally Euclidean of dimension n* if every point $p \in M$ has a neighborhood U such that there is a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. We call the pair $(U, \phi : U \rightarrow \mathbb{R}^n)$ a *chart*, U a *coordinate neighborhood* or a *coordinate open set*, and ϕ a *coordinate map* or a *coordinate system* on U . We say that a chart (U, ϕ) is *centered at $p \in U$* if $\phi(p) = 0$.

1.2 Topological Manifold

Definition 1.2.1. *Suppose M is a topological space. We say that M is a topological manifold of dimension n or a topological n -manifold if it has the following properties:*

1. *M is a Hausdorff space: for every pair of distinct points $p, q \in M$ there are disjoint open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$.*
2. *M is second-countable: there exists a countable basis for the topology of M .*
3. *M is locally Euclidean of dimension n .*

1.3 Smooth Manifold

Let M be a topological n -manifold. A coordinate chart on M is a pair (U, φ) , where U is an open subset of M and $\varphi : U \rightarrow \bar{U}$ is a homeomorphism from U to an open subset $\bar{U} = \varphi(U) \subseteq \mathbb{R}^n$. If (U, φ) and (V, ψ) are two charts such that $U \cap V \neq \emptyset$, the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)$ is called the transition map from φ to ψ . Two charts (U, φ) and (V, ψ) are said to be smoothly compatible if either $U \cap V = \emptyset$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism. An atlas $\mathcal{A}$ is the collection of charts whose domain cover M . If any two charts in $\mathcal{A}$ are smoothly compatible then $\mathcal{A}$ is said to be smooth atlas. A smooth atlas $\mathcal{A}$ on M is maximal if it is not properly contained in any larger smooth atlas.

Definition 1.3.1. *If M is a topological manifold, a smooth structure or differentiable structure ($\mathbb{C}^\infty$ -structure) on M is a maximal smooth atlas $\mathcal{A}$. A smooth manifold is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M .*

1.4 Riemannian Manifold

The Riemannian metric allows us to define the geometric concepts such as lengths, angles and distances on smooth manifolds. Similar to inner product on vector space, for manifold, the appropriate structure is a Riemannian metric, which is essentially a choice of inner product on each tangent space, varying smoothly from point to point.

Definition 1.4.1. *Let M be a smooth manifold with or without boundary. A Riemannian metric on M is a smooth symmetric covariant 2-tensor field on M that is positive definite at each point. A Riemannian manifold is a pair (M, g) , where M is a smooth manifold and g is a Riemannian metric on M .*

1.5 Connections on Riemannian Manifold

An affine or linear connection on a smooth manifold M is a R -bilinear mapping

$$\nabla : \chi(M) \times \chi(M) \rightarrow \chi(M),$$

which satisfies the following properties:

1. $\nabla_{fX}Y = f\nabla_XY$,
2. $\nabla_XfY = f\nabla_XY + (Xf)Y$,

for any $X, Y \in \chi(M)$ and smooth function f . On a Riemannian manifold M of dimension n , the affine connection ∇ is said to be Levi-Civita connection or Riemannian connection if it satisfies the following:

1. ∇ is symmetric or torsion-free i.e., $\nabla_XY - \nabla_YX = [X, Y]$ and
2. ∇ is a metric compatible i.e., $(\nabla_Xg)(Y, Z) = 0$ for all $X, Y, Z \in \chi(M)$.

Let M^n be an n -dimensional Riemannian manifold with Riemannian metric g . A linear connection $\bar{\nabla}$ is known as a quarter-symmetric connection if its torsion tensor T is given by

$$T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y]$$

satisfies

$$T(X, Y) = \Lambda(Y)\varphi X - \Lambda(X)\varphi Y,$$

where Λ is a 1-form and P is a vector field given by $\Lambda(X) = g(X, P)$ and φ is $(1, 1)$ -tensor. Qu and Wang (2015) introduced a special type of quarter-symmetric connection defined as:

$$\bar{\nabla}_X Y = \widehat{\nabla}_X Y + \psi_1 \Lambda(Y)X - \psi_2 g(X, Y)P, \quad (1.1)$$

where $\widehat{\nabla}$ denote the Levi-Civita connection. It is easy to see that the quarter-symmetric connection $\bar{\nabla}$ include the semi-symmetric metric connection ($\chi_1 = \chi_2 = 1$) and the semi-symmetric non-metric connection ($\psi_1 = 1, \psi_2 = 0$).

1.6 Symmetric Manifolds

Riemannian symmetric spaces have been a primary topic of research in differential geometry theories. It was Cartan, who first initiated the study of Riemannian symmetric spaces and gave its classification (Cartan, 1926). According to him, an n -dimensional Riemannian manifold M is said to be locally symmetric if its curvature tensor R satisfies $R_{hijk,l} = 0$, where “,” represent the covariant differentiation with respect to the metric tensor and R_{hijk} are the components of the curvature tensor of the manifold M . The notion of locally symmetric spaces has been extended by many geometers throughout the year, some of which are Recurrent (Walker, 1950), locally φ symmetric (Takahashi, 1977), pseudo sym-

metric (Chaki, 1987) and weakly symmetric (Tamássy and Binh, 1989). Sen and Chaki (1967) obtained an expression for the covariant derivative of the curvature tensor while studying conformally flat space of class one with certain curvature restrictions on the curvature tensor, which is as follows:

$$R_{ijk,l}^h = 2\lambda_l R_{ijk}^h + \lambda_i R_{ljk}^h + \lambda_j R_{ilk}^h + \lambda_k R_{ijl}^h + \lambda^h R_{ijk}^l, \quad (1.2)$$

where λ_i is a non-zero covariant vector. Later, Chaki (1987) introduced a manifold whose curvature tensor satisfies (1.2) and called it a pseudo symmetric manifold $(PS)_n$. Extending $(PS)_n$, De and Pal (2014) has introduced almost pseudo symmetric manifold $(APS)_n$. A Riemannian manifold $(M^n, g)(n > 2)$ is said to be an almost pseudo symmetric if its curvature tensor R of type (0,4) satisfies the following relation:

$$\begin{aligned} (\nabla_E R)(X, Y, W, V) &= [A(E) + B(E)]R(X, Y, W, V) + A(X)R(E, Y, W, V) \\ &+ A(Y)R(X, E, W, V) + A(W)R(X, Y, E, V) \\ &+ A(V)R(X, Y, W, X), \end{aligned} \quad (1.3)$$

for all $X, Y, W, V \in \chi(M)$ where A, B are non-zero 1-forms given by

$$g(E, \rho) = A(E), g(E, \sigma) = B(E), \quad (1.4)$$

for all vector held $E \in \chi(M)$ where $\chi(M)$ being the Lie algebra of vector field on M s. Further extending the notion of $(APS)_n$, Tamássy and Binh (1989) introduced weakly symmetric manifolds.

1.7 Recurrent Manifolds

Walker introduced the notion of a locally recurrent Riemannian manifold as an extension to locally symmetric spaces. A non flat Riemannian manifold is said to be locally recurrent (Walker, 1950) if there exists a non-zero 1-form A such

that

$$(\nabla_X R)(Y, Z)U = A(X)R(Y, Z)U, \quad (1.5)$$

for all $X, Y, Z, U \in \chi(M)$. Then, (De *et al.*, 2003) studied φ -recurrent Sasakian manifolds as an extension to locally φ -symmetric manifolds Takahashi (1977). As a weaker version of the locally recurrent Riemannian manifold, Dubey (1979) introduced the notion of the generalized recurrent manifold. A non-flat Riemannian manifold is said to be a generalized recurrent manifold if its curvature tensor R satisfies

$$\nabla R = A \otimes R + B \otimes G, \quad (1.6)$$

where A and B are non-vanishing 1-forms defined as (1.3) and the tensor G is defined by

$$G(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \quad (1.7)$$

for any $X, Y, Z \in \chi(M)$. Shaikh *et al.* (2022) extended this concept to generalized φ -recurrent Sasakian manifold. Hui (2018) studied generalized φ -recurrent generalized (κ, μ) -contact metric manifold and obtained interesting results. A non-flat Riemannian manifold is said to be a generalized φ -recurrent manifold if the curvature tensor R satisfies the condition

$$\varphi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z + B(W)G(X, Y)Z, \quad (1.8)$$

for any $X, Y, Z, W \in \chi(M)$.

1.8 Almost contact Metric Manifolds

A $(2n+1)$ -dimensional smooth manifold M is called an almost contact metric manifold if it admits a $(1, 1)$ -tensor field φ , a unit vector field ξ (called the Reeb

vector field) and a 1-form η such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \cdot \varphi = 0, \quad (1.9)$$

this is equivalent to a reduction of the structural group of the tangent bundle to $U(n) \times 1$ (see Sasaki (1960); Sasaki and Hatakeyama (1961)). A Riemannian metric g is said to be an associated (or compatible) metric if it satisfies

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (1.10)$$

for all $X, Y \in \chi(M)$. An almost contact manifold $M^{2n+1}(\varphi, \xi, \eta)$ together with a compatible metric g is known as an almost contact metric manifold (Blair *et al.*, 1995; Blair, 1976, 2010). Chinea and Gonzalez (1990) obtained a complete classification for almost contact metric manifolds through the study of the covariant derivative of the fundamental 2-form. The fundamental 2-form Φ of an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is defined by

$$\Phi(X, Y) = g(X, \varphi Y),$$

for all $X, Y \in \chi(M)$, and this form satisfies $\eta \wedge \Phi^n \neq 0$. This means that every almost contact metric manifold is orientable. Moreover, an almost contact metric manifold is said to be a contact metric manifold if $d\eta = \Phi$. The following formulas hold on a contact metric manifold (Blair, 2010):

$$\nabla_X \xi = -\varphi X - \varphi hX. \quad (1.11)$$

Further, we define two self-adjoint operators h and ℓ by $h = \frac{1}{2}(L_\xi \varphi)$, where $L_\xi \varphi$ denotes the Lie-derivative of φ along ξ and $\ell = R(\cdot, \xi)\xi$ respectively, where R is the Riemannian curvature of M . These operators satisfy

$$h\xi = \ell\xi = 0, \quad h\varphi + \varphi h = 0, \quad \text{Tr}.h = \text{Tr}.h\varphi = 0. \quad (1.12)$$

$$\text{Tr}.\ell = S(\xi, \xi) = 2n - \|h\|^2. \quad (1.13)$$

Here, “Tr.” denotes trace. When a unit vector ξ is Killing (i.e. $h = 0$ or $Tr.\ell = 2n$) then the contact metric manifold is called K -contact. On the K -contact manifold, we have

$$R(X, \xi)\xi = X - \eta(X)\xi, \quad (1.14)$$

An almost contact structure (φ, η, ξ) and almost contact manifold M is said to be normal if the almost complex structure on $M \times \mathbb{R}$ defined by $J(X, f d/dt) = (\varphi X - f\xi, \eta(X)d/dt)$, where f is a real function on $M \times \mathbb{R}$ and t a coordinate on $\mathbb{R}$, is integrable (Blair, 1976, 2010). The necessary and sufficient condition for the almost contact structure (φ, η, ξ) to be normal is

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where the pair $[\varphi, \varphi]$ is the Nijenhuis tensor of φ defined by

$$[\varphi, \varphi](X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y],$$

for all $X, Y \in \chi(M)$. A normal almost contact metric manifold is a Sasakian manifold. It was shown that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \quad (1.15)$$

for any $X, Y \in \chi(M)$. A Sasakian manifold is K -contact but the converse is true only in dimension 3. Olszak (1986) showed that a 3-dimensional almost contact metric manifold M is normal if and only if $\nabla \xi \cdot \varphi = \varphi \cdot \nabla \xi$, or, equivalently,

$$\nabla_X \xi = -\alpha \varphi X + \beta(X - \eta(X)\xi), \quad (1.16)$$

where $2\alpha = \text{div} \xi$ and $2\beta = \text{Tr}(\varphi \nabla \xi)$, $\text{div} \xi$ is the divergence of ξ defined by $\text{div} \xi = \text{Tr}\{X \rightarrow \nabla_X \xi\}$ and $\text{Tr}(\varphi \nabla \xi) = \text{Tr}\{X \rightarrow \varphi \nabla_X \xi\}$. On a 3-dimensional

normal almost contact metric manifold the following relations hold (Olszak, 1986).

$$S(X, \xi) = -X\alpha - (\varphi X)\beta - \{\xi\alpha + 2(\alpha^2 - \beta^2)\eta(X), \quad (1.17)$$

$$\xi\alpha + 2\alpha\beta = 0. \quad (1.18)$$

for any $X \in \chi(M)$.

1.9 (κ, ς) -contact metric Manifold

The (κ, ς) -nullity distribution of almost contact metric manifold $M(\varphi, \xi, \eta, g)$ is a distribution (Blair et al., 1995):

$$N(\kappa, \varsigma) : p \rightarrow N_p(\kappa, \varsigma) = \{Z \in \chi(M) : R(X, Y)Z = \kappa\{g(Y, Z)X - g(X, Z)Y\} + \varsigma\{g(Y, Z)hX - g(X, Z)hY\}\},$$

for any $X, Y, Z \in \chi(M)$ and real numbers κ and ς . If $\varsigma = 0$, the (κ, ς) -nullity distribution $N(\kappa, \varsigma)$ is called the κ -nullity distribution $N(\kappa)$ (Koufogiorgos, 1993). An almost contact metric manifold M with $\xi \in N(\kappa, \varsigma)$ is called a (κ, ς) -contact metric manifold. A (κ, ς) -contact metric manifold becomes Sasakian manifold for $\kappa = 1$ and $\varsigma = 0$. In a (k, ς) -contact metric manifold the following relations hold (Blair et al., 1995; Papantoniou, 1993)

$$h^2 = (k - 1)\varphi^2, \quad k \leq 1, \quad (1.19)$$

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \varsigma[\eta(Y)hX - \eta(X)hY], \quad (1.20)$$

$$S(X, Y) = [2(n - 1) - n\varsigma]g(X, Y) + [2(n - 1) + \mu]g(hX, Y) + [2(1 - n) + n(2k + \varsigma)]\eta(X)\eta(Y), \quad (1.21)$$

for any vector fields $X, Y \in \chi(M)$.

1.10 Generalized Sasakian space forms

A plane section π in the tangent bundle $T_p M$ at a point p of a Riemannian manifold is called a φ -section if it is spanned by X and φX , where X is a unit tangent vector orthogonal to ξ . The sectional curvature of a φ -section is called a φ -sectional curvature. A Sasakian manifold with constant φ -sectional curvature c is said to be a Sasakian space form and is denoted by $M(c)$ (Blair, 1976). Some examples of Sasakian space forms are $\mathbb{R}^{2n+1}$ and S^{2n+1} , with the standard Sasakian structures (Blair, 1976). Alegre *et al.* (2004) introduced generalized Sasakian space forms as an almost contact metric manifolds M whose curvature tensor R can be written as

$$\begin{aligned} R(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} \\ &+ f_2\{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}, \end{aligned} \quad (1.22)$$

for any vector fields X, Y, Z on M , where f_1, f_2 and f_3 are functions on M . Sasakian-space-forms appear as natural examples of generalized Sasakian-space-forms, with constant functions $f_1 = \frac{c+1}{4}, f_2 = f_3 = \frac{c-1}{4}$. Many authors have studied generalized Sasakian-space-forms in different context such as Alegre and Carriazo (2008), Rehman (2015), Sarkar *et al.* (2015) and Carriazo *et al.* (2020).

1.11 Curvatures on Riemannian Manifold

A conformal transformation is an angle preserving map. If g and $\bar{g}$ are two metrics of an n -dimensional Riemannian manifold M such that

$$\bar{g}(X, Y) = e^{2\sigma} g(X, Y),$$

for all vector fields X, Y on M and σ is a scalar function, then the angle between two tangent vectors at a point $p \in M$ does not change with respect to the change of metrics. Under such case M and $\bar{M}$ are conformally related and the corresponding between them is known as conformal transformation (Obata, 1970). One of the most important curvature tensors for analyzing the intrinsic properties of the Riemannian manifold is the Weyl conformal curvature tensor introduced by Yano and Kon (1984). This curvature is invariant under conformal transformation. The conformal curvature C of type $(1,3)$ on a $(2n + 1)$ -dimensional Riemannian manifold $(M, g), n > 1$, is defined by

$$C(X, Y)Z = R(X, Y)Z - \frac{1}{2n-1} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] + \frac{r}{2n(2n-1)} [g(Y, Z)X - g(X, Z)Y], \quad (1.23)$$

where r is the scalar curvature of M , Q is the Ricci operator and S the Ricci tensor.

It is known that a harmonic function is defined as a function whose Laplacian vanishes. In conformal transformation, harmonic functions are not invariant, in general. To tackle this, Ishi (1957) obtained the condition under which a harmonic function becomes invariant by introducing conharmonic transformation as a subgroup of conformal transformation (1.23) satisfying

$$\sigma_{,i}^i + \sigma_{,i}\sigma^i = 0.$$

The tensor H which remains invariant under conharmonic transformation is known as conharmonic curvature tensor. For a Riemannian manifold M of dimension- $(2n + 1)$, the conharmonic curvature tensor is given by

$$H(X, Y)Z = R(X, Y)Z - \frac{1}{(2n-1)} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY], \quad (1.24)$$

for all vector fields X, Y, Z on M .

According to Kim (2016), the semiconformal curvature tensor $\tilde{P}$ of type $(1, 3)$ on a Riemannian manifold (M^n, g) is defined as follows:

$$\tilde{P}(X, Y)W = -(n-2)bC(X, Y)W + [a + (n-2)b]H(X, Y)W, \quad (1.25)$$

where a, b are constants not simultaneously zero, $C(X, Y)W$ denotes the conformal curvature tensor of type $(1, 3)$, and $H(X, Y)W$ denotes the conharmonic curvature tensor of type $(1, 3)$.

Mantica and Suh (2013) introduced and studied $\tilde{Q}$ curvature tensor. In a $(2n+1)$ -dimensional Riemannian manifold (M, g) , the $\tilde{Q}$ curvature tensor is given by

$$\tilde{Q}(X, Y)Z = R(X, Y)Z - \frac{v}{2n}[g(Y, Z)X - g(X, Z)Y], \quad (1.26)$$

for any $X, Y, Z \in \chi(M)$ and v is an arbitrary scalar function on M . If $v = \frac{r}{2n+1}$, then $\tilde{Q}$ curvature tensor reduces to concircular curvature tensor (Yano, 1940).

1.12 Kenmotsu Manifold

To study the manifolds of negative curvature Bishop and O'Neill (1969) introduced the warped product as a generalization of the Riemannian product. Tanno (1969) gave a classification of connected $(2n+1)$ -dimensional almost contact metric manifold M based on its automorphism groups possessing the maximum dimension $(n+1)^2$. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, k (say). Then there are three classes:

- i) $k > 0$, M is a homogeneous Sasakian manifold of constant holomorphic sectional curvature.
- ii) $k = 0$, M is the global Riemannian product of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature.
- iii) $k < 0$, M is warped product space $\mathbb{R} \times_f \mathbb{C}^n$.

Kenmotsu (1972) studied the third case and obtained its geometric properties. The structure so obtained is now known as the Kenmotsu structure and the manifold with a Kenmotsu structure is called the Kenmotsu manifold (Janssens and Vanhecke, 1981). In general, a Kenmotsu manifold are not Sasakian. A Kenmotsu manifold can be defined as a normal almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. Kenmotsu proved that such a manifold is locally a warped product $I \times_f N^{2n}$, where I is an open interval with coordinate t , $f = ce^t$ is the warping function for some positive constant c and N^{2n} is a Kählerian manifold. It is well known that a Kenmotsu manifold can be characterized, through their Levi-Civita connection ∇ satisfying the following:

$$\nabla_X \xi = X - \eta(X)\xi, \quad (1.27)$$

$$(\nabla_X \varphi)Y = -\eta(Y)\varphi X - g(X, \varphi Y)\xi, \quad (1.28)$$

for any $X, Y \in \chi(M)$. On a Kenmotsu manifold M the following holds (Kenmotsu, 1972):

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (1.29)$$

$$Q\xi = -2n\xi, \quad (1.30)$$

for any vector fields X, Y on M .

1.13 Almost Kenmotsu Manifolds

Kim and Pak (2005) and Olszak (1989) studied almost contact metric manifolds such that η is closed and $d\Phi = 2\eta \wedge \Phi$ and called it as almost Kenmotsu manifold. A normal almost Kenmotsu manifold is a Kenmotsu manifold. In almost Kenmotsu manifold M , we have

$$\nabla_X \xi = -\varphi^2 X - \varphi hX, \quad (1.31)$$

for any vector field X on M . Dileo and Pastore (2009) studied almost Kenmotsu manifolds satisfying (κ, ς) -nullity distribution and $(\kappa, \varsigma)'$ -nullity distribution. Later, Pastore and Saltarelli (2011) extended it to generalized nullity distribution. An almost Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is said to be a generalized (κ, ς) -almost Kenmotsu manifold if ξ belongs to the generalized (κ, ς) -nullity distribution, i.e.,

$$R(X, Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \varsigma[\eta(Y)hX - \eta(X)hY], \quad (1.32)$$

for all vector fields X, Y on M , where κ, ς are smooth functions on M . An almost Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is said to be a generalized $(\kappa, \varsigma)'$ -almost Kenmotsu manifold if ξ belongs to the generalized $(\kappa, \varsigma)'$ -nullity distribution, i.e.,

$$R(X, Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \varsigma[\eta(Y)h'X - \eta(X)h'Y], \quad (1.33)$$

for all vector fields X, Y on M , where κ, ς are smooth functions on M . Moreover if both κ and ς are constants in (1.33), then M is called a $(\kappa, \varsigma)'$ -almost Kenmotsu manifold (Pastore and Saltarelli, 2011; Wang and Liu, 2016; Dileo and Pastore, 2009). On generalized (κ, ς) or $(\kappa, \varsigma)'$ -almost Kenmotsu manifold with $h \neq 0$ (equivalently, $h' \neq 0$), the following relations hold (Dileo and Pastore, 2009):

$$h'^2 = (\kappa + 1)\varphi^2, \quad h^2 = (\kappa + 1)\varphi^2, \quad (1.34)$$

$$Q\xi = 2n\kappa\xi. \quad (1.35)$$

1.14 Some Vector Fields

A vector field V is said to be harmonic vector field if it is a critical point of the energy functional E defined by

$$E(V) = \frac{1}{2} \int ||dV||^2 dM = \frac{n}{2} \text{vol}(M, g) + \frac{1}{2} \int_M ||\nabla V||^2 dM$$

on the space χ^1 of all unit vector fields on M . A contact metric manifold whose Reeb vector field is harmonic is called an H -contact manifold. Perrone (2004) proved that *a contact metric manifold is an H -contact manifold, that is ξ is a harmonic vector field, if and only if ξ is an eigenvector of the Ricci operator*. This implies $Q\xi = (Tr.l)\xi$. This is valid for K -contact manifolds, (k, ς) -contact manifolds and unit sphere S^{2n+1} with standard contact metric structure.

A vector field V on a contact metric manifold M is said to be contact if there exists a smooth function $\varrho : M \rightarrow \mathbb{R}$ satisfying

$$(L_V\eta)(Y) = \varrho\eta(Y), \quad (1.36)$$

for all $Y \in \chi(M)$ and if $\varrho = 0$, then the vector field V is called strict.

A smooth vector field X on a Riemannian manifold is said to be a conformal vector field if there exists a smooth function ψ on M that satisfies

$$L_X g = 2\psi g.$$

We say that X is non-trivial if X is not Killing, that is, $\psi \neq 0$.

A vector field ξ is called torse-forming (Blaga, 2018) if it satisfies

$$\nabla_X \xi = X + \eta(X)\xi, \quad (1.37)$$

for any $X \in \chi(M)$ and 1-form η .

1.15 Ricci-Yamabe Soliton

The theory of geometric flows plays a significant role in understanding the geometric structure in Riemannian geometry. The Ricci flow is a well-known geometric flow introduced by Hamilton, who used it to prove a three-dimensional sphere theorem (Hamilton, 1982). Ricci flow plays a crucial role in forming proof of Thurston's conjecture, including as a special case, the Poincare conjecture.

The Ricci soliton on a Riemannian manifold (M, g) are the self-similar solutions to Ricci flow and is defined by

$$\frac{1}{2}L_V g + S = \lambda g, \quad (1.38)$$

where $\mathcal{L}_V g$ denotes the Lie-derivative of g along potential vector field V , S is the Ricci curvature of M^{2n+1} and λ , a real constant. The soliton is called shrinking, steady and expanding if $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ respectively. When the vector field V is the gradient of a smooth function f on M^{2n+1} , that is, $V = \nabla f$, then we say that Ricci soliton is a gradient (For details see Cao (2009); Petersen and Wylie (2009)). According to Petersen and Wylie (2009), a gradient Ricci soliton is rigid if it is a flat $N \times_{\Gamma} \mathbb{R}^k$, where N is Einstein and gave certain classification. The notion of almost Ricci soliton was introduced by Pigola *et al.* (2011) by taking λ as a smooth function in the definition of Ricci soliton (1.38).

To tackle the Yamabe problem on manifolds of positive conformal Yamabe invariant, Hamilton (1982) introduced the geometric flow known as Yamabe flow. The Yamabe soliton is a self-similar solution to the Yamabe flow. On a Riemannian manifold (M, g) , a Yamabe soliton is given by

$$\frac{1}{2}L_V g = (R - \lambda)g, \quad (1.39)$$

where R is the scalar curvature of the manifold and λ , a real constant. The Yamabe soliton preserves the conformal class of the metric but the Ricci soliton does not in general. However, in dimension $n = 2$, both the solitons are similar. If λ is a smooth function in (1.39), then it is called almost Yamabe soliton.

Güler and Crasmareanu (2019) introduced a new type of geometric flow which is a scalar combination of Ricci flow and Yamabe flow under the name Ricci-Yamabe map and define the following:

Definition 1.15.1. (Güler and Crasmareanu (2019)) The map $RY^{(\alpha, \beta, g)} : I \rightarrow$

$T_2^s(M)$ given by:

$$RY^{(\alpha,\beta,g)} = \frac{\partial g}{\partial t}(t) + 2\alpha S(t) + \beta R(t)g(t),$$

is called the (α, β) -Ricci-Yamabe map of the Riemannian flow (M, g) . If

$$RY^{(\alpha,\beta,g)} \equiv 0,$$

then $g(\cdot)$ will be called an (α, β) -Ricci-Yamabe flow.

The Ricci-Yamabe flow can be Riemannian or semi-Riemannian or singular Riemannian flow due to the involvement of scalars α and β . These kind of different choices can be useful in some physical models such as relativity theory. The Ricci-Yamabe soliton emerges as the limit of the solution of Ricci-Yamabe flow.

Definition 1.15.2. A Riemannian manifold $(M^n, g), n > 2$ is said to admit almost Ricci-Yamabe soliton $(g, V, \lambda, \alpha, \beta)$ if there exist smooth function λ such that

$$L_V g + 2\alpha S = (2\lambda - \beta R)g, \quad (1.40)$$

where $\alpha, \beta \in \mathbb{R}$.

Almost Ricci-Yamabe soliton is of particular interest as it generalizes a large group of well-known solitons such as:

1. Ricci almost soliton $(\alpha = 1, \beta = 0)$.
2. almost Yamabe soliton $(\alpha = 0, \beta = 1)$.
3. Ricci-Bourguignon almost soliton $(\alpha = 1, \beta = -2\rho)$.

Also, if λ is constant, then it includes Ricci soliton, Yamabe soliton and Ricci-Bourguignon soliton among others.

If V is a gradient of some smooth function f on M , then the above notion is called gradient almost Ricci-Yamabe soliton and then (1.40) reduces to

$$\nabla^2 f + \alpha S = (\lambda - \frac{1}{2}\beta R)g, \quad (1.41)$$

where $\nabla^2 f$ is the Hessian of f .

The almost Ricci-Yamabe soliton (ARYS) is said to be expanding, shrinking or steady if $\lambda < 0$, $\lambda > 0$ or $\lambda = 0$ respectively. In particular, if λ is constant, then almost Ricci-Yamabe soliton reduces to Ricci-Yamabe soliton.

Extending the notion of Ricci soliton, Cho and Kimura (2009) introduced the η -Ricci soliton which is obtained by perturbing the equation 1.38 by a multiple of a certain $(0, 2)$ -tensor field $\eta \otimes \eta$. A more general extension is obtained by Siddiqi and Akyol (2020) and called such soliton as η -Ricci-Yamabe soliton of type (α, β) which is defined as:

$$L_V g + 2\alpha S + (2\mu - \beta R) + 2\omega \eta \otimes \eta = 0. \quad (1.42)$$

1.16 Submanifold

Let M and N be smooth manifolds, where $\dim(M) \leq \dim(N)$, let $F : M \rightarrow N$ be a smooth map, and let p be a point in M . We say that F is an immersion at p if the differential map $d_p(F) : T_p(M) \rightarrow T_{F(p)}(N)$ is injective, and that F is an immersion if it is an immersion at every p in M .

Definition 1.16.1. Suppose $(N, \tilde{g})$ is a Riemannian manifold of dimension m , M is a manifold of dimension n and $\iota : M \rightarrow N$ is an immersion. If M is given the induced Riemannian metric $g := \iota^* \tilde{g}$, then ι is said to be an isometric immersion. If in addition ι is injective, so that M is an immersed submanifold of N , then M is said to be a Riemannian submanifold of N .

The geometry of submanifolds in recent decades has become a topic of grow-

ing interest for its significant applications in applied mathematics and theoretical physics. The notion of invariant submanifolds can be used to discuss the properties of a non-linear autonomous system. In general, the invariant submanifolds inherit almost all the geometric properties of the ambient manifold. Another important type of submanifold is a totally geodesic submanifold. The significance of this submanifold is that the geodesics of the ambient manifolds remain geodesics in the submanifolds. Moreover, the notion of geodesics plays an important role in the theory of relativity.

Let M be an immersed submanifold of Riemannian manifold $\tilde{M}$ with induced metric g . By $\Gamma(TM)$ and $\Gamma(T^\perp M)$, we denote the tangent and normal subspaces of M in $\tilde{M}$. Also let ∇ and $\nabla^\perp$ denotes the induced connection on the tangent bundle TM and the normal bundle $T^\perp M$ of M respectively. Then the Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad (1.43)$$

and

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^\perp V, \quad (1.44)$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where σ and A are called the second fundamental form and shape operator of M respectively. They are related by

$$g(A_V X, Y) = g(\sigma(X, Y), V). \quad (1.45)$$

The mean curvature vector H of M^n is defined to be

$$H = \frac{1}{n} \text{Tr}(\sigma),$$

where Tr denotes the trace. A submanifold M in a Riemannian manifold is called minimal if its mean curvature vector vanishes identically. The submanifold M is called totally geodesic if $\sigma(X, Y) = 0$ for any $X, Y \in \Gamma(TM)$.

The covariant derivative of the second fundamental form σ is defined by

$$(\tilde{\nabla}_X \sigma)(Y, Z) = \nabla_X^\perp \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \quad (1.46)$$

for all $X, Y, Z \in \Gamma(TM)$, where $\tilde{\nabla}$ is called the Vander-Waerden-Bortolotti connection on M . Then $\tilde{\nabla}\sigma$ is a normal bundle valued tensor of type $(0,3)$ and is called the third fundamental form of M . Whenever $\tilde{\nabla}\sigma = 0$, then M is said to be have parallel second fundamental form. The Gauss equation for the Riemannian curvature R of the submanifold M is given by

$$\tilde{R}(X, Y)Z = R(X, Y)Z + A_{\sigma(X, Z)}Y - A_{\sigma(Y, Z)}X, \quad (1.47)$$

for any $X, Y, Z \in \Gamma(TM)$.

In submanifold theory, obtaining the relationships between the intrinsic invariant and extrinsic invariant has been the primary goal of many geometers in recent decades. Chen invariants were introduced by Chen (1993) to tackle the question raised by Chern's concerning the existence of minimal immersions into a Euclidean space of arbitrary dimension (Chern, 1968).

Suppose L is an r -dimensional subspace of $T_x M$, $x \in M$, $r \geq 2$ and $\{e_1, \dots, e_r\}$ an orthonormal basis of L . The scalar curvature τ of the r -plane section L is given by

$$\tau(L) = \sum_{1 \leq i < j \leq r} K_{ij}, \quad (1.48)$$

where K_{ij} is the sectional curvature of the plane section spanned by e_i and e_j at $x \in M$. Let $\Pi \subset T_x M$ be a 2-plane section and $K(\Pi)$ be the sectional curvature of M for a plane section Π in $T_x M$, $x \in M$. Then

$$K(\Pi) = \frac{1}{2}[R(e_1, e_2, e_2, e_1) - R(e_1, e_2, e_1, e_2)]. \quad (1.49)$$

The scalar curvature $\tau(x)$ of M at the point x is given by

$$\tau(x) = \sum_{i < j} K_{ij}, \quad (1.50)$$

where $\{e_1, \dots, e_m\}$ is an orthonormal basis for $T_x M$.

Chen's δ -invariant δ_M of a Riemannian manifold M introduced by Chen is

$$\delta_M(x) = \tau(x) - \inf\{K(\Pi) | \Pi \text{ is a plane section } \subset T_x M\}, \quad (1.51)$$

where τ is the scalar curvature of M .

1.17 Lorentzian Manifold

An n -dimensional pseudo-Riemannian (or sometimes called semi-Riemannian) manifold is a pair (M, g) , where M is an n -dimensional smooth manifold and g is a symmetric, nondegenerate 2-tensor field on M (called the metric). A pseudo-Riemannian manifold is said to be Riemannian if g has a signature $(+, \dots, +)$, and is said to be Lorentzian if g has signature $(-, +, \dots, +)$ or $(+, -, \dots, -)$.

1.18 F(R)-gravity theory

The relationship between spacetime and gravity constitutes a cornerstone of modern theoretical physics (Thorne *et al.*, 2000). Massive objects induce curvature in spacetime and create gravitational fields that controls the trajectories of nearby objects. This curvature alters the path of particles and compelling them to follow geodesics dictated by the underlying spacetime geometry. The recent detection of gravitational waves further confirms the profound link between spacetime and gravity. These waves propagate as ripples in spacetime which carry energy and provide insights into cataclysmic cosmic events.

General Relativity (GR) is an important theoretical framework for compre-

hending the Universe's large-scale structure (Hawking, 1973) yet it fails to explain the cosmic acceleration events. To solve this, researchers investigated theoretical revisions that include the ideas such as inflation and Dark Energy (DE). Einstein's gravitational field equations are altered to better account for observable cosmic dynamics. This often requires an introduction of an additional speculative component known as Dark Matter (Overdun and Wesson, 2004) which aims to reconcile theoretical predictions with observable data. Dark energy is believed to be the driving force behind cosmic expansion, while modified gravity theories propose alterations to the gravitational framework without dark energy. Both approaches aim to reconcile theoretical predictions with observational data but each presents unique challenges.

Beyond the fundamental Einstein-Hilbert action, Nojiri and Odintsov (2005) proposed $f(G)$ -gravity, Sotiriou and Faraoni (2010) discussed $f(R)$ -gravity and Cai *et al.* (2016) introduced $f(T)$ -gravity theory. These modified gravitational frameworks give novel insights that go beyond Einstein's gravitational theory. Furthermore, these alterations are expected to give effective approximations to the elusive domain of quantum gravity (Parker and Toms, 2009). The addition of a function $f(R)$ allows General Relativity (GR) to be extended into the world of $f(R)$ -gravity beyond the Einstein-Hilbert Lagrangian density. The Einstein-Hilbert action for $f(R)$ -gravity has the expression

$$\mathcal{H} = \frac{1}{\kappa^2} \int [f(R) + \mathcal{L}_m] \sqrt{-g} d^4x$$

where $f(R)$ represents an arbitrary function of the Ricci scalar R and $\mathcal{L}_m$ denotes the Lagrangian of the scalar field. The integration of higher-order curvature components causes the equations of motion within this framework to exhibit bigger degrees giving a solution to the dilemma encountered by massive neutron stars (Astashenok *et al.*, 2013, 2015, 2017).

1.19 Review of Literature

Recent years have seen considerable focus in academic circles on the study of Einstein manifolds and their numerous extensions. Sinha and Sharma (2011) initiated the investigation of Ricci solitons in the domain of contact geometry in their study. Subsequently, Tripathi (2008) and Bejan and Crasmareanu (2011), among others, performed comprehensive investigations on Ricci solitons in contact metric manifolds.

In the later part of the 20th century, Hamilton introduced the concept of Ricci flow. To elaborate, a one-parameter family is taken, which has a metric g defined on a Riemannian manifold M^d over an interval $I \subset \mathbb{R}$, where Ric represents the metric's Ricci tensor g , and the equation governing the Ricci flow is as follows:

$$\frac{\partial}{\partial t}g = -2Sg.$$

Hamilton (1982) established the existence of a unique solution g to the aforementioned equation for any smooth metric g defined on a compact Riemannian manifold M^d . This solution is defined over some $[0, \epsilon)$, where $\epsilon > 0$, which has the condition $g(0) = g_0$.

For the scenario of complete non-compact manifolds, Shi (1989) demonstrated the existence of a complete solution under the stipulation that the sectional curvatures of (M^d, g) remain bounded.

As a Riemannian metric, a Ricci soliton satisfies

$$S + \frac{1}{2}L_Vg - \lambda g = 0.$$

In the equation provided, the symbol L_V denotes the Lie derivative operator with respect to the vector field V . The tensor S signifies the Ricci tensor corresponding to the metric g , while λ represents a constant. It's evident that a trivial Ricci soliton corresponds to an Einstein metric where V can be zero or a

vector field that kills. When V is the gradient of a smooth function f , denoted as $V = df$, the soliton is termed a gradient Ricci soliton by Cao (2009), which is defined as

$$S + \nabla^2 f = \lambda g \quad (1.52)$$

where S denotes the Ricci tensor of the metric g , and λ is a constant. This characterization enables certain geometers to regard the equation as the defining condition for a Ricci soliton. For further insights into Ricci flow, readers are directed to consult (Chow *et al.* , 2006).

Gomes *et al.* (2017) introduced the concept of $\mathfrak{h}$ -almost Ricci soliton for the first time. The authors have conducted a comprehensive study on the properties and characterization of $\mathfrak{h}$ -almost Ricci solitons, contributing to the literature on this subject. The study involve discovering new findings, refining existing theorems, or presenting alternative proofs for known results. The author also discusses various aspects of $\mathfrak{h}$ -almost Ricci solitons, including their geometric interpretations, stability characteristics, classification outcomes, and connections to other geometric constructs. The authors also explore the potential applications of $\mathfrak{h}$ -almost Ricci solitons in mathematics and theoretical physics, highlighting their importance in understanding the geometric evolution and curvature properties of Riemannian manifolds. This research offers valuable insights into the theory of $\mathfrak{h}$ -almost Ricci solitons and their relevance to contemporary geometers. Also, in a subsequent paper, Ghosh (2008) studied the gradient Ricci soliton of a non-Sasakian (k, ς) -contact manifold. Inspired by these conditions, we investigate $\mathfrak{h}$ -almost Ricci solitons in the framework of almost Kenmotsu manifolds.

The definition of a $\mathfrak{h}$ -almost Ricci soliton, as determined by their findings, is characterized by a complete Riemannian manifold (M^d, g) furnished with a vector field $X \in \chi(M)$, a soliton function $\lambda : M \rightarrow \mathbb{R}$, and a function $\mathfrak{h} : M \rightarrow \mathbb{R}$. These

are all smooth and satisfy the equation as given in Gomes *et al.* (2017).

$$S + \frac{\mathfrak{h}}{2}L_V g = \lambda g. \quad (1.53)$$

For ease of reference, $\mathfrak{h}$ -almost Ricci soliton is denoted as $(M^d, g, X, \mathfrak{h}, \lambda)$. In the scenario where λ is constant, it is simply termed an $\mathfrak{h}$ -Ricci soliton. If $L_V g = L_{\nabla u} g$ holds true for some smooth function $u : M \rightarrow \mathbb{R}$, then $(M^d, g, \nabla u, \mathfrak{h}, \lambda)$ is referred to as a gradient $\mathfrak{h}$ -almost Ricci soliton having u as the potential function. Under this condition, we can express equation (1.53) as:

$$S + \mathfrak{h}\nabla^2 f = \lambda g, \quad (1.54)$$

the notation $\nabla^2 f$ signifies the Hessian of the function f .

Back in 1915, Albert Einstein introduced the geometric properties of gravity as a symmetric curvature of spacetime, known as the general theory of relativity (GR). Spacetime geometry is especially related to the symmetry of physical matter. The spacetime symmetries are used specifically for the study of exact solutions of Einstein's field equations of general relativity. The universe, including all its contents, such as stars, planets, galaxies, and other forms of matter, are all space, time, and energy. The general theory of relativity (GR) is a theoretical method for studying the universe in a structure of large-scale effectively. The acceleration of the Universe in the past and present cannot be described without taking into account the dark energy in the GR. Since it does not explain the gravity, the modification in order to obtain theories that admit inflation and intimates the Dark Energy (DE) is quite reasonable. The modification to Einstein's gravitational field equations (Sahni and Starobinsky, 2000; Peebles and Ratra, 2003) has provided the conventional method to analyze known cosmic dynamics. The greatest approximation to the observable data was given by Einstein's field equations, with the inclusion of a hypothetical element of the universe that was described as Dark Matter by Overdun and Wesson (2004).

Dark Energy (DE) is a strange component of the universe, which is supposed to be the main cause of the acceleration of the universe, extension and regulates the matter-energy ratio. These results in the construction of more advanced gravity theories from both the mathematicians and physicists, which grows as a result of the Einstein-Hilbert action and the use of modified gravity theories such as $F(R)$ -gravity (Sotiriou and Faraoni, 2010), Gauss-Bonnet $F(G)$ -gravity (Nojiri and Odintsov, 2005) and $F(T)$ theory (Cai *et al.*, 2016). These theories vary from Einstein's conventional gravity theory in that they might also have a closer approximation to quantum gravity (Parker and Toms, 2009).

After the discovery of dark energy in 1998, the modification of gravity to GR led to an alternate way of explaining the cosmic acceleration. $F(R)$ -theory based dark energy models have been studied to a great extent as the simplest modified gravity outlined to realize the late-time acceleration. A Lagrangian density-based model was proposed for dark energy in the metric formalism (Capozziello *et al.*, 2003; Carroll *et al.*, 2004). Although it was shown that this model aggravates by a matter of instability (Faraoni, 2006) as well as by the complicatedness to satisfy local gravity constraints (Olmo, 2005; Erickcek *et al.*, 2006; Navarro and Van Acoleyen, 2007). The model does not possess a standard matter-dominated epoch due to a large-scale coupling between dark energy and dark matter (Amendola *et al.*, 2007), which shows how non-trivial it is to get a viable $F(R)$ model. Amendola *et al.* (2007) derived the cosmological viability of $F(R)$ dark energy model conditions.

Corrections to GR were contemplated to be important to a few extents; truly, there are relevant studies: De and Loo (2021) showed that a Robertson–Walker spacetime is $(APRS)_4$ and vice versa under certain conditions imposed on its scale factor in almost pseudo-Ricci symmetric spacetime solutions in $F(R)$ -gravity. De *et al.* (2021) studied conformally flat generalized Ricci recurrent perfect fluid spacetime with constant Ricci scalar as a solution of modified $F(R)$ -gravity the-

ory and showed that a Robertson-Walker spacetime is a generalized Ricci recurrent if and only if it is Ricci symmetric. Also, in De *et al.* (2021), the authors proved that such a $(WRS)_4$ satisfying $F(R)$ -gravity field equations represents a perfect fluid with vanishing vorticity. Some energy conditions are studied under the current settings to constrain the functional form of $F(R)$.

Mantica and Suh (2012) defined a generalized (0,2) symmetric Z -tensor given by

$$Z_{ij} = R_{ij} + \phi g_{ij}, \quad (1.55)$$

where ϕ denotes an arbitrary scalar function.

If trace $Z = 0$, then (1.55) gives $\phi = -\frac{R}{n}$; it reduces to the classical Z -tensor, which is traceless. In our research, we considered the Z -tensor given by (1.55) with trace zero. In (Mantica and Suh, 2012, 2014), the authors studied the properties of pseudo-symmetric Riemannian manifolds $(PZS)_n$ and also considered $(PZS)_4$ spacetime. Afterward, several Z -symmetric spacetimes were studied by many geometers in the context of GR (for details see (Mantica and Suh, 2014; De *et al.*, 2021; De and Pal, 2014; Shanawy *et al.*, 2000)). Recently, Shanawy *et al.* (2000) extended the study of Z -tensor to $F(R)$ gravity. They investigated almost pseudo Z -symmetric spacetimes in $F(R)$ theory of gravity. De *et al.* (2022) studied conformally flat pseudo-symmetric spacetimes within the framework of $F(R)$ gravity theory.

Ricci solitons (1.38) are a naturally generalized class of Einstein manifolds and represent self-similar solutions to the Ricci flow equation.

$$\frac{\partial}{\partial t} g = -2S$$

which was defined by Hamilton (1982, 1995). Ricci solitons and some of their generalizations have been studied by many geometers in the recent years. For example, see (Cao, 2006; Eminenti *et al.*, 2008; Petersen and Wylie, 2009; Fernandez-

Lopez *et al.*, 2011; Chen, 2015, 2016; Karaca, 2018; Deshmukh and Al-Sodais, 2020; Blaga and Tastan, 2021; Blaga and Özgür, 2022; Güler and Ünal, 2022; Sharma and Deshmukh, 2022) and the references therein.

If the potential vector field is derived from the gradient of a smooth function u on M , the tuple $(M, g, \nabla u, \lambda)$ is known as a gradient Ricci soliton, and the equation (3.49) turns into

$$S + \text{Hess } u = \lambda g.$$

The study of the concept of the Ricci-Bourguignon soliton is introduced by Dwivedi (2021). They correspond to self-similar solutions of the Ricci-Bourguignon flow equation

$$\frac{\partial}{\partial t} g = -2(S - \rho Rg), \quad (1.56)$$

where R is the scalar curvature and $\rho \in \mathbb{R}$ is a constant. The flow in equation (3.50) was introduced by Bourguignon (1979). Equation (3.50) is precisely the Ricci flow for $\rho = 0$. As in the Ricci flow case, the following definition was given by Dwivedi (2021).

A Ricci-Bourguignon soliton (RBS) characterizes a semi-Riemannian manifold (M, g) along with a vector field X on M satisfying

$$S + \frac{1}{2} L_X g = \lambda g + \rho Rg, \quad (1.57)$$

where $L_X g$ signifies the Lie derivative of the metric g and $\lambda \in \mathbb{R}$ represents a constant, and it is denoted by (M, g, X, λ, ρ) . Suppose X corresponds to the gradient of a smooth function u defined on M . In such cases, $(M, g, \nabla u, \lambda, \rho)$ is denoted as a gradient Ricci-Bourguignon soliton, and the equation (3.51) turns into

$$S + \text{Hess } u = \lambda g + \rho Rg.$$

Dwivedi (2021) proved some results for the solitons of the Ricci-Bourguignon flow, generalizing the corresponding results for Ricci solitons. Soylu (2022) gave classification theorems for Ricci-Bourguignon solitons and almost solitons with concurrent potential vector fields. Ghosh (2022) studied Ricci-Bourguignon solitons and Ricci-Bourguignon almost solitons on a Riemannian manifold and proved some triviality results.

For a submanifold M of a Euclidean m -space $\mathbb{E}^m$, the most natural tangent vector field of M is the tangential component of the position vector field x of M (cf. Chen (2017a,b)). Ricci solitons on Euclidean submanifolds arisen from such a vector field have been studied recently by the authors in (Chen and Deshmukh, 2014, 2015).

One of the fundamental issues in submanifold theory is establishing criteria that suggest an isometric immersion of a product manifold must be an extrinsic product of isometric immersions. The first contribution to this problem was addressed by Moore (1971) for isometric immersions into Euclidean space and extended by Molzan (1983) for any space form as ambient space. They showed that an isometric immersion of a Riemannian product into a space form must be an extrinsic product of isometric immersions of the factors whenever its second fundamental form is adapted to the product structure of the manifold.

Garcia and Fernando (2024) studied the isometric immersions of Einstein manifolds into the product space $\mathbb{Q}_c^n \times \mathbb{R}$, with a flat normal bundle and a parallel mean curvature vector field. Consider the concept that a Riemannian manifold M is classified as an Einstein manifold when the Ricci tensor adheres to the condition

$$S(X, Y) = \lambda \langle X, Y \rangle,$$

for every $X, Y \in \chi(M)$, and for a certain constant $\lambda \in \mathbb{R}$. Riemannian manifolds M_c^n with constant sectional curvature c are the simplest examples of Einstein Manifolds, where $\lambda = (n - 1)c$.

An isometric immersion is said to be Einstein if the submanifold, endowed with the induced metric, is an Einstein manifold. For example, Ryan (1969) gave a local classification of Einstein hypersurfaces in any space form. In higher codimension, Di Scala (2003) proved that Einstein real Kahler submanifolds of a Euclidean space are totally geodesic provided that they are minimal, and the same conclusion still holds for minimal Einstein submanifolds in the Euclidean space with a flat normal bundle (Nolker, 1990).

Onti (2018) classified Einstein submanifolds into space forms, with flat normal bundle and parallel mean curvature vector field, extending a previous result due to Dajczer and Tojeiro (1993) for isometric immersions of Riemannian manifolds with constant sectional curvature. More recently, Leandro *et al.* (2021) classified Einstein hypersurfaces into product space $\mathbb{Q}_\epsilon^n \times \mathbb{R}$.

We commence an investigation into Ricci-Bourguignon solitons on Euclidean submanifolds, where the soliton fields correspond to the tangential components of their position vector fields. We establish several fundamental properties of these solitons. Notably, we provide a classification of such Ricci-Bourguignon solitons on Euclidean hypersurfaces.

The study of the theory of Ricci flow (Hamilton, 1982, 1988; Chow, *et.al.*, 2006) reached its highest magnitude and popularity soon after Perelman (2002, 2003) successfully applied it to solve the *Poincaré* conjecture. Ricci solitons were also studied by Hamilton, who viewed them as fixed or stationary points of the Ricci flow in the space of parameterized metrics $g(t)$ on scaling and M modulo diffeomorphisms. Since then, both topics have been extensively explored by numerous mathematicians, including Brendle (2014), Cao and Chow (1999), Chen (2015), and many others (see for instance, Calin and Crasmareanu (2010); Naik *et al.* (2020); Morgan and Tian (1999); Topping (2006); Ganguly *et al.* (2022)).

Recently, self-similar solutions, referred to as soliton solutions of various geometric flow equations, have been introduced and explored due to their potential

significance as models for singularities. Among these, Ricci solitons, defined as fixed points of the Ricci flow, stand as one of the most extensively studied and renowned classes. Substantial advancements have been achieved in this particular domain. Furthermore, Basu and Bhattacharyya (2015) established the Yamabe flow to address the Yamabe problem. This problem primarily involves the quest for a metric on a manifold with a dimension of $n \geq 3$, such that its scalar curvature remains constant. Hence, the Yamabe flow is precisely delineated by the metric $g(t)$ on a Riemannian manifold (M^n, g) , which adheres to the condition $\frac{\partial g(t)}{\partial t} = -Rg(t)$, where R denotes the scalar curvature of M . Notably, the two-dimensional scenario had already been resolved through the application of the Uniformization Theorem. For an in-depth exploration of this concept, we direct the reader to references (Daskalopoulou and Sesum, 2013), (Brendle, 2005), and (Brendle, 2007). It is pertinent to mention that this flow finds its solution through the lens of gradient Yamabe solitons, which are rigorously defined as follows:

A pseudo-Riemannian manifold (M^n, g) is said to be a gradient Yamabe soliton if there exists a smooth function ϕ on M and a constant λ satisfying

$$\text{Hess}(\phi) = (R - \lambda)g,$$

where $\text{Hess}(\phi)$ is the Hessian of ϕ , R is the scalar curvature, and g is the metric. A gradient Yamabe soliton on a pseudo-Riemannian manifold (M^n, g) is classified as shrinking if $\lambda > 0$, expanding if $\lambda < 0$, or steady if $\lambda = 0$. Notably, a seminal outcome pertaining to the resolution of the Yamabe problem is documented in (Daskalopoulou and Sesum, 2013), wherein it is demonstrated that the metric of any compact Yamabe gradient soliton is a metric characterized by constant scalar curvature.

Güler and Crasmareanu (2019) introduced the Ricci-Yamabe flow for a smooth n -dimensional manifold M^n . Later, the notion of the conformal Ricci-Yamabe

soliton equation was introduced by Zhang *et al.* (2022) as

$$L_X g + 2\alpha S = \left[2\lambda - \beta R - \left(p + \frac{2}{n} \right) \right] g, \quad (1.58)$$

where α, β, λ are constants and p is the conformal pressure. The equation is the generalization of the Ricci-Yamabe soliton equation, and it satisfies the conformal Ricci-Yamabe flow equation (For details, see Alkhaldi *et al.* (2021); Kim *et al.* (2022); Singh and Khatri (2021); Khatri *et al.* (2022)).

Bishop and O'Neill (1969) pioneered the warped product notion, which provides an adaptable foundation for complete manifolds of negative curvature. The origins of this notion may be traced back to the surfaces of revolution. Warped products are important in differential geometry and have applications in mathematical physics and general relativity. This multidisciplinary appeal has piqued the interest of mathematicians and physicists alike, resulting in a thriving field of research (Chen, 2013, 2017). Consider B and F Riemannian manifolds, as well as a positive smooth function f defined on B . The product manifold metric $B \times F$ is defined as follows:

$$g = \pi^* g_B + (f \circ \pi)^2 \sigma^* g_F, \quad (1.59)$$

where π and σ are the natural projections onto the base and fiber manifold respectively. The product manifold is designated by $M = B \times_f F$ and is said to be the warped product of B and F under this condition. In this context, manifolds B and F are referred to as the base and fiber respectively, where g_B and g_F are the induced metric on the base and fiber respectively. The function f is called the warping function.

Chapter 2

Certain Generalization of Ricci soliton

2.1 Introduction

In this chapter, we establish that if an almost Kenmotsu manifold enables a gradient $\mathfrak{h}$ -almost Ricci soliton, then it is a steady Ricci soliton and the manifold admits an Einstein manifold. We also establish that if a Kenmotsu manifold exhibits a non-trivial gradient $\mathfrak{h}$ -almost generalized Ricci soliton with a potential vector field, it is η -Einstein if it is pointwise collinear with the Reeb vector field ξ . As a consequence we obtain that if a Kenmotsu manifold features a non-trivial gradient $\mathfrak{h}$ -almost generalized Ricci soliton with a vanishing potential vector field and is Einstein, which are denoted by $QX = -2dX$ with $\lambda = -m$. Additionally, we demonstrate that if a (k, ς) '-almost Kenmotsu manifold with $h' \neq 0$ admits a metric of an $\mathfrak{h}$ -almost Ricci soliton with $\varphi V = 0$, then the soliton is expanding provided $\lambda = \frac{4d^2}{1+d} > 0$.

A smooth manifold M of dimension $(2d + 1)$ is designated as an almost contact metric manifold provided it accommodates a $(1, 1)$ -tensor field φ , a unit

¹Robert Sumlalsanga, Jay Prakash Singh and M. Sundararajan (2024), Generalized $\mathfrak{h}$ -almost Ricci Soliton in Almost Kenmotsu Manifold, *Science and Technology Journal of Mizoram University*, **12(2)**, 49-53.

vector field ξ (referred to as the Reeb vector field), and a 1-form η , satisfying the following conditions:

$$\varphi^2 X = -X + \eta(X)\xi, \eta(X) = g(X, \xi) \quad (2.1)$$

for all vector fields X on M . Additionally, a Riemannian metric, g , is considered an associated (or compatible) metric if it fulfills the condition:

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (2.2)$$

for all vector fields X, Y on M . The combination of an almost contact manifold $M^{2d+1}(\varphi, \xi, \eta)$ with a compatible metric g is referred to as an almost contact metric manifold (Blair, 2010).

An almost Kenmotsu manifold is characterized as an almost contact metric manifold satisfying $d\eta = 0$ and $\Phi = 2\eta \wedge \Phi$, where the fundamental 2-form Φ of the almost contact metric manifold is given by $\Phi(X, Y) = g(X, \varphi Y)$ for any vector fields X, Y on M (Vanhecke and Janssens, 1981). On the product $M^{2d+1} \times \mathbb{R}$ of an almost contact metric manifold M^{2d+1} and $\mathbb{R}$, an almost complex structure J is defined as:

$$J \left(X, f \frac{d}{dt} \right) = \left(\varphi X - f\xi, \eta(X) \frac{d}{dt} \right),$$

where t is the coordinate of $\mathbb{R}$, f is a C^∞ function on $M^{2d+1} \times \mathbb{R}$, and X is a vector field tangent to M^{2d+1} . If J is integrable, it is a normal structure in almost contact metric on the manifold M^{2d+1} . A normal almost Kenmotsu manifold is denoted as a Kenmotsu manifold (Kenmotsu, 1972). An almost Kenmotsu manifold transforms into a Kenmotsu manifold if and only if

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X$$

for any vector fields X, Y on M^{2d+1} . On a Kenmotsu manifold, the following conditions are satisfied (Kenmotsu, 1972):

$$\nabla_X \xi = X - \eta(X)\xi, \quad (2.3)$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (2.4)$$

$$Q\xi = -2d\xi \quad (2.5)$$

for any vector fields X, Y on M^{2d+1} . In this case, the $(1, 2)$ Ricci tensor S has a Ricci operator Q associated with it, where each and every vector field on M^{2d+1} , X, Y is represented by the notation $S(X, Y) = g(QX, Y)$. R stands for the curvature tensor of g . It has been demonstrated that a Kenmotsu manifold can be locally represented as a warped product $I \times_f N^{2d}$, where I is an open interval with coordinate t , $f = ce^t$ denotes the warping function for some positive constant c , and N^{2d} is a Kählerian manifold (Kenmotsu, 1972).

On an almost Kenmotsu manifold, the following formula holds true (Dileo and Pastore, 2007, 2009):

$$\nabla_X \xi = \varphi^2 X - \varphi hX \quad (2.6)$$

for any vector field X on M^{2d+1} . We introduce two operators, denoted by h and l , defined as $h = \frac{1}{2}L_\xi \varphi$ and $l = R(\cdot, \xi)\xi$, respectively. These operators satisfy the conditions: $h\xi = h'\xi = 0$, $Tr.h = Tr.h' = 0$, $h\varphi = -\varphi h$, where $h' = h \cdot \varphi$, and $Tr.$ denotes the trace operation.

2.2 Normal almost Kenmotsu manifold

Within the current section, exploration of Kenmotsu manifolds that exhibit a generalized $\mathfrak{h}$ -almost Ricci soliton structure is done. Initially, we give various illustrations of Kenmotsu manifolds accommodating a generalized $\mathfrak{h}$ -almost Ricci

metric.

Theorem 2.2.1. *If a gradient $\mathfrak{h}$ -almost generalized Ricci soliton is embodied by the Kenmotsu metric manifold $M^{2d+1}(\phi, \xi, \eta, g)$, then it either conforms to being η -Einstein or the soliton remains steady.*

Proof. From equation (1.54), we obtain

$$\mathfrak{h}\nabla_X Df = \lambda X - QX \quad (2.7)$$

Performing the covariant derivative of equation (2.7) with respect to an arbitrary vector field Y , we derive:

$$\begin{aligned} \mathfrak{h}\nabla_Y \nabla_X Df &= -\frac{1}{\mathfrak{h}}(\mathfrak{h}Y)(\lambda X - QX) + (\lambda Y)X + \lambda(\nabla_Y X) \\ &\quad - (\nabla_Y Q)X - Q(\nabla_Y X) \end{aligned} \quad (2.8)$$

Employing equations (2.7) and (2.8) in the relation

$$\mathfrak{h}R(X, Y)Df = \mathfrak{h}\nabla_X \nabla_Y Df - \mathfrak{h}\nabla_Y \nabla_X Df - \mathfrak{h}\nabla_{[X, Y]} Df$$

we derive

$$\begin{aligned} \mathfrak{h}R(X, Y)Df &= -\frac{1}{\mathfrak{h}}(X\mathfrak{h})(\lambda Y - QX) + (\lambda X)Y - (\nabla_X Q)Y \\ &\quad + \frac{1}{\mathfrak{h}}(Y\mathfrak{h})(\lambda X - QX) - (\lambda Y)X + (\nabla_Y Q)X \end{aligned} \quad (2.9)$$

By taking the inner product of equation (2.9) with ξ and utilizing equation (2.5), we obtain:

$$\begin{aligned} g(\mathfrak{h}R(X, Y)Df, \xi) &= (X\lambda)\eta(Y) - (Y\lambda)\eta(X) - g((\nabla_Y Q)X, \xi) \\ &\quad - g((\nabla_X Q)Y, \xi) + \frac{1}{\mathfrak{h}}(\lambda + 2d)[(Y\mathfrak{h})\eta(X) - (X\mathfrak{h})\eta(Y)]. \end{aligned} \quad (2.10)$$

Again, taking the inner product of equation (2.4) with Df and substituting it

into the equation (2.10), we get:

$$\begin{aligned} & \frac{1}{\mathfrak{h}}(\lambda + 2d)[(Y\mathfrak{h})(\eta(X) - (X\mathfrak{h})\eta(Y)) - (Y\lambda)\eta(X) + g(\nabla_X Q)Y, \xi) \\ & + (X\lambda)\eta(Y) + g[(\nabla_Y Q)\xi, X] - \eta(Y)g(X, Df) + \mathfrak{h}\eta(X)g(Y, Df)] = 0. \end{aligned} \quad (2.11)$$

Substituting Y with ξ in equation (2.11) and utilizing the relation $(\nabla_\xi)Y = -2QY - 4dY$ (see Lemma 2 of Ghosh (2019)), we have

$$[\sigma(\xi h) - h(\xi\lambda) + h^2(\xi f)]\eta(X) = \sigma(Xh) - h(X\lambda) + h^2(Xf). \quad (2.12)$$

where $\sigma = \lambda + 2d$. Contracting equation (2.9) along an arbitrary vector field X results in

$$\mathfrak{h}S(Df, Y) = \frac{1}{\mathfrak{h}}S(Y, D\mathfrak{h}) - 2d(Y\lambda) + \frac{1}{2}(Yr) + \frac{1}{\mathfrak{h}}(2d\lambda - r) \quad (2.13)$$

Substituting Y with ξ and employing equation (2.5) in equation (2.13), we obtain:

$$-2d\mathfrak{h}^2(\xi f) - (2d\sigma - (r + 2d(2d + 1)))(\xi\mathfrak{h}) + 2d\mathfrak{h}(\xi\lambda) - \frac{1}{2}\mathfrak{h}(\xi r) = 0. \quad (2.14)$$

Moreover, on the Kenmotsu manifold, we have $\xi r = -2(r + 2d(2d + 1))$ (Lemma 2 of Ghosh (2019)). Using into the above equation yields:

$$2d[\sigma(\xi\mathfrak{h}) - \mathfrak{h}(\xi\lambda) + \mathfrak{h}^2(\xi f)] = r + 2d(2d + 1)(\mathfrak{h} + \xi\mathfrak{h}). \quad (2.15)$$

Substituting Y with ξ in equation (2.9) and employing $\mathfrak{h}R(X, \xi)Y = \mathfrak{h}g\xi - \mathfrak{h}\eta(Y)X$, we get

$$\begin{aligned} & \frac{1}{\mathfrak{h}}[\sigma(X\mathfrak{h}) - \mathfrak{h}(X\lambda) + \mathfrak{h}^2(Xf)]\xi = \mathfrak{h}\eta(\xi f)X - (\xi\lambda)X + \frac{1}{\mathfrak{h}}(\lambda X)(\xi\mathfrak{h}) \\ & - \left(1 + \frac{\xi\mathfrak{h}}{\mathfrak{h}}\right)QX - 2dX. \end{aligned} \quad (2.16)$$

Combining equations (2.12), (2.15), and (2.16), we derive the following relation:

$$\begin{aligned} 2d[\mathfrak{h}(\xi f)X + \lambda X/\mathfrak{h}(\xi\lambda) - (\xi\lambda)X - \left(1 + \frac{\xi\mathfrak{h}}{\mathfrak{h}}\right)QX - 2dX] \\ = \frac{1}{\mathfrak{h}}(r + 2d(2d + 1))(\mathfrak{h} + \xi\mathfrak{h})\eta(X)\xi. \end{aligned} \quad (2.17)$$

Suppose $\mathfrak{h}$ is constant and $\xi\mathfrak{h} = 0$. Utilizing equation (2.15), we have:

$$(r + 2d(2d + 1))\eta(X)\xi = -\frac{2d\lambda}{\mathfrak{h}}(\xi\lambda)X + (r + 2d(2d + 1))X - 2dQX - 4d^2X.$$

Let $X = \xi$ and $\lambda(\xi\lambda) = 0$. If $\lambda = 0$, the soliton remains steady. If $\xi\lambda = 0$, then the manifold is η -Einstein. This brings the proof to a close. $\square$

Theorem 2.2.2. *A Kenmotsu manifold is η -Einstein if it has a non-trivial gradient $\mathfrak{h}$ -almost generalized Ricci soliton for which the potential vector field is pointwise collinear with the Reeb vector field ξ .*

Proof. Assume V which represents a potential vector field and the Reeb vector field ξ are pointwise collinear, such that $V = F\xi$, in which the smooth function F . By performing a covariant differentiation along an arbitrary vector field X of $V = F\xi$ and employing equation (2.3), we obtain:

$$\nabla_X V = (XF)\xi + F(-\varphi^2 X - \varphi\mathfrak{h}X). \quad (2.18)$$

Substituting equation (2.18) into equation (1.53) yields:

$$\begin{aligned} \frac{\mathfrak{h}}{2}[(XF)\eta(Y) + (YF)\eta(X)] + \mathfrak{h}Fg(\mathfrak{h}'X, Y) - \mathfrak{h}F\eta(X)\eta(Y) + S(X, Y) \\ = (\lambda - \mathfrak{h}F)g(X, Y) \end{aligned} \quad (2.19)$$

for any vector fields X and Y .

Substituting X, Y with ξ in equation (2.19) and employing equation (2.5), we get

$$\xi F = \frac{1}{\mathfrak{h}}(\lambda + 2d).$$

Furthermore, substituting Y with ξ and employing the final expression in equation

(2.19), we have

$$XF = \frac{1}{\mathfrak{h}}(\lambda + 2n)\eta(X). \quad (2.20)$$

Contracting equation (2.19) and inserting it into the preceding equation (2.20), we have

$$r = 2(\lambda - \mathfrak{h}F - 1). \quad (2.21)$$

As a consequence of equations (2.20) and (2.21), equation (2.19) simplifies to the following form:

$$QX = \left(\frac{r}{2d} + 1\right)X - \left(\frac{r}{2d} + 2d + 1\right)\eta(X)\xi \quad (2.22)$$

For any vector field X , thereby establishing the manifold as η -Einstein. $\square$

Assume that F is constant. From equation (2.20), it follows that $\lambda = -2d$. Substituting this value into equation (2.21), we deduce a constant r . Consequently, ξr vanishes, implying $r = -2d(2d + 1)$. Replacing r and λ with their respective values into equation (2.21), we find $F = 0$, which further implies $\lambda = -2d$.

Theorem 2.2.3. *If a Kenmotsu manifold has a non-trivial gradient $\mathfrak{h}$ -almost generalized Ricci soliton with a potential vector field that goes away, then it is Einstein, which is shown by $QX = -2dX$ with $\lambda = -m$.*

2.3 Non-normal almost Kenmotsu manifold

If ξ , that Reeb vector field, is contained within the generalized (k, ς) -nullity distribution, then the almost Kenmotsu manifold $M^{(2d+1)}$ is referred to as a generalized almost Kenmotsu manifold (k, ς) . i.e.,

$$k[\eta(Y)X - \eta(X)Y] + \varsigma[\eta(Y)hX - \eta(X)hY] = R(X, Y)\xi. \quad (2.23)$$

The condition holds true for all vector fields X, Y on M , with k and ς representing smooth functions defined on M . An almost Kenmotsu manifold $M^{(2d+1)}$ is termed a generalized $(k, \varsigma)'$ -almost Kenmotsu manifold if the Reeb vector field ξ is part of the generalized $(k, \varsigma)'$ -nullity distribution, which is defined as:

$$k[\eta(Y)X - \eta(X)Y] + \varsigma[\eta(Y)h'X - \eta(X)h'Y] = R(X, Y)\xi \quad (2.24)$$

this condition applies to all vector fields X, Y on M , where k, ς denote smooth functions defined on M , and $h' = h \circ \varphi$ Dileo and Pastore (2009). Furthermore, if both k and ς are constants in equation (2.24), then M is referred to as a $(k, \varsigma)'$ -almost Kenmotsu manifold Dileo and Pastore (2009); Wang and Liu (2016); Khatri and Singh (2023). The following relations hold on a generalized (k, ς) or $(k, \varsigma)'$ -almost Kenmotsu manifold with $h \neq 0$ (equivalently, $h' \neq 0$):

$$h'^2 = (k+1)\varphi^2, \quad h^2 = (k+1)\varphi^2, \quad (2.25)$$

$$Q\xi = 2dk\xi. \quad (2.26)$$

From equation (2.25), it follows that $k \leq -1$ and $v = \pm\sqrt{(-k-1)}$, where v represents an eigenvalue corresponding to the eigenvector $X \in D(D = \text{Ker}(\eta))$ of h' . The equality holds if and only if $h = 0$ (equivalently, $h' = 0$). Therefore, $h' \neq 0$ if and only if $k < -1$.

Lemma 2.3.1. (Wang and Liu, 2016) *Consider $M^{2d+1}(\varphi, \xi, \eta, g)$ to be a generalized $(k, \varsigma)'$ -almost Kenmotsu manifold with $h' \neq 0$. For $d > 1$, the Ricci operator Q of M can be expressed as:*

$$-2dX + 2d(k+1)\eta(X)\xi - [\varsigma - 2(d-1)h']X = QX$$

for any vector field X on M . Additionally, if k and ς are constants and $d \geq 1$, then $\varsigma = -2$, and thus

$$QX = -2dX + 2d(k+1)\eta(X)\xi - 2dh'X \quad (2.27)$$

The scalar curvature of M for any vector field X on M is $2d(k-2d)$ in both cases.

Theorem 2.3.2. *A soliton is said to be expanding if the metric of a $(k, \varsigma)'$ -almost Kenmotsu manifold with $h' \neq 0$ admits a $\mathfrak{h}$ -almost Ricci soliton with $\varphi V = 0$.*

Proof. Given the hypothesis $\varphi V = 0$, applying the operator φ yields $V = \eta(V)\xi$, that is, $V = F\xi$, where the function F is smooth. Equation (1.53) is obtained by calculating the final equation's covariant derivative along any vector field X and adding it:

$$\begin{aligned} S(X, Y) + \frac{\mathfrak{h}}{2}[(XF)\eta(Y) + (YF)\eta(X)] - Fg(h'X, Y) - \mathfrak{h}F\eta(X)\eta(Y) \\ = (\lambda - \mathfrak{h}F)g(X, Y). \end{aligned} \quad (2.28)$$

Substituting X with ξ in equation (2.28) yields

$$\frac{\mathfrak{h}}{2}(YF) = \left(\lambda + 2dk - \frac{\mathfrak{h}}{2}(\xi F) \right) \eta(Y) \quad (2.29)$$

for any vector field X on M . Contracting equation (2.26) and utilizing a lemma, we obtain:

$$\mathfrak{h}(\xi F) = (2d+1)(\lambda - 2\mathfrak{h}F) - \mathfrak{h}F(k+2) - \left(1 - \frac{\mathfrak{h}F}{2d} \right) (2d(k-2d)). \quad (2.30)$$

Substituting Y with ξ in equation (2.29) and combining it with equation (2.30) gives $F = \frac{d(\lambda+2d)}{\mathfrak{h}(3d+1)}$. By inserting equation (2.29) into equation (2.28) and utilizing it in the lemma, we derive

$$\left[\frac{d(\lambda+2+8d)}{(3d+1)}\mathfrak{h}X + (\lambda - 2\mathfrak{h}F)\varphi X \right] (k+1) = 0. \quad (2.31)$$

Substituting X with $h'X$ in equation (2.31) and considering $h' \neq 0$, we obtain $\lambda = \frac{4d^2}{1+d} > 0$. This concludes the proof, establishing that the soliton is expanding. $\square$

2.4 Conclusion

This chapter investigates almost Kenmotsu manifolds with gradient $\mathfrak{h}$ -almost Ricci solitons. We concluded that if a gradient is permitted by a manifold, $\mathfrak{h}$ -almost Ricci soliton, it must be an Einstein manifold, and the Ricci soliton must be steady. The analysis also showed that a Kenmotsu manifold with a non-trivial gradient $\mathfrak{h}$ -almost generalized η -Einstein is required for a Ricci soliton with a potential vector field collinear with the Reeb vector field. Additionally, if a manifold has a non-trivial gradient $\mathfrak{h}$ -almost generalized Ricci soliton with a vanishing potential vector field, then it becomes an Einstein manifold. We have also observed that the presence of an $\mathfrak{h}$ -almost Ricci soliton with certain conditions leads to an expanding soliton.

Chapter 3

Analysis on F(R) Gravity

3.1 Introduction

Motivated by the work of many geometers, we examine conformally harmonic Z -recurrent spacetime in $F(R)$ - theory of gravity. First, we analyze the equation of motion for a conformally harmonic Z -recurrent spacetime, and we found that such a spacetime obeying $F(R)$ - gravity with constant scalar curvature is a perfect fluid spacetime. We also take up the study of its cosmological implications in which the behavior of cosmological parameters such as the density, pressure and equation of state parameters are observed. The energy conditions within two $F(R)$ -gravity models are investigated and their graphical representations are obtained.

The notations $\dot{R} = A^k \nabla_k R$, $\ddot{R} = A^k \nabla_k \dot{R}$, $\dot{A}_i = A^k \nabla_k A_i$ are used throughout the chapter.

Einstein-Hilbert Lagrangian density to the function of $F(R)$, where R is the Ricci scalar, has extended the general theory of relativity (GR) to the $F(R)$ gravity. Huge neutron stars in the equations of motion of $F(R)$ gravity issues can be solved by the higher order curvature (Sotiriou and Faraoni, 2010; Astashenok *et al.*, 2013, 2017).

Even though experts try and decide to modify the theory of gravity for over the decades, it has not been an easy task. The theory of $F(R)$ gravity came in straightforwardly as a generalization by the action of the Lagrangian in the Einstein-Hilbert,

²Khatri, Mohan and Singh, Jay Prakash and Sumlalsanga, Robert and Chhakchhuak, Zosangzuala (2025), Classification of conformally harmonic Z -recurrent spacetime in $F(R)$ -gravity, *International Journal of Geometric Methods in Modern Physics*, **22(1)**, 2450257.

$$\mathcal{H} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R, \quad (3.1)$$

where $\kappa^2 \equiv 8\pi G$, G is the gravitational constant, g is the determinant of the metric and R is the Ricci scalar, to become a general function of R , *i.e.*,

$$\mathcal{H} = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} F(R) + \int L_m \sqrt{-g} d^4x, \quad (3.2)$$

this produces the field equations of $F(R)$ -gravity (Astashenok *et al.*, 2015) as follows:

$$F_R(R)R_{ij} - \frac{1}{2}F(R)g_{ij} + (g_{ij}\square - \nabla_i\nabla_j)F_R(R) = \kappa T_{ij}, \quad (3.3)$$

where $\square = \nabla^j\nabla_j$. The stress-energy tensor is calculated from the matter Lagrangian L_m by $T_{ij} = -\frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}L_m}{\delta g^{ij}}$.

The accelerated expansion of the Universe in the model $F(R) = R + \alpha R^2$ ($\alpha > 0$) is due to the presence of the αR^2 term. Starobinsky (1980) has proposed the first model of inflation. The above model is well conformable with the temperature anisotropies observed in the Cosmic Microwave Background (CMB); thus, it could be an applicable alternative to the scalar field models of inflation.

3.2 The equations of motion

Suppose that the spacetime is Z -recurrent, then we have

$$\nabla_k Z_{ij} = A_k Z_{ij}, \quad (3.4)$$

where A_k is a 1-form such that $A_k A^k = -1$.

Using the definition of Z -tensor(1.55), (3.4) can be rewritten as

$$\nabla_k R_{ij} = A_k Z_{ij} + \frac{\nabla_k R}{4} g_{ij}. \quad (3.5)$$

Interchanging i and k in (3.5), then subtracting with (3.5) gives

$$\nabla_k R_{ij} - \nabla_i R_{kj} = A_k Z_{ij} - A_i Z_{kj} + \frac{1}{4} (g_{ij} \nabla_k R - g_{kj} \nabla_i R). \quad (3.6)$$

For a conformally harmonic space the following relation holds:

$$\nabla_k R_{ij} - \nabla_i R_{kj} = \frac{1}{6} (g_{ij} \nabla_k R - g_{kj} \nabla_i R). \quad (3.7)$$

Making use of (3.7) in (3.6), we get

$$(A_k R_{ij} - A_i R_{kj}) - \frac{R}{4} (A_k g_{ij} - A_i g_{kj}) = -\frac{1}{12} (g_{ij} \nabla_k R - g_{kj} \nabla_i R). \quad (3.8)$$

Contracting (3.4) over k and i infers

$$A^k Z_{kj} = \frac{1}{4} \nabla_j R. \quad (3.9)$$

Transvecting the Z -tensor by A^i and using (3.9), we obtain

$$A^i R_{ij} = \frac{1}{4} (R A_j + \nabla_j R). \quad (3.10)$$

Again, transvecting (3.10) by A^j gives

$$4A^i A^j R_{ij} = -R + \dot{R}. \quad (3.11)$$

Making use of (3.10) in (3.8), we get

$$\begin{aligned} A_k R_{ij} - A_i R_{kj} - \frac{R}{4} (A_k g_{ij} - A_i g_{kj}) = & -\frac{1}{12} (4A^l R_{lk} + R A_k) g_{ij} \\ & + \frac{1}{12} (4A^l R_{il} - R A_i) g_{kj}. \end{aligned} \quad (3.12)$$

Transvecting the last expression (3.12) by A^j yields

$$A_k A^l R_{il} = A_i A^l R_{kl}. \quad (3.13)$$

Again, transvecting (3.13) by A^k , then using (3.11) gives

$$A^l R_{il} = \frac{R - \dot{R}}{4} A_i. \quad (3.14)$$

Combining (3.10) with (3.14) gives

$$\nabla_j R = -\dot{R} A_j. \quad (3.15)$$

Also, transvecting (3.12) by A^k and making use of (3.13) and (3.14), we obtain

$$R_{ij} = \frac{\dot{R} + 3R}{12} g_{ij} + \frac{\dot{R}}{3} A_i A_j. \quad (3.16)$$

Let us first consider the case when $\dot{R} = 0$, that is, R is constant from (3.15). In this case, we have

$$R_{ij} - \frac{R}{2}g_{ij} = \frac{k^2}{F_R(R)}T_{ij} + \frac{F(R) - RF_R(R)}{2F_R(R)}g_{ij}. \quad (3.17)$$

In view of (3.16) and (3.17), the energy-momentum tensor is perfect fluid. For a perfect fluid spacetime, the energy-momentum tensor T_{ij} is expressed as follows:

$$k^2T_{ij} = k^2(p + \rho)A_iA_j + k^2pg_{ij}, \quad (3.18)$$

Combining (3.16) and (3.17), along with the last expression gives

$$k^2(p + \rho)A_iA_j + k^2pg_{ij} = \frac{RF_R(R) - 2F(R)}{4}g_{ij}. \quad (3.19)$$

Transvecting (3.19) by A^i , we obtain the expression for density as follows:

$$\rho = \frac{1}{k^2} \left(\frac{2F(R) - RF_R(R)}{4} \right). \quad (3.20)$$

Also, contracting (3.19) and using (3.20), the expression for pressure is as follows:

$$p = \frac{1}{k^2} \left(\frac{2F(R) - RF_R(R)}{4} \right). \quad (3.21)$$

Therefore, we can state the following:

Theorem 3.2.1. *If a conformally harmonic Z-recurrent spacetime with constant scalar curvature obeys $F(R)$ -gravity theory, then the matter content is a perfect fluid with four velocity vector A^i , energy density (3.20) and isotropic pressure (3.21).*

Now we will consider the case when $\dot{R} \neq 0$. Taking the covariant derivative of (3.16) gives

$$\nabla_k R_{ij} = \frac{1}{12} \left(\nabla_k \dot{R} + 3\nabla_k R \right) g_{ij} + \frac{\nabla_k \dot{R}}{3} A_i A_j + \frac{\dot{R}}{3} ((\nabla_k A_i) A_j + A_i (\nabla_k A_j)). \quad (3.22)$$

Transvecting (3.22) by A^j , we obtain

$$A^j \nabla_k R_{ij} = \frac{1}{4} \left(\nabla_k R - \nabla_k \dot{R} \right) - \frac{\dot{R}}{3} (\nabla_k A_i). \quad (3.23)$$

Combining (3.5) and (3.15), then transvecting the obtained equation by A^j , we

get

$$A^j \nabla_k R_{ij} = -\frac{\dot{R}}{2} A_i A_k. \quad (3.24)$$

Transvecting (3.23) by A^k and using (3.24) gives

$$\frac{\dot{R}}{3} \dot{A} = -\frac{1}{4} (\dot{R} + \ddot{R}) A_i,$$

which on transvecting by A^i implies $\ddot{R} + \dot{R} = 0$ and hence by the previous equation, we get $\dot{R} \dot{A} = 0$. But, by our hypothesis, $\dot{R} \neq 0$, hence $\dot{A} = 0$.

Taking the covariant derivative of (3.15), we obtain

$$\nabla_i \nabla_j R = -(\nabla_i \dot{R}) A_j - \dot{R} (\nabla_i A_j). \quad (3.25)$$

Interchanging i and j in (3.25) yields

$$\nabla_j \nabla_i R = -(\nabla_j \dot{R}) A_i - \dot{R} (\nabla_j A_i). \quad (3.26)$$

Subtracting (3.25) and (3.26) gives

$$(\nabla_j \dot{R}) A_i - (\nabla_i \dot{R}) A_j + \dot{R} (\nabla_j A_i - \nabla_i A_j) = 0. \quad (3.27)$$

Transvecting (3.27) by A^j and using the fact that $\dot{A} = 0$, we obtain

$$\nabla_i \dot{R} = -\ddot{R} A_i, \quad (3.28)$$

In consequence, Eq. (3.27) becomes $\dot{R} (\nabla_j A_i - \nabla_i A_j) = 0$, which implies A is a closed 1-form.

Combining (3.15), (3.23) and (3.28), we get

$$\nabla_k A_i = \frac{3}{4\dot{R}} (\dot{R} + \ddot{R}) A_i A_k. \quad (3.29)$$

Inserting (3.29) and (3.28) in (3.26), (3.26) becomes

$$\nabla_j \nabla_i R = \frac{\ddot{R} - 3\dot{R}}{4} A_i A_j. \quad (3.30)$$

For any analytic function $F(R)$, we have

$$\nabla_i \nabla_j F_R(R) = F_{RR}(R) \nabla_i \nabla_j R + F_{RRR}(R) (\nabla_i R) (\nabla_j R). \quad (3.31)$$

Making use of (3.15) and (3.30) in (3.31), we obtain

$$\nabla_i \nabla_j F_R(R) = \frac{(-3\dot{R} + \ddot{R})F_{RR}(R) + 4\dot{R}^2 F_{RRR}(R)}{4} A_i A_j. \quad (3.32)$$

Then contracting (3.32) gives

$$\square F_R(R) = -\frac{(-3\dot{R} + \ddot{R})F_{RR}(R) + 4\dot{R}^2 F_{RRR}(R)}{4}. \quad (3.33)$$

Making use of (3.32) and (3.33) in (3.18), we obtain the following relation:

$$\begin{aligned} k^2 T_{ij} &= \frac{(\dot{R} + 3R)F_R(R) - 6F(R) - 3(-3\dot{R} + \ddot{R})F_{RR}(R) - 12\dot{R}^2 F_{RRR}(R)}{12} g_{ij} \\ &+ \frac{4\dot{R}F_R(R) - 3(-3\dot{R} + \ddot{R})F_{RR}(R) - 12\dot{R}^2 F_{RRR}(R)}{12} A_i A_j. \end{aligned} \quad (3.34)$$

Thus, the matter content is perfect fluid. Inserting (3.18) in (3.34) yields

$$\begin{aligned} k^2(p + \rho)A_i A_j + k^2 p g_{ij} &= \frac{1}{12} \left\{ (\dot{R} + 3R)F_R(R) - 6F(R) \right. \\ &\quad \left. - 3(-3\dot{R} + \ddot{R})F_{RR}(R) - 12\dot{R}^2 F_{RRR}(R) \right\} g_{ij} \\ &+ \frac{1}{12} \left\{ 4\dot{R}F_R(R) - 3(-3\dot{R} + \ddot{R})F_{RR}(R) \right. \\ &\quad \left. - 12\dot{R}^2 F_{RRR}(R) \right\} A_i A_j. \end{aligned} \quad (3.35)$$

Transvecting (3.35) by A^i , the expression for density is as follows:

$$\rho = \frac{1}{k^2} \left(\frac{F(R)}{2} + \frac{\dot{R} - R}{4} F_R(R) \right). \quad (3.36)$$

Contracting (3.35) and using (3.36), the expression for pressure is as follows:

$$p = \frac{1}{k^2} \left(\frac{(\dot{R} + 3R)F_R(R) - 6F(R) + 3(3\dot{R} - \ddot{R})F_{RR}(R) - 12\dot{R}^2 F_{RRR}(R)}{12} \right). \quad (3.37)$$

Therefore, we can state the following:

Theorem 3.2.2. *In a conformally harmonic Z -recurrent spacetime with non-constant scalar curvature obeying $F(R)$ -gravity theory, if the four velocity vector is identical with A^i , then the matter content is a perfect fluid with energy density (3.36) and isotropic pressure (3.37).*

In (Mantica *et al.*, 2016), the authors showed that a Lorentzian manifold M

of dimension $n > 3$ with Ricci curvature tensor $R_{ij} = Ag_{ij} + Bu_iu_j$, where A and B are scalar fields, u is a closed timelike unit vector field and with a conformally harmonic curvature tensor is a generalized Robertson-Walker (GRW) spacetime with a Einstein fibre. In consequence of Theorem 3.2.2, we can state the following:

Corollary 3.2.3. *If a conformally harmonic Z -recurrent spacetime with non-constant scalar curvature obeys $F(R)$ -gravity theory with four velocity vector field A^i , then it is a GRW spacetime.*

3.3 Cosmological Study

Hubble's law is considered the first observational basis for the expansion of the universe and the rate of expansion of the universe is measured extensively by the Hubble's parameter, $H = \frac{\dot{a}}{a}$, where a is the scale factor and $(\dot{})$ above the letter represents the derivative with respect to time. In cosmological setting, the higher derivative of the scale factor a are dimensionless parameters known as deceleration (q), jerk (j) and snap (s) (Visser, 2004):

$$\begin{aligned} q &= -\frac{1}{H^2} \frac{\ddot{a}}{a} \\ j &= \frac{1}{H^3} \frac{\ddot{\dot{a}}}{a} \\ s &= \frac{1}{H^4} \frac{\ddot{\ddot{a}}}{a} \end{aligned}$$

We note that the Ricci scalar and its derivatives for a spatially flat FLRW geometry can be expressed in terms of the deceleration (q), jerk (j) and snap (s) parameters (Santos, *et.al.*, 2007):

$$R = -6H^2(1 - q) \quad (3.38)$$

$$\dot{R} = -6H^3(j - q - 2) \quad (3.39)$$

$$\ddot{R} = -6H^4(s + q^2 + 8q + 6) \quad (3.40)$$

Moreover, the Equation of State (EoS) parameter ω is defined as $\omega = \frac{p}{\rho}$, the ratio of the isotropic pressure to the energy density of the perfect fluid. We can use the EoS parameter to observe different phases of the universe (Saini *et al.*, 2000). The different phases of the EoS parameter are as follows: $\omega = 0$ represent the dust phase, $\omega = \frac{1}{3}$ shows the radiation dominated phase, $\omega = -1$ is the Λ CDM and $\omega < -\frac{1}{3}$ supports the cosmic speed up. Also, $\omega < -1$ indicates the phantom era. It also includes the quintessence era when $-1 < \omega < 0$.

Case-I : When $\dot{R} \neq 0$

We obtain the density and pressure in terms of deceleration parameter, jerk and snap parameters using (3.38), (3.39) and (3.40) as

$$\rho = \frac{1}{2k^2} (F(R) + 3H^2(1 - q - H(j - q - 2))F_R(R)), \quad (3.41)$$

and

$$\begin{aligned} p = \frac{1}{2k^2} \{ & -H^2(3(1 - q) + H(j - q - 2))F_R(R) - F(R) \\ & + 3H^3(H(s + q^2 + 8q + 6) - 3(j - q - 2))F_{RR}(R) \\ & - 72H^6(j - q - 2)^2F_{RRR}(R) \}. \end{aligned} \quad (3.42)$$

3.3.1 Model-I

For our first test model, we assume a logarithmic functional form $F(R) = R + \alpha \log(\beta R)$ (Gironés *et al.*, 2009; De *et al.*, 2022). Clearly, $F(R)$ is continuous and smooth for all $\beta R \geq 0$. As Ricci scalar R is negative for present value of q , we take $\beta \leq 0$. Moreover, for $\alpha = 0$ we see that $F(R) = R$ which corresponds to the GR case. We will consider the present value of parameters as $q_0 = -0.55$, $j_0 = 1$, $s_0 = -0.35$ and $H_0 = 67.9$ km/s/Mpc (Capozziello *et al.*, 2019; Aghanim *et al.*, 2020). Making use of (3.38)-(3.42), the expression for pressure p and density ρ in this model are obtained as follows:

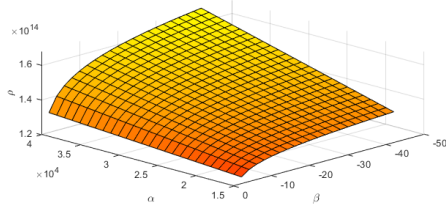
$$\begin{aligned} \rho = \frac{1}{2k^2} (& -6H^2(1 - q) + \alpha \log(-6\beta H^2(1 - q)) \\ & + 3H^2(1 - q - H(j - q - 2)) \left(1 - \frac{\alpha}{6H^2(1 - q)} \right)), \end{aligned} \quad (3.43)$$

and

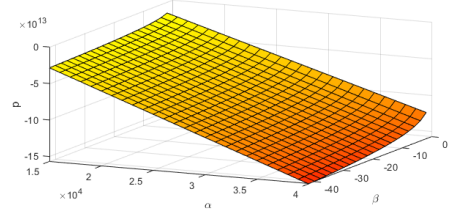
$$\begin{aligned} p = \frac{1}{12k^2} \left(& \frac{(\alpha - 6H^2(1 - q))(3(1 - q) + H(j - q - 2))}{1 - q} + \frac{4\alpha(j - q - 2)^2}{(1 - q)^3} \right. \\ & + 6(6H^2(1 - q) - \alpha \log(-6\beta H^2(1 - q))) \\ & \left. - \frac{\alpha(H(s + q^2 + 8q + 6) - 3(j - q - 2))}{2H(1 - q)^2} \right). \end{aligned} \quad (3.44)$$

As $F(R) = R + \alpha \log(\beta R)$ is continuous and smooth for $\beta R \geq 0$, we take $\beta \leq 0$ since the Ricci scalar R is negative for present value of $q_0 = -0.55$. The bounds for α and β are obtained from the behavior of density which is always positive as shown in Figure 3.1a and pressure which is negative as shown in Figure 3.1b.

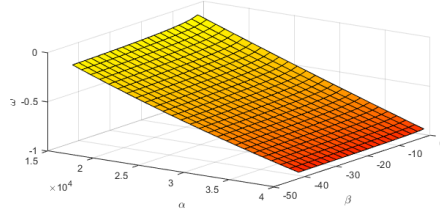
It can be seen that β influence the growth of density and pressure more than α , ρ increases for more negative β and decreases for less negative β . Initially, the density increases sharply for $\beta \in (-1, -10)$, then it follows a slow rate of increase. The behavior of the EoS parameter for our considered model is shown in Figure 3.1c. The plot of EoS shows a transition from positive to negative supporting an accelerated expansion of the universe and lies in the quintessence phase $-1 < w < 0$.



(a) The density parameter.



(b) The pressure parameter.



(c) The EoS parameter.

Figure 3.1: The behavior of the density, pressure and the EoS parameters considering $F(R) = R + a \log(\beta R)$ for Case-I with $15000 \leq \alpha \leq 40000$ and $-42 \leq \beta \leq -1$.

3.3.2 Model-II

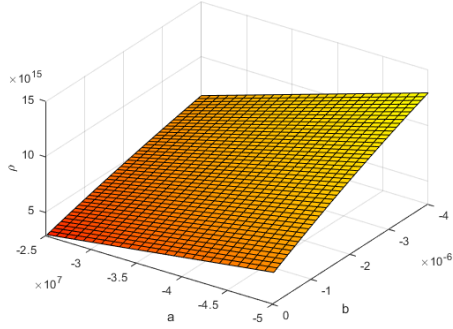
For the second model, we consider the exponential functional form $F(R) = R - a \exp(-bR - 1)$ (Cognola *et al.*, 2008). It is obvious that for $a = 0$ we get $F(R) = R$, which corresponds to the GR case. We will consider the present value of parameters as $q_0 = -0.55$, $j_0 = 1$, $s_0 = -0.35$ and $H_0 = 67.9$ km/s/Mpc (Capozziello *et al.*, 2019; Aghanim *et al.*, 2020). Making use of (3.38)-(3.42), the expressions for pressure p and density ρ in this model are obtained as follows:

$$\rho = \frac{1}{2k^2} \left(-6H^2(1 - q) - a \exp(6bH^2(1 - q) - 1) + 3H^2(1 - q - H(j - q - 2)) (1 + ab \exp(6bH^2(1 - q) - 1)) \right), \quad (3.45)$$

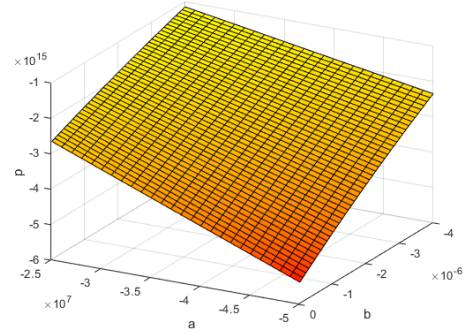
and

$$\begin{aligned}
 p = \frac{1}{2k^2} & \left(-H^2(3(1-q) + H(j-q-2)) (1 + ab \exp(6bH^2(1-q) - 1)) \right. \\
 & + 6H^2(1-q) + a \exp(6bH^2(1-q) - 1) \\
 & - 72H^6(j-q-2)^2 ab^3 \exp(6bH^2(1-q) - 1) \\
 & - 3H^3 (H(s+q^2+8q+6) - 3(j-q-2)) \\
 & \left. (ab^2 \exp(6bH^2(1-q) - 1)) \right). \tag{3.46}
 \end{aligned}$$

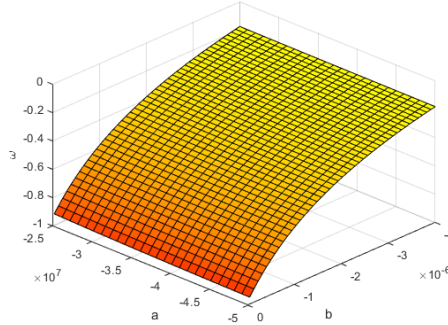
As seen in Figure 3.2, the ranges for a and b are obtained from the behavior of the density parameter as shown in Figure 3.2a, which is always positive and the behavior of pressure as shown in Figure 3.2b. The density increases more with the increase in the negative value of b . The behavior of the EoS parameter for our considered model is shown in Figure 3.2c. The plot of EoS shows a transition from positive to negative supporting an acceleration in the universe and lies in the quintessence phase $-1 < w < 0$.



(a) The density parameter.



(b) The pressure parameter.



(c) The EoS parameter.

Figure 3.2: The behavior of the density, pressure and the EoS parameters considering $F(R) = R - a \exp(-bR - 1)$ for Case-I with $-5 \times 10^7 \leq a \leq -2.5 \times 10^7$ and $-4 \times 10^{-6} \leq b \leq -10^{-9}$.

Case-II : When $\dot{R} = 0$

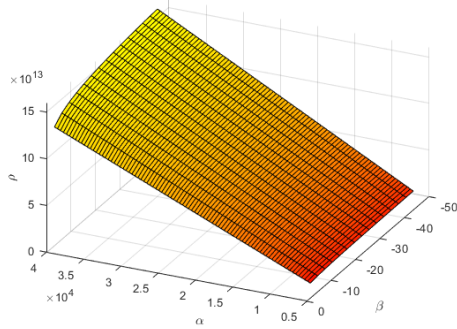
In the similar manner as case-I, the expressions for pressure and density in terms of deceleration parameter, jerk and snap parameters using (3.38), (3.39) and (3.40) are as follows:

$$\rho = \frac{1}{4k^2} (2F(R) + 6H^2(1 - q)F_R(R)) , \quad (3.47)$$

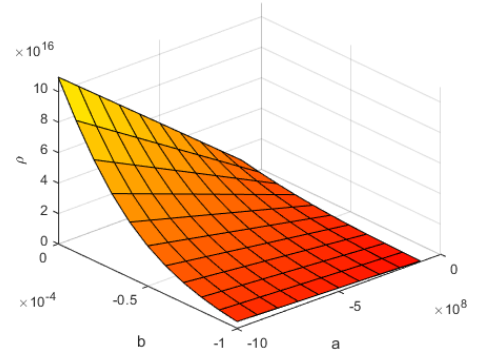
and

$$p = -\frac{1}{4k^2} (2F(R) + 6H^2(1 - q)F_R(R)) . \quad (3.48)$$

It is obvious from (3.47) and (3.48), that $\rho + p = 0$ and the EoS parameter, $\omega = -1$. In this case, our model coincide with the Λ CDM model. For $F(R) = R + \alpha \log(\beta R)$, the bounds for α and β are obtained from the behavior of density as shown in Figure 3.3a which increases more with the increase in the value of α . Similarly, the bounds of a and b are obtained from the behavior of density parameter for $F(R) = R - a \exp(-bR - 1)$ as shown in Figure 3.3b, which is always positive. The density ρ increases with the increase in the value of b .



(a) The behavior of density parameter considering $F(R) = R + \alpha \log(\beta R)$ with $5500 \leq \alpha \leq 40000$ and $-45 \leq \beta \leq -2$.



(b) The behavior of density parameter considering $F(R) = R - a \exp(-bR - 1)$ with $-10^9 \leq a \leq -10^4$ and $-10^{-4} \leq b \leq 0$.

Figure 3.3: The behavior of density parameter for Case-II.

Therefore, from the figures above for Model I and II, we conclude with the following table:

MODEL	CASES	PARAMETERS AND THEIR RANGES	EQUATION OF STATE (EoS)
Model I: $F(R) =$ $R + \alpha \log(\beta R)$	$\dot{R} = 0$	$\alpha \in [5500, 40000],$ $\beta \in [-45, -2]$	$\omega = -1$
	$\dot{R} \neq 0$	$\alpha \in [15000, 40000],$ $\beta \in [-42, -1]$	$-1 < \omega < 0$
Model II: $F(R) =$ $R - a \exp(-bR - 1)$	$\dot{R} = 0$	$a \in [-10^9, -10^4], b \in$ $[-10^{-4}, 0]$	$\omega = -1$
	$\dot{R} \neq 0$	$a \in [-5 \times 10^7, -2.5 \times$ $10^7], b \in [-4 \times$ $10^{-6}, -10^{-9}]$	$-1 < \omega < 0$

Table 3.1: Constraints on models' parameters a, b, α, β to support an acceleration of the universe.

3.4 Energy Conditions in $F(R)$ -gravity

Energy conditions were the primary tool for deriving general results that hold for a variety of situations such as Hawking-Penrose singularity theorems, second law of black hole thermodynamics, phantom fields potentials (Santos and Alcaniz, 2015), expansion history of the universe (see (Visser, 1997; Santos *et al.*, 2006; Santos, *et al.*, 2007a,b; Sen and Scherrer, 2008)), as well as evolution of the deceleration parameter and their confront with supernovae observations (Gong *et al.*, 2007; Gong and Wang, 2007). The Raychaudhuri equations along with the requirement that gravity is attractive gave rise to the concept of energy conditions (for details, see Kar and Sengupta (2007)). Let u^i be the tangent vector field to a congruence of timelike geodesics in a spacetime manifold endowed with a metric g_{ij} , then the Raychaudhuri's equation is given as:

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \rho_{ij}\rho^{ij} + \omega_{ij}\omega^{ij} - R_{ij}u^i u^j,$$

where R_{ij} , θ , ρ_{ij} and ω_{ij} are respectively the Ricci tensor, expansion, shear and rotation associated to the congruence defined by the vector field u^i . Due to the attractive nature of gravity and $\omega_{ij} = 0$, $R_{ij}u^i u^j \geq 0$, which by virtue of Einstein's equation implies $(T_{ij} - \frac{T}{2}g_{ij}) u^i u^j \geq 0$, where T is the trace of energy-momentum tensor.

In the context of the perfect fluid spacetime in GR, the energy conditions are defined as:

- Null energy condition (NEC): $\rho + p \geq 0$
- Strong energy condition (SEC): $\rho + 3p \geq 0$ and $\rho + p \geq 0$
- Weak energy condition (WEC): $\rho + p \geq 0$ and $\rho \geq 0$
- Dominant energy condition (DEC): $\rho \pm p \geq 0$ and $\rho \geq 0$.

The concept of energy conditions has been extended to $F(R)$ -gravity (for details, see Santos, *et.al.* (2007); Wang *et al.* (2010)). The freedom of selecting different functional form of $F(R)$ gives rise to the problem of how to constrain from theoretical and observational aspects of many possible $F(R)$ -gravities. For this purpose, the energy conditions may be used as a constraint for selecting the best possible $F(R)$ -gravities. The effective energy-momentum tensor in $F(R)$ -gravity is given as (see for details: Hwang (1990); Capozziello (2002); Capozziello *et al.* (2003); Olmo (2007)):

$$T_{ij}^{eff} = \frac{1}{F_R(R)} \left(T_{ij} + \frac{1}{2}(F(R) - RF_R(R))g_{ij} + (\nabla_i \nabla_j - g_{ij} \square)F_R(R) \right). \quad (3.49)$$

Then from (3.49), the expression for effective density and pressure are respectively

$$\rho_{eff} = \frac{1}{F_R(R)} \left(\rho + \frac{1}{2}(F(R) - RF_R(R)) - 3\dot{R}H F_{RR}(R) \right), \quad (3.50)$$

and

$$p_{eff} = \frac{1}{F_R(R)} \left(p + \frac{1}{2}(F(R) - RF_R(R)) + (\ddot{R} + 2\dot{R}H)F_{RR}(R) + \dot{R}^2 F_{RRR}(R) \right). \quad (3.51)$$

For a perfect fluid FLRW, the energy conditions in the context of $F(R)$ -gravity are defined as (Mantica *et al.*, 2016):

- Null energy condition (NEC): $\rho_{eff} + p_{eff} \geq 0$
- Strong energy condition (SEC): $\rho_{eff} + 3p_{eff} \geq 0$ and $\rho_{eff} + p_{eff} \geq 0$
- Weak energy condition (WEC): $\rho_{eff} + p_{eff} \geq 0$ and $\rho_{eff} \geq 0$
- Dominant energy condition (DEC): $\rho_{eff} \pm p_{eff} \geq 0$ and $\rho_{eff} \geq 0$.

It is obvious that for $F(R) = R$, the energy conditions reduces to the case of GR.

3.4.1 Model-I

The equations obtained for Case I and Case II will be used again by changing the model in Model II.

Case-I: When $\dot{R} \neq 0$

The energy conditions for the functional form of $F(R)$ are obtained from (3.36), (3.37), (3.50) and (3.51) as:

$$\rho_{eff} = \frac{1}{F_R(R)} \left[\left(\frac{1+k^2}{2k^2} \right) F(R) + \left(\frac{\dot{R} - (1+2k^2)R}{4k^2} \right) F_R(R) - 3\dot{R}H F_{RR}(R) \right] \quad (3.52)$$

$$\begin{aligned} \rho_{eff} + p_{eff} = & \frac{1}{F_R(R)} \left[F(R) + \left(\frac{\dot{R} - 3k^2 R}{3k^2} \right) F_R(R) \right. \\ & \left. + \left(\frac{(4k^2 - 1)\ddot{R} + (3 - 4k^2 H)\dot{R}}{4k^2} \right) F_{RR}(R) + \left(\frac{k^2 - 1}{k^2} \right) \dot{R}^2 F_{RRR}(R) \right] \end{aligned} \quad (3.53)$$

$$\begin{aligned} \rho_{eff} - p_{eff} = & \frac{1}{F_R(R)} \left[\frac{1}{2k^2} (2F(R) - R F_R(R)) \right. \\ & \left. + \left(\frac{(3 - 20k^2 H)\dot{R} - (1 + 4k^2)\ddot{R}}{4k^2} \right) F_{RR}(R) - \left(\frac{k^2 + 1}{k^2} \right) \dot{R}^2 F_{RRR}(R) \right] \end{aligned} \quad (3.54)$$

$$\begin{aligned} \rho_{eff} + 3p_{eff} = & \frac{1}{F_R(R)} \left[\left(\frac{2k^2 - 1}{k^2} \right) F(R) + \frac{\dot{R} + (1 - 4k^2)R}{2k^2} F_R(R) \right. \\ & \left. + \left(\frac{(12k^2 - 3)\ddot{R} + 3(4k^2 H + 3)\dot{R}}{4k^2} \right) F_{RR}(R) + \left(\frac{3(k^2 - 1)}{k^2} \right) \dot{R}^2 F_{RRR}(R) \right] \end{aligned} \quad (3.55)$$

Taking into account our model $F(R) = R + \alpha \log(\beta R)$ and using (3.38)-(3.40) in the foregoing equations, we obtain the graph for this case. The bounds for α and β can be chosen from effective density parameter which is always positive as shown in Figure 3.4a. From the Figure 3.4b, 3.4c and Figure 3.4a, it is obvious that the conditions for NEC, DEC and WEC are satisfied for the range $21500 \leq \alpha \leq 40000$ and $-42 \leq \beta \leq -1$. However, in the same range, the SEC condition is violated as shown in Figure 3.4d. There may be a violation of energy conditions in some instances without the system becoming unacceptable. The success of the hypothesis of inflation and recent observations of cosmic acceleration points to a violation of SEC in the universe.

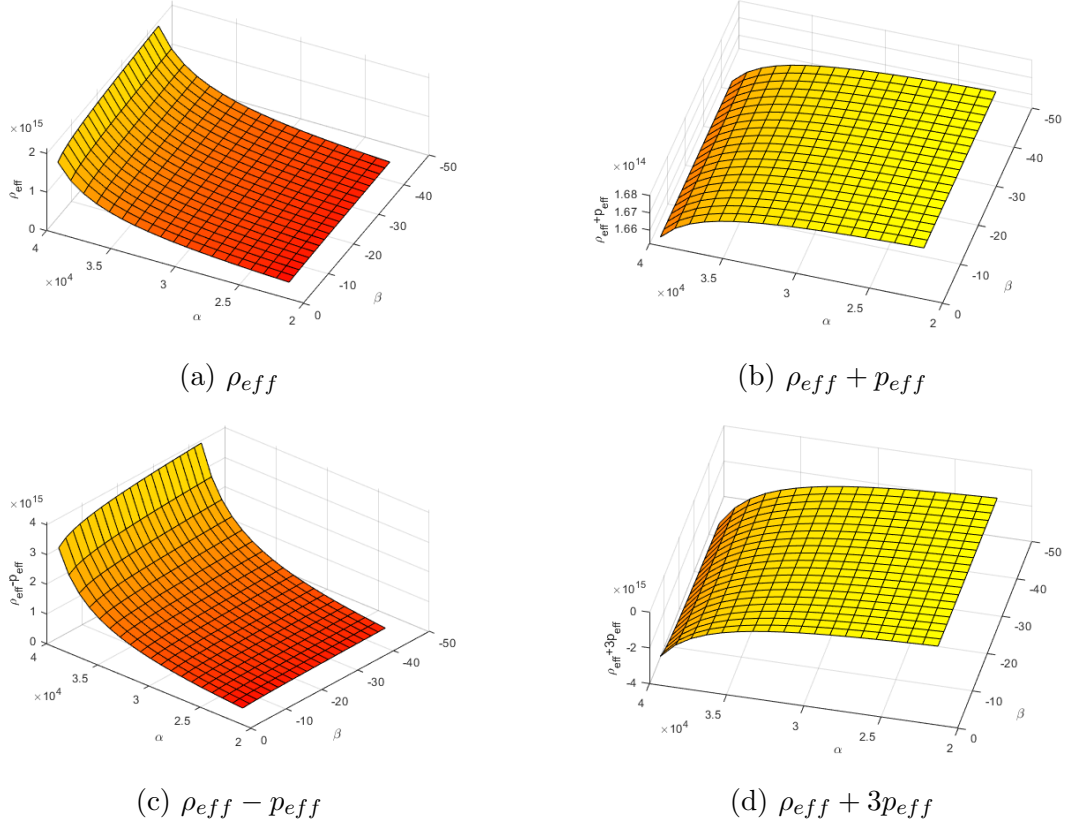


Figure 3.4: The behavior of energy conditions considering $F(R) = R + \alpha \log(\beta R)$ for Case-I with $21500 \leq \alpha \leq 40000$ and $-42 \leq \beta \leq -1$.

Case-II: When $\dot{R} = 0$.

In view of (3.47), (3.48), (3.50) and (3.51), we get the energy conditions for the functional $F(R)$ as below:

$$\rho_{eff} = \frac{1}{F_R(R)} \left[\left(\frac{1+k^2}{2k^2} \right) F(R) - \left(\frac{1+2k^2}{4k^2} \right) F_R(R) - 3\dot{R}H F_{RR}(R) \right] \quad (3.56)$$

$$\rho_{eff} + p_{eff} = \frac{1}{F_R(R)} [F(R) - R F_R(R) + (\ddot{R} - \dot{R}H) F_{RR}(R) + \dot{R}^2 F_{RRR}(R)] \quad (3.57)$$

$$\rho_{eff} - p_{eff} = \frac{1}{F_R(R)} \left[\frac{1}{2k^2} (2F(R) - R F_R(R)) - (\ddot{R} + 5\dot{R}H) F_{RR}(R) - \dot{R}^2 F_{RRR}(R) \right] \quad (3.58)$$

$$\begin{aligned} \rho_{eff} + 3p_{eff} = \frac{1}{F_R(R)} & \left[\left(\frac{2k^2-1}{k^2} \right) F(R) - \left(\frac{4k^2+1}{2k^2} \right) R F_R(R) \right. \\ & \left. + 3(\ddot{R} + \dot{R}H) F_{RR}(R) + 3\dot{R}^2 F_{RRR}(R) \right] \end{aligned} \quad (3.59)$$

Substituting the values of $R, \dot{R}$ and $\ddot{R}$ and taking into account our model

$F(R) = R + \alpha \log(\beta R)$ in the above equations, we obtain the plots of the behavior of energy conditions for this case as shown in Figure 3.5. Similar to Case-I, we obtained the range of α and β from the effective density parameter ($\rho_{eff} \geq 0$) and its behavior is shown in Figure 3.5a. Under this range, Figure 3.5b and Figure 3.5c shows positive behavior whereas Figure 3.5d execute negative behavior. Therefore the energy conditions for NEC, WEC and DEC are satisfied, while there is a violation of SEC in the range.

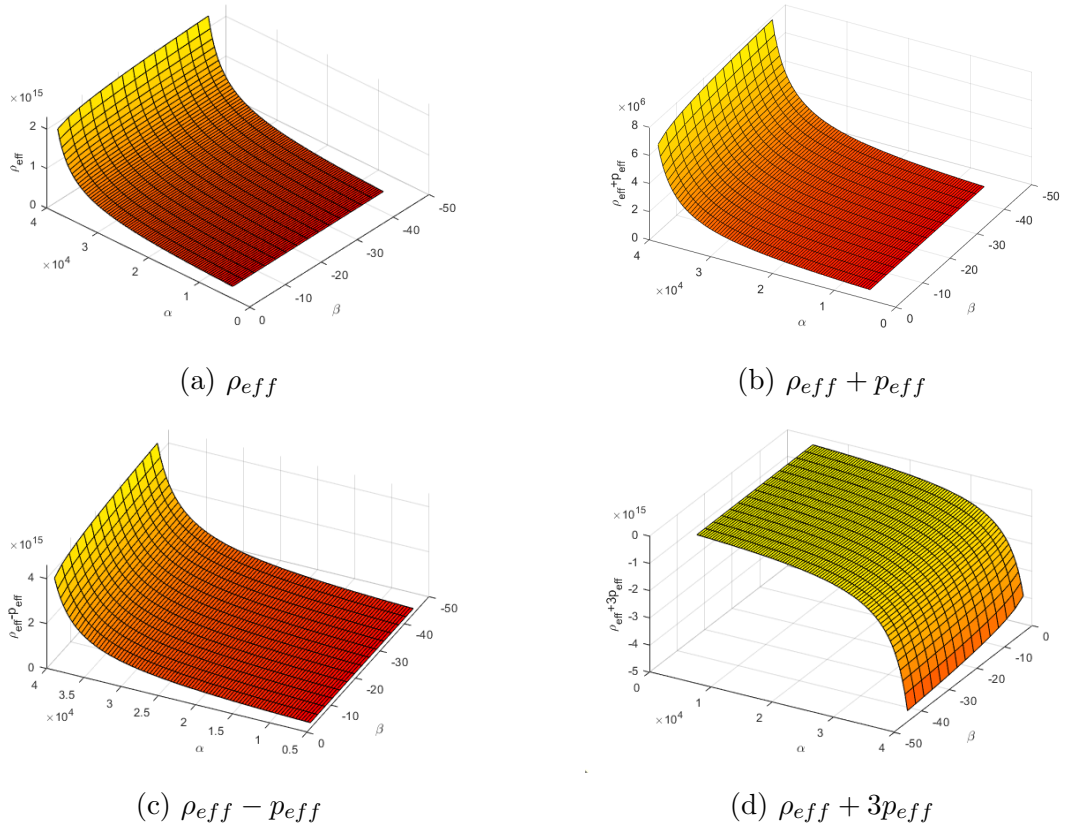


Figure 3.5: The behavior of energy conditions considering $F(R) = R + \alpha \log(\beta R)$ for Case-II with $5500 \leq \alpha \leq 40000$ and $-45 \leq \beta \leq -2$.

3.4.2 Model-II

Case-I: When $\dot{R} \neq 0$

The energy conditions for the functional form $F(R) = R - a \exp(-bR - 1)$ are shown in the following plots.

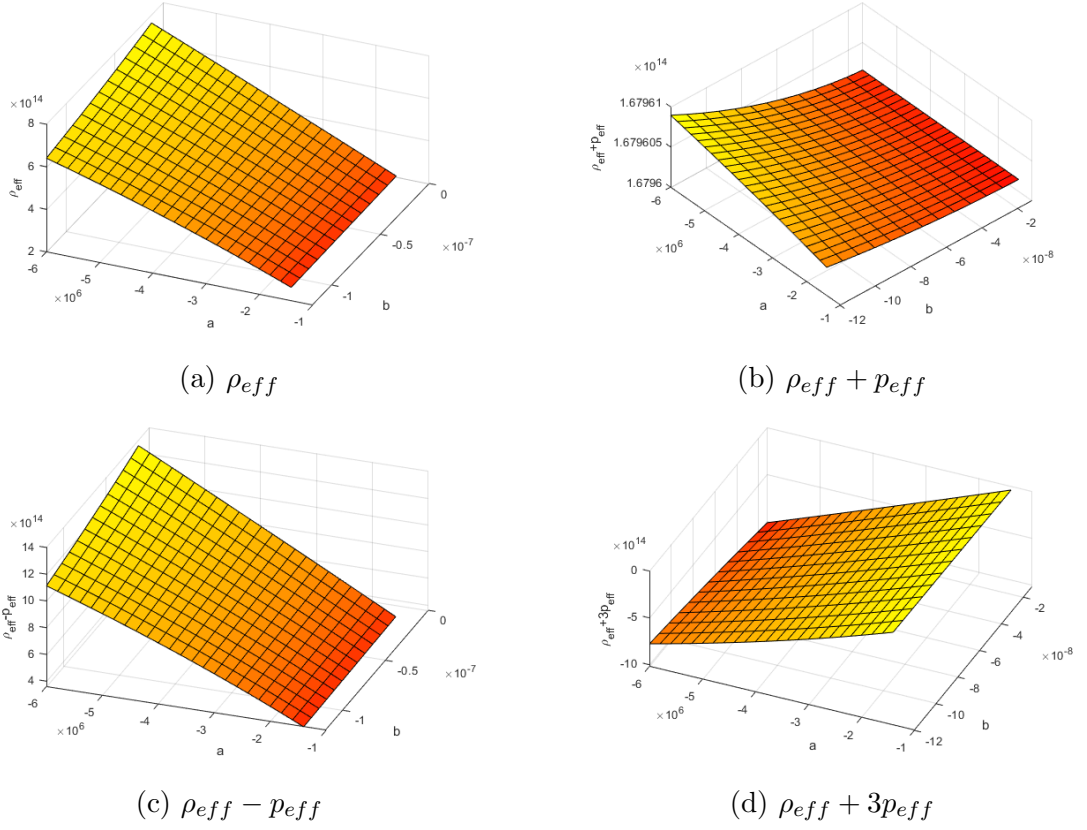


Figure 3.6: The behavior of energy conditions considering $F(R) = R - a \exp(-bR - 1)$ for Case-I with $-6 \times 10^6 \leq a \leq -1.3 \times 10^6$ and $-1.2 \times 10^{-7} \leq b \leq -1.2 \times 10^{-8}$.

The plot 3.6 reveal that $\rho_{eff} \geq 0$ along with $\rho_{eff} + p_{eff} \geq 0$ and $\rho_{eff} - p_{eff} \geq 0$. Thus, the conditions of NEC, WEC and DEC are satisfied for $-6 \times 10^6 \leq a \leq -1.3 \times 10^6$ and $-1.2 \times 10^{-7} \leq b \leq -1.2 \times 10^{-8}$ as shown in plot 3.6a, 3.6b and 3.6c. However, the plot 3.6d indicate the violation of SEC conditions confirming that our universe experiences an accelerated expansion of the universe.

Case-II: When $\dot{R} = 0$

The energy conditions for the functional form $F(R) = R - a \exp(-bR - 1)$ in this case is shown in the plot 3.7. Similar behaviors of the energy conditions as in Case-I are obtained within the suitable range for a and b . Also in this case, conformation of an accelerated expansion of the universe is shown by the violations of the SEC condition.

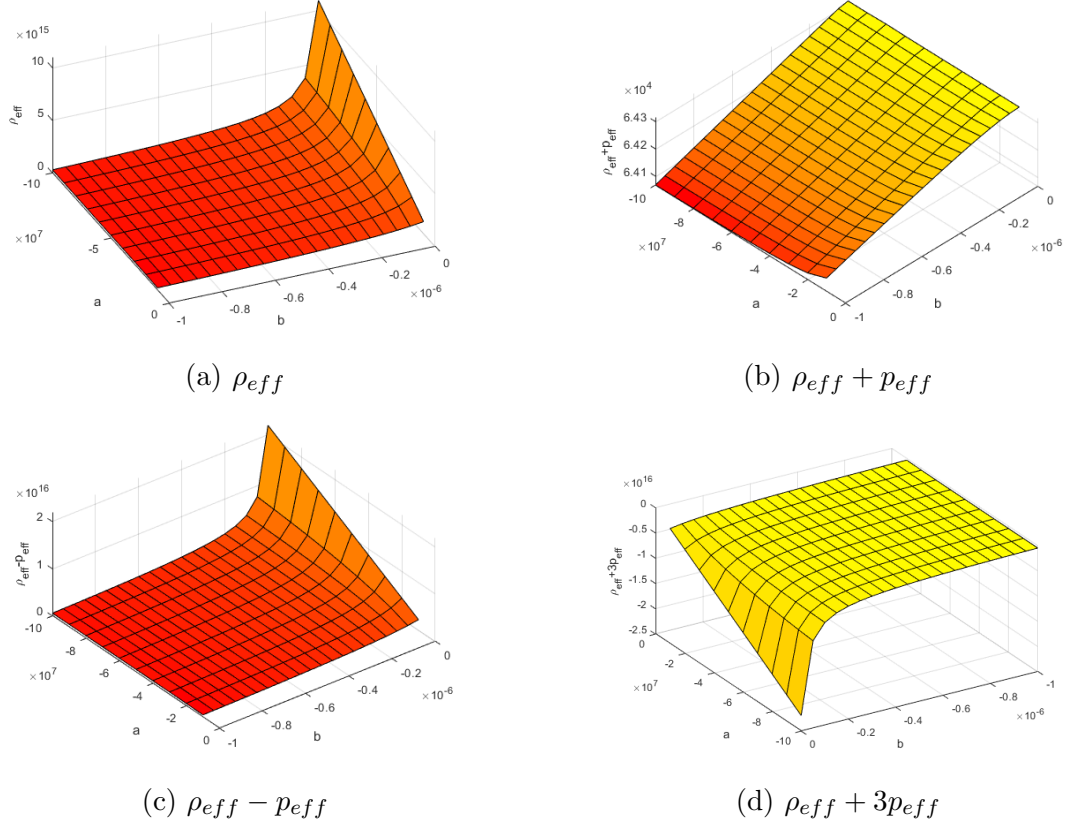


Figure 3.7: The behavior of energy conditions considering $F(R) = R - a \exp(-bR - 1)$ for Case-II with $-10^8 \leq a \leq -1$ and $-10^{-6} \leq b \leq 0$.

From the above figures of Model I and II, we see that the parameters are constraint on particular intervals to satisfy different energy conditions which we give in the following table:

MODEL	CASES	PARAMETERS AND THEIR RANGES	ENERGY CONDI- TIONS
Model I: $F(R) = R + \alpha \log(\beta R)$	$\dot{R} = 0$	$\alpha \in [5500, 40000],$ $\beta \in [-45, -2]$	NEC, WEC and DEC are satisfied but SEC is violated.
	$\dot{R} \neq 0$	$\alpha \in [21500, 40000],$ $\beta \in [-42, -1]$	NEC, WEC and DEC are satisfied but SEC is violated.
Model II: $F(R) = R - a \exp(-bR - 1)$	$\dot{R} = 0$	$a \in [-10^8, -1], b \in$ $[-10^{-6}, 0]$	NEC, WEC and DEC are satisfied but SEC is violated.
	$\dot{R} \neq 0$	$a \in [-6 \times 10^6, -1.3 \times$ $10^6], b \in [-1.2 \times$ $10^{-7}, -1.2 \times 10^{-8}]$	NEC, WEC and DEC are satisfied but SEC is violated.

Table 3.2: Constraints on models' parameters a, b, α, β to satisfy different energy conditions.

3.5 Conclusion

The recent observations indicates an accelerated expansion of the universe. It is not clear, till this point, about what could be the cause of this accelerated expansion and it has been the main topic of research. In this article, a detailed classification of conformally harmonic Z -recurrent spacetime is done in the theory of $F(R)$ -gravity. We assume the functional form $F(R) = R + \alpha \log(\beta R)$, where α and β are constants and $F(R) = R - a \exp(-bR - 1)$ with constants a and b . In both the models with non-negative scalar curvature, the density shows positive behavior, whereas the EoS parameter shows negative behavior, supporting an accelerated expansion of the universe and lies in the quintessence phase. However, when scalar curvature is constant, the models coincide with Λ CDM model. Further, the behavior of energy conditions are investigated and we obtain the bounds of model parameters α, β, a and b showing the viability of the considered

models. In both the cases, the energy conditions NEC, WEC and DEC are satisfying their derived conditions whereas SEC is violating its condition supporting the accelerated expansion of the universe.

The implications of our results are quite profound and have significant importance in cosmology, particularly in understanding the accelerated expansion of the universe. Following are some points on the importance of our results:

1. **Understanding Accelerated Expansion:** The accelerated expansion of the universe, first discovered through observations of distant supernovae, remains one of the most intriguing phenomena in cosmology. This research provides insights into the behavior of gravity at cosmological scales, suggesting that modified gravity theories, such as $F(R)$ gravity, can account for this acceleration.
2. **Confirmation of Quintessence-like Behavior:** Quintessence refers to a hypothetical form of dark energy with a dynamic equation of state (EoS) parameter, causing the accelerated expansion. By observing positive behavior in density and negative behavior in the EoS parameter in the considered models, this research provides evidence supporting the quintessence phase of dark energy, which has implications for understanding the nature of the universe's expansion.
3. **Coincidence with Λ CDM Model:** The Λ CDM (Lambda Cold Dark Matter) model has been remarkably successful in describing various cosmological observations, including the accelerated expansion. The finding that the considered models coincide with Λ CDM when the scalar curvature is constant highlights the compatibility of these modified gravity theories with well-established cosmological frameworks.
4. **Viability of Model Parameters:** Investigating the behavior of energy conditions and deriving bounds on model parameters (α , β , a , and b) demonstrate the viability of the considered models within the constraints of observational data and theoretical consistency. This not only validates the theoretical framework but also guides future research by narrowing down the parameter space.
5. **Violation of SEC and Accelerated Expansion:** The violation of the Strong Energy Condition (SEC) in the considered models supports the accelerated expansion of the universe. This is consistent with our under-

standing that exotic forms of energy, such as dark energy or modifications to gravity, are required to drive this accelerated expansion.

This study offers a framework for evaluating and comparing various cosmological models, including mainstream models like Λ CDM and modified gravity theories like $F(R)$ gravity. It helps to better comprehend dark energy, estimate parameters, and test gravity at cosmic scales. The discoveries have important consequences for our understanding of the universe's fast expansion, dark energy nature, and gravity behavior at cosmic scales.

Chapter 4

Geometry of Submanifolds in almost contact manifold

4.1 Introduction

In this chapter, we begin an investigation into Ricci-Bourguignon solitons on Euclidean submanifolds, with a focus on cases where the soliton fields correspond to the tangential components of the position vector fields. We derive several fundamental properties of these solitons, shedding light on their geometry and curvature. A significant portion of our research is dedicated to the classification of Ricci-Bourguignon solitons on Euclidean hypersurfaces. We analyze the soliton equation and curvature conditions to identify specific classes of hypersurfaces that support such solitons. For general references on Riemannian submanifolds, we refer to (Chen, 1973, 2011, 2015).

Consider an n -dimensional Riemannian manifold (M, g) . Let $(e_1, \dots, e_n)$ be an orthonormal basis for $T_p M$ the tangent space at a point $p \in M$. Let the sectional curvature of the plane section spanned by the vectors e_i and e_j (where $i \neq j$) be represented by the notation K_{ij} .

$$R = \sum_{1 \leq i \neq j \leq n} K_{ij}. \quad (4.1)$$

Consider an isometric immersion $\phi : (M, g) \rightarrow \mathbb{E}^m$, where M is an n -dimensional Riemannian manifold equipped with the metric g , and $\mathbb{E}^m$ is an n -dimensional

³Robert Sumlalsanga, Jay Prakash Singh and M. Sundararajan, Ricci Borguignon Soliton on Euclidean Submanifold (Communicated).

Euclidean space with the metric $\tilde{g}$.

Let ∇ and $\tilde{\nabla}$ denote the Levi-Civita connections corresponding to the Riemannian manifold (M, g) and the Euclidean space $(\mathbb{E}^m, \tilde{g})$, respectively.

The Gauss equation and the Weingarten map expressions are presented for tangent vector fields X and Y on M , along with a normal vector field η to M , as follows:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (4.2)$$

$$\tilde{\nabla}_X \eta = -A_\eta X + D_X \eta. \quad (4.3)$$

In these case, $\tilde{\nabla}_X Y$ represents the tangential component, while $h(X, Y)$ represents the normal component in (4.2). Likewise, $-A_\eta X$ represent the tangential component and $D_X \eta$ represent the normal component in (4.3). These formulas describe the normal connection D of the manifold M concerning the surrounding space $\mathbb{E}^m$, the shape operator A , and the second fundamental form h .

The fact that every A_η is a self-adjoint endomorphism is well known. Its shape operator A and the second basic form h are related in the following way:

$$\tilde{g}(h(X, Y), \eta) = g(A_\eta X, Y). \quad (4.4)$$

The mean curvature vector H associated with M within the ambient space $\mathbb{E}^m$ is established through

$$H = \left(\frac{1}{n} \right) Tr h, \quad (4.5)$$

for all vector fields X and Y on the manifold M -dimensional $(2n + 1)$. Here, Q represents the Ricci operator connected to the $(1, 2)$ Ricci tensor S , and R represents the curvature tensor of the metric g , stated for all vector fields X and Y on the $(2n + 1)$ -dimensional manifold M as $S(X, Y) = g(QX, Y)$.

Recent findings have shown that a Kenmotsu manifold can be locally described as a warped product $I \times_f N^{2n}$, where I is an open interval with the coordinate t , $f = ce^t$ represents the warping function for a positive constant c and N^{2n} is a Kählerian manifold (Kenmotsu, 1972).

On an almost Kenmotsu manifold, the following expression holds true (Dileo

and Pastore, 2007, 2009):

$$\nabla_X \xi = \varphi^2 X - \varphi hX \quad (4.6)$$

If the vector field of its mean curvature disappears identically, a submanifold M is said to be minimal. If the second fundamental form is such that $h(X, Y) = g(X, Y)H$ for tangent vectors X, Y , it is referred to as fully umbilical. A hypersurface of a Euclidean $(n+1)$ -space $\mathbb{E}^{n+1}$ is called a quasi-umbilical hypersurface if its shape operator has an eigenvalue κ of multiplicity $\text{mult}(\kappa) \geq n-1$ (cf. Chen (1973)). On the subset U of M on which $\text{mult}(\kappa) = n-1$, an eigenvector with eigenvalue of multiplicity one is called a distinguished direction of the quasi-umbilical hypersurface.

Gauss's equations are provided by

$$\begin{aligned} g(R(X, Y)Z, W) = & \tilde{g}(R(X, Y)Z, W) + \tilde{g}(h(X, W), h(Y, Z)) \\ & - \tilde{g}(h(X, Z), h(Y, W)) \end{aligned} \quad (4.7)$$

for the vectors tangent to M that are X, Y, Z and W .

We write ∇f for the gradient of a function f defined on M and H^f for its Hessian. Consequently, we have

$$g(\nabla f, X) = Xf, \quad (4.8)$$

$$H^f(X, Y) = XYf - (\nabla_X Y)f. \quad (4.9)$$

Consider a Riemannian m -manifold (M, g) . Define a $(1, 1)$ -tensor Q in relation to the Ricci tensor S as follows:

$$g(Q(X), Y) = S(X, Y).$$

As a tensor field of type $(1, 3)$, the Weyl conformal curvature tensor C is defined by

$$\begin{aligned} C(X, Y)Z = & R(X, Y)Z + \frac{1}{m} \{S(X, Z)Y - S(Y, Z)X + \langle X, Z \rangle QY \\ & - \langle Y, Z \rangle QX\} - \frac{2\tau}{m(m+1)} \{\langle X, Z \rangle Y - \langle Y, Z \rangle X\}. \end{aligned}$$

A well-known result of H. Weyl (1918) states that a Riemannian manifold M of dimension ≥ 4 is conformally flat if and only if the conformal curvature tensor C vanishes identically.

4.2 Main result

The position vector field x of M in $\mathbb{E}^m$ has tangential and normal components, respectively, x^T and $x^\perp$, given an isometric immersion $\phi : (M, g) \rightarrow \mathbb{E}^m$ of a n -dimensional Riemannian manifold (M, g) into m -dimensional Euclidean space $\mathbb{E}^m$. Thus, we get

$$x = x^T + x^\perp \quad (4.10)$$

Theorem 4.2.1. *A Ricci-Bourguignon soliton with x^T as its soliton field can be recognized as a Euclidean submanifold (M, g) in $\mathbb{E}^m$ if and only when the second fundamental form h of M meets the following condition*

$$S = (\rho R + \lambda - 1)g(V, W) - \tilde{g}(h(V, W), x^\perp) \quad (4.11)$$

for tangent vectors V, W on M , where R and λ represent the scalar curvature and a constant respectively.

Proof. Let the isometric immersion be denoted by $\phi : (M, g) \rightarrow \mathbb{E}^m$. The position vector field x of M in $\mathbb{E}^m$ is known to be a concurrent vector field, meaning that x satisfies

$$\tilde{\nabla}_V X = V, \quad (4.12)$$

for any tangent vector V on M .

From (4.10), (4.12) and formulas of Gauss and Weingarten that

$$\begin{aligned} V &= \tilde{\nabla}_V(X) = \tilde{\nabla}_V(x^T + x^\perp) = \tilde{\nabla}_V x^T + \tilde{\nabla}_V x^\perp \\ &= \nabla_V x^T + h(V, x^T) - A_x N V + D_V x^\perp \end{aligned} \quad (4.13)$$

for any tangent vector V to M . Examining the tangential and normal components

from equation (4.13), we observe

$$\nabla_V x^T = A_{x^\perp} V + V, \quad (4.14)$$

$$h(V, x^T) = -D_V x^\perp. \quad (4.15)$$

Using the definition of the Lie derivative along with equation (4.14), we derive

$$\begin{aligned} (L_{x^T} g)(V, W) &= g(\nabla_V x^T, W) + g(\nabla_W x^T, V) \\ &= 2g(V, W) + 2g(A_{x^\perp} V, W) \\ &= 2g(V, W) + \tilde{g}(h(V, W), x^\perp) \end{aligned} \quad (4.16)$$

for the tangent V and W to M . As a result, using (3.50) and (4.15), we obtain

$$\lambda g + \rho Rg - S = g + \tilde{g}(h(V, W), x^\perp).$$

After rearranging the equation above, we may infer that if and only if (4.11) remains the same for some constant λ , then (M, g) is a Ricci-Bourguignon Soliton with x^T as its soliton field. $\square$

The following is a significant use of Theorem 4.2.1:

Theorem 4.2.2. *Assume that a Riemannian manifold is (M, g) . If and only if (M, g) has constant scalar curvature R , then an isometric immersion $\phi : (M, g) \rightarrow S_o^{m-1}(r) \subset \mathbb{E}^m$ of M into the hypersphere $S_o^{m-1}(r)$ of radius r with center at the origin o is a Ricci-Borguignon soliton with x^T as its soliton field.*

Proof. Assume that a Riemannian manifold is (M, g) . An isometric immersion of M into $S_o^{m-1}(r)$ is represented as $\phi : (M, g) \rightarrow S_o^{m-1}(r) \subset \mathbb{E}^m$. In this case, $x = x^\perp$. Also in this case it follows from Lemma of Chen (2011) that the second fundamental form of M in $\mathbb{E}^m$ satisfies

$$h(V, W) = h'(V, W) - \frac{g(V, W)}{r^2} x \quad (4.17)$$

for vectors V, W tangent to M , where h' denotes the second fundamental form of M in $S_o^{m-1}(r)$. It is evident that (4.17) indicates:

$$\tilde{g}(h(V, W), x^\perp) = -g(V, W). \quad (4.18)$$

Therefore, if and only if $R = -\frac{\lambda}{\rho}$ holds, then condition (4.11) in Theorem 4.2.1 holds. Thus, (M, g, x^T, λ) with $\lambda = -R\rho$ constitutes a Ricci-Bourguignon soliton if and only if the manifolds (M, g) possesses constant scalar curvature $-R\rho$. $\square$

Remark 4.2.3. *Theorem 4.2.2 states that there are several examples of Ricci-Bourguignon solitons with x^T as the soliton field.*

The subsequent outcome categorizes Ricci-Bourguignon solitons on Euclidean hypersurfaces featuring x^T as their soliton fields.

Corollary 4.2.4. *Consider (M, g) as a Euclidean hypersurface in $\mathbb{E}^{n+1}$. Subsequently, (M, g, x^T, λ) qualifies as a Ricci-Bourguignon soliton if and only if either*

1. *when $\lambda = 1$, and hypersurface M is an open section of a hyperplane in $\mathbb{E}^{n+1}$, or*
2. *when $\lambda = R < 0$ and hypersurface M represents an open section of a hypersphere within $\mathbb{E}^{n+1}$ centered at the origin.*

Proof. Let M be a Euclidean hypersurface of $\mathbb{E}^{n+1}$. Assume that (M, g, x^T, λ) is a Ricci-Bourguignon soliton with x^T as its soliton field. Then we have (4.11) by Theorem 4.2.1.

Case (a): $x^\perp \equiv 0$. In this case, In this instance, the hypersurface M and the position vector field x always align tangentially. Thus M is an open portion of a hyperplane encompassing the $\mathbb{E}^{n+1}$ origin. Therefore, according to equation (4.7) of Gauss, M represents a flat space embedded as a completely geodesic hypersurface within $\mathbb{E}^{n+1}$. Thus, it can be deduced from equation (4.16) that $\lambda = 1$. As a result, the Ricci-Bourguignon soliton (M, g) exhibits expansion. This corresponds to Case (1) in the corollary.

Case (b): $x^\perp \equiv x$. In this case, the hypersurface M is perpendicular to the position vector field x everywhere. Thus, M is an open segment of a hypersphere centered at the origin with a radius of, let's say, r . So, in this case we find from (see, e.g. Lemma of Chen (2011)) that

$$h(V, W) = \frac{g(V, W)}{r^2} x \quad (4.19)$$

for V, W tangent to M . After substituting (4.19) into (4.16) we obtain $\lambda = \rho R > 0$. Thus, the Ricci-Bourguignon soliton undergoes expansion. This corresponds to Case (2) in the corollary.

Case (c): $x^\perp \neq 0, x$. From equation (4.16), it can be inferred that M is completely umbilical within $\mathbb{E}^m$. Therefore, the scalar curvature R remains constant. From (4.16) we get

$$\tilde{g}(h(U, U), x^\perp) = \rho R + \lambda - 1 = \text{constant} \quad (4.20)$$

for any unit vector U tangent to M .

Assume for the moment that hypersurface M is fully geodesic inside $\mathbb{E}^{n+1}$. Thus, (4.20) simplifies to $\lambda = 1$. Therefore, we encounter Case (1) once more.

Let us first suppose that in $\mathbb{E}^{n+1}$, M is completely umbilical but not wholly geodesic. Then, a hypersphere of radius, let's say r , centered at $x_o \neq 0$, contains M . Thus we have (see, e.g. Lemma of Chen (2011)):

$$h(U, U) = \frac{x - x_o}{r^2} \quad (4.21)$$

for any unit vector U tangent to M . By substituting (4.21) into (4.16), we get

$$\tilde{g}(x - x_o, x^\perp) = \text{constant}.$$

The identities $x^\perp = |x^\perp|N$ and $x - x_o = rN$, where N is a unit normal vector field of M , are readily apparent. Since the origin of $\mathbb{E}^{n+1}$ is not the center of the hypersurface, $|xN|$ is thus constant on M , which is not possible. $\square$

A unit normal vector field ξ of a Euclidean submanifold M is called a parallel (resp., non-parallel) normal section if $D\xi = 0$ (resp., $D\xi \neq 0$) everywhere on M (cf. Chen (2011); Chen and Yano (1971, 1973)).

Corollary 4.2.5. *Let (M, g) be an n -dimensional submanifold of $\mathbb{E}^{n+2}$ with $n > 3$ and $x^\perp \neq 0$. Assume that (M, g, x^T, λ) is a Ricci-Borguignon soliton. Then we have:*

1. *If $\frac{x^\perp}{|x^\perp|}$ represents a non-parallel normal section, then (M, g) forms a conformally flat space. Furthermore, under this condition, M represents the locus of $(n - 1)$ -spheres.*

2. If $\frac{x^\perp}{|x^\perp|}$ constitutes a parallel normal section, then (M, g) resides either in a hyperplane or within a hypersphere of $\mathbb{E}^{n+2}$.

Proof. Given $n > 3$ and $x^\perp \neq 0$, let (M, g) be an n -dimensional submanifold of $\mathbb{E}^{n+2}$. Given a Ricci-Borguignon soliton (M, g, x^T, λ) , M is umbilical with respect to the normal direction $x^\perp$, as per Theorem 4.2.1.

1. $\frac{x^\perp}{|x^\perp|}$ is a normal section that is non-parallel. It follows from (Chen and Yano, 1973) that (M, g) is a conformally flat space. Moreover, from (Chen and Yano, 1973) we know that the hypersurface is a locus of $(n-1)$ -spheres in $\mathbb{E}^{n+1}$.
2. $\frac{x^\perp}{|x^\perp|}$ is a parallel normal section. It follows from (Chen and Yano, 1971) that M lies either in a hyperplane or in a hypersphere of $\mathbb{E}^{n+2}$.

□

4.3 Conclusion

In conclusion, this study investigates Ricci-Bourguignon solitons on Euclidean submanifolds, specifically examining soliton fields related to the tangential components of the position vector fields. Key findings include the identification of conditions under which a Euclidean submanifold qualifies as a Ricci-Bourguignon soliton, characterized by a specific relation involving the second fundamental form, Ricci tensor, and scalar curvature. Additionally, we classified Ricci-Bourguignon solitons on Euclidean hypersurfaces, demonstrating that they must be an open part of either a hyperplane or a hypersphere, contingent on the soliton parameter λ . This classification aligns with existing submanifold theory and provides a clear framework for identifying such solitons. This work lays a foundational understanding of Ricci-Bourguignon solitons on Euclidean submanifolds.

Chapter 5

Properties of Warped product manifold

5.1 Introduction

Our investigation is centered around identifying the conditions that render the warped product a conformal Ricci-Yamabe soliton. To commence, let's begin by revisiting a pivotal result (cf. Bishop and O'Neill (1969); Singh and Zosangzuala (2022)) that will prove essential for our subsequent discussions.

Lemma 5.1.1. *(Bishop and O'Neill, 1969) Let $(M, g) = (B \times_f B, g_B \oplus f^2 g_F)$ be a warped product of two Riemannian manifolds B and F with $\dim B = m$ and $\dim F = n$. Then, for all $X, Y \in \chi(B)$ and $U, V \in \chi(F)$*

1. $D_X U = D_U X = \frac{X(f)}{f} U,$
2. $S(X, U) = 0,$
3. $S(X, Y) = S^B(X, Y) - \frac{n}{f} H^f(X, Y),$
4. $S(U, V) = S^F(U, V) - \left(\frac{\Delta f}{f} + (n-1) \frac{\|\nabla f\|^2}{f^2} \right) g(U, V),$

where $D_X Y$ is the lift of $\nabla_X Y$ on B and S^B, S^F are the lifts of the Ricci tensors on the base B and the fiber F respectively.

⁴Jay Praksah Singh and Robert Sumlalsanga (2024), Conformal Ricci-Yamabe Solitons on Warped Product Manifolds, *Filomat*, **38(11)**, 3791-3802.

Motivated by their result (Ganguly *et al.*, 2022; Khatri and Singh, 2023), we extend our exploration to the realm of conformal Ricci-Yamabe solitons with $\alpha \neq 0$ on warped product manifolds. We delve into the investigations surrounding conformal Ricci-Yamabe solitons on warped product manifolds, examining the impact on both the base and fiber manifolds when a warped product manifold admits a conformal Ricci-Yamabe soliton. The influence of specific types of smooth vector fields such as Killing vector fields, conformal vector fields, and concurrent vector fields on conformal Ricci-Yamabe solitons within warped product spaces are investigated. Also the admittance of conformal Ricci-Yamabe soliton with a concurrent vector field in a warped product manifold is analysed.

5.2 Investigation of Conformal Ricci-Yamabe Soliton on Warped Product Manifolds

This section presents an exploration into the realm of conformal Ricci-Yamabe solitons on warped product manifolds. Our central objective here is to meticulously examine the repercussions of a warped product manifold adopting the characteristics of a conformal Ricci-Yamabe soliton. In particular, we are keen to elucidate how this alignment influences both the base manifold and the fiber manifold, effectively isolating the precise conditions that facilitate their transformation into conformal Ricci-Yamabe solitons. To facilitate our investigation, we consider the manifold $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, representing a warped product of two Riemannian manifolds B and F with $\dim B = m$ and $\dim F = n$ respectively. Proceeding further, let (M, g, μ, ξ) exemplify a conformal Ricci-Yamabe soliton, with $\mu = [2\lambda - \beta R - (p + \frac{2}{n})]$. Consequently, by invoking equation (1.58), we obtain:

$$L_\xi g + 2\alpha S = \left[2\lambda - \beta R - \left(p + \frac{2}{n} \right) \right] g = \mu g, \quad (5.1)$$

where $\mu = [2\lambda - \beta R - (p + \frac{2}{n})]$. Recalling Lemma 5.1.1 from the preceding section, we readily deduce the following two widely recognized formulas tailored for warped product manifolds:

$$L_\xi g = L_{\xi_B}^B g_B + f^2 L_{\xi_F}^F g_F + 2f\xi_B(f)g_F, \quad (5.2)$$

$$S = S^B - \frac{n}{f}H^f + S^F - \tilde{f}g_F, \quad (5.3)$$

where $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$. Now, applying the definition of warped metric from equation (5.1) and using (5.2) and (5.3), we have

$$\begin{aligned} \mu(g_B + f^2g_F) &= \mu g = L_\xi g + 2\alpha S \\ &= L_{\xi_B}^B + f^2L_{\xi_F}^F g_F + 2f\xi_B(f)g_F + 2\alpha S^B - 2\alpha\frac{n}{f}H^f + 2\alpha S^F - 2\alpha\tilde{f}g_F. \end{aligned} \quad (5.4)$$

Again, for all $U, V \in \chi(B)$, we can write

$$(L_{\xi_B}^B)(U, V) = g_B(D_U^B\xi_B, V) + g_B(U, D_V^B\xi_B). \quad (5.5)$$

By definition of Hessian and (5.5), we get

$$\left(L_{\xi_B}^B g_B - 2\alpha\frac{n}{f}H^f\right)(U, V) = g_B(D_U^B\xi_B, V) + g_B(U, D_V^B\xi_B) - 2\alpha\frac{n}{f}g_B(D_U^B\nabla^B f, V).$$

The above equation can be reduced as

$$\begin{aligned} &\left(L_{\xi_B}^B g_B - 2\alpha\frac{n}{f}H^f\right)(U, V) \\ &= g_B(D_U^B\xi_B, V) - \alpha\frac{n}{f}g_B(D_U^B\nabla^B f, V) + (g_B(U, D_V^B\xi_B) - \alpha\frac{n}{f}g_B(D_U^B\nabla^B f, V)) \\ &= g_B(D_U^B(\xi_B - \alpha n\nabla^B \ln f), V) + g_B(U, D_V^B(\xi_B - \alpha n\nabla^B \ln f)) \\ &= (L_{\xi_B - \alpha n\nabla^B \ln f}^B g_B)(U, V), \forall U, V \in \chi(B). \end{aligned} \quad (5.6)$$

Since it is true for all $U, V \in \chi(B)$, in operator notation we can write

$$L_{\xi_B}^B g_B - 2\alpha\frac{n}{f}H^f = L_{\xi_B - \alpha n\nabla^B \ln f}^B g_B. \quad (5.7)$$

Putting the value of (5.7) in (5.4) gives

$$\begin{aligned} &(L_{\xi_B - \alpha n\nabla^B \ln f}^B g_B + 2\alpha S^B) + (f^2L_{\xi_F}^F g_F + 2\alpha S^F) \\ &= \mu g_B + (\mu f^2 - 2f\xi_B(f) + 2\tilde{f})g_F. \end{aligned} \quad (5.8)$$

Hence, we can state the following theorem.

Theorem 5.2.1. *Suppose a warped product of two Riemannian manifolds B and F is $(M, g) = (B \times_f F, g_B \oplus f^2g_F)$, with the warping function denoted as f . We have $\dim B = m$ and $\dim F = n$. Assuming that (M, g, μ, ξ) forms*

a conformal Ricci-Yamabe soliton, it follows that both the base $(B, g_B, \mu, \xi_B - \alpha n \nabla^B \ln f)$ and the fiber $(F, g_F, \mu f^2 - 2f \xi_B(f) + 2\tilde{f}, f^2 \xi_F)$ exhibit the characteristics of conformal Ricci-Yamabe solitons. Here, $\tilde{f} = f \Delta f + (n-1)|\nabla f|_B^2$ and $\mu = [2\lambda - \beta R - (p + \frac{2}{n})]$, where λ , β and p represent the soliton constants and the conformal pressure respectively.

Now, let us consider the soliton vector field ξ of the conformal Ricci-Yamabe soliton (M, g, μ, ξ) to be a gradient of some smooth function ϕ , i.e., when $\xi = \text{grad } \phi = \nabla \phi$. The function ϕ is referred to as the potential function of the soliton, and the soliton is known as a conformal gradient Ricci-Yamabe soliton. To maintain clarity in notation, A conformal gradient Ricci-Yamabe soliton is designated as (M, g, μ, ϕ) , where the final term identifies the soliton's potential function.

Let us consider a warped product of two Riemannian manifolds B and F be $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$. We have $\dim B = m$ and $\dim F = n$. Now, if (M, g, μ, ϕ) forms a conformal gradient Ricci-Yamabe soliton, for any vector fields $X, Y \in \chi(M)$, then (5.1) provides

$$\begin{aligned} 2H^\phi(X, Y) + 2\alpha S(X, Y) &= \left[2\lambda - \beta R - \left(p + \frac{2}{n} \right) \right] g(X, Y) \\ &= \mu g(X, Y). \end{aligned} \quad (5.9)$$

Again, we consider $X = X_B$ and $Y = Y_B$, with X_B and Y_B representing the lifts of the vector fields X and Y in $\chi(B)$, then (5.9) reduces to

$$2H^\phi(X_B, Y_B) + 2\alpha S(X_B, Y_B) = \mu g(X_B, Y_B).$$

By substituting the value of the Ricci tensor for the base manifold as provided in Lemma 5.1.1, the equation above can be expressed as

$$2H_B^{\phi_B}(X_B, Y_B) + 2\alpha S^B(X_B, Y_B) - 2\alpha \frac{n}{f} H_B^f(X_B, Y_B) = \mu g_B(X_B, Y_B),$$

where $\phi_B = \phi$ at a specific point of the fiber F . Finally, by employing the properties of the Hessian in the equation above, we obtain

$$2H_B^{\phi_B - \alpha n \ln f}(X_B, Y_B) + 2\alpha S^B(X_B, Y_B) = \mu g_B(X_B, Y_B). \quad (5.10)$$

This demonstrates that $(B, g_B, \mu, \phi_B - \alpha \ln f)$ qualifies as a conformal gradient Ricci-Yamabe soliton.

Furthermore, if we set $(X = X_F)$ and $(Y = Y_F)$, with X_F and Y_F representing the lifts of the vector fields X and Y in $\chi(F)$, then (5.9) yields

$$2H^\phi(X_F, Y_F) + 2\alpha S(X_F, Y_F) = \mu g(X_F, Y_F).$$

Using (5.3) and Lemma 5.1.1, the above equation becomes

$$2H_F^{\phi_F}(X_F, Y_F) + 2\alpha S^F(X_F, Y_F) - \tilde{f}g_F(X_F, Y_F) = \mu f^2 g_F(X_F, Y_F),$$

where $\phi_F = \phi$ at a fixed point of the base B and $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$. From the above equation, we get

$$2H_F^{\phi_F}(X_F, Y_F) + 2\alpha S^F(X_F, Y_F) = (\mu f^2 + \tilde{f})g_F(X_F, Y_F).$$

Hence, given the constancy of the warping function f , the term $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$ disappears from the right-hand side of the above equation, we get

$$2H_F^{\phi_F}(X_F, Y_F) + 2\alpha S^F(X_F, Y_F) = \mu f^2 g_F(X_F, Y_F). \quad (5.11)$$

Thus, $(F, g_F, \mu f^2, \phi_F)$ is a conformal gradient Ricci-Yamabe soliton. Hence, from the above observations and (5.10) and (5.11), we can state the following:

Theorem 5.2.2. *Suppose a warped product of two Riemannian manifolds B and F is $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, with the warping function denoted as f . We have $\dim B = m$ and $\dim F = n$. Assuming that (M, g, μ, ξ) forms a conformal Ricci-Yamabe soliton, then*

1. *the base $(B, g_B, \mu, \phi_B - n \ln f)$ forms a conformal gradient Ricci-Yamabe soliton with $\phi_B = \phi$ at a point within the fiber F .*
2. *the fiber $(F, g_F, \mu f^2, \phi_f)$ forms a conformal gradient Ricci-Yamabe soliton with $\phi_F = \phi$ at a point within the base B , given that the warping function f is constant.*

Remark 5.2.3. *For particular values of $\alpha = 1, \beta = 0$, the above theorem reduces to Theorem 2.2 of Ganguly et al. (2022).*

5.3 Influence of Distinctive Vector Field Types on Conformal Ricci-Yamabe Solitons in Warped Product Manifolds

This section's major goal is to investigate how particular forms of smooth vector fields affect conformal Ricci-Yamabe solitons in warped product spaces. We shall pay special attention to the conformal and killing vector fields, which are defined as follows:

Definition 5.3.1. *A smooth vector field X on a Riemannian manifold (M, g) is called*

1. *Killing vector field or an infinitesimal isometry, if the local 1-parameter group of transformations generated by X in a neighbourhood of each point of M consists of local isometries, or in other words, if X satisfies $L_X g = 0$ and*
2. *conformal vector field if X satisfies $L_X g = \rho g$,*

Here, ρ represents a smooth function defined on the manifold M and $L_X g$ refers the Lie derivative of the Riemannian metric g along the vector field X .

As per the definition outlined above, our initial result in this section is presented below:

Proposition 5.3.2. *Let $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$ represent a warped product of two Riemannian manifolds B and F denoted as a warping function f . Here, $\dim B = m$ and $\dim F = n$. Assuming that (M, g, μ, ξ) forms a conformal Ricci-Yamabe soliton, and under the condition that any one of the following conditions holds*

1. *$\xi = \xi_B$ and ξ_B is a Killing vector field on the base B .*
2. *$\xi = \xi_F$ and ξ_F is a Killing vector field on the fiber F .*

Then, under any one of the above conditions, the manifold (M, g) transforms into an Einstein manifold.

Proof. Based on our assumption that (M, g, μ, ξ) constitutes a conformal Ricci-Yamabe soliton, it complies with equation (5.1) gives

$$L_\xi g + 2\alpha S = \mu g. \quad (5.12)$$

Now, let $\xi = \xi_B$, and ξ_B is a Killing on B , we get $L_{\xi_B}^B g_B = 0$. Then, (5.2) gives $L_\xi g = 0$. Therefore, (5.12) yields $S = \frac{\mu}{2\alpha} g$ and this implies (M, g) is an Einstein manifold.

Furthermore, if we consider $\xi = \xi_F$ and note that ξ_F is a Killing vector field on F , we find that $L_{\xi_F}^F g_F = 0$. Proceeding similarly to the previous part of the proof, we get the same result. $\square$

Theorem 5.3.3. *Suppose a warped product of two Riemannian manifolds B and F is $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, with the warping function denoted as f . We have $\dim B = m$ and $\dim F = n$. Assuming that (M, g, μ, ξ) forms a conformal Ricci-Yamabe soliton, it follows that ξ_B is a Killing vector field on the base B , then the base $(B, g_B, \mu, -n\alpha \ln f)$ forms a conformal Ricci-Yamabe soliton, where ξ_B is the lift of the vector field ξ to $\chi(B)$.*

Proof. Considering the given information that (M, g, μ, ξ) is a conformal Ricci-Yamabe soliton as per Theorem (5.2.1), we can conclude that the base $(B, g_B, \mu, \xi_B - n\alpha \nabla^B \ln f)$ also qualifies as a conformal Ricci-Yamabe soliton and hence it satisfies (5.1). Thus, we can write

$$L_{\xi_B - n\alpha \nabla^B \ln f}^B g_B + 2\alpha S^B = \mu g_B. \quad (5.13)$$

Again, using (5.7) in the above equation, (5.13) becomes

$$L_{\xi_B}^B g_B - 2\alpha \frac{n}{f} H^f + 2\alpha S^B = \mu g_B.$$

Given that ξ_B is a Killing vector field on the base B , we can assert that $L_{\xi_B}^B g_B = 0$. Then, the above equation gives us

$$-2\alpha \frac{n}{f} H^f + 2\alpha S^B = \mu g_B.$$

Thus, using the properties of Hessian, the above equation results in

$$2H^{-\alpha n \ln f} + 2\alpha S^B = \mu g_B. \quad (5.14)$$

Hence, comparing (5.14) with (5.9) completes the proof. $\square$

The following conclusion is reached from the study of Killing vector fields on conformal Ricci-Yamabe soliton warped product manifolds:

Theorem 5.3.4. *Suppose a warped product of two Riemannian manifolds B and F is $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, with the warping function denoted as f . We have $\dim B = m$ and $\dim F = n$. Assuming that (M, g, μ, ξ) forms a conformal Ricci-Yamabe soliton and the lifts ξ_B and ξ_F are both Killing on the base B and the fiber F respectively. Then, the manifold (M, g) is Einstein provided $\xi_B(f) = 0$.*

Proof. As provided, both ξ_B and ξ_F are Killing vector fields, implying $L_{\xi_B}^B g_B = 0$ and $L_{\xi_F}^F g_F = 0$. Utilizing these values in equation (5.2), we get

$$L_\xi g = 2f\xi_B(f)g_F. \quad (5.15)$$

Again, assuming that (5.1) is true and that (M, g, μ, ξ) is a conformal Ricci-Yamabe soliton, we obtain

$$L_\xi g + 2\alpha S = \mu g.$$

Now, using (5.15) in the above equation gives us

$$2f\xi_B(f)g_F + 2\alpha S = \mu g. \quad (5.16)$$

The proof is complete if $\xi_B(f) = 0$, in which case the above equation (5.16) produces $S = \frac{\mu g}{2\alpha}$, indicating that the manifold (M, g) is Einstein. $\square$

Now, our attention turns towards examining the impact of conformal vector fields on warped product manifolds that admit conformal Ricci-Yamabe solitons. An immediate result is given below:

Proposition 5.3.5. *Suppose a warped product of two Riemannian manifolds B and F is $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, with the warping function denoted as f . We have $\dim B = m$ and $\dim F = n$. Assuming that (M, g, μ, ξ) forms a*

conformal Ricci-Yamabe soliton. Then, the manifold (M, g) transforms into an Einstein manifold with the factor $(\frac{\mu}{2\alpha} - \frac{\rho}{\alpha})$ if and only if the vector field ξ is conformal with a factor of 2ρ .

Proof. Since (M, g, μ, ξ) being a conformal Ricci-Yamabe soliton, equation (5.1) gives

$$L_\xi g + 2\alpha S = \mu g. \quad (5.17)$$

Assuming that the vector field ξ is conformal with a factor of 2ρ , according to Definition 5.3.1, we have $L_\xi g = 2\rho g$, where ρ is a smooth function. Therefore, by substituting this value into equation (5.17), we obtain

$$S = \left(\frac{\mu}{2\alpha} - \frac{\rho}{\alpha} \right) g. \quad (5.18)$$

As a result, (M, g) is an Einstein manifold from this. The reverse calculation method can also be used to demonstrate that (M, g) is an Einstein manifold with factor $(\frac{\mu}{2\alpha} - \frac{\rho}{\alpha})$. Afterwards, ξ conforms to factor 2ρ .

The evidence is now complete. $\square$

It should be noted that the vector field ξ was assumed to be conformal when discussing conformal Ricci-Yamabe solitons in the aforementioned result. It follows that the question of whether it is necessary to consider ξ to be conformal as a whole or if a weaker condition exists is a natural one. The following theorem may help to clarify the situation.

Theorem 5.3.6. *Suppose a warped product of two Riemannian manifolds B and F is $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, with the warping function denoted as f . We have $\dim B = m$ and $\dim F = n$. Assuming that (M, g, μ, ξ) forms a conformal Ricci-Yamabe soliton, it follows that the lifts ξ_B and ξ_F are both conformal on the base B and the fiber F with the factors $2\rho_B$ and $2\rho_F$ respectively. Here ρ_B and ρ_F are two smooth functions. Under this conditions, if $\rho_B = \rho_F + \xi_B(\ln f)$, then the manifold (M, g) turns out to be Einstein.*

Proof. Given that ξ_B is conformal on the base B with a factor of $2\rho_B$, it follows that $L_{\xi_B}^B g_B = 2\rho_B g_B$. Similarly, for ξ_F being conformal with a factor of $2\rho_F$, we

obtain $L_{\xi_F}^F g_F = 2\rho_F g_F$. Then, using these values in (5.2), we get

$$L_{\xi} g = 2(\rho_B g_B + f^2 \rho_F g_F + f \xi_B(f) g_F). \quad (5.19)$$

Again, (M, g, μ, ξ) being a conformal Ricci-Yamabe soliton, from (5.1) and the above equation (5.19), we have

$$2(\rho_B g_B + f^2 \rho_F g_F + f \xi_B(f) g_F) = \mu g.$$

Now, the above equation gives us

$$S = \frac{1}{\alpha} \left(\frac{\mu g}{2} - \rho_B g_B - f^2(\rho_F + \xi_B(\ln f)) g_F \right). \quad (5.20)$$

Setting $\rho_B = \rho_F + \xi_B(\ln f)$ and using (1.59), the above equation (5.20) yields $S = \left(\frac{\mu}{2\alpha} - \frac{\rho_B}{\alpha} \right) g$. This implies that the manifold (M, g) is Einstein and this concludes the proof. $\square$

Our final theorem, which provides the converse of the prior theorem, brings this section to a close. The conformal Ricci-Yamabe soliton (M, g, μ, ξ) was described in the previous result, and the next result outlines the circumstances in which a warped product manifold (M, g) admits a conformal Ricci-Yamabe soliton.

Theorem 5.3.7. *Assume (B, g_B, μ, ξ_B) be a conformal Ricci-Yamabe soliton, F be an Einstein manifold with factor β . We have $\dim B = m$ and $\dim F = n$. Assume that $(M, g) = ((B \times_f F, g_B \oplus f^2 g_F))$ is a warped product of two Riemannian manifolds B and F with a warping function f and ξ_F is conformal vector field with factor 2ρ , if $H^f = 0$ and the warping function f satisfies the quadratic equation*

$$(2\rho - \mu)f^2 + 2f\xi_B(f) + 2\beta + 2(1 - n)k^2 = 0,$$

where $k^2 = \|\nabla f\|_B^2 = g_B(\nabla f, \nabla f)$ for some real number k , then (M, g, μ, ξ) is a conformal Ricci-Yamabe soliton.

Proof. Let (B, g_B, μ, ξ_B) be a conformal Ricci-Yamabe soliton, from (5.1) we get

$$L_{\xi_B}^B g_B + 2\alpha S^B = \mu g_B. \quad (5.21)$$

Since F is an Einstein manifold with a factor of β , we can express the Ricci tensor as $S^F = \beta g_F$. By substituting this value into equation (5.3), we get

$$S = S^B - \alpha \frac{n}{f} H^f + \beta g_F - \tilde{f} g_F, \quad (5.22)$$

where $\tilde{f} = f \Delta f + (n-1) \|\nabla f\|_B^2$. Using (5.21) in (5.2), we have

$$L_\xi g = \mu g_B - 2\alpha S^B + f^2 L_{\xi_F}^F g_F + 2f \xi_B(f) g_F. \quad (5.23)$$

Multiplying both sides of the equation (5.22) by 2 and then adding it with (5.23) gives

$$L_\xi g + 2\alpha S = \mu g_B + f^2 L_{\xi_F}^F g_F + 2f \xi_B(f) g_F + 2 \left(-\frac{n}{f} H^f + \beta g_F - \tilde{f} g_F \right).$$

Again, considering the vector field ξ_F is conformal with a factor of 2ρ , i.e., $L_{\xi_F}^F g_F = 2\rho g_F$, we get

$$L_\xi g + 2\alpha S = \mu g_B + 2\alpha f^2 \rho g_F + 2\alpha f \xi_B(f) g_F + 2\alpha \left(-\frac{n}{f} H^f + \beta g_F - \tilde{f} g_F \right). \quad (5.24)$$

Given that $H^f = 0$, it follows that $\Delta f = 0$, and consequently, $\tilde{f} = f \Delta f + (n-1) \|\nabla f\|_B^2$ simplifies to $\tilde{f} = (n-1) \|\nabla f\|_B^2 = (n-1)k^2$, where $k^2 = \|\nabla f\|_B^2 = g_B(\nabla f, \nabla f)$ for some real number k . Thus, utilizing these results in equation (5.24), we obtain

$$\begin{aligned} L_\xi g + 2\alpha S &= \mu g_B + 2\alpha f^2 \rho g_F + 2\alpha f \xi_B(f) g_F + 2\alpha(\beta g_F - (n-1)k^2 g_F) \\ &= \mu(g_B + 2f^2 g_F) + \{2\alpha f^2 \rho - \mu f^2 + 2\alpha f \xi_B(f) + 2\alpha(\beta - (n-1)k^2)\} g_F. \end{aligned}$$

Thus, if $2\alpha f^2 \rho - \mu f^2 + 2\alpha f \xi_B(f) + 2\alpha(\beta - (n-1)k^2) = 0$ i.e., if f satisfies the quadratic equation $(2\alpha\rho - \mu)f^2 + 2\alpha f \xi_B(f) + 2\alpha\beta + 2\alpha(1-n)k^2 = 0$, the foregoing equation reduces to

$$L_\xi g + 2\alpha S = \mu(g_B + f^2 g_F) = \mu g. \quad (5.25)$$

Hence, we can establish that (M, g, μ, ξ) indeed forms a conformal Ricci-Yamabe soliton, thereby concluding the proof. $\square$

5.4 Conformal Ricci-Yamabe Soliton on Warped Product Manifolds with Concurrent Vector Field

The concept of concircular vector fields is introduced by Yano (1940) as a means to investigate concircular mappings, which essentially correspond to conformal mappings that uphold the integrity of geodesic circles. These vector fields find a multitude of applications in mathematical physics and general relativity. Chow, *et.al.* (2006) established the theorem that a Lorentzian manifold can be classified as a generalized Robertson-Walker spacetime if and only if it accommodates a timelike concircular vector field. A vector field ξ on a Riemannian manifold M satisfying

$$\nabla_X \xi = \alpha X, \quad (5.26)$$

for all vector fields $X \in \chi(M)$, is called a concircular vector field (Chen, 2015), where α is a non-trivial function on M . In particular, if the function α is constant, then the vector field ξ is called a concurrent vector field. Thus, we have the following definition (Chen, 2015):

Definition 5.4.1. *A vector field ξ defined on a Riemannian manifold M is termed a concurrent vector field if it satisfies the following equation for all vector fields $X \in \chi(M)$:*

$$\nabla_X \xi = X. \quad (5.27)$$

The soliton vector field ξ is a circular (and concurrent) vector field, and we study conformal Ricci-Yamabe solitons based on the definition given above. Our initial result can be stated as follows:

Theorem 5.4.2. *Consider (M, g, μ, ξ) as a conformal Ricci-Yamabe soliton situated on an n -dimensional Riemannian manifold (M, g) , and the soliton vector field ξ possesses a concircular factor of α , then*

1. *the manifold (M, g) is an Einstein manifold with factor $(\frac{\mu}{2\alpha} - 1)$ and*

2. the soliton is expanding if $2\left(1 - \frac{1}{n}\right)\alpha + \beta R + \left(p + \frac{2}{n}\right) < 0$, steady if $2\left(1 - \frac{1}{n}\right)\alpha + \beta R + \left(p + \frac{2}{n}\right) = 0$ or shrinking if $2\left(1 - \frac{1}{n}\right)\alpha + \beta R + \left(p + \frac{2}{n}\right) > 0$.

Proof. Based on our assumption, the soliton vector field ξ is considered to be concircular with a factor of α . Therefore, from equation (5.26), we obtain

$$(L_{\xi}g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = g(\alpha X, Y) + g(X, \alpha Y) = 2\alpha g(X, Y), \quad (5.28)$$

for all vector fields $X, Y \in \chi(M)$. Again, considering (M, g, μ, ξ) as a conformal Ricci-Yamabe soliton, utilize (5.28) in equation (5.1), we get

$$S(X, Y) = \left(\frac{\mu}{2\alpha} - 1\right) g(X, Y), \quad (5.29)$$

for all vector fields $X, Y \in \chi(M)$, and with $\mu = [\lambda - \beta R - (p + \frac{2}{n})]$, equation (5.29) demonstrates that (M, g) is an Einstein manifold with a factor of $(\frac{\mu}{2\alpha} - 1)$. This concludes the initial portion of the theorem.

Furthermore, it is worth noting that for conformal Ricci flow, the scalar curvature $r(g) = -1$. Therefore, taking an orthonormal basis $\{e_i : 1 \leq i \leq n\}$ of the manifold M and summing over $1 \leq i \leq n$ on both sides of equation (5.29) yields

$$-1 = r(g) = n \left(\frac{\mu}{2\alpha} - 1\right).$$

Lastly, by substituting the value $\mu = [2\lambda - \beta R - (p + \frac{2}{n})]$ into the equation above and simplifying, we obtain

$$\lambda = \left(1 - \frac{1}{n}\right)\alpha + \frac{\beta R}{2} + \frac{p}{2} + \frac{1}{n}. \quad (5.30)$$

We know that the soliton is expanding if $\lambda < 0$, steady if $\lambda = 0$ or shrinking if $\lambda > 0$. Therefore, applying this to equation (5.30) concludes the proof. $\square$

We then get a concurrent vector field result that directly follows the aforementioned theorem.

Corollary 5.4.3. *Assume (M, g, μ, ξ) is a conformal Ricci-Yamabe soliton with a concurrent soliton vector field ξ . Under these circumstances*

1. the manifold (M, g) becomes an Einstein with factor $(\frac{\mu}{2} - 1)$ and

2. the soliton is undergoing expansion as $(1 + \frac{p}{n} + \frac{\beta R}{2}) < 0$, steady as $(1 + \frac{p}{n} + \frac{\beta R}{2}) = 0$ or shrinking as $(1 + \frac{p}{n} + \frac{\beta R}{2}) > 0$.

Proof. The proof is finished by proceeding as in Theorem (5.4.2) and then entering $\alpha = 1$ into equations (5.29) and (5.30). $\square$

We conclude this section with the concurrent vector field theorem shown below:

Theorem 5.4.4. *Suppose a warped product of two Riemannian manifolds B and F is $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, with the warping function denoted as f . We have $\dim B = m$ and $\dim F = n$. Assuming that (M, g, μ, ξ) forms a conformal Ricci-Yamabe soliton with concurrent vector field ξ . If a warping function f is constant and both lifts ξ_B and ξ_F on the base B and the fiber F are concurrent, then*

1. the soliton (M, g, μ, ξ) is expanding as $(\frac{p}{2} + \frac{1}{n} + \frac{\beta R}{2} + \alpha) < 0$, steady as $(\frac{p}{2} + \frac{1}{n} + \frac{\beta R}{2} + \alpha) = 0$ or shrinking as $(\frac{p}{2} + \frac{1}{n} + \frac{\beta R}{2} + \alpha) > 0$,
2. all the three manifolds M, B and F are Ricci flat manifolds and
3. all the three manifolds M, B and F admit conformal gradient Ricci-Yamabe solitons.

Proof. Given that (M, g, μ, ξ) is a conformal Ricci-Yamabe soliton on M with a concurrent vector field ξ , the first part of the corollary 5.4.3 can be used to prove this.

$$S(X, Y) = \left(\frac{\mu}{2\alpha} - 1 \right) g(X, Y), \quad (5.31)$$

for all vector fields $X, Y \in \chi(M)$. Suppose we take $X = X_F$ and $Y = Y_F$. Then, by utilizing Lemma 5.1.1 and equation (5.3), we obtain

$$S(X_F, Y_F) = S^F(X_F, Y_F) - \tilde{f} g_F(X_F, Y_F), \quad (5.32)$$

where $\tilde{f} = f \Delta f + (n-1) \|\nabla f\|_B^2$. Now, using equations (5.31) and (1.59) in the above equation (5.32) yields

$$S^F(X_F, Y_F) = \tilde{f} g_F(X_F, Y_F) + \left(\frac{\mu}{2\alpha} - 1 \right) f^2 g_F(X_F, Y_F),$$

where $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$. According to the given condition that f is constant, denoted as $f = c$ for some constant c , it follows that $\tilde{f} = 0$. Consequently, the equation above simplifies to

$$S^F(X_F, Y_F) = c^2 \left(\frac{\mu}{2\alpha} - 1 \right) g_F(X_F, Y_F), \quad (5.33)$$

for all vector fields $X_F, Y_F \in \chi(F)$. Therefore, based on the above equation (5.33), we can conclude that F is Einstein. Given that equation (5.33) holds true for any vector field in $\chi(F)$, by substituting $X_F = Y_F = \xi_F$ into the equation (5.33), we obtain

$$S^F(\xi_F, \xi_F) = c^2 \left(\frac{\mu}{2\alpha} - 1 \right) g_F(\xi_F, \xi_F) = c^2 \left(\frac{\mu}{2\alpha} - 1 \right) \|\xi_F\|_F^2. \quad (5.34)$$

Consider $\xi_F, e_1, e_2, e_3, \dots, e_{n-1}$ as an orthonormal basis of $\chi(F)$. In this basis, the curvature tensor of the manifold F can be expressed as

$$R^F(\xi_F, e_i, \xi_F, e_i) = g_F(R^F(\xi_F, e_i)\xi_F, e_i).$$

Utilizing the formula for the curvature tensor, we can rewrite the above equation as

$$R^F(\xi_F, e_i, \xi_F, e_i) = g_F(\nabla_{\xi_F}^F \nabla_{e_i}^R \xi_F - \nabla_{e_i}^R \nabla_{\xi_F}^F \xi_F - \nabla_{[\xi_F, e_i]}^F \xi_F, e_i). \quad (5.35)$$

Furthermore, given that ξ_F is a concurrent vector field, equation (5.27) yields $\nabla_X \xi_F = X$ for all $X \in \chi(F)$. Utilizing this result in equation (5.35), we obtain

$$R^F(\xi_F, e_i, \xi_F, e_i) = g_F(\nabla_{\xi_F}^F e_i - \nabla_{e_i}^F \xi_F - [\xi_F, e_i], e_i) = 0.$$

This implication leads to $S^F(\xi_F, \xi_F) = 0$, and consequently, from equation (5.34), we obtain $\mu = 2\alpha$, i.e., $\mu = [2\lambda - \beta R - (p + \frac{2}{n})] = 2\alpha$. After simplification, this gives $\lambda = (\frac{p}{2} + \frac{1}{n} + \frac{\beta R}{2} + \alpha)$ and the soliton is shrinking if $\lambda > 0$, steady $\lambda = 0$ or expanding $\lambda < 0$. This demonstrates the theorem's initial proposition.

We now have $S = S^F = 0$ using the value $\mu = 2\alpha$ in equations (5.33) and (5.31). This establishes the Ricci flatness of M and F .

Now, assuming that $X = X_B$ and $Y = Y_B$, we can write from Lemma 5.1.1

$$S(X_B, Y_B) = S^B(X_B, Y_B) - \frac{n}{f} H^f(X_B, Y_B), \quad (5.36)$$

for all $X_B, Y_B \in \chi(B)$, the equation above, given that $S = 0$, simplifies to

$$S^B(X_B, Y_B) = \frac{n}{f} H^f(X_B, Y_B). \quad (5.37)$$

As a result of our presumption that f is constant, $H^f = 0$ implies that the above equation (5.37) yields $S^B(X_B, Y_B) = 0$, for all $X_B, Y_B \in \chi(B)$. As a result, we arrive at $S^B = 0$, which establishes the Ricci flatness of the manifold B . This concludes the second part of the theorem's proof.

Let's suppose that $\phi = \frac{1}{2}g(\xi, \xi)$ in order to prove the final portion of the theorem. Then

$$g(X, \text{grad } \phi) = X(\phi) = g(\nabla_X \xi, \xi), \quad (5.38)$$

for all $X \in \chi(M)$. Again, as ξ being concurrent, using equation (5.27) in equation (5.38), we get

$$g(X, \text{grad } \phi) = g(X, \xi),$$

for all $X \in \chi(M)$. We can conclude that $\xi = \text{grad } \phi$ because the above equation holds true for any vector field $X \in \chi(M)$. As a result, (M, g) admits a Ricci-Yamabe soliton with a conformal gradient.

Taking $\phi_B = \frac{1}{2}g(\xi_B, \xi_B)$ and $\phi_F = \frac{1}{2}g(\xi_F, \xi_F)$ and proceeding in the same manner, we can show that $\xi_B = \text{grad } \phi_B$ and $\phi_F = \text{grad } \phi_F$. Also, from Theorem 5.2.1, we know that since (M, g) is a conformal Ricci-Yamabe soliton, B and F are both conformal Ricci-Yamabe soliton. Therefore, we can say that the conformal gradient Ricci-Yamabe soliton is admissible on both the manifolds B and F . $\square$

5.5 Conclusion

This chapter investigates the properties of conformal Ricci-Yamabe solitons on warped product manifolds, which extend the concept of Riemannian products. We analyze how the base and fiber manifolds interact to admit these solitons, with emphasis on conditions that render the warped product Ricci-flat or Einstein. The study explores the roles of Killing, conformal, and concurrent vector fields in shaping the soliton structure and establishes criteria for both gradient and

concurrent solitons. We provide several theorems linking the soliton parameters to manifold geometry and examines their implications in differential geometry. These findings extend the theoretical framework of geometric flows and warped product manifolds.

Chapter 6

Summary and Conclusions

In this thesis, we extend the concept of Ricci solitons and conduct an analysis of $F(R)$ - gravity. Furthermore, we provide a classification of almost contact metric manifolds admitting specific geometric structures, along with a detailed study of their submanifolds. Additionally, the properties of warped product manifolds are investigated. The research is structured around the following key objectives:

1. Certain generalization of Ricci Soliton.
2. Analysis on $F(R)$ - gravity.
3. Geometry of Submanifolds in almost contact manifold.
4. Properties of warped product manifold.

The first chapter provides a general introduction to the thesis, encompassing a review of classical results, fundamental definitions, and an overview of soliton theory. It also includes an analysis of modified gravity, submanifold theory, warped products and their applications, as well as a review of the literature.

Chapter 2 is the study of generalized $\mathfrak{h}$ -almost Ricci solitons in almost Kenmotsu manifolds, offering significant insights into their geometric properties. By extending the concept of Ricci solitons and analyzing warped product structures, the work demonstrates that such manifolds either conform to Einstein structures or exhibit steady, expanding solitons. Key results include the classification of η -Einstein manifolds, the role of potential vector fields, and the intricate relationship between soliton dynamics and curvature.

Chapter 3 classifies conformally harmonic Z-recurrent spacetimes within $F(R)$ -gravity theory by examining two models: one with a logarithmic form and the other with an exponential functional form. The study demonstrates that both models support the accelerated expansion of the universe, driven by dark energy, as shown by the positive energy density and the negative Equation of State (EoS) parameter, placing them in the quintessence phase. The author also derives key energy conditions, confirming that the Null Energy Condition (NEC), Weak Energy Condition (WEC), and Dominant Energy Condition (DEC) are satisfied, while the Strong Energy Condition (SEC) is violated, further supporting the acceleration. When the scalar curvature is constant, the models align with the Λ CDM model, making them consistent with widely accepted cosmological theories and providing a strong basis for further research in modified gravity and dark energy.

In Chapter 4, we explore the geometry of Ricci-Bourguignon solitons on Euclidean submanifolds, focusing on cases where the soliton field corresponds to the tangential component of the position vector field. It establishes key geometric properties and provides a classification of Ricci-Bourguignon solitons on Euclidean hypersurfaces, demonstrating that such hypersurfaces must either be open parts of hyperplanes or hyperspheres, depending on the soliton parameter. The finding links the existence of these solitons to conditions involving the Ricci tensor, scalar curvature, and second fundamental form, contributing to the broader understanding of solitons in differential geometry. The findings enhance submanifold theory and geometric flow studies by offering a clear framework for identifying soliton structures and their geometric constraints.

Chapter 5 provides a comprehensive exploration of conformal Ricci-Yamabe solitons within the framework of warped product manifolds, extending the classical understanding of Ricci-Yamabe solitons. The authors rigorously examine the conditions under which such solitons exist, particularly focusing on the interaction of Killing and conformal vector fields with the soliton structure. The result presents key results, such as the classification of warped product manifolds admitting these solitons and their transformation into Einstein manifolds under specific conditions involving concurrent vector fields. The study is mathematically robust, offering well-supported theorems and detailed proofs that contribute

to the broader understanding of solitonic structures in geometric flows.

Chapter 6 is a summary and conclusion.

Finally, a list of references is given at the end.

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List of Publications

1. **Robert Sumlalsanga**, Jay Prakash Singh and M. Sundararajan. Generalized $\mathfrak{h}$ -almost Ricci Soliton in Almost Kenmotsu Manifold (Communicated).
2. Khatri, Mohan and Singh, Jay Prakash and **Sumlalsanga, Robert** and Chhakchhuak, Zosangzuala (2025), Classification of conformally harmonic Z-recurrent spacetime in F(R)-gravity, *International Journal of Geometric Methods in Modern Physics*, **22(1)**, 2450257.
3. **Robert Sumlalsanga**, Jay Prakash Singh and M. Sundararajan, Ricci Borguignon Soliton on Euclidean Submanifold (Communicated).
4. Jay Praksah Singh and **Robert Sumlalsanga** (2024), Conformal Ricci-Yamabe Solitons on Warped Product Manifolds, *Filomat*, **38(11)**, 3791-3802.

Conferences/Seminars/Workshops

1. Attended Short Term Course on Research and Publication Ethics organized UGC-Human Resource Development Centre, Mizoram University, Aizawl on 1st-7th October, 2021.
2. Attended the National Workshop On Ancient Indian Mathematics organized by Department of Mathematics and Statistics, Central University of Punjab, Bathinda on 14th-16th March, 2022.
3. Attended the One Week Training Program on Mathematical Modelling and Computing organized by Department of Mathematics and Computer Science, Mizoram University on 6th April,- 2nd May, 2022.
4. Attended the International Faculty Programme on Mathematical Modelling of Biosystem with Special Focus on Epidemiology organized by Department of Mathematics and Computer Science, Mizoram University, Aizawl-796 004, Mizoram on 22th-27th August, 2022.
5. Attended Teacher Enrichment Workshop: Advance Training in Mathematical Science, Department of Mathematics and Computer Science, Mizoram University, Aizawl-796 004, Mizoram on 2nd-17th December, 2022.
6. Attended and presented a paper entitled with 'Classification of Conformally Harmonic Z-Recurrent Spacetime In F(R)-Gravity' in the National Conference on Computational Mathematics organized by Srinivasa Ramanujan Department of Mathematics National Institute of Technology Puducherry, Karaikal-609 609 on 22nd-23rd December, 2022.
7. Attended and presented a paper entitled with 'Characterization of F(R)-gravity admitting conformally harmonic Z-Recurrent spacetime' in the National Conference on Recent Developments in Mathematics and Computer

Science (NCRDMCS23) jointly organized by Mizoram Mathematical Research Association (MMRA) and Department of Mathematics, Govt Champhai College, Mizoram held during 13th- 14th March, 2023.

8. Attended and presented a paper entitled with ‘Warped Product Manifolds with conformal Ricci-Yamabe Solitons’ in the National Seminar on Role of Number Theory and Graph Theory in Mathematical Science jointly organized by Department of Mathematics, Pachhunga University College and Mizoram University, Mizoram on 12th-13th June, 2023.
9. Attended the and presented a paper entitled ‘Conformal Ricci-Yamabe Solitons on Warped Product Manifolds’ in the 2nd International Conference on Recent Trends in Applied Sciences and Computing Engineering organized by VIT Bhopal University in association with National Institute of Technology, Warangal on 7th-9th July, 2023.
10. Attended the Lecture series on Number Theory organized by Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal, Punjab, India on 10th-13th August, 2023.
11. Attended and presented a paper entitled ‘Generalized $\mathfrak{h}$ -almost Ricci Soliton in Almost Kenmotsu Manifold’ in the International Conference on Recent Advances in Mathematical, Physical, and Chemical Sciences on Mathematical modelling and Computing organized by School of Physical Sciences, Mizoram University, Aizawl during 21st February - 23rd February, 2024.

PARTICULARS OF THE CANDIDATE

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DEPARTMENT : MATHEMATICS AND COMPUTER
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TITLE OF THESIS : A STUDY ON CONTACT STRUCTURES,
WARPED PRODUCT AND THEIR
APPLICATIONS

DATE OF ADMISSION : 16.09.2021

APPROVAL OF RESEARCH PROPOSAL:

1. DRC : 13.04.2022
2. BOS : 24.05.2022
3. SCHOOL BOARD : 07.06.2022

MZU REGISTRATION NO. : 2108632

Ph.D. REGISTRATION NO.: MZU/Ph.D./1779 of 16.09.2021

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Conformal Ricci-Yamabe solitons on warped product manifolds

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Abstract. Self-similar solutions of the conformal Ricci-Yamabe flow equation are called conformal Ricci-Yamabe solitons. This paper mainly concerned with the study of conformal Ricci-Yamabe solitons within the structure of warped product manifolds, which extend the notion of usual Riemannian product manifolds. First, the proof is provided that the base and the fiber sharing the same property implies the existence of a warped product manifold admitting a conformal Ricci-Yamabe soliton. In the next section, warped product manifolds are used to study the characterization of conformal Ricci-Yamabe solitons in terms of Killing and conformal vector fields. Finally, we prove that a conformal Ricci-Yamabe soliton with a concurrent potential vector field admitted on a warped product manifold is Ricci flat.

1. Introduction

The study of the theory of Ricci flow (1982) by Hamilton [2, 26, 27] reached its highest magnitude and popularity soon after Perelman [10, 11] successfully applied it to solve the *Poincaré* conjecture. Ricci solitons were also studied by Hamilton, who viewed them as fixed or stationary points of the Ricci flow in the space of parameterized metrics $g(t)$ on scaling and M modulo diffeomorphisms. Since then, both topics have been extensively explored by numerous mathematicians, including Brendle [30], Cao [13], Chen [6] and many others (see for instance [7, 9, 14, 18, 21]).

A smooth manifold M furnished with a Riemannian metric g is known to be a Ricci soliton if, for some constant λ , there exists a smooth vector field X on M satisfying the following equation:

$$\text{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g,$$

where $\mathcal{L}_X$ denotes the Lie derivative and Ric is the Ricci tensor. The Ricci soliton is called shrinking, steady and expanding if $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ respectively.

Recently, self-similar solutions, referred as soliton solutions of various geometric flow equations, have been introduced and explored due to their potential significance as models for singularities. Among these, Ricci soliton defined as fixed points of the Ricci flow stand as one of the most extensively studied and renowned classes. Substantial advancements have been achieved in this particular domain. Furthermore, Hamilton [20] established the Yamabe flow to address the Yamabe problem. This problem primarily

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involves the quest for a metric on a manifold with a dimension of $n \geq 3$, such that its scalar curvature remains constant. Hence, the Yamabe flow is precisely delineated by the metric $g(t)$ on a Riemannian manifold (M^n, g) , which adheres to the condition $\frac{\partial g(t)}{\partial t} = -Rg(t)$, where R denotes the scalar curvature of M . Notably, the two-dimensional scenario had already been resolved through the application of the Uniformization Theorem. For an in-depth exploration of this concept, we direct the reader to references [22], [28], and [29]. It is pertinent to mention that this flow finds its solution through the lens of gradient Yamabe solitons, which are rigorously defined as follows:

A pseudo-Riemannian manifold (M^n, g) is said to be a gradient Yamabe soliton if there exists a smooth function ϕ on M and a constant λ satisfying

$$\text{Hess}(\phi) = (R - \lambda)g,$$

where $\text{Hess}(\phi)$ is the Hessian of ϕ , R is the scalar curvature and g is the metric.

A gradient Yamabe soliton on a pseudo-Riemannian manifold (M^n, g) is classified as shrinking if $\lambda > 0$, expanding if $\lambda < 0$ or steady if $\lambda = 0$. Notably, a seminal outcome pertaining to the resolution of the Yamabe problem is documented in [22], wherein it is demonstrated that the metric of any compact Yamabe gradient soliton is a metric characterized by constant scalar curvature.

In 2019, Güler and Crasmareanu [12] introduced the Ricci-Yamabe flow for a smooth n -dimensional manifold M^n . Later, the notion of the conformal Ricci-Yamabe soliton equation was introduced by Zhang et.al, [23] as

$$\mathcal{L}_X g + 2\alpha \text{Ric} = \left[2\lambda - \beta R - \left(p + \frac{2}{n} \right) \right] g, \quad (1)$$

where α, β, λ are constants and p is the conformal pressure. The equation is the generalization of the Ricci-Yamabe soliton equation and it satisfies the conformal Ricci-Yamabe flow equation (For details, see [1, 16, 17, 19]).

Bishop and O'Neill [24] pioneered the warped product notion, which provides an adaptable foundation for complete manifolds of negative curvature. The origins of this notion may be traced back to the surfaces of revolution. Warped products are important in differential geometry and have applications in mathematical physics and general relativity. This multidisciplinary appeal has piqued the interest of mathematicians and physicists alike, resulting in a thriving field of research [4, 5]. Consider B and F Riemannian manifolds, as well as a positive smooth function f defined on B . The product manifold metric $B \times F$ is defined as follows:

$$g = \pi^* g_B + (f \circ \pi)^2 \sigma^* g_F, \quad (2)$$

where π and σ are the natural projections onto the base and fiber manifold respectively. The product manifold is designated by $M = B \times_f F$ and is said to be the warped product of B and F under this condition. In this context, manifolds B and F are referred to as the base and fiber respectively, where g_B and g_F are the induced metric on base and fiber respectively. The function f is called the warping function.

Our investigation is centered around identifying the conditions that render the warped product a conformal Ricci-Yamabe soliton. To commence, let's begin by revisiting a pivotal result (for detailed information, refer to [25]) that will prove essential for our subsequent discussions.

Lemma 1.1. [25] Let $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$ be a warped product of two Riemannian manifolds B and F with $\dim B = m$ and $\dim F = n$. Then, for all $X, Y \in \mathfrak{X}(B)$ and $U, V \in \mathfrak{X}(F)$

- (i) $D_X U = D_U X = \frac{X(f)}{f} U$,
- (ii) $\text{Ric}(X, U) = 0$,
- (iii) $\text{Ric}(X, Y) = \text{Ric}^B(X, Y) - \frac{n}{f} H^f(X, Y)$,
- (iv) $\text{Ric}(U, V) = \text{Ric}^F(U, V) - \left(\frac{\Delta f}{f} + (n-1) \frac{\|\nabla f\|^2}{f^2} \right) g(U, V)$,

where $D_X Y$ is the lift of $\nabla_X Y$ on B and Ric^B , Ric^F are the lifts of the Ricci tensors on the base B and the fiber F respectively.

Motivated by their research [18], we extend our exploration to the realm of conformal Ricci-Yamabe solitons with $\alpha \neq 0$ on warped product manifolds. Our paper is structured as follows: In Section 2, we delve into the investigations surrounding conformal Ricci-Yamabe solitons on warped product manifolds, examining the impact on both the base and fiber manifolds when a warped product manifold admits a conformal Ricci-Yamabe soliton. Section 3 centers on the influence of specific types of smooth vector fields such as Killing vector fields, conformal vector fields, and concurrent vector fields on conformal Ricci-Yamabe solitons within warped product spaces. Section 4 is devoted to the study of the admittance of conformal Ricci-Yamabe soliton with a concurrent vector field in a warped product manifold.

2. Investigation of Conformal Ricci-Yamabe Soliton on Warped Product Manifolds

This section presents an exploration into the realm of conformal Ricci-Yamabe solitons on warped product manifolds. Our central objective here is to meticulously examine the repercussions of a warped product manifold adopting the characteristics of a conformal Ricci-Yamabe soliton. In particular, we are keen to elucidate how this alignment influences both the base manifold and the fiber manifold, effectively isolating the precise conditions that facilitate their transformation into conformal Ricci-Yamabe solitons. To facilitate our investigation, we consider the manifold $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, representing a warped product of two Riemannian manifolds B and F with $\dim B = m$ and $\dim F = n$ respectively. Proceeding further, let (M, g, μ, ξ) exemplify a conformal Ricci-Yamabe soliton, with $\mu = \left[2\lambda - \beta R - \left(p + \frac{2}{n}\right)\right]$. Consequently, by invoking equation (1), we obtain:

$$\mathcal{L}_\xi g + 2\alpha Ric = \left[2\lambda - \beta R - \left(p + \frac{2}{n}\right)\right] g = \mu g, \quad (3)$$

where $\mu = \left[2\lambda - \beta R - \left(p + \frac{2}{n}\right)\right]$. Recalling Lemma 1.1 from the preceding section, we readily deduce the following two widely recognized formulas tailored for warped product manifolds:

$$\mathcal{L}_\xi g = \mathcal{L}_{\xi_B}^B g_B + f^2 \mathcal{L}_{\xi_F}^F g_F + 2f \xi_B(f) g_F, \quad (4)$$

$$Ric = Ric^B - \frac{n}{f} H^f + Ric^F - \tilde{f} g_F, \quad (5)$$

where $\tilde{f} = f \Delta f + (n-1) \|\nabla f\|_B^2$. Now, applying the definition of warped metric from equation (3) and using (4) and (5), we have

$$\begin{aligned} \mu(g_B + f^2 g_F) &= \mu g = \mathcal{L}_\xi g + 2\alpha Ric \\ &= \mathcal{L}_{\xi_B}^B g_B + f^2 \mathcal{L}_{\xi_F}^F g_F + 2f \xi_B(f) g_F + 2\alpha Ric^B - 2\alpha \frac{n}{f} H^f + 2\alpha Ric^F - 2\alpha \tilde{f} g_F. \end{aligned} \quad (6)$$

Again, for all $U, V \in \mathfrak{X}(B)$, we can write

$$\left(\mathcal{L}_{\xi_B}^B\right)(U, V) = g_B(D_U^B \xi_B, V) + g_B(U, D_V^B \xi_B). \quad (7)$$

By definition of Hessian and (7), we get

$$\left(\mathcal{L}_{\xi_B}^B g_B - 2\alpha \frac{n}{f} H^f\right)(U, V) = g_B(D_U^B \xi_B, V) + g_B(U, D_V^B \xi_B) - 2\alpha \frac{n}{f} g_B(D_U^B \nabla^B f, V).$$

The above equation can be reduced as

$$\begin{aligned} \left(\mathcal{L}_{\xi_B}^B g_B - 2\alpha \frac{n}{f} H^f\right)(U, V) &= g_B(D_U^B \xi_B, V) - \alpha \frac{n}{f} g_B(D_U^B \nabla^B f, V) + (g_B(U, D_V^B \xi_B) - \alpha \frac{n}{f} g_B(D_U^B \nabla^B f, V)) \\ &= g_B(D_U^B (\xi_B - \alpha n \nabla^B \ln f), V) + g_B(U, D_V^B (\xi_B - \alpha n \nabla^B \ln f)) \\ &= (\mathcal{L}_{\xi_B - \alpha n \nabla^B \ln f}^B g_B)(U, V), \forall U, V \in \mathfrak{X}(B). \end{aligned} \quad (8)$$

Since it is true for all $U, V \in \mathfrak{X}(B)$, in operator notation we can write

$$\mathcal{L}_{\xi_B}^B g_B - 2\alpha \frac{n}{f} H^f = \mathcal{L}_{\xi_B - \alpha n \nabla^B \ln f}^B g_B. \quad (9)$$

Putting the value of (9) in (6) gives

$$(\mathcal{L}_{\xi_B - \alpha n \nabla^B \ln f} g_B + 2\alpha \text{Ric}^B) + (f^2 \mathcal{L}_{\xi_F}^F g_F + 2\alpha \text{Ric}^F) = \mu g_B + (\mu f^2 - 2f \xi_B(f) + 2\tilde{f}) g_F. \quad (10)$$

Hence, we can state the following theorem.

Theorem 2.1. Suppose a warped product of two Riemannian manifolds B and F is $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, with the warping function denoted as f . We have $\dim B = m$ and $\dim F = n$. Assuming that (M, g, μ, ξ) forms a conformal Ricci-Yamabe soliton, it follows that both the base $(B, g_B, \mu, \xi_B - \alpha n \nabla^B \ln f)$ and the fiber $(F, g_F, \mu f^2 - 2f \xi_B(f) + 2\tilde{f}, f^2 \xi_F)$ exhibit the characteristics of conformal Ricci-Yamabe solitons. Here, $\tilde{f} = f \Delta f + (n-1)|\nabla f|_B^2$ and $\mu = \left[2\lambda - \beta R - \left(p + \frac{2}{n}\right)\right]$, where λ, β and p represent the soliton constants and the conformal pressure respectively.

Now, let us consider the soliton vector field ξ of the conformal Ricci-Yamabe soliton (M, g, μ, ξ) to be a gradient of some smooth function ϕ , i.e., when $\xi = \text{grad} \phi = \nabla \phi$. The function ϕ is referred to as the potential function of the soliton, and the soliton is known as a conformal gradient Ricci-Yamabe soliton. To maintain clarity in notation, A conformal gradient Ricci-Yamabe soliton is designated as (M, g, μ, ϕ) , where the final term identifies the soliton's potential function.

Let us consider a warped product of two Riemannian manifolds B and F be $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$. We have $\dim B = m$ and $\dim F = n$. Now, if (M, g, μ, ϕ) forms a conformal gradient Ricci-Yamabe soliton, for any vector fields $X, Y \in \mathfrak{X}(M)$, then (3) provides

$$2H^\phi(X, Y) + 2\alpha \text{Ric}(X, Y) = \left[2\lambda - \beta R - \left(p + \frac{2}{n}\right)\right] g(X, Y) = \mu g(X, Y). \quad (11)$$

Again, we consider $X = X_B$ and $Y = Y_B$, with X_B and Y_B representing the lifts of the vector fields X and Y in $\mathfrak{X}(B)$, then (11) reduces to

$$2H^\phi(X_B, Y_B) + 2\alpha \text{Ric}(X_B, Y_B) = \mu g(X_B, Y_B).$$

By substituting the value of the Ricci tensor for the base manifold as provided in Lemma 1.1, the equation above can be expressed as

$$2H_B^{\phi_B}(X_B, Y_B) + 2\alpha \text{Ric}^B(X_B, Y_B) - 2\alpha \frac{n}{f} H_B^f(X_B, Y_B) = \mu g_B(X_B, Y_B),$$

where $\phi_B = \phi|_B$ at a specific point of the fiber F . Finally, by employing the properties of the Hessian in the equation above, we obtain

$$2H_B^{\phi_B - \alpha n \ln f}(X_B, Y_B) + 2\alpha \text{Ric}^B(X_B, Y_B) = \mu g_B(X_B, Y_B). \quad (12)$$

This demonstrates that $(B, g_B, \mu, \phi_B - \alpha n \ln f)$ qualifies as a conformal gradient Ricci-Yamabe soliton.

Furthermore, if we set $(X = X_F)$ and $(Y = Y_F)$, with X_F and Y_F representing the lifts of the vector fields X and Y in $\mathfrak{X}(F)$, then (11) yields

$$2H^\phi(X_F, Y_F) + 2\alpha \text{Ric}(X_F, Y_F) = \mu g(X_F, Y_F).$$

Using (5) and Lemma 1.1, the above equation becomes

$$2H_F^{\phi_F}(X_F, Y_F) + 2\alpha \text{Ric}^F(X_F, Y_F) - \tilde{f} g_F(X_F, Y_F) = \mu f^2 g_F(X_F, Y_F),$$

where $\phi_F = \phi$ at a fixed point of the base B and $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$. From the above equation, we get

$$2H_F^{\phi_F}(X_F, Y_F) + 2\alpha \text{Ric}^F(X_F, Y_F) = (\mu f^2 + \tilde{f})g_F(X_F, Y_F).$$

Hence, given the constancy of the warping function f , the term $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$ disappears from the right-hand side of the above equation, we get

$$2H_F^{\phi_F}(X_F, Y_F) + 2\alpha \text{Ric}^F(X_F, Y_F) = \mu f^2 g_F(X_F, Y_F). \quad (13)$$

Thus, $(F, g_F, \mu f^2, \phi_F)$ is a conformal gradient Ricci-Yamabe soliton. Hence, from the above observations and (12) and (13), we can state the following:

Theorem 2.2. Suppose a warped product of two Riemannian manifolds B and F is $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, with the warping function denoted as f . We have $\dim B = m$ and $\dim F = n$. Assuming that (M, g, μ, ξ) forms a conformal Ricci-Yamabe soliton, then

- (i) the base $(B, g_B, \mu, \phi_B - n\ln f)$ forms a conformal gradient Ricci-Yamabe soliton with $\phi_B = \phi$ at a point within the fiber F .
- (ii) the fiber $(F, g_F, \mu f^2, \phi_f)$ forms a conformal gradient Ricci-Yamabe soliton with $\phi_F = \phi$ at a point within the base B , given that the warping function f is constant.

Remark 2.3. For particular values of $\alpha = 1, \beta = 0$, the above theorem reduces to Theorem 2.2 of [18].

3. Influence of Distinctive Vector Field Types on Conformal Ricci-Yamabe Solitons in Warped Product Manifolds

This section's major goal is to investigate how particular forms of smooth vector fields affect conformal Ricci-Yamabe solitons in warped product spaces. We shall pay special attention to the conformal and killing vector fields, which are defined as follows:

Definition 3.1. A smooth vector field X on a Riemannian manifold (M, g) is called

- (i) Killing vector field or an infinitesimal isometry, if the local 1-parameter group of transformations generated by X in a neighbourhood of each point of M consists of local isometries, or in other words, if X satisfies $\mathcal{L}_X g = 0$ and
- (ii) conformal vector field if X satisfies $\mathcal{L}_X g = \rho g$,

Here, ρ represents a smooth function defined on the manifold M and $\mathcal{L}_X g$ refers the Lie derivative of the Riemannian metric g along the vector field X .

As per the definition outlined above, our initial result in this section is presented below:

Proposition 3.2. Let $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$ represent a warped product of two Riemannian manifolds B and F denoted as a warping function f . Here, $\dim B = m$ and $\dim F = n$. Assuming that (M, g, μ, ξ) forms a conformal Ricci-Yamabe soliton, and under the condition that any one of the following conditions holds

- (i) $\xi = \xi_B$ and ξ_B is a Killing vector field on the base B .
- (ii) $\xi = \xi_F$ and ξ_F is a Killing vector field on the fiber F .

Then, under any one of the above conditions, the manifold (M, g) transforms into an Einstein manifold.

Proof. Based on our assumption that (M, g, μ, ξ) constitutes a conformal Ricci-Yamabe soliton, it complies with equation (3) gives

$$\mathcal{L}_\xi g + 2\alpha \text{Ric} = \mu g. \quad (14)$$

Now, let $\xi = \xi_B$, and ξ_B is a Killing on B , we get $\mathcal{L}_{\xi_B}^B g_B = 0$. Then, (4) gives $\mathcal{L}_\xi g = 0$. Therefore, (14) yields $\text{Ric} = \frac{\mu}{2\alpha} g$ and this implies (M, g) is an Einstein manifold.

Furthermore, if we consider $\xi = \xi_F$ and note that ξ_F is a Killing vector field on F , we find that $\mathcal{L}_{\xi_F}^F g_F = 0$. Proceeding similarly to the previous part of the proof, we get the same result. $\square$

Theorem 3.3. Suppose a warped product of two Riemannian manifolds B and F is $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, with the warping function denoted as f . We have $\dim B = m$ and $\dim F = n$. Assuming that (M, g, μ, ξ) forms a conformal Ricci-Yamabe soliton, it follows that ξ_B is a Killing vector field on the base B , then the base $(B, g_B, \mu, -n\alpha \ln f)$ forms a conformal Ricci-Yamabe soliton, where ξ_B is the lift of the vector field ξ to $\mathfrak{X}(B)$.

Proof. Considering the given information that (M, g, μ, ξ) is a conformal Ricci-Yamabe soliton as per Theorem 2.1, we can conclude that the base $(B, g_B, \mu, \xi_B - n\alpha \nabla^B \ln f)$ also qualifies as a conformal Ricci-Yamabe soliton and hence it satisfies (3). Thus, we can write

$$\mathcal{L}_{\xi_B - n\alpha \nabla^B \ln f}^B g_B + 2\alpha \text{Ric}^B = \mu g_B. \quad (15)$$

Again, using (9) in the above equation, (15) becomes

$$\mathcal{L}_{\xi_B}^B g_B - 2\alpha \frac{n}{f} H^f + 2\alpha \text{Ric}^B = \mu g_B.$$

Given that ξ_B is a Killing vector field on the base B , we can assert that $\mathcal{L}_{\xi_B}^B g_B = 0$. Then, the above equation gives us

$$-2\alpha \frac{n}{f} H^f + 2\alpha \text{Ric}^B = \mu g_B.$$

Thus, using the properties of Hessian, the above equation results in

$$2H^{-\alpha \ln f} + 2\alpha \text{Ric}^B = \mu g_B. \quad (16)$$

Hence, comparing (16) with (11) completes the proof. $\square$

The following conclusion is reached from the study of Killing vector fields on conformal Ricci-Yamabe soliton warped product manifolds:

Theorem 3.4. Suppose a warped product of two Riemannian manifolds B and F is $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, with the warping function denoted as f . We have $\dim B = m$ and $\dim F = n$. Assuming that (M, g, μ, ξ) forms a conformal Ricci-Yamabe soliton and the lifts ξ_B and ξ_F are both Killing on the base B and the fiber F respectively. Then, the manifold (M, g) is Einstein provided $\xi_B(f) = 0$.

Proof. As provided, both ξ_B and ξ_F are Killing vector fields, implying $\mathcal{L}_{\xi_B}^B g_B = 0$ and $\mathcal{L}_{\xi_F}^F g_F = 0$. Utilizing these values in equation (4), we get

$$\mathcal{L}_{\xi} g = 2f \xi_B(f) g_F. \quad (17)$$

Again, assuming that (3) is true and that (M, g, μ, ξ) is a conformal Ricci-Yamabe soliton, we obtain

$$\mathcal{L}_{\xi} g + 2\alpha \text{Ric} = \mu g.$$

Now, using (17) in the above equation gives us

$$2f \xi_B(f) g_F + 2\alpha \text{Ric} = \mu g. \quad (18)$$

The proof is complete if $\xi_B(f) = 0$, in which case the above equation (18) produces $\text{Ric} = \frac{\mu g}{2\alpha}$, indicating that the manifold (M, g) is Einstein. $\square$

Now, our attention turns towards examining the impact of conformal vector fields on warped product manifolds that admit conformal Ricci-Yamabe solitons. An immediate result is given below:

Proposition 3.5. Suppose a warped product of two Riemannian manifolds B and F is $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, with the warping function denoted as f . We have $\dim B = m$ and $\dim F = n$. Assuming that (M, g, μ, ξ) forms a conformal Ricci-Yamabe soliton. Then, the manifold (M, g) transforms into an Einstein manifold with the factor $\left(\frac{\mu}{2\alpha} - \frac{\rho}{\alpha}\right)$ if and only if the vector field ξ is conformal with a factor of 2ρ .

Proof. Since (M, g, μ, ξ) being a conformal Ricci-Yamabe soliton, equation (3) gives

$$\mathcal{L}_\xi g + 2\alpha \text{Ric} = \mu g. \quad (19)$$

Assuming that the vector field ξ is conformal with a factor of 2ρ , according to Definition 3.1, we have $\mathcal{L}_\xi g = 2\rho g$, where ρ is a smooth function. Therefore, by substituting this value into equation (19), we obtain

$$\text{Ric} = \left(\frac{\mu}{2\alpha} - \frac{\rho}{\alpha}\right) g. \quad (20)$$

As a result, (M, g) is an Einstein manifold from this. The reverse calculation method can also be used to demonstrate that (M, g) is an Einstein manifold with factor $\left(\frac{\mu}{2\alpha} - \frac{\rho}{\alpha}\right)$. Afterwards, ξ conforms to factor 2ρ . The evidence is now complete. $\square$

It should be noted that the vector field ξ was assumed to be conformal when discussing conformal Ricci-Yamabe solitons in the aforementioned result. It follows that the question of whether it is necessary to consider ξ to be conformal as a whole or if a weaker condition exists is a natural one. The following theorem may help to clarify the situation.

Theorem 3.6. Suppose a warped product of two Riemannian manifolds B and F is $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, with the warping function denoted as f . We have $\dim B = m$ and $\dim F = n$. Assuming that (M, g, μ, ξ) forms a conformal Ricci-Yamabe soliton, it follows that the lifts ξ_B and ξ_F are both conformal on the base B and the fiber F with the factors $2\rho_B$ and $2\rho_F$ respectively. Here ρ_B and ρ_F are two smooth functions. Under this conditions, if $\rho_B = \rho_F + \xi_B(\ln f)$, then the manifold (M, g) turns out to be Einstein.

Proof. Given that ξ_B is conformal on the base B with a factor of $2\rho_B$, it follows that $\mathcal{L}_{\xi_B}^B g_B = 2\rho_B g_B$. Similarly, for ξ_F being conformal with a factor of $2\rho_F$, we obtain $\mathcal{L}_{\xi_F}^F g_F = 2\rho_F g_F$. Then, using these values in (4), we get

$$\mathcal{L}_\xi g = 2(\rho_B g_B + f^2 \rho_F g_F + f \xi_B(f) g_F). \quad (21)$$

Again, (M, g, μ, ξ) being a conformal Ricci-Yamabe soliton, from (3) and the above equation (21), we have

$$2(\rho_B g_B + f^2 \rho_F g_F + f \xi_B(f) g_F) = \mu g.$$

Now, the above equation gives us

$$\text{Ric} = \frac{1}{\alpha} \left(\frac{\mu g}{2} - \rho_B g_B - f^2 (\rho_F + \xi_B(\ln f)) g_F \right). \quad (22)$$

Setting $\rho_B = \rho_F + \xi_B(\ln f)$ and using (2), the above equation (22) yields $\text{Ric} = \left(\frac{\mu}{2\alpha} - \frac{\rho_B}{\alpha}\right) g$. This implies that the manifold (M, g) is Einstein and this concludes the proof. $\square$

Our final theorem, which provides the converse of the prior theorem, brings this section to a close. The conformal Ricci-Yamabe soliton (M, g, μ, ξ) was described in the previous result, and the next result outlines the circumstances in which a warped product manifold (M, g) admits a conformal Ricci-Yamabe soliton.

Theorem 3.7. Assume (B, g_B, μ, ξ_B) be a conformal Ricci-Yamabe soliton, F be an Einstein manifold with factor β . We have $\dim B = m$ and $\dim F = n$. Assume that $(M, g) = ((B \times_f F, g_B \oplus f^2 g_F))$ is a warped product of two Riemannian manifolds B and F with a warping function f and ξ_F is conformal vector field with factor 2ρ , if $H^f = 0$ and the warping function f satisfies the quadratic equation

$$(2\rho - \mu)f^2 + 2f\xi_B(f) + 2\beta + 2(1 - n)k^2 = 0,$$

where $k^2 = \|\nabla f\|_B^2 = g_B(\nabla f, \nabla f)$ for some real number k , then (M, g, μ, ξ) is a conformal Ricci-Yamabe soliton.

Proof. Let (B, g_B, μ, ξ_B) be a conformal Ricci-Yamabe soliton, from (3) we get

$$\mathcal{L}_{\xi_B}^B g_B + 2\alpha \text{Ric}^B = \mu g_B. \quad (23)$$

Since F is an Einstein manifold with a factor of β , we can express the Ricci tensor as $\text{Ric}^F = \beta g_F$. By substituting this value into equation (5), we get

$$\text{Ric} = \text{Ric}^B - \alpha \frac{n}{f} H^f + \beta g_F - \tilde{f} g_F, \quad (24)$$

where $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$. Using (23) in (4), we have

$$\mathcal{L}_{\xi} g = \mu g_B - 2\alpha \text{Ric}^B + f^2 \mathcal{L}_{\xi_F}^F g_F + 2f\xi_B(f)g_F. \quad (25)$$

Multiplying both sides of the equation (24) by 2 and then adding it with (25) gives

$$\mathcal{L}_{\xi} g + 2\alpha \text{Ric} = \mu g_B + f^2 \mathcal{L}_{\xi_F}^F g_F + 2f\xi_B(f)g_F + 2\left(-\frac{n}{f}H^f + \beta g_F - \tilde{f}g_F\right).$$

Again, considering the vector field ξ_F is conformal with a factor of 2ρ , i.e., $\mathcal{L}_{\xi_F}^F g_F = 2\rho g_F$, we get

$$\mathcal{L}_{\xi} g + 2\alpha \text{Ric} = \mu g_B + 2\alpha f^2 \rho g_F + 2\alpha f\xi_B(f)g_F + 2\alpha\left(-\frac{n}{f}H^f + \beta g_F - \tilde{f}g_F\right). \quad (26)$$

Given that $H^f = 0$, it follows that $\Delta f = 0$, and consequently, $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$ simplifies to $\tilde{f} = (n-1)\|\nabla f\|_B^2 = (n-1)k^2$, where $k^2 = \|\nabla f\|_B^2 = g_B(\nabla f, \nabla f)$ for some real number k . Thus, utilizing these results in equation (26), we obtain

$$\begin{aligned} \mathcal{L}_{\xi} g + 2\alpha \text{Ric} &= \mu g_B + 2\alpha f^2 \rho g_F + 2\alpha f\xi_B(f)g_F + 2\alpha(\beta g_F - (n-1)k^2 g_F) \\ &= \mu(g_B + f^2 g_F) + \{2\alpha f^2 \rho - \mu f^2 + 2\alpha f\xi_B(f) + 2\alpha(\beta - (n-1)k^2)\}g_F. \end{aligned}$$

Thus, if $2\alpha f^2 \rho - \mu f^2 + 2\alpha f\xi_B(f) + 2\alpha(\beta - (n-1)k^2) = 0$ i.e., if f satisfies the quadratic equation $(2\alpha\rho - \mu)f^2 + 2\alpha f\xi_B(f) + 2\alpha\beta + 2\alpha(1-n)k^2 = 0$, the foregoing equation reduces to

$$\mathcal{L}_{\xi} g + 2\alpha \text{Ric} = \mu(g_B + f^2 g_F) = \mu g. \quad (27)$$

Hence, we can establish that (M, g, μ, ξ) indeed forms a conformal Ricci-Yamabe soliton, thereby concluding the proof. $\square$

4. Conformal Ricci-Yamabe Soliton on Warped Product Manifolds with Concurrent Vector Field

The concept of concircular vector fields is introduced by Yano [15] as a means to investigate concircular mappings, which essentially correspond to conformal mappings that uphold the integrity of geodesic circles. These vector fields find a multitude of applications in mathematical physics and general relativity. Chen [3] established the theorem that a Lorentzian manifold can be classified as a generalized Robertson-Walker spacetime if and only if it accommodates a timelike concircular vector field. A vector field ξ on a Riemannian manifold M satisfying

$$\nabla_X \xi = \alpha X, \quad (28)$$

for all vector fields $X \in \mathfrak{X}(M)$, is called a concircular vector field [6], where α is a non-trivial function on M . In particular, if the function α is constant, then the vector field ξ is called a concurrent vector field. Thus, we have the following definition [6]:

Definition 4.1. A vector field ξ defined on a Riemannian manifold M is termed a concurrent vector field if it satisfies the following equation for all vector fields $X \in \mathfrak{X}(M)$:

$$\nabla_X \xi = X. \quad (29)$$

The soliton vector field ξ is a circular (and concurrent) vector field, and we study conformal Ricci-Yamabe solitons based on the definition given above. Our initial result can be stated as follows:

Theorem 4.2. Consider (M, g, μ, ξ) as a conformal Ricci-Yamabe soliton situated on an n -dimensional Riemannian manifold (M, g) , and the soliton vector field ξ possesses a concircular factor of α , then

- (i) the manifold (M, g) is an Einstein manifold with factor $\left(\frac{\mu}{2\alpha} - 1\right)$ and
- (ii) the soliton is expanding if $2\left(1 - \frac{1}{n}\right)\alpha + \beta R + \left(p + \frac{2}{n}\right) < 0$, steady if $2\left(1 - \frac{1}{n}\right)\alpha + \beta R + \left(p + \frac{2}{n}\right) = 0$ or shrinking if $2\left(1 - \frac{1}{n}\right)\alpha + \beta R + \left(p + \frac{2}{n}\right) > 0$.

Proof. Based on our assumption, the soliton vector field ξ is considered to be concircular with a factor of α . Therefore, from equation (28), we obtain

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = g(\alpha X, Y) + g(X, \alpha Y) = 2\alpha g(X, Y), \quad (30)$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Again, considering (M, g, μ, ξ) as a conformal Ricci-Yamabe soliton, utilize (30) in equation (3), we get

$$\text{Ric}(X, Y) = \left(\frac{\mu}{2\alpha} - 1\right)g(X, Y), \quad (31)$$

for all vector fields $X, Y \in \mathfrak{X}(M)$, and with $\mu = \left[2\lambda - \beta R - \left(p + \frac{2}{n}\right)\right]$, equation (31) demonstrates that (M, g) is an Einstein manifold with a factor of $\left(\frac{\mu}{2\alpha} - 1\right)$. This concludes the initial portion of the theorem.

Furthermore, it is worth noting that for conformal Ricci flow, the scalar curvature $r(g) = -1$. Therefore, taking an orthonormal basis $\{e_i : 1 \leq i \leq n\}$ of the manifold M and summing over $1 \leq i \leq n$ on both sides of equation (31) yields

$$-1 = r(g) = n\left(\frac{\mu}{2\alpha} - 1\right).$$

Lastly, by substituting the value $\mu = \left[2\lambda - \beta R - \left(p + \frac{2}{n}\right)\right]$ into the equation above and simplifying, we obtain

$$\lambda = \left(1 - \frac{1}{n}\right)\alpha + \frac{\beta R}{2} + \frac{p}{2} + \frac{1}{n}. \quad (32)$$

We know that the soliton is expanding if $\lambda < 0$, steady if $\lambda = 0$ or shrinking if $\lambda > 0$. Therefore, applying this to equation (32) concludes the proof. $\square$

We then get a concurrent vector field result that directly follows the aforementioned theorem.

Corollary 4.3. Assume (M, g, μ, ξ) is a conformal Ricci-Yamabe soliton with a concurrent soliton vector field ξ . Under these circumstances

- (i) the manifold (M, g) becomes an Einstein with factor $\left(\frac{\mu}{2} - 1\right)$ and
- (ii) the soliton is undergoing expansion as $\left(1 + \frac{p}{n} + \frac{\beta R}{2}\right) < 0$, steady as $\left(1 + \frac{p}{n} + \frac{\beta R}{2}\right) = 0$ or shrinking as $\left(1 + \frac{p}{n} + \frac{\beta R}{2}\right) > 0$.

Proof. The proof is finished by proceeding as in Theorem 4.1 and then entering $\alpha = 1$ into equations (31) and (32). $\square$

We conclude this section with the concurrent vector field theorem shown below:

Theorem 4.4. Suppose a warped product of two Riemannian manifolds B and F is $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, with the warping function denoted as f . We have $\dim B = m$ and $\dim F = n$. Assuming that (M, g, μ, ξ) forms a conformal Ricci-Yamabe soliton with concurrent vector field ξ . If a warping function f is constant and both lifts ξ_B and ξ_F on the base B and the fiber F are concurrent, then

- (i) the soliton (M, g, μ, ξ) is expanding as $\left(\frac{p}{2} + \frac{1}{n} + \frac{\beta R}{2} + \alpha\right) < 0$, steady as $\left(\frac{p}{2} + \frac{1}{n} + \frac{\beta R}{2} + \alpha\right) = 0$ or shrinking as $\left(\frac{p}{2} + \frac{1}{n} + \frac{\beta R}{2} + \alpha\right) > 0$,
- (ii) all the three manifolds M, B and F are Ricci flat manifolds and
- (iii) all the three manifolds M, B and F admit conformal gradient Ricci-Yamabe solitons.

Proof. Given that (M, g, μ, ξ) is a conformal Ricci-Yamabe soliton on M with a concurrent vector field ξ , the first part of the corollary 4.3 can be used to prove this.

$$\text{Ric}(X, Y) = \left(\frac{\mu}{2\alpha} - 1\right)g(X, Y), \quad (33)$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Suppose we take $X = X_F$ and $Y = Y_F$. Then, by utilizing Lemma 1.1 and equation (5), we obtain

$$\text{Ric}(X_F, Y_F) = \text{Ric}^F(X_F, Y_F) - \tilde{f}g_F(X_F, Y_F), \quad (34)$$

where $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$. Now, using equations (33) and (2) in the above equation (34) yields

$$\text{Ric}^F(X_F, Y_F) = \tilde{f}g_F(X_F, Y_F) + \left(\frac{\mu}{2\alpha} - 1\right)f^2g_F(X_F, Y_F),$$

where $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$. According to the given condition that f is constant, denoted as $f = c$ for some constant c , it follows that $\tilde{f} = 0$. Consequently, the equation above simplifies to

$$\text{Ric}^F(X_F, Y_F) = c^2 \left(\frac{\mu}{2\alpha} - 1\right)g_F(X_F, Y_F), \quad (35)$$

for all vector fields $X_F, Y_F \in \mathfrak{X}(F)$. Therefore, based on the above equation (35), we can conclude that F is Einstein. Given that equation (35) holds true for any vector field in $\mathfrak{X}(F)$, by substituting $X_F = Y_F = \xi_F$ into the equation (35), we obtain

$$\text{Ric}^F(\xi_F, \xi_F) = c^2 \left(\frac{\mu}{2\alpha} - 1\right)g_F(\xi_F, \xi_F) = c^2 \left(\frac{\mu}{2\alpha} - 1\right)\|\xi_F\|_F^2. \quad (36)$$

Consider $\xi_F, e_1, e_2, e_3, \dots, e_{n-1}$ as an orthonormal basis of $\mathfrak{X}(F)$. In this basis, the curvature tensor of the manifold F can be expressed as

$$R^F(\xi_F, e_i, \xi_F, e_i) = g_F(R^F(\xi_F, e_i)\xi_F, e_i).$$

Utilizing the formula for the curvature tensor, we can rewrite the above equation as

$$R^F(\xi_F, e_i, \xi_F, e_i) = g_F(\nabla_{\xi_F}^F \nabla_{e_i}^R \xi_F - \nabla_{e_i}^R \nabla_{\xi_F}^F \xi_F - \nabla_{[\xi_F, e_i]}^F \xi_F, e_i). \quad (37)$$

Furthermore, given that ξ_F is a concurrent vector field, equation (29) yields $\nabla_X \xi_F = X$ for all $X \in \mathfrak{X}(F)$. Utilizing this result in equation (37), we obtain

$$R^F(\xi_F, e_i, \xi_F, e_i) = g_F(\nabla_{\xi_F}^F e_i - \nabla_{e_i}^F \xi_F - [\xi_F, e_i], e_i) = 0.$$

This implication leads to $\text{Ric}^F(\xi_F, \xi_F) = 0$, and consequently, from equation (36), we obtain $\mu = 2\alpha$, i.e., $\mu = \left[2\lambda - \beta R - \left(p + \frac{2}{n}\right)\right] = 2\alpha$. After simplification, this gives $\lambda = \left(\frac{p}{2} + \frac{1}{n} + \frac{\beta R}{2} + \alpha\right)$ and the soliton is

shrinking if $\lambda > 0$, steady $\lambda = 0$ or expanding $\lambda < 0$. This demonstrates the theorem's initial proposition.

We now have $Ric = Ric^F = 0$ using the value $\mu = 2\alpha$ in equations (35) and (33). This establishes the Ricci flatness of M and F .

Now, assuming that $X = X_B$ and $Y = Y_B$, we can write from Lemma 1.1

$$Ric(X_B, Y_B) = Ric^B(X_B, Y_B) - \frac{n}{f} H^f(X_B, Y_B), \quad (38)$$

for all $X_B, Y_B \in \mathfrak{X}(B)$, the equation above, given that $Ric = 0$, simplifies to

$$Ric^B(X_B, Y_B) = \frac{n}{f} H^f(X_B, Y_B). \quad (39)$$

As a result of our presumption that f is constant, $H^f = 0$ implies that the above equation (39) yields $Ric^B(X_B, Y_B) = 0$, for all $X_B, Y_B \in \mathfrak{X}(B)$. As a result, we arrive at $Ric^B = 0$, which establishes the Ricci flatness of the manifold B . This concludes the second part of the theorem's proof.

Let's suppose that $\phi = \frac{1}{2}g(\xi, \xi)$ in order to prove the final portion of the theorem. Then

$$g(X, \text{grad}\phi) = X(\phi) = g(\nabla_X \xi, \xi), \quad (40)$$

for all $X \in \mathfrak{X}(M)$. Again, as ξ being concurrent, using equation (29) in equation (40), we get

$$g(X, \text{grad}\phi) = g(X, \xi),$$

for all $X \in \mathfrak{X}(M)$. We can conclude that $\xi = \text{grad}\phi$ because the above equation holds true for any vector field $X \in \mathfrak{X}(M)$. As a result, (M, g) admits a Ricci-Yamabe soliton with a conformal gradient.

Taking $\phi_B = \frac{1}{2}g(\xi_B, \xi_B)$ and $\phi_F = \frac{1}{2}g(\xi_F, \xi_F)$ and proceeding in the same manner, we can show that $\xi_B = \text{grad}\phi_B$ and $\phi_F = \text{grad}\phi_F$. Also, from Theorem 2.1, we know that since (M, g) is a conformal Ricci-Yamabe soliton, B and F are both conformal Ricci-Yamabe soliton. Therefore, we can say that the conformal gradient Ricci-Yamabe soliton is admissible on both the manifolds B and F . $\square$

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Classification of conformally harmonic Z -recurrent spacetime in $F(R)$ -gravity

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In this paper, we give some classification of conformally harmonic Z -recurrent spacetime in the context of $F(R)$ -gravity theory based on its scalar curvature and obtained the expression for energy density and isotopic pressure. Their behaviors along with EoS parameter are analyzed by considering two $F(R)$ -gravity models. Moreover, the energy conditions for the considered models are studied and found to support the accelerated expansion of the universe.

Keywords: Z -recurrent; $F(R)$ -gravity; GRW spacetime; perfect fluid spacetime.

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1. Introduction

In 1915, Albert Einstein introduced the geometric properties of gravity as a symmetric curvature of spacetime, known as general theory of relativity (GR). Spacetime geometry is specially related with the symmetry of physical matter. The spacetime symmetries are used specifically for the study of exact solutions of Einstein's field

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equations of general relativity. The universe includes all its contents such as stars, planets, galaxies and the other forms of matter are space, time and energy. The general theory of relativity (GR) is a theoretical method for studying the universe in a structure of large-scale effectively. The acceleration of the universe in the past and present cannot be described without taking into account the dark energy in the GR. Since it does not explain the gravity, the modification, in order to obtain theories that admit inflation and intimate the Dark Energy (DE), is quite reasonable. The modification to Einstein's gravitational field equations [22, 23] has provided the conventional method to analyze known cosmic dynamics. The greatest approximation to the observable data was given by Einstein's field equations, with the inclusion of a hypothetical element of the universe which was described as Dark Matter [24].

Dark Energy (DE) is a strange component of the universe, which is supposed to be the main cause of the acceleration of the universe, extension and regulates the matter-energy ratio. These results in construction of more advanced gravity theories from both the mathematicians and physicists, which grows as a result of the Einstein–Hilbert action and the use of modified gravity theories such as $F(R)$ -gravity [25], Gauss–Bonnet $F(G)$ -gravity [26] and $F(T)$ theory [27]. These theories vary from Einstein's conventional gravity theory in which they might also have a closer approximation to quantum gravity [28].

Einstein–Hilbert Lagrangian density to the function of $F(R)$, where R is the Ricci scalar, extended the general theory of relativity (GR) to the $F(R)$ -gravity. Huge neutron stars in the equations of motion of $F(R)$ -gravity issues can be solved by the higher order curvature [29–31].

Even though experts try and decide to modify the theory of gravity for over the decades, it has not been an easy task. Theory of $F(R)$ -gravity came straightforward as a generalization by the action of the Lagrangian in the Einstein–Hilbert

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R, \quad (1.1)$$

where $\kappa^2 \equiv 8\pi G$, G is the gravitational constant, g is the determinant of the metric and R is the Ricci scalar to become a general function of R , i.e.

$$S = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} F(R) + \int L_m \sqrt{-g} d^4x. \quad (1.2)$$

This produces the field equations of $f(R)$ -gravity [32] as follows:

$$F_R(R)R_{ij} - \frac{1}{2}F(R)g_{ij} + (g_{ij}\square - \nabla_i\nabla_j)F_R(R) = \kappa T_{ij}, \quad (1.3)$$

where $\square = \nabla^j\nabla_j$. The stress-energy tensor is calculated from the matter Lagrangian L_m by $T_{ij} = -\frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}L_m}{\delta g^{ij}}$.

The accelerated expansion of the universe in the model $F(R) = R + \alpha R^2$ ($\alpha > 0$) is due to the presence of the αR^2 term. Starobinsky in 1980 [33] proposed the first model of inflation. The above model is well conformable with the temperature

anisotropies observed in CMB, thus it could be an applicable alternative to the scalar field models of inflation.

After the discovery of dark energy in 1998, the modification of gravity to GR led to an alternate way of explaining the cosmic acceleration. $F(R)$ theory-based dark energy models have been studied to a great extent as the simplest modified gravity outlined to realize the late-time acceleration. Lagrangian density-based model was proposed for dark energy in the metric formalism [34, 35]. Although it was shown that this model aggravates by a matter of instability [36] as well as by the complicatedness to satisfy local gravity constraints [37–39]. The model does not possess a standard matter-dominated epoch due to a large-scale coupling between dark energy and dark matter [40], which shows how nontrivial it is to get a viable $F(R)$ model. Amendola *et al.* [41] derived the cosmological viability of $F(R)$ dark energy model conditions.

Corrections to GR were contemplated to be important for a few extent, truly, there are relevant studies: De and Loo [45] showed that a Robertson–Walker spacetime is $(\text{APRS})_4$ and vice versa under certain condition imposed on its scale factor in almost pseudo-Ricci symmetric spacetime solutions in $F(R)$ -gravity, De *et al.* [48] studied conformally flat generalized Ricci recurrent perfect fluid spacetime with constant Ricci scalar as a solution of modified $F(R)$ -gravity theory and show that a Robertson–Walker spacetime is a generalized Ricci recurrent if and only if it is Ricci symmetric. Also, in [49], the authors proved that such a $(\text{WRS})_4$ satisfying $F(R)$ -gravity field equations represents a perfect fluid with vanishing vorticity. Some energy conditions are studied under the current settings to constrain the functional form of $F(R)$.

Mantica and Suh [42] defined a generalized $(0,2)$ symmetric Z -tensor given by

$$Z_{ij} = R_{ij} + \phi g_{ij}, \quad (1.4)$$

where ϕ denotes an arbitrary scalar function.

If trace $Z = 0$, then (1.4) gives $\phi = -\frac{R}{n}$, reducing (1.4) to the classical Z -tensor which is traceless. Throughout this paper, we will consider Z -tensor given by (1.4) with trace zero. In [42, 43], authors studied the properties of pseudo symmetric Riemannian manifolds $(\text{PZS})_n$ and also consider $(\text{PZS})_4$ spacetime. Afterward, several Z -symmetric spacetimes were studied by many geometers in the context of GR (for detail see [50–52, 44]). Recently, Shenawy *et al.* [44] extended the study of Z -tensor to $F(R)$ -gravity. They investigated almost pseudo Z -symmetric spacetimes in $F(R)$ theory of gravity. In 2022, De *et al.* [47] studied a conformally flat pseudo-symmetric spacetimes within the framework of $F(R)$ -gravity theory.

Motivated by the above work, we examine conformally harmonic Z -recurrent spacetime in $F(R)$ theory of gravity. This paper is arranged as follows. In Sec. 2, we analyze the equation of motion for a conformally harmonic Z -recurrent spacetime and we found that such spacetime obeying $F(R)$ -gravity with constant scalar curvature is a perfect fluid spacetime. Section 3 is dedicated to the cosmological study in which the behavior of cosmological parameters such as the density, pressure

and equation of state parameters are observed. We investigate the energy conditions within two $F(R)$ -gravity models in Sec. 4. Further, discussions and conclusions are drawn in the last section.

The notations $\dot{R} = A^k \nabla_k R$, $\ddot{R} = A^k \nabla_k \dot{R}$, $\dot{A}_i = A^k \nabla_k A_i$ are used throughout the paper.

2. The Equations of Motion

Suppose that the spacetime is Z -recurrent, then we have

$$\nabla_k Z_{ij} = A_k Z_{ij}, \quad (2.1)$$

where A_k is 1-form such that $A_k A^k = -1$.

Using the definition of Z -tensor, (2.1) can be rewritten as

$$\nabla_k R_{ij} = A_k Z_{ij} + \frac{\nabla_k R}{4} g_{ij}. \quad (2.2)$$

Interchanging i and k in (2.2), then subtracting with (2.2) will give

$$\nabla_k R_{ij} - \nabla_i R_{kj} = A_k Z_{ij} - A_i Z_{kj} + \frac{1}{4} (g_{ij} \nabla_k R - g_{kj} \nabla_i R). \quad (2.3)$$

For a conformally harmonic space, the following relation holds:

$$\nabla_k R_{ij} - \nabla_i R_{kj} = \frac{1}{6} (g_{ij} \nabla_k R - g_{kj} \nabla_i R). \quad (2.4)$$

Making use of (2.4) in (2.3), we get

$$(A_k R_{ij} - A_i R_{kj}) - \frac{R}{4} (A_k g_{ij} - A_i g_{kj}) = -\frac{1}{12} (g_{ij} \nabla_k R - g_{kj} \nabla_i R). \quad (2.5)$$

Contracting (2.1) over k and i infers

$$A^k Z_{kj} = \frac{1}{4} \nabla_j R. \quad (2.6)$$

By transvecting the Z -tensor by A^i and using (2.6), we obtain

$$A^i R_{ij} = \frac{1}{4} (R A_j + \nabla_j R). \quad (2.7)$$

Again, transvecting (2.7) by A^j gives

$$4A^i A^j R_{ij} = -R + \dot{R}. \quad (2.8)$$

By making use of (2.7) in (2.5), we get

$$\begin{aligned} & A_k R_{ij} - A_i R_{kj} - \frac{R}{4} (A_k g_{ij} - A_i g_{kj}) \\ &= -\frac{1}{12} (4A^l R_{lk} + R A_k) g_{ij} + \frac{1}{12} (4A^l R_{il} - R A_i) g_{kj}. \end{aligned} \quad (2.9)$$

Transvecting the last expression (2.9) by A^j yields

$$A_k A^l R_{il} = A_i A^l R_{kl}. \quad (2.10)$$

Again, transvecting (2.10) by A^k , then using (2.8) gives

$$A^l R_{il} = \frac{R - \dot{R}}{4} A_i. \quad (2.11)$$

Combining (2.7) with (2.11) gives

$$\nabla_j R = -\dot{R} A_j. \quad (2.12)$$

Also, transvecting (2.9) by A^k and making use of (2.10) and (2.11), we obtain

$$R_{ij} = \frac{\dot{R} + 3R}{12} g_{ij} + \frac{\dot{R}}{3} A_i A_j. \quad (2.13)$$

Let us first consider the case when $\dot{R} = 0$, that is, R is constant from (2.12). In this case, we have

$$R_{ij} - \frac{R}{2} g_{ij} = \frac{k^2}{F_R(R)} T_{ij} + \frac{F(R) - RF_R(R)}{2F_R(R)} g_{ij}. \quad (2.14)$$

In view of (2.13) and (2.14), the energy-momentum tensor is perfect fluid. For a perfect fluid spacetime, the energy-momentum tensor T_{ij} is expressed as follows:

$$k^2 T_{ij} = k^2 (p + \rho) A_i A_j + k^2 p g_{ij}. \quad (2.15)$$

Combining (2.13) and (2.14) along with the last expression gives

$$k^2 (p + \rho) A_i A_j + k^2 p g_{ij} = \frac{RF_R(R) - 2F(R)}{4} g_{ij}. \quad (2.16)$$

Transvecting (2.16) by A^i , we obtain the expression for density as follows:

$$\rho = \frac{1}{k^2} \left(\frac{2F(R) - RF_R(R)}{4} \right). \quad (2.17)$$

Also, contracting (2.16) and using (2.17), the expression for pressure is as follows:

$$p = \frac{1}{k^2} \left(\frac{2F(R) - RF_R(R)}{4} \right). \quad (2.18)$$

Therefore, we can state the following.

Theorem 2.1. *If a conformally harmonic Z -recurrent spacetime with constant scalar curvature obeys $F(R)$ -gravity theory, then the matter content is a perfect fluid with four velocity vector A^i , energy density (2.17) and isotropic pressure (2.18).*

Now we will consider the case when $\dot{R} \neq 0$. Taking covariant derivative of (2.13) gives

$$\nabla_k R_{ij} = \frac{1}{12} (\nabla_k \dot{R} + 3\nabla_k R) g_{ij} + \frac{\nabla_k \dot{R}}{3} A_i A_j + \frac{\dot{R}}{3} ((\nabla_k A_i) A_j + A_i (\nabla_k A_j)). \quad (2.19)$$

Transvecting (2.19) by A^j , we obtain

$$A^j \nabla_k R_{ij} = \frac{1}{4}(\nabla_k R - \nabla_k \dot{R}) - \frac{\dot{R}}{3}(\nabla_k A_i). \quad (2.20)$$

Combining (2.2) and (2.12), then transvecting the obtained equation by A^j , we get

$$A^j \nabla_k R_{ij} = -\frac{\dot{R}}{2} A_i A_k. \quad (2.21)$$

Transvecting (2.20) by A^k and using (2.21) gives

$$\frac{\dot{R}}{3} \dot{A} = -\frac{1}{4}(\dot{R} + \ddot{R}) A_i,$$

which on transvecting by A^i implies $\ddot{R} + \dot{R} = 0$ and hence by previous equation, we get $\dot{R}\dot{A} = 0$. However, by our hypothesis, $\dot{R} \neq 0$, hence $\dot{A} = 0$.

Taking the covariant derivative of (2.12), we obtain

$$\nabla_i \nabla_j R = -(\nabla_i \dot{R}) A_j - \dot{R}(\nabla_i A_j). \quad (2.22)$$

Interchanging i and j in (2.22) yields

$$\nabla_j \nabla_i R = -(\nabla_j \dot{R}) A_i - \dot{R}(\nabla_j A_i). \quad (2.23)$$

Subtracting (2.22) and (2.23) gives

$$(\nabla_j \dot{R}) A_i - (\nabla_i \dot{R}) A_j + \dot{R}(\nabla_j A_i - \nabla_i A_j) = 0. \quad (2.24)$$

Transvecting (2.24) by A^j and using the fact that $\dot{A} = 0$, we obtain

$$\nabla_i \dot{R} = -\ddot{R} A_i. \quad (2.25)$$

In consequence, Eq. (2.24) becomes $\dot{R}(\nabla_j A_i - \nabla_i A_j) = 0$, which implies A is a closed 1-form.

Combining (2.12), (2.20) and (2.25), we get

$$\nabla_k A_i = \frac{3}{4\dot{R}}(\dot{R} + \ddot{R}) A_i A_k. \quad (2.26)$$

Inserting (2.26) and (2.25) in (2.23), (2.23) becomes

$$\nabla_j \nabla_i R = \frac{\ddot{R} - 3\dot{R}}{4} A_i A_j. \quad (2.27)$$

For any analytic function $F(R)$, we have

$$\nabla_i \nabla_j F_R(R) = F_{RR}(R) \nabla_i \nabla_j R + F_{RRR}(R) (\nabla_i R) (\nabla_j R). \quad (2.28)$$

Making use of (2.12) and (2.27) in (2.28), we obtain

$$\nabla_i \nabla_j F_R(R) = \frac{(-3\dot{R} + \ddot{R}) F_{RR}(R) + 4\dot{R}^2 F_{RRR}(R)}{4} A_i A_j. \quad (2.29)$$

Then contracting (2.29) gives

$$\square F_R(R) = -\frac{(-3\dot{R} + \ddot{R})F_{RR}(R) + 4\dot{R}^2 F_{RRR}(R)}{4}. \quad (2.30)$$

Making use of (2.29) and (2.30) in (2.15), we obtain the following relation:

$$\begin{aligned} k^2 T_{ij} &= \frac{(\dot{R} + 3R)F_R(R) - 6F(R) - 3(-3\dot{R} + \ddot{R})F_{RR}(R) - 12\dot{R}^2 F_{RRR}(R)}{12} g_{ij} \\ &+ \frac{4\dot{R}F_R(R) - 3(-3\dot{R} + \ddot{R})F_{RR}(R) - 12\dot{R}^2 F_{RRR}(R)}{12} A_i A_j. \end{aligned} \quad (2.31)$$

Thus, the matter content is perfect fluid. Inserting (2.15) in (2.31) yields

$$\begin{aligned} &k^2(p + \rho)A_i A_j + k^2 p g_{ij} \\ &= \frac{1}{12} \{(\dot{R} + 3R)F_R(R) - 6F(R) - 3(-3\dot{R} + \ddot{R})F_{RR}(R) - 12\dot{R}^2 F_{RRR}(R)\} g_{ij} \\ &+ \frac{1}{12} \{4\dot{R}F_R(R) - 3(-3\dot{R} + \ddot{R})F_{RR}(R) - 12\dot{R}^2 F_{RRR}(R)\} A_i A_j. \end{aligned} \quad (2.32)$$

Transvecting (2.32) by A^i , the expression for density is as follows:

$$\rho = \frac{1}{k^2} \left(\frac{F(R)}{2} + \frac{\dot{R} - R}{4} F_R(R) \right). \quad (2.33)$$

Contracting (2.32) and using (2.33), the expression for pressure is as follows:

$$p = \frac{1}{k^2} \left(\frac{(\dot{R} + 3R)F_R(R) - 6F(R) + 3(3\dot{R} - \ddot{R})F_{RR}(R) - 12\dot{R}^2 F_{RRR}(R)}{12} \right). \quad (2.34)$$

Therefore, we can state the following.

Theorem 2.2. *In a conformally harmonic Z -recurrent spacetime with non-constant scalar curvature obeying $F(R)$ -gravity theory, if the four-velocity vector is identical with A^i , then the matter content is a perfect fluid with energy density (2.33) and isotropic pressure (2.34).*

In [21], the authors showed that a Lorentzian manifold M of dimension $n > 3$ with Ricci curvature tensor $R_{ij} = Ag_{ij} + Bu_i u_j$, where A and B are scalar fields, u is a closed timelike unit vector field and with a conformally harmonic curvature tensor is a generalized Robertson–Walker (GRW) spacetime with a Einstein fibre. In consequence of Theorem 2.2, we can state the following.

Corollary 2.3. *If a conformally harmonic Z -recurrent spacetime with non-constant scalar curvature obeys $F(R)$ -gravity theory with four velocity vector field A^i , then it is a GRW spacetime.*

3. Cosmological Study

Hubble's law is considered the first observational basis for the expansion of the universe and the rate of expansion of the universe is measured extensively by the Hubble's parameter, $H = \frac{\dot{a}}{a}$, where a is the scale factor and $(\dot{})$ above the letter represents the derivative with respect to time. In cosmological setting, the higher derivatives of the scale factor a are dimensionless parameters known as deceleration (q), jerk (j) and snap (s) [1]:

$$q = -\frac{1}{H^2} \frac{\ddot{a}}{a},$$

$$j = \frac{1}{H^3} \frac{\dddot{a}}{a},$$

$$s = \frac{1}{H^4} \frac{\ddddot{a}}{a}.$$

We note that the Ricci scalar and its derivatives for a spatially flat FLRW geometry can be expressed in terms of the deceleration (q), jerk (j) and snap (s) parameters [2]:

$$R = -6H^2(1 - q), \quad (3.1)$$

$$\dot{R} = -6H^3(j - q - 2), \quad (3.2)$$

$$\ddot{R} = -6H^4(s + q^2 + 8q + 6). \quad (3.3)$$

Moreover, the Equation of State (EoS) parameter ω is defined as $\omega = \frac{p}{\rho}$, the ratio of the isotropic pressure to the energy density of the perfect fluid. We can use the EoS parameter to observe different phases of the universe [46]. The different phases of the EoS parameter are as follows: $\omega = 0$ represents the dust phase, $\omega = \frac{1}{3}$ shows the radiation dominated phase, $\omega = -1$ is the Λ CDM and $\omega < -\frac{1}{3}$ supports the cosmic speed up. Also, $\omega < -1$ indicates the phantom era. It also includes the quintessence era when $-1 < \omega < 0$.

Case-I: When $\dot{R} \neq 0$.

We obtain the density and pressure in terms of deceleration parameter, jerk and snap parameters using (3.1), (3.2) and (3.3) as

$$\rho = \frac{1}{2k^2} (F(R) + 3H^2(1 - q - H(j - q - 2))F_R(R)) \quad (3.4)$$

and

$$p = \frac{1}{2k^2} \{-H^2(3(1 - q) + H(j - q - 2))F_R(R) - F(R) + 3H^3(H(s + q^2 + 8q + 6) - 3(j - q - 2))F_{RR}(R) - 72H^6(j - q - 2)^2F_{RRR}(R)\}. \quad (3.5)$$

3.1. Model-I

For our first test model, we assume a logarithmic functional form $F(R) = R + \alpha \log(\beta R)$ [53, 47]. Clearly, $F(R)$ is continuous and smooth for all $\beta R \geq 0$. As Ricci scalar R is negative for present value of q , we take $\beta \leq 0$. Moreover, for $\alpha = 0$, we see that $F(R) = R$ which corresponds to the GR case. We will consider the present value of parameters as $q_0 = -0.55$, $j_0 = 1$, $s_0 = -0.35$ and $H_0 = 67.9$ km/s/Mpc [3, 4]. Making use of (3.1)–(3.5), the expression for pressure p and density ρ in this model is obtained as follows:

$$\rho = \frac{1}{2k^2} \left(-6H^2(1-q) + \alpha \log(-6\beta H^2(1-q)) + 3H^2(1-q - H(j-q-2)) \left(1 - \frac{\alpha}{6H^2(1-q)} \right) \right) \quad (3.6)$$

and

$$p = \frac{1}{12k^2} \left(\frac{(\alpha - 6H^2(1-q))(3(1-q) + H(j-q-2))}{1-q} + \frac{4\alpha(j-q-2)^2}{(1-q)^3} + 6(6H^2(1-q) - \alpha \log(-6\beta H^2(1-q))) - \frac{\alpha(H(s+q^2+8q+6) - 3(j-q-2))}{2H(1-q)^2} \right). \quad (3.7)$$

As $F(R) = R + \alpha \log(\beta R)$ is continuous and smooth for $\beta R \geq 0$, we take $\beta \leq 0$ since the Ricci scalar R is negative for present value of $q_0 = -0.55$. The bounds

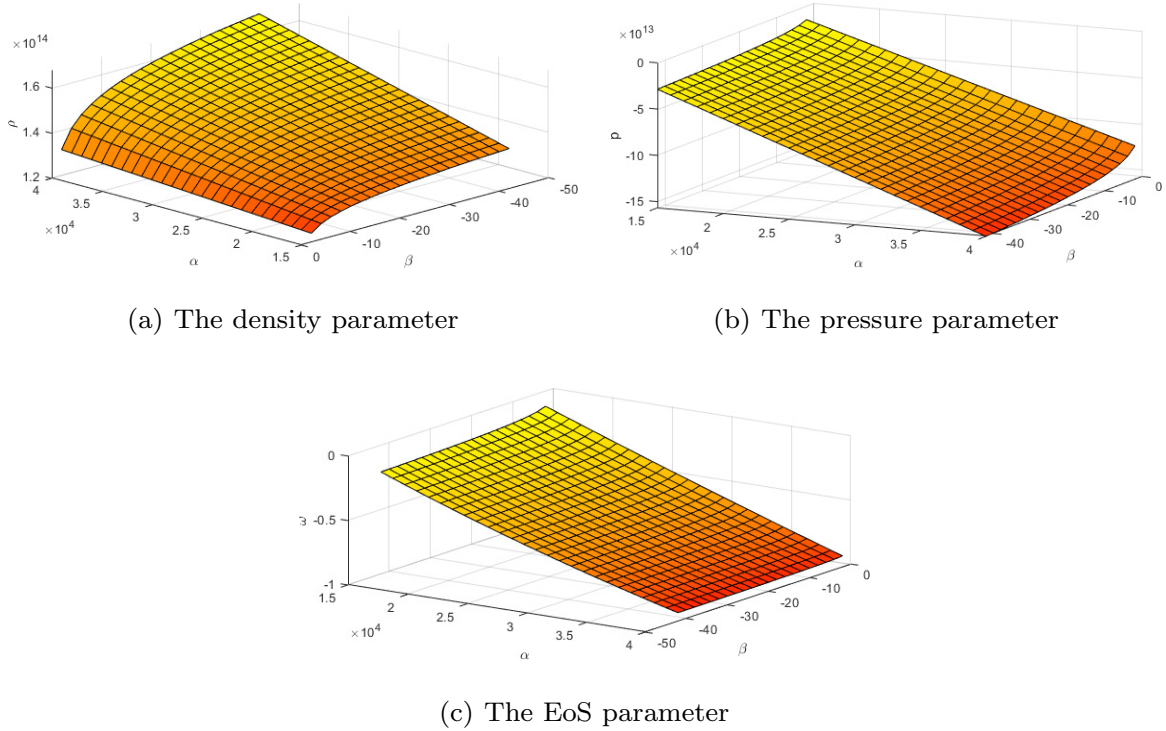


Fig. 1. The behavior of the density, pressure and the EoS parameters considering $F(R) = R + \alpha \log(\beta R)$ for Case-I with $15000 \leq \alpha \leq 40000$ and $-42 \leq \beta \leq -1$.

for α and β are obtained from the behavior of density which is always positive, as shown in Fig. 1(a) and pressure which is negative as shown in Fig. 1(b). It can be seen that β influences the growth of density and pressure more than α , ρ increases for more negative β and decreases for less negative β . Initially, the density increases sharply for $\beta \in (-1, -10)$, then it follows a slow rate of increase. The behavior of the EoS parameter for our considered model is shown in Fig. 1(c). The plot of EoS shows a transition from positive to negative supporting an accelerated expansion of the universe and lies in the quintessence phase $-1 < w < 0$.

3.2. Model-II

For the second model, we consider the exponential functional form $F(R) = R - a \exp(-bR - 1)$ [7]. It is obvious that for $a = 0$ we get $F(R) = R$, which corresponds to the GR case. We will consider the present value of parameters as $q_0 = -0.55$, $j_0 = 1$, $s_0 = -0.35$ and $H_0 = 67.9$ km/s/Mpc [3, 4]. Making use of (3.1)–(3.5), the expression for pressure p and density ρ in this model is obtained as follows:

$$\begin{aligned} \rho = & \frac{1}{2k^2}(-6H^2(1-q) - a \exp(6bH^2(1-q) - 1) \\ & + 3H^2(1-q - H(j-q-2))(1 + ab \exp(6bH^2(1-q) - 1))) \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} p = & \frac{1}{2k^2}(-H^2(3(1-q) + H(j-q-2))(1 + ab \exp(6bH^2(1-q) - 1)) + 6H^2(1-q) \\ & + a \exp(6bH^2(1-q) - 1) - 72H^6(j-q-2)^2 ab^3 \exp(6bH^2(1-q) - 1) \\ & - 3H^3(H(s+q^2+8q+6) - 3(j-q-2))(ab^2 \exp(6bH^2(1-q) - 1))). \end{aligned} \quad (3.9)$$

As seen in Fig. 2, the range for a and b is obtained from the behavior of density parameter, as shown in Fig. 2(a), which is always positive and the behavior of pressure, as shown in Fig. 2(b). The density increases more with the increase in the negative value of b . The behavior of the EoS parameter for our considered model is shown in Fig. 2(c). The plot of EoS shows a transition from positive to negative supporting an acceleration in the universe and lies in the quintessence phase $-1 < w < 0$.

Case-II: When $\dot{R} = 0$.

In the similar manner as Case-I, the expressions for pressure and density in terms of deceleration parameter, jerk and snap parameters using (3.1), (3.2) and (3.3) are as follows:

$$\rho = \frac{1}{4k^2}(2F(R) + 6H^2(1-q)F_R(R)) \quad (3.10)$$

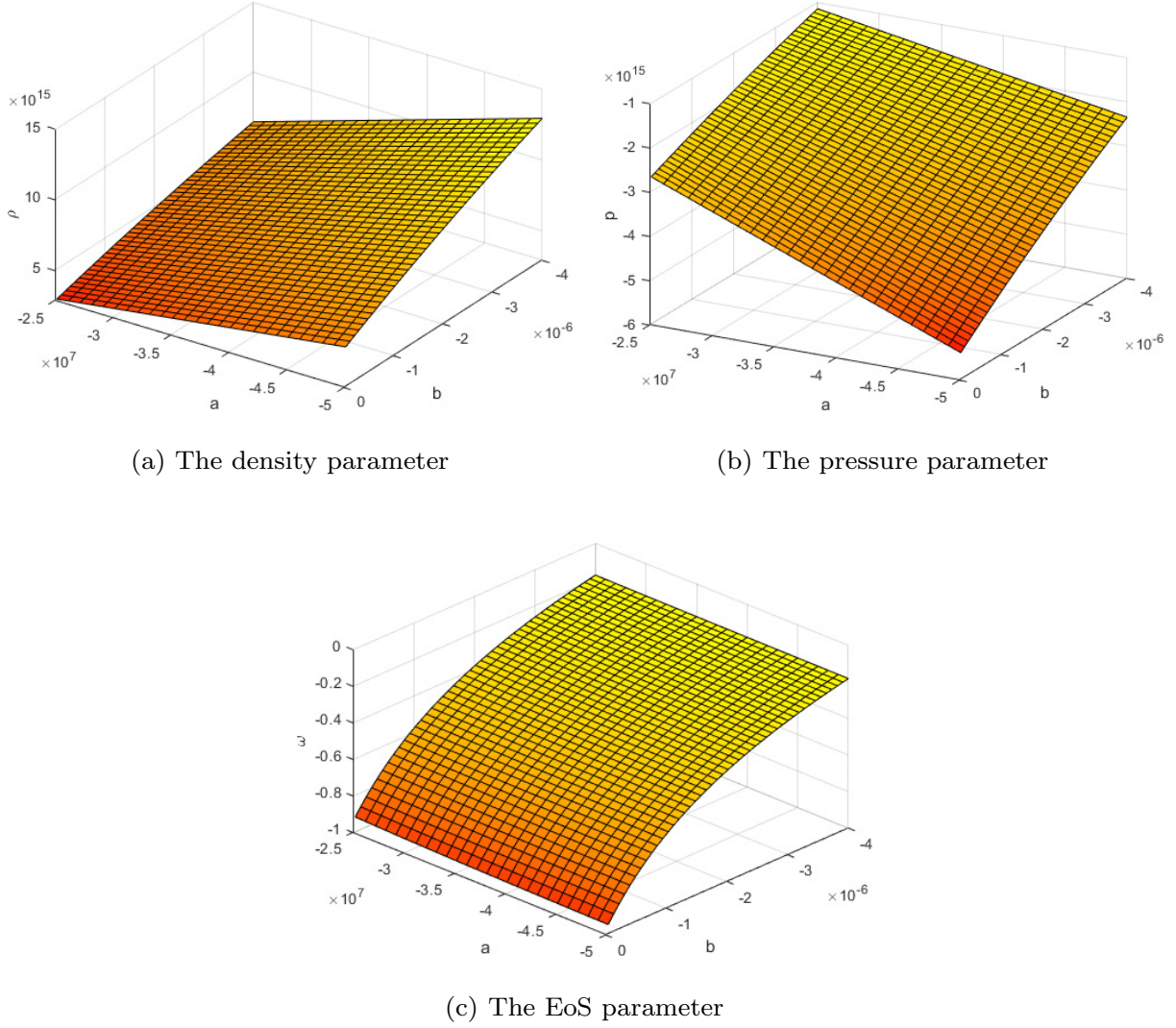


Fig. 2. The behavior of the density, pressure and the EoS parameters considering $F(R) = R - a \exp(-bR - 1)$ for Case-I with $-5 \times 10^7 \leq a \leq -2.5 \times 10^7$ and $-4 \times 10^{-6} \leq b \leq -10^{-9}$.

and

$$p = -\frac{1}{4k^2}(2F(R) + 6H^2(1 - q)F_R(R)). \quad (3.11)$$

It is obvious from (3.10) and (3.11) that $\rho + p = 0$ and the EoS parameter, $\omega = -1$. In this case, our model coincides with the Λ CDM model. For $F(R) = R + \alpha \log(\beta R)$, the bounds for α and β are obtained from the behavior of density, as shown in Fig. 3(a), which increases more with the increase in the value of α . Similarly, the bounds of a and b are obtained from the behavior of density parameter for $F(R) = R - a \exp(-bR - 1)$, as shown in Fig. 3(b), which is always positive. The density ρ increases with the increase in the value of b .

Therefore, from the figures above for Model-I and Model-II, we conclude with Table 1.

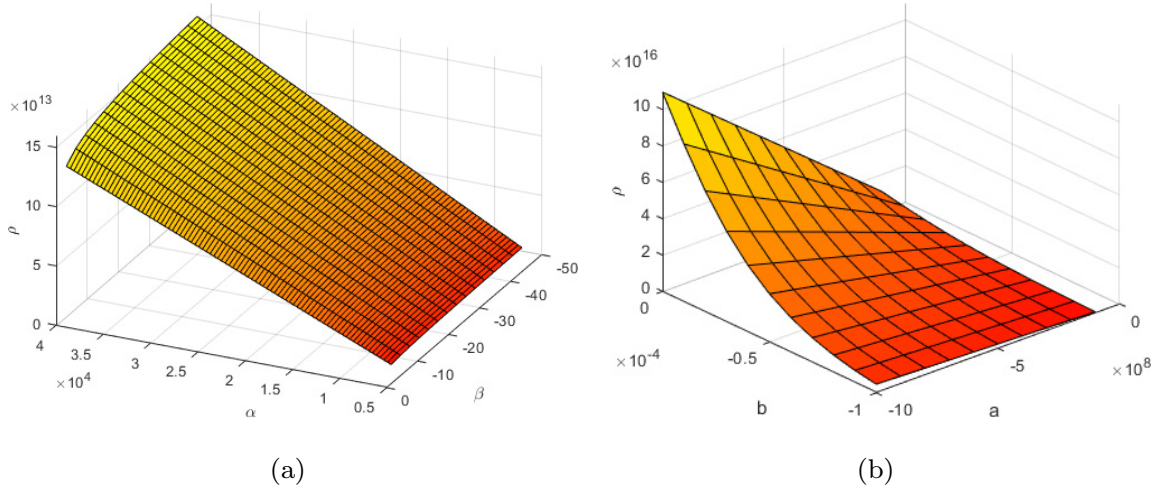


Fig. 3. The behavior of density parameter for Case-II. (a) The behavior of density parameter considering $F(R) = R + \alpha \log(\beta R)$ with $5500 \leq \alpha \leq 40000$ and $-45 \leq \beta \leq -2$ and (b) the behavior of density parameter considering $F(R) = R - a \exp(-bR - 1)$ with $-10^9 \leq a \leq -10^4$ and $-10^{-4} \leq b \leq 0$.

Table 1. Constraints on models' parameters a, b, α, β to support an acceleration of the universe.

Model	Cases	Parameters and their ranges	Equation of state (EoS)
Model I:	$\dot{R} = 0$	$\alpha \in [5500, 40000], \beta \in [-45, -2]$	$\omega = -1$
$F(R) = R + \alpha \log(\beta R)$	$\dot{R} \neq 0$	$\alpha \in [15000, 40000], \beta \in [-42, -1]$	$-1 < \omega < 0$
Model II:	$\dot{R} = 0$	$a \in [-10^9, -10^4], b \in [-10^{-4}, 0]$	$\omega = -1$
$F(R) = R - a \exp(-bR - 1)$	$\dot{R} \neq 0$	$a \in [-5 \times 10^7, -2.5 \times 10^7],$ $b \in [-4 \times 10^{-6}, -10^{-9}]$	$-1 < \omega < 0$

4. Energy Conditions in $F(R)$ -Gravity

Energy conditions were the primary tool for deriving general results that hold for a variety of situations such as Hawking–Penrose singularity theorems, second law of black hole thermodynamics, phantom fields potentials [5], expansion history of the universe (see [6, 8–11]), as well as evolution of the deceleration parameter and their confront with supernovae observations [12, 13]. The Raychaudhuri equations along with the requirement that gravity is attractive gave rise to the concept of energy conditions (for details, see [14]). Let u^i be the tangent vector field to a congruence of timelike geodesics in a spacetime manifold endowed with a metric g_{ij} , then Raychaudhuri's equation is given as

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \rho_{ij}\rho^{ij} + \omega_{ij}\omega^{ij} - R_{ij}u^i u^j,$$

where R_{ij} , θ , ρ_{ij} and ω_{ij} are, respectively, the Ricci tensor, expansion, shear and rotation associated to the congruence defined by the vector field u^i . Due to the

attractive nature of gravity and $\omega_{ij} = 0$, $R_{ij}u^i u^j \geq 0$, which by virtue of Einstein's equation implies $(T_{ij} - \frac{T}{2}g_{ij})u^i u^j \geq 0$, where T is the trace of energy-momentum tensor.

In the context of the perfect fluid spacetime in GR, the energy conditions are defined as

- Null energy condition (NEC): $\rho + p \geq 0$.
- Strong energy condition (SEC): $\rho + 3p \geq 0$ and $\rho + p \geq 0$.
- Weak energy condition (WEC): $\rho + p \geq 0$ and $\rho \geq 0$.
- Dominant energy condition (DEC): $\rho \pm p \geq 0$ and $\rho \geq 0$.

The concept of energy conditions has been extended to $F(R)$ -gravity (for details, see [15, 16]). The freedom of selecting different functional form of $F(R)$ gives rise to the problem of how to constrain from theoretical and observational aspects of many possible $F(R)$ -gravities. For this purpose, the energy conditions may be used as a constraint for selecting the best possible $F(R)$ -gravities. The effective energy-momentum tensor in $F(R)$ -gravity is given as (see for details [17–20])

$$T_{ij}^{\text{eff}} = \frac{1}{F_R(R)} \left(T_{ij} + \frac{1}{2}(F(R) - RF_R(R))g_{ij} + (\nabla_i \nabla_j - g_{ij}\square)F_R(R) \right). \quad (4.1)$$

Then from (4.1), the expression for effective density and pressure is, respectively,

$$\rho_{\text{eff}} = \frac{1}{F_R(R)} \left(\rho + \frac{1}{2}(F(R) - RF_R(R)) - 3\dot{R}H F_{RR}(R) \right) \quad (4.2)$$

and

$$p_{\text{eff}} = \frac{1}{F_R(R)} \left(p + \frac{1}{2}(F(R) - RF_R(R)) + (\ddot{R} + 2\dot{R}H)F_{RR}(R) + \dot{R}^2 F_{RRR}(R) \right). \quad (4.3)$$

For a perfect fluid FLRW, the energy conditions in the context of $F(R)$ -gravity are defined as [15]

- Null energy condition (NEC): $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$.
- Strong energy condition (SEC): $\rho_{\text{eff}} + 3p_{\text{eff}} \geq 0$ and $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$.
- Weak energy condition (WEC): $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$ and $\rho_{\text{eff}} \geq 0$.
- Dominant energy condition (DEC): $\rho_{\text{eff}} \pm p_{\text{eff}} \geq 0$ and $\rho_{\text{eff}} \geq 0$.

It is obvious that for $F(R) = R$, the energy conditions reduce to the case of GR.

4.1. Model-I

The equations obtained for Case-I and Case-II will be used again by changing the model in Model-II.

Case-I: When $\dot{R} \neq 0$.

The energy conditions for the functional form of $F(R)$ are obtained from (2.33), (2.34), (4.2) and (4.3) as

$$\rho_{\text{eff}} = \frac{1}{F_R(R)} \left[\left(\frac{1+k^2}{2k^2} \right) F(R) + \left(\frac{\dot{R} - (1+2k^2)R}{4k^2} \right) F_R(R) - 3\dot{R}H F_{RR}(R) \right], \quad (4.4)$$

$$\begin{aligned} \rho_{\text{eff}} + p_{\text{eff}} = & \frac{1}{F_R(R)} \left[F(R) + \left(\frac{\dot{R} - 3k^2 R}{3k^2} \right) F_R(R) \right. \\ & \left. + \left(\frac{(4k^2 - 1)\ddot{R} + (3 - 4k^2 H)\dot{R}}{4k^2} \right) F_{RR}(R) + \left(\frac{k^2 - 1}{k^2} \right) \dot{R}^2 F_{RRR}(R) \right], \end{aligned} \quad (4.5)$$

$$\begin{aligned} \rho_{\text{eff}} - p_{\text{eff}} = & \frac{1}{F_R(R)} \left[\frac{1}{2k^2} (2F(R) - R F_R(R)) \right. \\ & + \left(\frac{(3 - 20k^2 H)\dot{R} - (1 + 4k^2)\ddot{R}}{4k^2} \right) F_{RR}(R) \\ & \left. - \left(\frac{k^2 + 1}{k^2} \right) \dot{R}^2 F_{RRR}(R) \right], \end{aligned} \quad (4.6)$$

$$\begin{aligned} \rho_{\text{eff}} + 3p_{\text{eff}} = & \frac{1}{F_R(R)} \left[\left(\frac{2k^2 - 1}{k^2} \right) F(R) + \frac{\dot{R} + (1 - 4k^2)R}{2k^2} F_R(R) \right. \\ & + \left(\frac{(12k^2 - 3)\ddot{R} + 3(4k^2 H + 3)\dot{R}}{4k^2} \right) F_{RR}(R) \\ & \left. + \left(\frac{3(k^2 - 1)}{k^2} \right) \dot{R}^2 F_{RRR}(R) \right]. \end{aligned} \quad (4.7)$$

Taking into account our model $F(R) = R + \alpha \log(\beta R)$ and using (3.1)–(3.3) in the foregoing equations, we obtain the graph for this case. The bounds for α and β can be chosen from effective density parameter which is always positive, as shown in Fig. 4(a). From Figs. 4(a)–4(c), it is obvious that the conditions for NEC, DEC and WEC are satisfied for the range $21500 \leq \alpha \leq 40000$ and $-42 \leq \beta \leq -1$. However, in the same range, the SEC condition is violated, as shown in Fig. 4(d). There may be a violation of energy conditions in some instances without the system becoming unacceptable. The success of the hypothesis of inflation and recent observations of cosmic acceleration points to a violation of SEC in the universe.

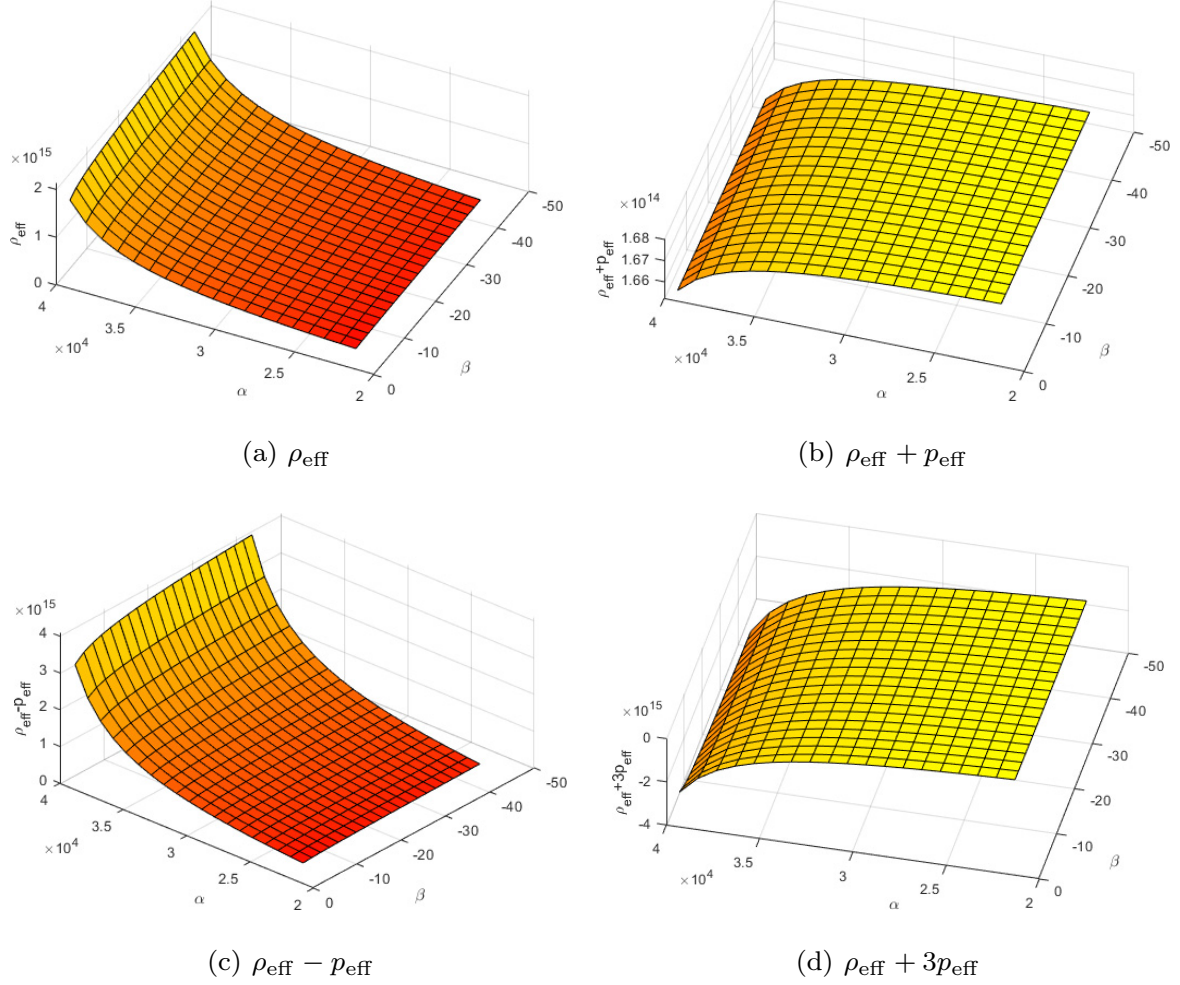


Fig. 4. The behavior of energy conditions considering $F(R) = R + \alpha \log(\beta R)$ for Case-I with $21500 \leq \alpha \leq 40000$ and $-42 \leq \beta \leq -1$.

Case-II: When $\dot{R} = 0$.

In view of (3.10), (3.11), (4.2) and (4.3), we get the energy conditions for the functional $F(R)$ as follows:

$$\rho_{\text{eff}} = \frac{1}{F_R(R)} \left[\left(\frac{1+k^2}{2k^2} \right) F(R) - \left(\frac{1+2k^2}{4k^2} \right) F_R(R) - 3\dot{R}H F_{RR}(R) \right], \quad (4.8)$$

$$\rho_{\text{eff}} + p_{\text{eff}} = \frac{1}{F_R(R)} [F(R) - R F_R(R) + (\ddot{R} - \dot{R}H) F_{RR}(R) + \dot{R}^2 F_{RRR}(R)], \quad (4.9)$$

$$\rho_{\text{eff}} - p_{\text{eff}} = \frac{1}{F_R(R)} \left[\frac{1}{2k^2} (2F(R) - R F_R(R)) - (\ddot{R} + 5\dot{R}H) F_{RR}(R) - \dot{R}^2 F_{RRR}(R) \right], \quad (4.10)$$

$$\begin{aligned} \rho_{\text{eff}} + 3p_{\text{eff}} = & \frac{1}{F_R(R)} \left[\left(\frac{2k^2-1}{k^2} \right) F(R) - \left(\frac{4k^2+1}{2k^2} \right) R F_R(R) \right. \\ & \left. + 3(\ddot{R} + \dot{R}H) F_{RR}(R) + 3\dot{R}^2 F_{RRR}(R) \right]. \end{aligned} \quad (4.11)$$

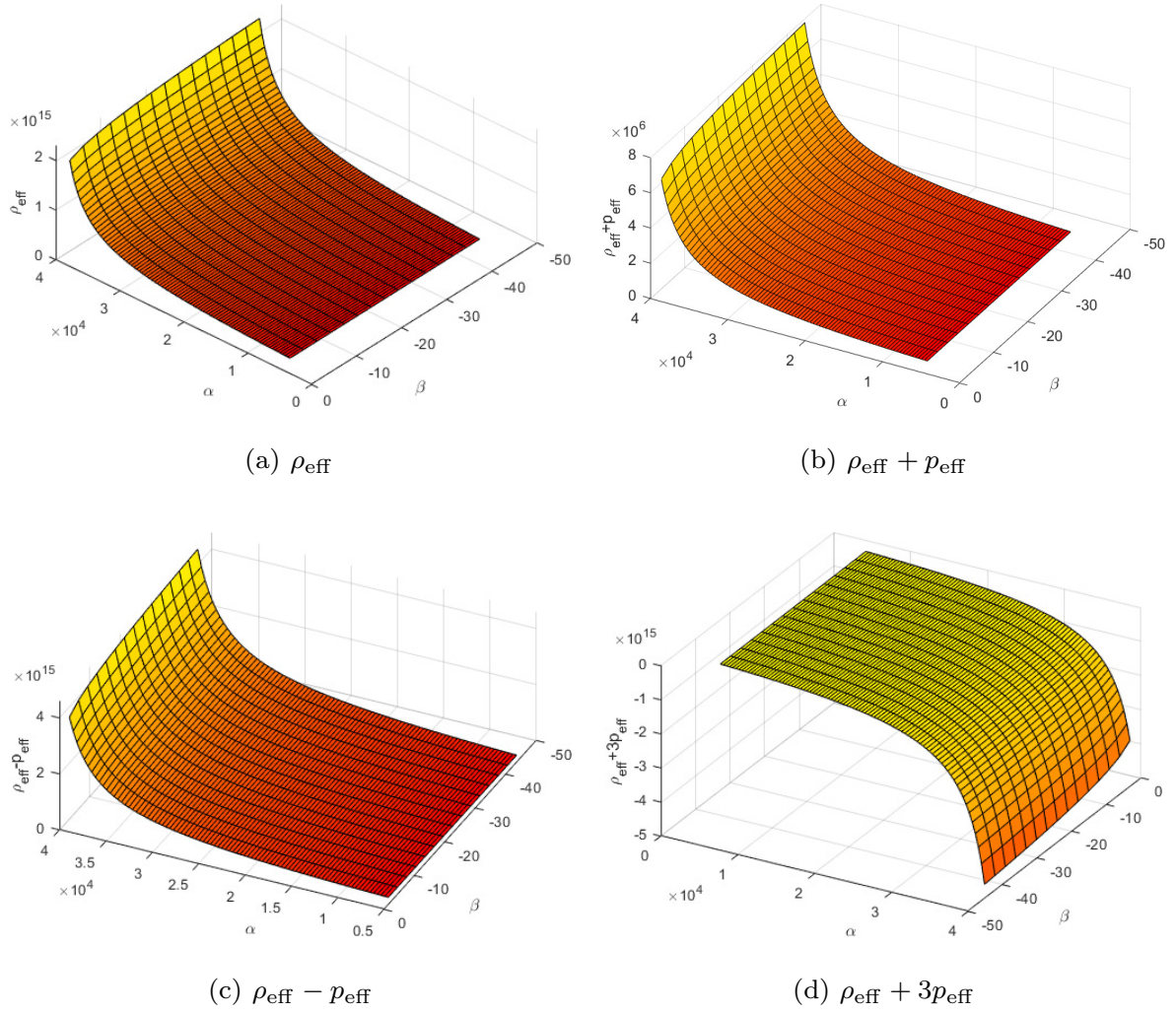


Fig. 5. The behavior of energy conditions considering $F(R) = R + \alpha \log(\beta R)$ for Case-II with $5500 \leq \alpha \leq 40000$ and $-45 \leq \beta \leq -2$.

Substituting the values of $R, \dot{R}$ and $\ddot{R}$ and taking into account our model $F(R) = R + \alpha \log(\beta R)$ in the above equations, we obtain the plots of the behavior of energy conditions for this case, as shown in Fig. 5. Similar to Case-I, we obtained the range of α and β from the effective density parameter ($\rho_{\text{eff}} \geq 0$) and its behavior is shown in Fig. 5(a). Under this range, Figs. 5(b) and 5(c) show positive behavior, whereas Fig. 5(d) executes negative behavior. Therefore, the energy conditions for NEC, WEC and DEC are satisfied, while there is a violation of SEC in the range.

4.2. Model-II

Case-I: When $\dot{R} \neq 0$.

The energy conditions for the functional form $F(R) = R - a \exp(-bR - 1)$ are shown in the following plots.

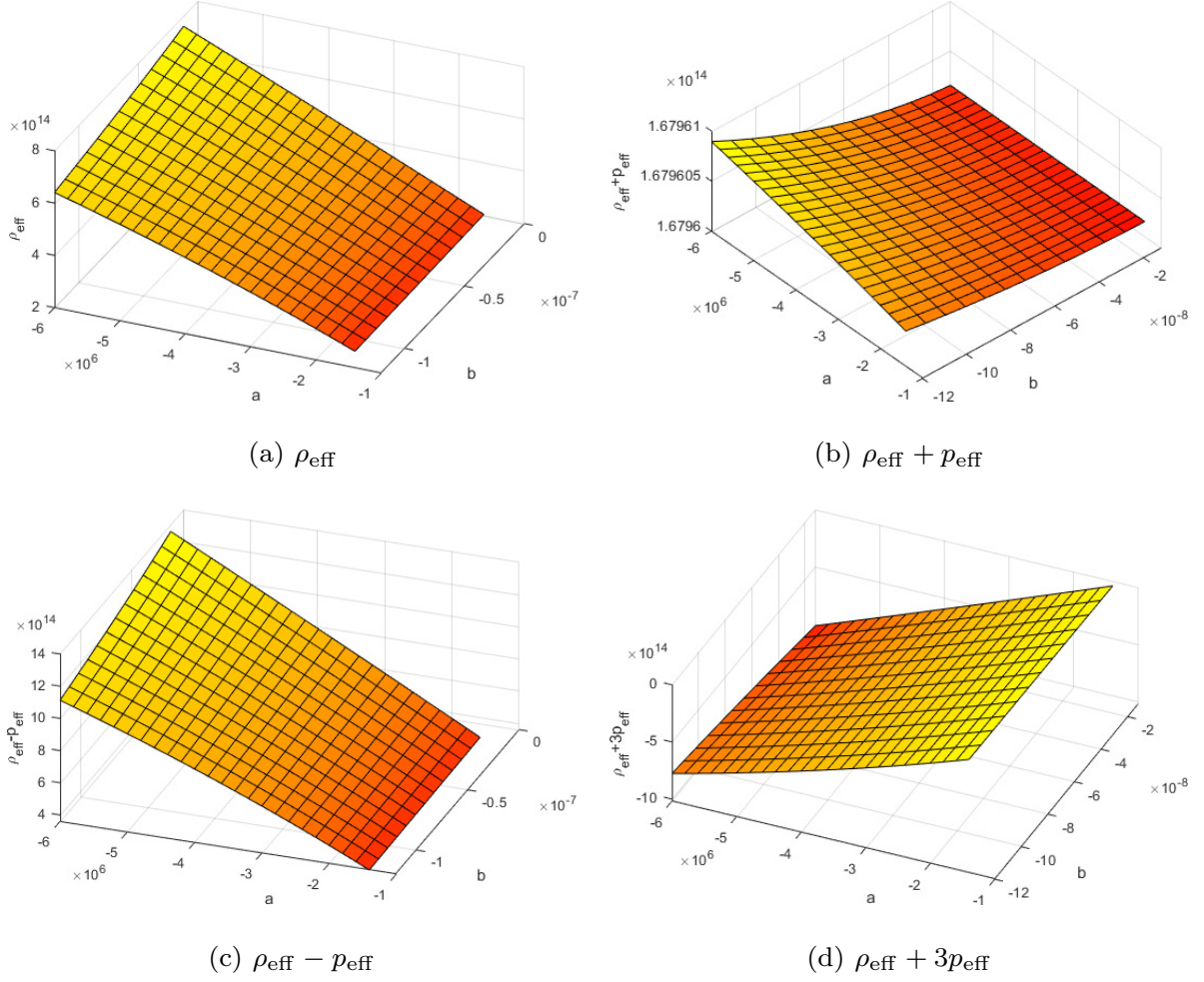


Fig. 6. The behavior of energy conditions considering $F(R) = R - a \exp(-bR - 1)$ for Case-I with $-6 \times 10^6 \leq a \leq -1.3 \times 10^6$ and $-1.2 \times 10^{-7} \leq b \leq -1.2 \times 10^{-8}$.

Plot 6 reveals that $\rho_{\text{eff}} \geq 0$ along with $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$ and $\rho_{\text{eff}} - p_{\text{eff}} \geq 0$. Thus, the conditions of NEC, WEC and DEC are satisfied for $-6 \times 10^6 \leq a \leq -1.3 \times 10^6$ and $-1.2 \times 10^{-7} \leq b \leq -1.2 \times 10^{-8}$ as shown in Plots 6(a), 6(b) and 6(c). However, Plot 6(d) indicates the violation of SEC conditions confirming that our universe experiences an accelerated expansion of the universe.

Case-II: When $\dot{R} = 0$.

The energy conditions for the functional form $F(R) = R - a \exp(-bR - 1)$ in this case are shown in Plot 7. Similar behaviors of the energy conditions as in Case-I are obtained within the suitable range for a and b . Also in this case, conformation of an accelerated expansion of the universe is shown by the violations of the SEC condition.

From the above figures of Model-I and Model-II, we see that the parameters are constraint on particular intervals to satisfy different energy conditions, which we give in Table 2.

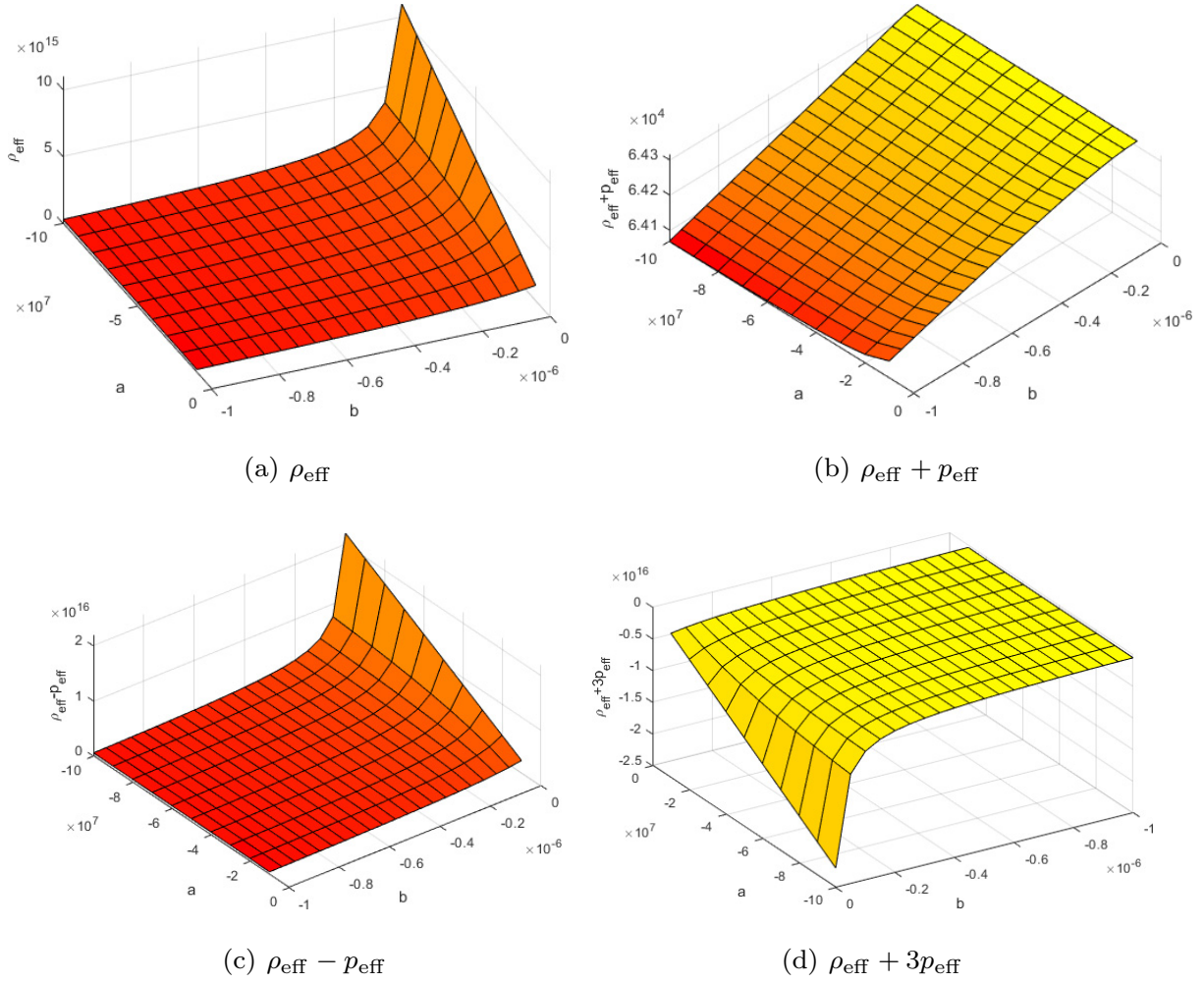


Fig. 7. The behavior of energy conditions considering $F(R) = R - a \exp(-bR - 1)$ for Case-II with $-10^8 \leq a \leq -1$ and $-10^{-6} \leq b \leq 0$.

Table 2. Constraints on models' parameters a, b, α, β to satisfy different energy conditions.

Model	Cases	Parameters and their ranges	Energy conditions
Model I: $F(R) = R + \alpha \log(\beta R)$	$\dot{R} = 0$	$\alpha \in [5500, 40000],$ $\beta \in [-45, -2]$	NEC, WEC and DEC are satisfied but SEC is violated.
	$\dot{R} \neq 0$	$\alpha \in [21500, 40000],$ $\beta \in [-42, -1]$	NEC, WEC and DEC are satisfied but SEC is violated.
Model II: $F(R) = R - a \exp(-bR - 1)$	$\dot{R} = 0$	$a \in [-10^8, -1],$ $b \in [-10^{-6}, 0]$	NEC, WEC and DEC are satisfied but SEC is violated.
	$\dot{R} \neq 0$	$a \in [-6 \times 10^6, -1.3 \times 10^6],$ $b \in [-1.2 \times 10^{-7}, -1.2 \times 10^{-8}]$	NEC, WEC and DEC are satisfied but SEC is violated.

5. Conclusion

The recent observations indicate an accelerated expansion of the universe. It is not clear, till this point, about what could be the cause of this accelerated expansion and it has been the main topic of research. In this paper, a detailed classification of conformally harmonic Z -recurrent spacetime is done in the theory of $F(R)$ -gravity. We assume the functional form $F(R) = R + \alpha \log(\beta R)$, where α and β are constants and $F(R) = R - a \exp(-bR - 1)$ with constants a and b . In both the models with non-negative scalar curvature, the density shows positive behavior, whereas the EoS parameter shows negative behavior, supporting an accelerated expansion of the universe and lies in the quintessence phase. However, when scalar curvature is constant, the models coincide with Λ CDM model. Further, the behavior of energy conditions is investigated and we obtain the bounds of model parameters α, β, a and b showing the viability of the considered models. In both the cases, the energy conditions NEC, WEC and DEC satisfy their derived conditions, whereas SEC violates its condition supporting the accelerated expansion of the universe.

The implications of our results are quite profound and have significant importance in cosmology, particularly in understanding the accelerated expansion of the universe. Following are some points on the importance of our results:

- (1) **Understanding Accelerated Expansion:** The accelerated expansion of the universe, first discovered through observations of distant supernovae, remains one of the most intriguing phenomena in cosmology. This research provides insights into the behavior of gravity at cosmological scales, suggesting that modified gravity theories, such as $F(R)$ -gravity, can account for this acceleration.
- (2) **Confirmation of Quintessence-like Behavior:** Quintessence refers to a hypothetical form of dark energy with a dynamic equation of state (EoS) parameter, causing the accelerated expansion. By observing positive behavior in density and negative behavior in the EoS parameter in the considered models, this research provides evidence supporting the quintessence phase of dark energy, which has implications for understanding the nature of the universe's expansion.
- (3) **Coincidence with Λ CDM Model:** The Λ CDM (Lambda Cold Dark Matter) model has been remarkably successful in describing various cosmological observations, including the accelerated expansion. The finding that the considered models coincide with Λ CDM when the scalar curvature is constant highlights the compatibility of these modified gravity theories with the well-established cosmological frameworks.
- (4) **Viability of Model Parameters:** Investigating the behavior of energy conditions and deriving bounds on model parameters (α, β, a , and b) demonstrate the viability of the considered models within the constraints of observational data and theoretical consistency. This not only validates the theoretical framework, but also guides future research by narrowing down the parameter space.

- (5) **Violation of SEC and Accelerated Expansion:** The violation of the Strong Energy Condition (SEC) in the considered models supports the accelerated expansion of the universe. This is consistent with our understanding that exotic forms of energy, such as dark energy or modifications to gravity, are required to drive this accelerated expansion.

This study offers a framework for evaluating and comparing various cosmological models, including mainstream models like Λ CDM and modified gravity theories like $F(R)$ -gravity. It helps to better comprehend dark energy, estimate parameters, and test gravity at cosmic scales. The discoveries have important consequences for our understanding of the universe's fast expansion, dark energy nature, and gravity behavior at cosmic scales. Future studies can utilize observational data to tighten these parameters and test improved gravity theories.

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ABSTRACT

A STUDY ON CONTACT STRUCTURES, WARPED PRODUCT AND THEIR APPLICATIONS

**AN ABSTRACT SUBMITTED IN PARTIAL
FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY**

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**DEPARTMENT OF MATHEMATICS AND
COMPUTER SCIENCE
SCHOOL OF PHYSICAL SCIENCES
DECEMBER, 2024**

ABSTRACT

**A STUDY ON CONTACT STRUCTURES, WARPED PRODUCT
AND THEIR APPLICATIONS**

BY

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Supervisor : Prof. M. Sundararajan

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Submitted

**In partial fulfillment of the requirement of the Degree of Doctor of
Philosophy in Mathematics of Mizoram University, Aizawl.**

ABSTRACT

We extend the concept of Ricci solitons and conduct an analysis of $F(R)$ -gravity. Furthermore, we provide a classification of almost contact metric manifolds admitting specific geometric structures, along with a detailed study of their submanifolds. Additionally, the properties of warped product manifolds are investigated. The research is structured around the following key objectives:

1. Certain generalization of Ricci Soliton.
2. Analysis on $F(R)$ - gravity.
3. Geometry of Submanifolds in almost contact manifold.
4. Properties of warped product manifold.

The first chapter provides a general introduction to the thesis, encompassing a review of classical results, fundamental definitions, and an overview of soliton theory. It also includes an analysis of modified gravity, submanifold theory, warped products and their applications, as well as a review of the literature.

Chapter 2 is the study of generalized $\mathfrak{h}$ -almost Ricci solitons in almost Kenmotsu manifolds, offering significant insights into their geometric properties. By extending the concept of Ricci solitons and analyzing warped product structures, the work demonstrates that such manifolds either conform to Einstein structures or exhibit steady, expanding solitons. Key results include the classification of η -Einstein manifolds, the role of potential vector fields, and the intricate relationship between soliton dynamics and curvature. The research enriches the understanding of soliton theory, submanifold geometry, and modified gravity, contributing valuable perspectives to the fields of differential geometry and geometric analysis.

Chapter 3 classifies conformally harmonic Z -recurrent spacetimes within $F(R)$ -gravity theory by examining two models: one with a logarithmic form and the other with an exponential functional form. The study demonstrates that both models support the accelerated expansion of the universe, driven by dark energy, as shown by the positive energy density and the negative Equation of State (EoS) parameter, placing them in the quintessence phase. The author also derives key energy conditions, confirming that the Null Energy Condition (NEC), Weak Energy Condition (WEC), and Dominant Energy Condition (DEC) are satisfied, while the Strong Energy Condition (SEC) is violated, further supporting the acceleration. When the scalar curvature is constant, the models align with the

Λ CDM model, making them consistent with widely accepted cosmological theories and providing a strong basis for further research in modified gravity and dark energy.

In Chapter 4, we explore the geometry of Ricci-Bourguignon solitons on Euclidean submanifolds, focusing on cases where the soliton field corresponds to the tangential component of the position vector field. It establishes key geometric properties and provides a classification of Ricci-Bourguignon solitons on Euclidean hypersurfaces, demonstrating that such hypersurfaces must either be open parts of hyperplanes or hyperspheres, depending on the soliton parameter. The finding links the existence of these solitons to conditions involving the Ricci tensor, scalar curvature, and second fundamental form, contributing to the broader understanding of solitons in differential geometry. The findings enhance submanifold theory and geometric flow studies by offering a clear framework for identifying soliton structures and their geometric constraints.

Chapter 5 provides a comprehensive exploration of conformal Ricci-Yamabe solitons within the framework of warped product manifolds, extending the classical understanding of Ricci-Yamabe solitons. The authors rigorously examine the conditions under which such solitons exist, particularly focusing on the interaction of Killing and conformal vector fields with the soliton structure. The result presents key results, such as the classification of warped product manifolds admitting these solitons and their transformation into Einstein manifolds under specific conditions involving concurrent vector fields. The study is mathematically robust, offering well-supported theorems and detailed proofs that contribute to the broader understanding of solitonic structures in geometric flows.

Chapter 6 is a summary and conclusion.

Finally, a list of references is given at the end.