

**A STUDY OF CERTAIN STRUCTURES ON
PARA-CONTACT AND LORENTZIAN
PARA-CONTACT MANIFOLDS**

**A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY**

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**A STUDY OF CERTAIN STRUCTURES ON
PARA-CONTACT AND LORENTZIAN
PARA-CONTACT MANIFOLDS**

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Submitted

**In partial fulfillment of the requirement of the Degree of Doctor of
Philosophy in Mathematics of Mizoram University, Aizawl**

Dedicated to,

*To my beloved father, Rochungnunga(L),
whose words of encouragement to pursue this
journey continue to resonate with me every
day. Your unwavering belief in my potential
and your wisdom guide me still, even in your
absence. I dedicate this work to the memory of
your strength, love, and the dream you planted
in my heart. And to my dearest mother,
Zothanpari, who stands beside me with endless
support, love, and encouragement. Your
sacrifices and constant faith in me have been
the pillars that carried me through this
endeavor. This achievement is as much yours
as it is mine. With all my love and gratitude,
Lalnunenga Colney*



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CERTIFICATE

This is to certify that the thesis entitled “A Study of Certain Structures on Para-contact and Lorentzian Para-contact Manifolds” submitted by Lalnunenga Colney (Registration No: MZU/Ph. D./1782 of 16.09.2021) for the degree of Doctor of Philosophy (Ph.D.) of the Mizoram University, embodies the record of original investigation carried out by him under my supervision. He has been duly registered and the thesis presented is worthy of being considered for the award of the Ph. D. degree. This work has not been submitted for any degree of any other universities.

(Dr. RAJESH KUMAR)

Supervisor

DECLARATION

MIZORAM UNIVERSITY

JUNE, 2025

I, Lalnunenga Colney, hereby declare that the subject matter of this thesis is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in any other University/Institute.

This is being submitted to the Mizoram University for the degree of Doctor of Philosophy in Mathematics.

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PREFACE

The present thesis entitled “**A Study of Certain Structures on Para-contact and Lorentzian Para-contact Manifolds**” is an outcome of the research carried out by the author under the supervision of Dr. Rajesh Kumar, Associate Professor, Department of Mathematics, Pachhunga University College, Mizoram University, Aizawl, Mizoram.

This thesis has been divided into seven chapters and each chapter is subdivided into smaller sections. The first chapter is the general introduction which includes the literature reviews and the basic definitions such as topological manifold, smooth manifold, Vector field, tangent vector, tangent space, covariant derivative, Lie brackets, Lie derivative, exterior derivative, Riemannian manifold, connection on Riemannian manifold, tangent bundle, vertical and complete lifts, curvature tensor, Ricci tensor, important curvature tensors on Riemannian manifold, Ricci soliton, submanifold, almost contact metric manifold, Einstein manifold and η -Einstein manifold, Sasakian manifold, Kenmotsu manifold, 3-dimensional Quasi-Sasakian manifold, Almost Para-contact metric manifold, Lorentzian manifold, Lorentzian Para-contact metric manifold, LP-Sasakian manifold, LP-Kenmotsu manifold, statistical manifold, Sasakian statistical manifold, statistical submanifold.

The second chapter focuses on exploring a new class of semi-symmetric non-metric connections on an LP-Sasakian manifolds. It is structured into seven sections. The first section provides an introduction of a new class of semi-symmetric non-metric connections admitting two 1-forms on a Riemannian manifold. Section 2 provides the existence of the connection on a Riemannian manifold. In section 3, the curvature tensors of LP-Sasakian manifolds are examined with respect to the new connection. Section 4 investigates the conditions under which the manifold becomes an Einstein manifold in relation to the new connection. Section 5 delves into the study of deformation algebra. In section 6, we explore the application of LP-Sasakian manifolds in the general theory of relativity with the newly introduced connection. Lastly, section 7 provides an example to validate our findings.

In the third chapter, we study the geometrical properties of LP-Sasakian manifolds with quarter symmetric non-metric connection (briefly QSNMC) in the tangent bundle. In the first section, we introduce a QSNMC defined by

Prakash and Narain (2011). The complete lifts of LP-Sasakian manifolds and QSNMC in an LP-Sasakian manifolds to its tangent bundle are investigated in sections 2 and 3. In sections 4 and 5, the complete lifts of the curvature tensor, symmetric and skew-symmetric condition of the Ricci tensor in an LP-Sasakian manifolds endowed with QSNMC to its tangent bundle are investigated. The skew-symmetric properties of the projective Ricci tensor and Einstein manifold endowed with QSNMC in an LP-Sasakian manifolds to its tangent bundle are studied in sections 6 and 7. Lastly, an example of the lift of four-dimensional LP-Sasakian manifold to its tangent bundle is shown in section 8.

In the fourth chapter, we study the properties of statistical manifolds and Sasakian statistical manifolds in the tangent bundle. This chapter is divided into two sections. The first section is devoted to the study of the lifts of statistical manifolds and its submanifold in the tangent bundle with semi symmetric non-metric connection (briefly SSNMC) introduced by Yildirim (2022). The second section deals with the study of the lifts of Sasakian statistical manifolds in the tangent bundle with semi symmetric metric connection (briefly SSMC) introduced by Yano (1970).

In the fifth chapter, we obtain some properties of the Kenmotsu manifolds in the tangent bundle, This chapter is divided into two sections , the first section is concerned with the study of the lifts of the Kenmotsu manifolds in the tangent bundle with non-symmetric non-metric connection (briefly NSNMC) introduced by Singh et. al (2022). The second section deals with the study of the lifts of the Kenmotsu manifolds in the tangent bundle with generalized Tanaka Webster connection (briefly GTWC) introduced by Ghosh and De (2017).

The sixth chapter presents a detailed study of the CL -transformation within the frameworks of LP-Kenmotsu manifolds and 3-dimensional quasi-Sasakian manifolds. This chapter is structured into two sections. The first section addresses the CL -transformation on LP-Kenmotsu manifolds, whereas the second section is dedicated to the analysis of the CL -transformation on 3-dimensional quasi-Sasakian manifolds, with particular attention to their Ricci soliton.

In chapter 7, we give the summary and the conclusion. The references of the mention papers have been given with the surname of the author and the years of the publication, which are decoded in chronological order in the Bibliography.

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Chapter 1

Introduction

1.1 Topological Manifold

Definition 1.1. *Suppose M is a topological space. We say that M is a topological manifold of dimension n or a topological n -manifold if it has the following properties:*

1. M is a Hausdorff space.
2. M is second-countable.
3. M is locally Euclidean of dimension n .

1.2 Smooth Manifold

Let M be a topological n -dimensional manifold. A coordinate chart on M is a pair (U, ϕ) , where U is an open subset of M and $\phi : U \rightarrow \bar{U}$ is a homeomorphism from U to an open subset $\bar{U} = \phi(U) \subseteq \mathbb{R}^n$. If (U, ϕ) and (V, ψ) are two charts such that $U \cap V \neq \emptyset$, the composite map $\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V)$ is called the transition map from ϕ to ψ . Two charts (U, ϕ) and (V, ψ) are said to be smoothly compatible if either $U \cap V = \emptyset$ or the transition map $\psi \circ \phi^{-1}$ is a

diffeomorphism. An atlas $\mathcal{A}$ is the collection of charts whose domain cover M . If any two charts in $\mathcal{A}$ are smoothly compatible then $\mathcal{A}$ is said to be smooth atlas. A smooth atlas $\mathcal{A}$ on M is maximal if it is not properly contained in any larger smooth atlas.

Definition 1.2. *If M is a topological manifold, a smooth structure or differentiable structure ($\mathbb{C}^\infty$ -structure) on M is a maximal smooth atlas $\mathcal{A}$. A smooth manifold is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M .*

Example 1.1. *For each non-negative integer n , the Euclidean space $\mathbb{R}^n$ is a smooth n -manifold with the smooth structure determined by the atlas consisting of the single chord $(\mathbb{R}^n, Id \mathbb{R}^n)$.*

Example 1.2. *Let $M(m+n, \mathbb{R}^n)$ be the set of $m+n$ matrices with real entries. Since it is a real vector space of dimension mn under matrix addition and scalar multiplication so $M(m+n, \mathbb{R}^n)$ is a smooth mn -dimension manifold. Similarly, $M(m+n, \mathbb{R}^n)$ of $n \times m$ complex matrices is a vector space of dimension $2nm$ over $\mathbb{R}^n$, and thus it is a smooth manifold of dimension $2mn$.*

Example 1.3. *The general linear group $GL(n, \mathbb{R})$ is the set of invertible $n+n$ matrices with real entries. So $GL(n, \mathbb{R})$ is a smooth n^2 -dimensional manifold.*

1.3 Vector Field

A vector field X on a set A is a mapping that assigns to each p in A a vector X_p in $T_{(p)}$. A vector field X is C^∞ on A , if

1. A is open, and
2. The function Xf at p is C^∞ on $A \cap M$, f being C^∞ real valued function on M^n .

1.4 Tangent Vector

Let M^n be an n -dimensional differentiable manifold and $p \in M^n$ and $C^\infty(p)$ be the set of all real valued C^∞ function on some neighbourhood $\cup$ of p . Let us consider a vector X at p such that

1. $X \in M^n$, $f \in C^\infty(p)$ implies that $Xf \in C^\infty(p)$,
2. $X(f + g) = Xf + Xg$, $f, g \in C^\infty(p)$,
3. $X(fg) = f(Xg) + (Xf)g$, and
4. $X(af) = a(Xf)$, $a \in \mathbb{R}$, then X is called a tangent vector to M^n at p .

1.5 Tangent Space

The set of all tangent vectors at p with operation addition "+" satisfying

$$(X + Y)f = Xf + Yf$$

and operation of scalar multiplication "." satisfying

$$(f \cdot X)g = f \cdot (Xg)$$

is a vector space and this space is called the tangent space to M^n at the point p and is denoted by T_p . The basis of T_p with respect to co-ordinate system $(x^1, x^2, \dots, x^n)$ is $(\frac{\partial}{\partial x^i})$, $i = 1, 2, \dots, n$. Let $T_{(p)}^*$ be the dual space of $T_{(p)}$ whose basis with respect to the basis $(\frac{\partial}{\partial x^i})$ is $(dx^1, dx^2, \dots, dx^n)$. We observe that the elements $T_{(p)}$ are the contravariant vectors and elements of $T_{(p)}^*$ are the covariant vectors with respect to the basis of $T_{(p)}$.

1.6 Covariant Derivative

A linear affine connection on M is a function

$$\nabla : \phi(M) * \phi(M) \rightarrow \phi(M)$$

such that

$$\nabla_{(fX+gY)}Z = f(\nabla_X Z) + g(\nabla_Y Z), \quad (1.1)$$

$$\nabla_X f = Xf, \quad (1.2)$$

$$\nabla_X(fY + gZ) = f(\nabla_X Y) + g(\nabla_X Z) + (Xf)Y + (Xg)Z, \quad (1.3)$$

for arbitrary vector fields X, Y, Z and smooth function f, g on M . ∇_X is called covariant derivative operator and $\nabla_X Y$ is called covariant derivative of Y with respect to X . The covariant derivative of a 1-form η is given by

$$(\nabla_X \eta)(Y) = X(\eta(Y)) - \eta(\nabla_X Y).$$

1.7 Lie Brackets

Let X and Y be arbitrary C^∞ vector field of M^n , then their Lie-bracket is a mapping $[\]: M^n \times M^n \rightarrow M^n$ such that

$$[X, Y]f = X(Y)f - Y(Xf), \quad (1.4)$$

where f is a C^∞ -function. The Lie-brackets has the following properties:

$$[X, Y](f_1 + f_2) = [X, Y]f_1 + [X, Y]f_2, \quad (1.5)$$

$$[X, Y](f_1, f_2) = f_1[X, Y]f_2 + f_2[X, Y]f_1, \quad (1.6)$$

$$[X, Y] + [Y, X] = 0, \quad \text{skew-symmetry} \quad (1.7)$$

$$[X + Y, Z] = [X, Z] + [Y, Z], \quad \text{bilinear} \quad (1.8)$$

$$[X, [Y + Z]] + [Y, [Z + X]] + [Z, [X, Y]] = 0, \quad \text{Jacobian identity} \quad (1.9)$$

and

$$[f_1 X, f_2 Y] = f_1 f_2 [X, Y] + f_1 (X f_2) Y - f_2 (Y f_1) X. \quad (1.10)$$

1.8 Lie Derivative

Let X be a C^∞ vector field on an open set A of M^n . An operator L_X is called the Lie derivative along the vector field X if it is a type preserving mapping

$$L_X : T_s^r \rightarrow T_s^r \quad \text{such that}$$

$$L_X f = X f, \quad f \in F, \quad (1.11)$$

$$L_X a = 0, \quad a \in \mathbb{R}, \quad (1.12)$$

$$L_X Y = [X, Y], \quad Y \in T(p), \quad (1.13)$$

$$(L_X A)(Y) = X(A(Y)) - A([X, Y]), \quad A \in T_{(p)}^* \quad (1.14)$$

and

$$\begin{aligned} & (L_X P)(A_1, \dots, A_r, X_1, \dots, X_s) \\ &= X(P(A_1, \dots, A_r, X_1, \dots, X_s)) \\ &- P(L_X(A_1, \dots, A_r, X_1, \dots, X_s)) \\ &- P((A_1, \dots, A_r, [X, X_1], X_2, \dots, X_s)) \\ &\dots \dots \dots \\ &- P(A_1, \dots, A_r, X_{s-1}, [X, X_s]), P \in T_s^r, \end{aligned} \quad (1.15)$$

where f is a C^∞ function, $X_1, X_2, \dots, X_s$ are vector fields, $A_1, A_2, \dots, A_r$ are 1-forms and P is a tensor field of type (r, s) is called Lie differentiation with respect to X and $L_X P$ is called Lie derivative of P with respect to X .

1.9 Exterior Derivative

Let V_p be the set of all C^∞ P -forms on an open set A . Then the mapping

$$d : V_p \rightarrow V_{p+1}$$

is given by

$$(df)(X) = Xf, \quad X \in T_{(p)}, \quad f \in F \quad (1.16)$$

and

$$\begin{aligned} (dA)(X_1, \dots, X_{p+1}) &= X_1(A(X_2, \dots, X_{p+1})) \\ &- X_2(A(X_1, X_3, \dots, X_{p+1})) \\ &+ X_3(A(X_1, X_2, X_4, \dots, X_{p+1})) \dots \\ &- A([X_1, X_2], X_3, \dots, X_{p+1}) \\ &+ A([X_1, X_3], X_2, \dots, X_{p+1}) \\ &- A([X_2, X_3], X_1, X_4, \dots, X_{p+1}) + \dots \end{aligned} \quad (1.17)$$

for arbitrary C^∞ vector fields $X' \in V^1$ and $A \in V_p$ is called the exterior derivative.

1.10 Riemannian Manifold

The Riemannian metric allows us to define the geometric concepts such as lengths, angles and distances on smooth manifolds. Similar to inner product on vector space, for manifold, the appropriate structure is a Riemannian metric, which is essentially a choice of inner product on each tangent space, varying smoothly from point to point.

Let us consider a C^∞ realvalued, bilinear symmetric, non-singular positive definite function g on the ordered pair $X, Y \in T_{(p)}$ at a point $p \in M^n$, such that

$$g(X, Y) \quad \text{is a real number} \quad (1.18)$$

$$g \text{ is symmetric } \implies g(X, Y) = g(Y, X), \quad (1.19)$$

$$g \text{ is non-similar i.e. } g(X, Y) = 0, \forall Y \neq 0 \implies 0. \quad (1.20)$$

$$g \text{ is positive definite i.e. } g(X, Y) > 0, \forall X \neq 0 \implies 0. \quad (1.21)$$

and

$$g(X + Y, Z) = g(X, Z) + g(Y, Z), \quad (1.22)$$

$$g(aX, Z) = ag(X, Z), \quad a \in R \quad (\text{bilinear}) \quad (1.23)$$

then g is said to be Riemannian metric tensor.

The manifold M^n with a Riemannian metric is called a Riemannian manifold and its geometry is called a Riemannian geometry.

A Riemannian manifold M^n is said to be Einstein manifold if

$$S(X, Y) = \frac{r}{n}g(X, Y). \quad (1.24)$$

A Riemannian manifold M^n is said to be flat manifold if

$$R(X, Y)Z = 0.$$

1.11 Connections on Riemannian Manifold

An affine or linear connection on a smooth manifold M is a R -bilinear mapping

$$\nabla : \chi(M) \times \chi(M) \rightarrow \chi(M),$$

which satisfies the following properties:

$$(i) \quad \nabla_{fX}Y = f\nabla_XY,$$

$$(ii) \quad \nabla_XfY = f\nabla_XY + (Xf)Y,$$

for any $X, Y \in \chi(M)$ and smooth function f . On a Riemannian manifold M

of dimension n , the affine connection ∇ is said to be Levi-Civita connection or Riemannian connection if it satisfies the following:

- (i) ∇ is symmetric or torsion-free i.e., $\nabla_X Y - \nabla_Y X = [X, Y]$ and
- (ii) ∇ is a metric compatible i.e., $(\nabla_X g)(Y, Z) = 0$ for all $X, Y, Z \in \chi(M)$.

Let M^n be an n -dimensional Riemannian manifold with Riemannian metric g . A linear connection $\check{\nabla}$ with torsion tensor $\check{T}$ given by

$$\check{T}(X, Y) = \check{\nabla}_X Y - \check{\nabla}_Y X - [X, Y]. \quad (1.25)$$

satisfies the form

- (i) $\check{T}(X, Y) = \eta(Y)X - \eta(X)Y$ and $(\check{\nabla}_X g)(Y, Z) = 0$, then it is called semi-symmetric metric connection (Friedmann and Schouten, 1924).
- (ii) $\check{T}(X, Y) = \eta(Y)X - \eta(X)Y$ and $(\check{\nabla}_X g)(Y, Z) \neq 0$, then it is called semi-symmetric non-metric connection (Friedmann and Schouten, 1924).
- (iii) $\check{T}(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y$ and $(\check{\nabla}_X g)(Y, Z) \neq 0$, then it is called quarter-symmetric non-metric connection (Friedmann and Schouten, 1924).
- (iv) $\check{T}(X, Y) = 2g(\phi X, Y)\xi$ and $(\check{\nabla}_X g)(Y, Z) \neq 0$, then it is called non-symmetric non-metric connection (Friedmann and Schouten, 1924; Singh et. al, 2022).
- (v) $\check{T}(X, Y) = \eta(X)Y - \eta(Y)X - \eta(X)\phi Y + \eta(Y)\phi X$ and $(\check{\nabla}_X g)(Y, Z) = 0$, then it is called generalized Tanaka Webster connection (Tanno, 1989).

where η is a 1-form and ξ is a vector field given by $\eta(X) = g(X, \xi)$ and ϕ is $(1, 1)$ -tensor field.

1.12 Tangent Bundle

Let M^n be a differentiable manifold and $TM^n = \bigcup_{p \in M^n} T_p M^n$ be the tangent bundle, where $T_p M^n$ is the tangent space at a point $p \in M^n$ and $\pi : TM^n \rightarrow M^n$ is the natural bundle structure of TM^n over M^n . For any co-ordinate system (Q, x^h) in M^n , where (x^h) is a local co-ordinate system in the neighborhood Q , then $(\pi^{-1}(Q), x^h, y^h)$ is a co-ordinate system in TM^n , where (x^h, y^h) is an induced co-ordinate system in $\pi^{-1}(Q)$ from (x^h) (Yano and Ishihara, 1973).

1.13 Vertical and Complete Lifts

Let us define a vector field X , a tensor field f of type $(1, 1)$, a function f , a 1-form ω , and an affine connection ∇ in M^n ; their vertical and complete lifts are denoted by f^v , X^v , ω^v , f^c , ∇^v , and f^c , X^c , ω^c , f^c , ∇^c , respectively. The following formulas of complete and vertical lifts are defined by (Yano and Ishihara, 1973; Khan and Jun, 2017)

$$(fX)^v = f^v X^v, \quad (fX)^c = f^c X^v + f^v X^c, \quad (1.26)$$

$$X^v f^v = 0, \quad X^v f^c = X^c f^v = (Xf)^v, \quad X^c f^c = (Xf)^c, \quad (1.27)$$

$$\omega(f^v) = 0, \quad \omega^v(X^c) = \omega^c(X^v) = \omega(X)^v, \quad \omega^c(X^c) = \omega(X)^c, \quad (1.28)$$

$$f^v X^c = (fX)^v, \quad f^c X^c = (fX)^c, \quad (1.29)$$

$$[X, Y]^v = [X^c, Y^v] = [X^v, Y^c], \quad [X, Y]^c = [X^c, Y^c], \quad (1.30)$$

$$\nabla^c_{X^c} Y^c = (\nabla_X Y)^c, \quad \nabla^c_{X^c} Y^v = (\nabla_X Y)^v. \quad (1.31)$$

Suppose TM is the tangent bundle and let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M , then its vertical and complete lifts in the term of partial differential equations are

$$X^v = X^i \frac{\partial}{\partial y^i} \quad \text{and} \quad X^c = X^i \frac{\partial}{\partial x^i} + \frac{\partial X^i}{\partial x^j} y^j \frac{\partial}{\partial y^i}.$$

1.14 Curvature Tensors

The curvature tensor R with respect to the Riemannian connection ∇ is given by

$$R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

is called Riemann-Christoffel curvature tensor of the second kind *i.e.* of the type $(1, 3)$.

Let $'R$ be the associate curvature tensor of the type $(0, 4)$ of the curvature tensor R . Then

$$'R(X, Y, Z, W) = g(R(X, Y, Z), W),$$

$'R$ is called the Riemann-Christoffel curvature tensor of the first kind.

The following identities are satisfied by associate curvature tensor $'R$: $'R$ is skew-symmetric in first two slot *i.e.*

$$'R(X, Y, Z, W) = - 'R(Y, X, Z, W)$$

$'R$ is skew-symmetric in last two slot *i.e.*

$$'R(X, Y, Z, W) = - 'R(X, Y, W, Z)$$

$'R$ is symmetric in two pair of slot *i.e.*

$$'R(X, Y, Z, W) = 'R(Z, W, X, Y)$$

$'R$ satisfies Bianchi's first identities *i.e.*

$$'R(X, Y, Z, W) + 'R(Y, Z, X, W) + 'R(Z, X, Y, W) = 0$$

and $'R$ satisfies Bianchi's second identities *i.e.*

$$(\nabla_X 'R)(Y, Z, W, V) + (\nabla_Y 'R)(Z, X, W, V) + (\nabla_Z 'R)(X, Y, W, V) = 0.$$

1.15 Ricci Tensor

The tensor defined by

$$S(Y, Z) \stackrel{\text{def}}{=} (C_1^1 R)(Y, Z) - (C_2^1 R)(Z, Y)$$

is called the Ricci-tensor of type $(0, 2)$ where C_1^1 and C_2^1 denote respective contractions. It is symmetric tensor *i.e.*

$$S(X, Y) = S(Y, X).$$

The linear map Q of the type $(1, 1)$ defined by

$$g(QX, Y) \stackrel{\text{def}}{=} S(X, Y)$$

is called a Ricci-map. It is self-adjoint *i.e.*

$$g(QX, Y) = g(X, QY).$$

The scalar r defined by

$$r \stackrel{\text{def}}{=} (C_1^1 R)$$

is called the scalar curvature of M^n at the point p .

1.16 Important Curvature Tensors on Riemannian Manifold

The projective curvature tensor $'P$ of the type $(0, 4)$, $(n > 2)$ is defined by

$$\begin{aligned} 'P(X, Y, Z, W) = 'R(X, Y, Z, W) - \frac{1}{n-1} \{ S(Y, Z)g(X, W) \\ - S(X, Z)g(Y, W) \}. \end{aligned} \quad (1.32)$$

The projective curvature tensor $'P$ satisfies the following identities

- (i) $'P(X, Y, Z, W) = -'P(Y, X, Z, W)$,
- (ii) $C_1^1 P = C_2^1 P = C_3^1 P = 0$,
- (iii) $'P(X, Y, Z, W) + 'P(Y, Z, X, W) + 'P(Z, X, Y, W) = 0$,

where

$$'P(X, Y, Z, W) = g(P(X, Y, Z), W).$$

The concircular curvature tensor $'V$ of type $(0, 4)$, $(n > 3)$ is given by

$$\begin{aligned} 'V(X, Y, Z, W) = 'R(X, Y, Z, W) - \frac{r}{n(n-1)} \{ g(Y, Z)g(X, W) \\ - g(X, Z)g(Y, W) \}. \end{aligned} \quad (1.33)$$

It satisfied the following properties

- (i) $'V(X, Y, Z, W) = -'V(Y, X, Z, W)$,
- (ii) $'V(X, Y, Z, W) = -'V(X, Y, W, Z)$,
- (iii) $'V(X, Y, Z, W) = -'V(Z, W, X, Y)$,
- (iv) $'V(X, Y, Z, W) + 'V(Y, Z, X, W) + 'V(Z, X, Y, W) = 0$,

where

$$'V(X, Y, Z, W) = g(V(X, Y, Z), W).$$

The conharmonic curvature tensor $'L$ of the type $(0, 4)$, $(n > 3)$ is defined as follows

$$\begin{aligned} 'L(X, Y, Z, W) = & 'R(X, Y, Z, W) - \frac{1}{n-1} \{S(Y, Z)g(X, W) \\ & - S(X, Z)g(Y, W) + S(X, W)g(Y, Z) \\ & - S(Y, W)g(X, Z)\}. \end{aligned} \quad (1.34)$$

It satisfies the following properties

- (i) $'L(X, Y, Z, W) = -'L(Y, X, Z, W)$,
- (ii) $'L(X, Y, Z, W) = -'L(X, Y, W, Z)$,
- (iii) $'L(X, Y, Z, W) = -'L(Z, W, X, Y)$,
- (iv) $'L(X, Y, Z, W) + 'L(Y, Z, X, W) + 'L(Z, X, Y, W) = 0$,

where

$$'L(X, Y, Z, W) = g(L(X, Y, Z), W).$$

The conformal curvature tensor $'C$ of the type $(0, 4)$, $(n > 3)$ is defined as

$$\begin{aligned} 'C(X, Y, Z, W) = & R(X, Y, Z, W) - \frac{1}{n-1} \{S(Y, Z)g(X, W) \\ & - S(X, Z)g(Y, W) + S(X, W)g(Y, Z) \\ & - S(Y, W)g(X, Z)\} + \frac{r}{(n-1)(n-2)} \{g(Y, Z)g(X, W) \\ & - g(X, Z)g(Y, W)\}. \end{aligned} \quad (1.35)$$

It satisfies the following identities

- (i) $'C(X, Y, Z, W) = -'C(Y, X, Z, W)$,
- (ii) $'C(X, Y, Z, W) = -'C(X, Y, W, Z)$,
- (iii) $'C(X, Y, Z, W) = 'C(Z, W, X, Y)$,
- (iv) $'C(X, Y, Z, W) + 'C(Y, Z, X, W) + 'C(Z, X, Y, W) = 0$.

where

$$'C(X, Y, Z, W) = g(C(X, Y, Z), W).$$

1.17 Ricci Soliton

A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by (Singh et. al, 2022)

$$L_V g + 2S + 2\lambda g = 0, \quad (1.36)$$

where $L_V g$ is a Lie derivative of Riemannian metric g associated with vector field V and real constant λ . It is known to be shrinking, steady, and expanding if $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$ accordingly.

1.18 Submanifold

Let M and N be smooth manifolds, where $\dim(M) \leq \dim(N)$, let $F : M \rightarrow N$ be a smooth map, and let p be a point in M . We say that F is an immersion at p if the differential map $d_p(F) : T_p(M) \rightarrow T_{F(p)}(N)$ is injective, and that F is an immersion if it is an immersion at every p in M .

Definition 1.3. Suppose $(N, \tilde{g})$ is a Riemannian manifold of dimension m , M is a manifold of dimension n and $\iota : M \rightarrow N$ is an immersion. If M is given the induced Riemannian metric $g := \iota^* \tilde{g}$, then ι is said to be an isometric immersion. If in addition ι is injective, so that M is an immersed submanifold of N , then M is said to be a Riemannian submanifold of N .

1.19 Almost contact Metric Manifolds

A $(2n+1)$ -dimensional smooth manifold M is called an almost contact metric manifold if it admits a $(1, 1)$ -tensor field ϕ , a unit vector field ξ (called the Reeb

vector field) and a 1-form η such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \cdot \phi = 0, \quad (1.37)$$

this is equivalent to a reduction of the structural group of the tangent bundle to $U(n) \times 1$ (Sasaki, 1960; Sasaki and Hatakeyama, 1961). A Riemannian metric g is said to be an associated (or compatible) metric if it satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (1.38)$$

for all $X, Y \in \chi(M)$. An almost contact manifold $M^{2n+1}(\phi, \xi, \eta)$ together with a compatible metric g is known as almost contact metric manifold (Blair, 1976).

1.20 Einstein manifold and η -Einstein manifold

An almost-contact metric manifold M is said to be an η -Einstein manifold if there exists the real valued functions λ_1, λ_2 such that (Sengupta et. al, 2000)

$$S(X, Y) = \lambda_1 g(X, Y) + \lambda_2 \eta(X)\eta(Y). \quad (1.39)$$

For $\lambda_2 = 0$, the manifold M is an Einstein manifold.

1.21 Sasakian Manifold

Let M be a $(2n + 1)$ dimensional differentiable manifold and it is known to admit an almost contact Riemannian structure (ϕ, η, ξ, g) , where ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M

such that

$$\phi\xi = 0, \quad \eta(\xi) = 1, \quad g(\xi, X) = \eta(X), \quad (1.40)$$

$$\phi^2 X = -X + \eta(X)\xi, \quad (1.41)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (1.42)$$

for all vector fields X, Y on M . Also, if (ϕ, η, ξ, g) satisfy

$$d\eta = 0, \quad \nabla_X \xi = \phi X, \quad (1.43)$$

$$(\nabla_X \phi)Y = \eta(Y)X - g(X, Y)\xi, \quad (1.44)$$

where ∇ is the Levi-Civita connection. Then, M is called a Sasakian Manifold (Blair, 1976; Furuhata et al., 2017).

1.22 Kenmotsu Manifold

Consider M as a manifold of dimension $(2n + 1)$ equipped with an almost contact metric structure (ϕ, ξ, η, g) . This structure consists of a $(1, 1)$ tensor field ϕ representing a specific linear transformation on the tangent spaces, a vector field ξ , a 1-form η , and Riemannian metric g on M satisfying

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi(X)) = 0, \quad g(X, \xi) = \eta(X), \quad (1.45)$$

$$\phi^2(X) = -X + \eta(X)\xi, \quad g(X, \phi Y) = -g(\phi X, Y), \quad (1.46)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \quad (1.47)$$

A manifold with an almost contact metric structure (ϕ, ξ, η, g) is classified as a Kenmotsu manifold if and only if it satisfies the following condition (Kenmotsu, 1972):

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \quad (1.48)$$

where ∇ is the Levi-civita connection.

1.23 3-dimensional quasi Sasakian Manifold

Let M be a connected almost contact metric manifold of odd dimension $(2n + 1)$ with an almost contact metric structure (ϕ, ξ, η, g) , where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is an 1-form and g is a Riemannian metric on M . The structure satisfies (Blair, 1976, 2002; Akgun and Acet, 2023):

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0, \quad \phi\xi = 0, \quad (1.49)$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad (1.50)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (1.51)$$

for all vector fields X and Y on M . Blair (Blair, 1967) was the first who introduced quasi Sasakian manifold, that an odd dimensional differentiable manifold M endowed with an almost contact metric structure (ϕ, ξ, η, g) is said to be quasi Sasakian manifold if the almost contact structure (ϕ, ξ, η, g) is normal and the fundamental 2-form Φ defined by $\Phi(X, Y) = g(X, \phi Y)$ is closed *i.e.* $d\Phi = 0$. Specifically, a 3-dimensional almost contact metric manifold M is quasi Sasakian if (Majhi et. al, 2017):

$$\nabla_X \xi = -\beta \phi X, \quad (1.52)$$

for some certain function β on M , where $\xi\beta = 0$ and ∇ is the Levi-Civita connection on M .

1.24 Almost Para-Contact Metric Manifold

Let M be an n -dimensional C^∞ -manifold. If there exist in M^n a tensor field ϕ of the type $(1, 1)$, consisting of a vector field ξ and 1-form η in M^n satisfying

$$\phi^2(X) = X - \eta(X)\xi, \quad \phi(\xi) = 0, \quad \eta(\xi) = 1, \quad (1.53)$$

then, M^n is called an almost Para-contact manifold.

Let g be the Riemannian metric satisfying

$$\eta(X) = g(X, \xi), \quad \eta(\phi X) = 0, \quad (1.54)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (1.55)$$

then the structure (ϕ, ξ, η, g) of (1.53) to (1.55) is called an almost Para-contact Riemannian structure. The manifold with such structure is called an almost Para-contact Riemannian manifold (Sato and Matsumoto, 1976).

1.25 Lorentzian Manifold

An n -dimensional pseudo-Riemannian (or sometime called semi-Riemannian) manifold is a pair (M, g) , where M is an n -dimensional smooth manifold and g is a symmetric, nondegenerate 2-tensor field on M (called the metric). A pseudo-Riemannian manifold is said to be Riemannian if g has signature $(+\dots+)$, and is said to be Lorentzian if g has signature $(- + \dots +)$.

1.26 Lorentzian Para-Contact Metric Manifold

Let M^n be an n -dimensional C^∞ -manifold. If there exist in M^n a tensor field ϕ of the type $(1, 1)$, consisting of a vector field ξ , a 1-form η in M^n and a

Lorentzian metric g satisfying

$$\phi^2(X) = X + \eta(X)\xi, \quad (1.56)$$

$$\eta(\xi) = -1, \quad (1.57)$$

$$\eta(X) = g(X, \xi), \quad \eta(\phi X) = 0, \quad (1.58)$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \quad (1.59)$$

for arbitrary vector field X and Y , then M^n is called a Lorentzian Para-contact (LP-contact) manifold and the structure (ϕ, ξ, η, g) is called the Lorentzian Para-contact structure (Matsumoto, 1989).

1.27 LP-Sasakian Manifold

Let M^n be a Lorentzian Para-contact manifold with structure (ϕ, ξ, η, g) . Then it satisfies

$$\phi(\xi) = 0, \quad \eta(\phi X) = 0, \quad (1.60)$$

$$\text{rank}(\phi) = n - 1. \quad (1.61)$$

A LP-contact manifold is called a Lorentzian Para-Sasakian manifold if (Matsumoto and Mihai, 1988)

$$\nabla_X \xi = \phi X, \quad (1.62)$$

and

$$(\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \quad (1.63)$$

where ∇_X denotes the covariant differentiation with respect to g . If we take

$$\Phi(X, Y) = g(X, \phi Y) \quad (1.64)$$

for any vector fields X and Y , then the tensor field $\Phi(X, Y)$ is a symmetric $(0, 2)$ tensor field (Matsumoto, 1989). Since the 1-form η is closed in an LP-Sasakian

manifold, we have (Matsumoto, 1989; Mihai et. al, 1999)

$$(\nabla_X \eta)(Y) = \Phi(X, Y), \quad \Phi(X, \xi) = 0 \quad (1.65)$$

for all $X, Y \in \chi(M)$, and ∇ is the Levi-civita connection.

1.28 LP-Kenmotsu Manifold

In a Lorentzian almost Para-contact manifold, for any vector fields X, Y on M^n , we have

$$\Phi(X, Y) = \Phi(Y, X), \quad (1.66)$$

where $\Phi(X, Y) = g(X, \phi Y)$ is a symmetric $(0, 2)$ tensor field.

A Lorentzian almost Para-contact manifold M^n is called Lorentzian Para-Kenmotsu manifold if (Haseeb and Prasad, 2021)

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \quad (1.67)$$

for all $X, Y \in \chi(M^n)$, where $\chi(M^n)$ is the set of all differentiable vector field on M^n and ∇ is known to be the operator of covariant differentiation with respect to the Lorentzian metric g .

1.29 Statistical Manifold

In an n -dimensional Riemannian manifold (M^n, g) with Riemannian metric g , we consider $\ddot{\nabla}$ be its affine connection and let ∇ be the Levi-Civita connection. The structure $(M, \ddot{\nabla}, g)$ is said to be a statistical manifold if $\ddot{\nabla}$ is an affine and torsion-free connection and $\ddot{\nabla}g$ satisfied the Codazzi equation

$$(\ddot{\nabla}_X g)(Y, Z) = (\ddot{\nabla}_Y g)(X, Z), \quad (1.68)$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the set of all tangent vector fields on M . We know that there exists an affine connection $\ddot{\nabla}^*$ which is the dual of $\ddot{\nabla}$ with respect to g such that (Yildirim, 2022)

$$Xg(Y, Z) = g(\ddot{\nabla}_X Z) + g(Y, \ddot{\nabla}_X^* Z). \quad (1.69)$$

Also the pair of connections $\ddot{\nabla}$ and $\ddot{\nabla}^*$ satisfies $(\ddot{\nabla}^*)^* = \ddot{\nabla}$, one can obtain

$$\nabla = \frac{1}{2}(\ddot{\nabla} + \ddot{\nabla}^*). \quad (1.70)$$

The tensor field K of type $(1, 2)$ on $(M, \ddot{\nabla}, g)$ is defined by

$$K_X Y = \ddot{\nabla}_X Y - \nabla_X Y, \quad K_X Y = \frac{1}{2}(\ddot{\nabla}_X Y - \ddot{\nabla}_X^* Y), \quad (1.71)$$

and K is symmetric which gives

$$g(K_X Y, Z) = g(K_X Z, Y), \quad K_X Y = K_Y X. \quad (1.72)$$

The statistical curvature tensor field with respect to $\ddot{\nabla}$ is defined as

$$\ddot{R}(X, Y)Z = \ddot{\nabla}_X \ddot{\nabla}_Y Z - \ddot{\nabla}_Y \ddot{\nabla}_X Z - \ddot{\nabla}_{[X, Y]} Z. \quad (1.73)$$

By replacing $\ddot{\nabla}$ with $\ddot{\nabla}^*$, we can obtain the statistical curvature tensor field $\ddot{R}^*$.

The curvature tensor fields $\ddot{R}$ and $\ddot{R}^*$ satisfy

$$\ddot{R}(X, Y)Z = -\ddot{R}(Y, X)Z, \quad \ddot{R}^*(X, Y)Z = -\ddot{R}^*(Y, X)Z, \quad (1.74)$$

$$g(\ddot{R}(X, Y)Z, W) = -g(\ddot{R}^*(X, Y)W, Z), \quad (1.75)$$

$$\ddot{R}(X, Y)Z + \ddot{R}(Y, Z)X + \ddot{R}(Z, X)Y = 0.$$

1.30 Sasakian Statistical Manifold

In an n dimensional statistical and Sasakian manifolds defined in (1.68-1.44), a quadruple $(\ddot{\nabla}, g, \phi, \xi)$ is known as Sasakian statistical structure on M , if $(\ddot{\nabla}, g)$

is a statistical structure and (g, ϕ, ξ) is a Sasakian structure on M and the formula

$$K(X, \phi Y) + \phi K(X, Y) = 0, \quad (1.76)$$

holds for any vector fields X and Y on M (Furuhata et. al, 2017). In a statistical structure $(\ddot{\nabla}, g)$ and an almost contact metric structure (g, ϕ, ξ) on M , the structure $(\ddot{\nabla}, g, \phi, \xi)$ is known as Sasakian statistical structure if and only if it satisfies the following formulas (Kazan and Kazan, 2018):

$$\ddot{\nabla}_X \phi Y - \phi \ddot{\nabla}_X^* Y = g(Y, \xi)X - g(Y, X)\xi, \quad (1.77)$$

$$\ddot{\nabla}_X \xi = \phi X + g(\ddot{\nabla}_X \xi, \xi)\xi. \quad (1.78)$$

1.31 Statistical submanifold

In a statistical manifold M let N be its submanifold with g which is an induced metric. Also $\nabla^\odot$ and $h^\odot$ are the induced connection and the second form on N respectively. Then the Gauss and Weingarten formulas for the Levi-Civita connection are

$$\nabla_X Y = \nabla_X^\odot Y + h^\odot(X, Y), \quad \nabla_X Q = -A_Q^\odot X + D_X^\odot Q, \quad (1.79)$$

for all $X, Y \in \chi(N)$ and $Q \in \chi(N^\perp)$ in which $A^\odot$ is the shape operator and $D^\odot$ is the normal connection on $N^\perp$. According to the statistical connections on $\ddot{\nabla}$ and $\ddot{\nabla}^*$, the Gauss and Weingarten are given by (Aydin et. al, 2017)

$$\ddot{\nabla}_X Y = \acute{\nabla}_X Y + h(X, Y), \quad \ddot{\nabla}_X Q = -A_Q X + D_X Q, \quad (1.80)$$

$$\ddot{\nabla}_X^* Y = \acute{\nabla}_X^* Y + h^*(X, Y), \quad \ddot{\nabla}_X^* Q = -A_Q^* X + D_X^* Q, \quad (1.81)$$

for all $X, Y \in \chi(N)$ and $Q \in \chi(N^\perp)$ in which $\acute{\nabla}, \acute{\nabla}^*$ are statistical and dual connections on N (Calin and Udriste, 2014). A and A^* denotes the shape operators D and D^* are the normal connections on $N^\perp$ and h, h^* are second fundamental

forms on N respectively. From (1.80) and (1.81) we get

$$g(A_Q X, Y) = g(h(X, Y), Q), \quad g(A_Q^* X, Y) = g(h^*(X, Y), Q). \quad (1.82)$$

If the second fundamental forms vanish then N is regarded to be as totally geodesic submanifold with respect to $\ddot{\nabla}$ and $\ddot{\nabla}^*$. Furthermore, if

$$h(X, Y) = Hg(X, Y), \quad h^*(X, Y) = H^*g(X, Y), \quad (1.83)$$

then the submanifold is called totally umbilical submanifold, where H and H^* are the mean curvature vectors admitting $\ddot{\nabla}$ and $\ddot{\nabla}^*$.

1.32 Review of Literature

The study of semi-symmetric connections (SSC) on differentiable manifolds was initiated by Friedman and Schouten (1924). Agashe (1992) introduced a semi-symmetric non-metric connection (SSNMC) on Riemannian manifolds, which was further investigated by several authors, including De and Kamilya (1995), De et al. (1996), Biswas and De (1997), Singh (2011), Singh and Pandey (2013), among others. In the early 1930s, Friedman and Schouten (1927) and Hayden (1932) began studying semi-symmetric metric and linear connections on differentiable manifolds. A connection $\check{\nabla}$ is called symmetric or torsion-free if its torsion tensor $\check{T}$ vanishes; otherwise, it is non-symmetric. In an n -dimensional manifold M^n , if the torsion tensor of a linear connection satisfies $\check{T}(X, Y) = 2g(\Phi X, Y)$ for all vector fields X, Y on M^n , the connection is considered non-symmetric. If, in addition, $\check{\nabla}g = 0$, it is called a metric connection; otherwise, it is non-metric.

The concept of a quarter-symmetric connection (QSC) in a Riemannian manifold with an affine connection was introduced by Golab (1975) and has since been studied by various researchers such as Yano and Imai (1982), Rastogi (1978, 1987), Mishra and Pandey (1980), Mukhopadhyay et al. (1991), Biswas and De

(1997a), Sengupta and Biswas (2003), Singh and Pandey (2007), among others.

The canonical affine connection known as the Tanaka–Webster connection was established on non-degenerate pseudo-Hermitian CR-manifolds by Tanaka (1976) and Webster (1978). Tanno (1989) extended this by introducing the generalized Tanaka–Webster connection (GTWC) for contact metric manifolds, which coincides with the classical version when the CR-structure is integrable. The GTWC has been applied in studying real hypersurfaces in complex space forms (Takagi, 1975). Kenmotsu manifolds, introduced by Kenmotsu (1972), have been the subject of recent GTWC-related investigations by various authors (Ghosh and De, 2017; Acet et al., 2013; Prakasha and Hadimani, 2018). A wide range of connection types have been defined and studied in works such as (Aktan, 2008; Beyendi et al., 2020, 2020a; Chaubey, 2007; Chaubey et al., 2020; Kumar et al., 2011; Mishra and Pandey, 1980; Pankaj et al., 2020; Rastogi, 1978; Sengupta et al., 2000; Siddiqi, 2018; Tripathi, 2008; Yano, 1970; Yano and Imai, 1982; Yildirim, 2022; Yoldas et al., 2019, 2023).

The concept of an LP-Sasakian manifolds, a subclass of almost Para-contact metric manifolds, was introduced by Matsumoto (1989) and independently by Mihai and Roşca (1992), who derived several key results. Lorentzian Para-Sasakian manifold have been explored by Matsumoto and Mihai (1988), Mihai et al. (1999), Shaikh and Biswas (2004), Tripathi and De (2001), Singh and Pandey (2013), Prasad and Haseeb (2016), Kumar and Bagewadi (2015), Perktas et al. (2010, 2010a), Chaubey and De (2019a), Prakash and Narain (2011), Barman (2018), and others.

Kenmotsu (1972) introduced a new class of almost contact manifolds, now known as Kenmotsu manifolds, which have since been studied by Sinha and Pandey (1991), Cihan and De (2006), among others. Sinha and Prasad (1995) introduced Para-Kenmotsu (P-Kenmotsu) and special Para-Kenmotsu (SP-Kenmotsu) manifolds as analogues of P-Sasakian and SP-Sasakian manifolds. In 2021, Haseeb

and Prasad introduced Lorentzian Para-Kenmotsu (LP-Kenmotsu) manifolds, and Prasad et al. (2021) examined ϕ -semi-symmetric LP-Kenmotsu manifolds with quarter-symmetric non-metric connection (QSNMC) admitting Ricci solitons.

Quasi-Sasakian manifolds, introduced by Blair (1967) to unify Sasakian and cosymplectic structures, have been examined by Gonzalez and Chinea (1989), Kanemaki (1977, 1979), De and Sarkar (2009), De et al. (2012), Turan et al. (2019), and others. In three dimensions, Olszak (1986) defined a structure function β for such manifolds and later (1996) established conditions for conformal flatness. He showed that with constant β , the manifold is either locally a product of $\mathbb{R}$ and a 2-dimensional Kähler space (cosymplectic case), or it has constant positive curvature (non-cosymplectic, homothetic to Sasakian).

S. Amari (1985) pioneered the use of differential geometry in statistical inference, introducing the notion of statistical manifolds, where each point represents a probability distribution. Such manifold may even be infinite-dimensional. Applications of statistical manifolds span time series, convex analysis, dynamical systems, PDEs, and more (Abdel-All et al., 2003). Kurose (2004) extended this to holomorphic statistical structures, while Furuhata (2009), Furuhata et al. (2017a), and Bagher et al. (2021) studied Sasakian and Kenmotsu statistical structures. Kazan and Kazan (2018) investigated semi-symmetric metric connection (SSMC) on Sasakian statistical manifolds. More recently, Mustafa (2022) studied semi-symmetric non-metric connection (SSNMC) on statistical manifolds.

In the theory of relativity, spacetime is modeled as a 4-dimensional Lorentzian manifold equipped with a Lorentzian metric, which encodes curvature and causality. The Riemann curvature tensor, Ricci tensor, and scalar curvature are fundamental in describing gravity. Cosmological observations suggest that matter in spacetime is isotropic and homogeneous. De et al. (2017) demonstrated that

under certain conditions, spacetime is filled with an isotropic and homogeneous fluid. Chaubey and De (2019) studied LP-Sasakian manifolds admitting a new type of quarter-symmetric non-metric ξ -connection in the context of general relativity. Other contributions in this area include those of Güler and Demirbağ (2016), Mantica et al. (2019), and Zengin (2012).

Tangent bundle geometry is a classical yet continuously evolving topic in differential geometry. The theory of vertical, complete, and horizontal lifts from a manifold to its tangent bundle was formalized by Yano and Ishihara (1973). Related studies have been conducted by Yano and Kobayashi (1966), Tani (1969), Pandey and Chaturvedi (2018), and Khan (2020, 2021). Metallic structure lifts on tangent bundles have been explored by Khan et al. (2022), and Khan and De (2022, 2022a). Tangent bundles equipped with quarter-symmetric and semi-symmetric connections on various manifolds (Hermitian, Kähler, Kenmotsu, Sasakian, Para-Sasakian, statistical, and their submanifolds) have been studied in (De and Khan, 2022; Khan, 2016, 2020, 2021a, 2022, 2022a, 2023, 2023a; Khan and Jun, 2017; Kumar et al., 2023b; Khan et al., 2023a).

A loxodrome is a curve on the unit sphere intersecting all meridians at a constant angle, traditionally used in navigation and also known as a rhumb line. Tashiro and Tachibana (1963) extended this concept to C-loxodromes in almost contact manifolds with affine connections. A C-loxodrome intersects the geodesic trajectories of the characteristic vector field of a Sasakian manifold at a constant angle. A conformal transformation preserves angles but not necessarily distances or curvature; the conformal curvature tensor remains invariant under such transformations (Weyl, 1978). The CL -transformation was introduced by Tashiro and Tachibana (1963), where the angle between two C-loxodromes remains invariant. Koto and Nagao (1966) identified a tensor field invariant under CL -transformations. Takamatsu and Mizusawa (1966) further showed that infinitesimal CL -transformations on compact Sasakian manifolds are necessar-

ily projective. Matsumoto and Mihai (1988) investigated invariant tensors under CL -transformations on LP-Sasakian manifolds. More recently, Shaikh and Ahmad (2014) studied such transformations in Lorentzian concircular structure manifolds, obtaining new invariant tensors.

Ricci solitons generalize Einstein manifolds and are intimately related to the Ricci flow, introduced by Hamilton (1988), defined as $\frac{\partial g}{\partial t} = -2S$. A Ricci soliton (g, V, γ) on a Riemannian manifold (M^{2n+1}, g) satisfies

$$L_V g + 2S + 2\lambda g = 0, \quad (1.84)$$

where $L_V g$ is the Lie derivative of g along the vector field V , S is the Ricci tensor, and λ is a constant. Based on the sign of λ , the soliton is classified as shrinking ($\lambda < 0$), steady ($\lambda = 0$), or expanding ($\lambda > 0$). Sharma (2008) investigated Ricci solitons in contact geometry, while Ghosh et al. (2008) studied gradient Ricci solitons on non-Sasakian (κ, μ) -contact manifolds. Since Sasaki (1960) introduced Sasakian manifolds, many researchers have studied Ricci solitons on such spaces (He and Zhu, 2011; Singh et al., 2023; Pandey et al., 2022; Roy et al., 2019; Krishnendu, 2020; Yoldas et al., 2023). More recently, Ricci solitons on 3-dimensional Quasi-Sasakian manifolds have been explored by Sarkar et al. (2019), De and Mondal (2012), and Prakasha et al. (2022). Motivated by these studies, we investigate Ricci solitons under CL -transformations on 3-dimensional quasi Sasakian manifolds.

Chapter 2

A new class of semi-symmetric non-metric connection (SSNMC) on LP-Sasakian manifolds

2.1 Introduction

The linear connection $\check{\nabla}$ in an n -dimensional Riemannian manifold M is said to be semi-symmetric connection (briefly SSC) if its torsion tensor $\check{T}(X, Y) = \check{\nabla}_X Y - \check{\nabla}_Y X - [X, Y]$, satisfies the condition $\check{T}(X, Y) = \lambda(Y)X - \lambda(X)Y$, for all arbitrary vector fields X, Y on M and λ is 1-form (Friedman and Schouten, 1924). Moreover, if the Riemmanian metric g satisfies $(\check{\nabla}g) \neq 0$, then the SSC $\check{\nabla}$ is called non-metric connection (Prvanovic, 1975: Andonie, 1976).

Let us define a linear connection on an n -dimensional Riemannian manifold M by

$$\check{\nabla}_X Y = \nabla_X Y + \eta(Y)X - a(X)Y - g(X, Y)\xi + g(X, Y)A \quad (2.1)$$

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where η and a are 1-form associated with vector field ξ and A on M^n given by

$$g(X, \xi) = \eta(X), \quad (2.2)$$

$$g(X, A) = a(X). \quad (2.3)$$

for all vector field $X \in \chi(M^n)$, where $\chi(M^n)$ is the set of all differentiable vector fields on M^n . Also the 1-forms η and a are related by

$$\lambda(X) = \eta(X) + a(X) \quad (2.4)$$

The torsion tensor $\check{T}$ associated with the connection of equation (2.1) is given by

$$\check{T}(X, Y) = \lambda(Y)X - \lambda(X)Y \quad (2.5)$$

Any linear connection satisfying equation (2.5) is referred to semi-symmetric connection by Friedman and Schouten (1924). Further by using equation (2.1), we obtain

$$(\check{\nabla}_X g)(Y, Z) = 2a(X)g(Y, Z) - a(Z)g(X, Y) - a(Y)g(X, Z) \quad (2.6)$$

Any linear connection defined by equation (2.1) meets the conditions of the equations (2.5) and (2.6). So, we refer $\check{\nabla}$ as semi-symmetric non-metric connection (briefly SSNMC).

Conversely we can obtain equation (2.1) from the equations (2.5) and (2.6).

We carry on as follows:

$$\begin{aligned} g(\check{\nabla}_X Y, Z) &= \check{\nabla}_X g(Y, Z) - (\check{\nabla}_X g)(Y, Z) - g(Y, \check{\nabla}_X Z) \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - 2a(X)g(Y, Z) \\ &\quad + a(Z)g(X, Y) + a(Y)g(X, Z) - g(Y, \check{T}(X, Z)) \\ &\quad + \check{\nabla}_Z X + [X, Z]. \end{aligned} \quad (2.7)$$

Using equation (2.4), we obtain

$$\begin{aligned}
g(\breve{\nabla}_X Y, Z) &= g(\nabla_X Y, Z) - 2a(X)g(Y, Z) + a(Z)g(X, Y) \\
&\quad + a(Y)g(X, Z) - \lambda(Z)g(Y, X) + \lambda(X)g(Y, Z) \\
&\quad - g(Y, \breve{\nabla}_Z X) + g(Y, \nabla_Z X).
\end{aligned} \tag{2.8}$$

Continuing in the same manner as above, we obtain

$$\begin{aligned}
g(\breve{\nabla}_Z X, Y) &= g(\nabla_Z X, Y) - 2a(Z)g(X, Y) + a(Y)g(Z, X) \\
&\quad + a(X)g(Z, Y) - \lambda(Y)g(X, Z) + \lambda(Z)g(X, Y) \\
&\quad - g(X, \breve{\nabla}_Y Z) + g(X, \nabla_Y Z).
\end{aligned} \tag{2.9}$$

Using equation (2.9) in equation (2.8), we obtain

$$\begin{aligned}
g(\breve{\nabla}_X Y, Z) &= g(\nabla_X Y, Z) - 3a(X)g(Y, Z) + 3a(Z)g(X, Y) \\
&\quad - 2\lambda(Z)g(X, Y) + \lambda(X)g(Y, Z) + \lambda(Y)g(X, Z) \\
&\quad + g(X, \breve{\nabla}_Y Z) - g(X, \nabla_Y Z).
\end{aligned} \tag{2.10}$$

Adding the value of $g(X, \breve{\nabla}_Y Z)$ to equation (2.10) as well, we obtain

$$\begin{aligned}
g(\breve{\nabla}_X Y, Z) &= g(\nabla_X Y, Z) - 3a(X)g(Y, Z) + 3a(Z)g(X, Y) \\
&\quad - 2\lambda(Z)g(X, Y) + \lambda(X)g(Y, Z) + \lambda(Y)g(X, Z) \\
&\quad + g(\nabla_Y Z, X) + \eta(Z)g(Y, X) - a(Y)g(Z, X) \\
&\quad + a(X)g(Y, Z) - \eta(X)g(Y, Z) - g(X, \nabla_Y Z).
\end{aligned} \tag{2.11}$$

Using equation (2.4) in equation (2.11), we have

$$\begin{aligned}
g(\breve{\nabla}_X Y, Z) &= g(\nabla_X Y, Z) + \eta(Y)g(X, Z) - a(X)g(Y, Z) \\
&\quad - \eta(Z)g(X, Y) + a(Z)g(X, Y).
\end{aligned} \tag{2.12}$$

Thus

$$\breve{\nabla}_X Y = \nabla_X Y + \eta(Y)X - a(X)Y - g(X, Y)\xi + g(X, Y)A. \tag{2.13}$$

2.2 Existence of a SSNMC $\overset{\circ}{\nabla}$ admitting two 1-form on a Riemannian manifold

Let X, Y, Z be any three vector fields on a Riemannian manifold M^n , the SSNMC $\overset{\circ}{\nabla}$ is determined by the following

$$\begin{aligned}
2g(\overset{\circ}{\nabla}_X Y, Z) = & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\
& - g([X, Z], Y) - g([Y, X], Z) + g([Z, Y], X) \\
& + g(\eta(Y)Z - \eta(Z)Y, X) + g(\eta(Y)X - \eta(X)Y, Z) \\
& + g(\eta(X)Z - \eta(Z)X, Y) + g(a(Z)Y - a(Y)Z, X) \\
& + g(a(Z)X - a(X)Z, Y) + g(a(Y)X - a(X)Y, Z)
\end{aligned} \tag{2.14}$$

which holds for all vector fields $X, Y, Z \in \chi(M^n)$. It can be easily verified that the mapping

$$\overset{\circ}{\nabla} : (X, Y) \rightarrow \overset{\circ}{\nabla}_X Y$$

satisfies the following equalities

$$\overset{\circ}{\nabla}_X (Y + Z) = \overset{\circ}{\nabla}_X Y + \overset{\circ}{\nabla}_X Z, \tag{2.15}$$

$$\overset{\circ}{\nabla}_{X+Y} Z = \overset{\circ}{\nabla}_X Z + \overset{\circ}{\nabla}_Y Z, \tag{2.16}$$

$$\overset{\circ}{\nabla}_{fX} Y = f \overset{\circ}{\nabla}_X Y, \tag{2.17}$$

$$\overset{\circ}{\nabla}_X (fY) = f \overset{\circ}{\nabla}_X Y + (Xf)Y. \tag{2.18}$$

for all $X, Y, Z \in \chi(M^n)$ and for all $f \in F(M^n)$, the set of all differentiable mapping over M^n . From equations (2.15), (2.16), (2.17) and (2.18) we can conclude that $\overset{\circ}{\nabla}$ determines a linear connection on M^n .

Now, from equation (2.14), we have

$$\begin{aligned}
2g(\overset{\vee}{\nabla}_X Y, Z) - 2g(\overset{\vee}{\nabla}_Y X, Z) &= 2g([X, Y], Z) + 2\eta(Y)g(X, Z) \\
&\quad - 2\eta(X)g(Y, Z) + 2a(Y)g(X, Z) \\
&\quad - 2a(X)g(Y, Z).
\end{aligned} \tag{2.19}$$

Hence,

$$\overset{\vee}{\nabla}_X Y - \overset{\vee}{\nabla}_Y X - [X, Y] = \eta(Y)X - \eta(X)Y + a(Y)X - a(X)Y, \tag{2.20}$$

or

$$\overset{\vee}{T}(X, Y) = \lambda(Y)X - \lambda(X)Y. \tag{2.21}$$

Also, we have from equation (2.14)

$$\begin{aligned}
2g(\overset{\vee}{\nabla}_X Y, Z) + 2g(\overset{\vee}{\nabla}_X Z, Y) &= 2Xg(Y, Z) + 2a(Z)g(X, Y) \\
&\quad + 2a(Y)g(X, Z) - 4a(X)g(Y, Z)
\end{aligned} \tag{2.22}$$

which gives

$$(\overset{\vee}{\nabla}_X g)(Y, Z) = 2a(X)g(Y, Z) - a(Z)g(X, Y) - a(Y)g(X, Z). \tag{2.23}$$

From equations (2.21) and (2.23) it follows that $\overset{\vee}{\nabla}$ determines a SSNMC $\overset{\vee}{\nabla}$ admitting two 1-forms on (M^n, g) . It can be easily verified that $\overset{\vee}{\nabla}$ determines a unique SSNMC $\overset{\vee}{\nabla}$ admitting two 1-forms on a Riemannian manifold M . Hence, we can state the following theorem

Theorem 2.1. *Let M^n be an n -dimensional Riemannian manifold. Then there exists a unique linear connection $\overset{\vee}{\nabla}$ satisfying equations (2.5) and (2.6)*

The above theorem proves the existence of a SSNMC $\overset{\vee}{\nabla}$ admitting two 1-forms on a Riemannian manifold.

2.3 SSNMC on LP-Sasakian manifolds

In an n dimensional LP-Sasakian manifold defined in equations (1.56-1.65), the following relations hold (Matsumoto, 1989; Mihai et. al, 1999)

$$g(R(X, Y)Z, \xi) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \quad (2.24)$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \quad (2.25)$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad (2.26)$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \quad (2.27)$$

$$S(X, \xi) = (n - 1)\eta(X), \quad (2.28)$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y), \quad (2.29)$$

for any arbitrary vector fields X, Y, Z on M^n , where R and S are the Riemannian curvature and the Ricci tensors of the manifold, respectively.

Let $\check{R}$ and R be the curvature tensor of the connection $\check{\nabla}$ and ∇ respectively, then

$$\check{R}(X, Y)Z = \check{\nabla}_X \check{\nabla}_Y Z - \check{\nabla}_Y \check{\nabla}_X Z - \check{\nabla}_{[X, Y]}Z. \quad (2.30)$$

Using equations (2.1) and (2.30), we obtain

$$\begin{aligned} \check{R}(X, Y)Z &= R(X, Y)Z - s(Y, Z)X + s(X, Z)Y - g(Y, Z)BX \\ &\quad + g(X, Z)BY - 2da(Y, X)Z, \end{aligned} \quad (2.31)$$

where s is a tensor field of type $(0, 2)$ defined by

$$s(Y, Z) = g(\phi Y, Z) - \eta(Y)\eta(Z) - \frac{1}{2}\eta(A)g(Y, Z) - \frac{1}{2}g(Y, Z) \quad (2.32)$$

and

$$\begin{aligned} BY &= \phi Y - \frac{1}{2}Y + a(Y)\xi - \eta(Y)\xi - \nabla_Y A - \frac{1}{2}\eta(A)Y - a(Y)A \\ &\quad + \eta(Y)A. \end{aligned} \quad (2.33)$$

The Ricci tensor $\check{S}$ of SSNMC $\check{\nabla}$ is defined by

$$\check{S}(Y, Z) = \text{Trace of the map: } X \rightarrow R(X, Y)Z.$$

Then from equation (2.31), we obtain

$$\begin{aligned} \check{S}(Y, Z) &= S(Y, Z) - (n-1)s(Y, Z) - g(Y, Z)\{div \xi - div A \\ &\quad - \frac{1}{2}(n-4) - \frac{1}{2}(n-4)\eta(A)\} + g(BY, Z) + 2da(Y, Z). \end{aligned} \quad (2.34)$$

From equation (2.34), it follows that

$$\begin{aligned} \check{r} &= r - (n-1)trace(s) - (n-1)\{div \xi - div A - \frac{1}{2}(n-4) \\ &\quad - \frac{1}{2}(n-4)\eta(A)\}. \end{aligned} \quad (2.35)$$

Theorem 2.2. *If an LP-Sasakian manifold admitting SSNMC $\check{\nabla}$ defined by equation (2.1) containing two 1-forms, then the curvature tensor $\check{R}$, Ricci tensor $\check{S}$ and the scalar curvature $\check{r}$ of the manifold M with respect to the SSNMC $\check{\nabla}$ on an LP-Sasakian manifold are given by equations (2.34) and (2.35).*

Lemma 2.1. *In an LP-Sasakian manifold M^n with the SSNMC $\check{\nabla}$ admitting two 1-forms. We have*

- (i) $(\check{\nabla}_X \phi)Y = (\nabla_X \phi)Y - \eta(Y)\phi X - g(X, \phi Y)\xi + g(X, \phi Y)A,$
- (ii) $\check{\nabla}_X \xi = -X + \phi X - \lambda(X)\xi + \eta(X)A,$
- (iii) $(\check{\nabla}_X \eta)Y = \Phi(X, Y) - \eta(X)\eta(Y) + a(X)\eta(Y) - g(X, Y) \\ - g(X, Y)\eta(A).$

Proof. We know

$$(\check{\nabla}_X \phi)Y = \check{\nabla}_X \phi Y - \phi(\check{\nabla}_X Y). \quad (2.36)$$

Using equation (2.1) in equation (2.36) we get (i). Using equation (2.1), we get (ii). We know

$$\begin{aligned} (\breve{\nabla}_X \eta)Y &= \breve{\nabla}_X \eta(Y) - \eta(\breve{\nabla}_X Y) \\ &= (\breve{\nabla}_X g)(Y, \xi) + g(\breve{\nabla}_X Y, \xi) + g(Y, \breve{\nabla}_X \xi) - \eta(\breve{\nabla}_X Y). \end{aligned} \quad (2.37)$$

And, we also know

$$(\breve{\nabla}_X g)(Y, \xi) = 2a(X)\eta(Y) - a(Y)\eta(X) - g(X, Y)\eta(A). \quad (2.38)$$

Putting equation (2.38) in equation (2.37), we get (iii). $\square$

Corollary 2.1. *The tensor field s in an LP-Sasakian manifold satisfies*

$$s(X, \xi) = \frac{1}{2}\eta(X)\{1 - \eta(A)\} \quad (2.39)$$

Proof. Using equation (2.32) and equation (2.2), we get equation (2.39). $\square$

Let R and $\breve{R}$ be the curvature tensors of type (0,4) given by

$$R(X, Y, Z, U) = g(R(X, Y)Z, U) \quad (2.40)$$

and

$$\breve{R}(X, Y, Z, U) = g(\breve{R}(X, Y)Z, U). \quad (2.41)$$

Theorem 2.3. *In an LP-Sasakian manifold with SSNMC $\breve{\nabla}$ containing two 1-forms we have*

$$\begin{aligned} (i) \quad & \breve{R}(X, Y)Z = -\breve{R}(Y, X)Z, \\ (ii) \quad & \breve{R}(X, Y)Z + \breve{R}(Y, Z)X + \breve{R}(Z, X)Y = 0, \\ (iii) \quad & \breve{K}(X, Y, Z, U) + \breve{K}(X, Y, U, Z) = s(X, Z)g(Y, U) - s(Y, Z)g(X, U) \\ & + s(X, U)g(Y, Z) - s(Y, U)g(X, Z) + g(X, Z)g(BY, U) - g(Y, Z)g(BX, U) \\ & + g(X, U)g(BY, Z) - g(Y, U)g(BX, Z) - 4da(Y, X)g(Z, U), \end{aligned}$$

$$\begin{aligned}
(iv) \quad \check{K}(X, Y, Z, U) - \check{K}(Z, U, X, Y) &= -s(Y, Z)g(X, U) + s(U, X)g(Z, Y) \\
&\quad - g(X, Z)\{g(BU, Y) - g(BY, U)\} + g(U, X)g(BZ, Y) - g(Y, Z)g(BX, U) \\
&\quad + 2da(U, Z)g(X, Y) - 2da(Y, X)g(Z, U),
\end{aligned}$$

$$(v) \quad \check{K}(X, Y, Z, U) + \check{K}(X, Z, U, Y) + \check{K}(X, U, Y, Z) = 0.$$

Proof. Interchanging X and Y in equation (2.30) and comparing it with equation (2.30), we get (i). Using equation (2.30) we have

$$\begin{aligned}
\check{R}(X, Y)Z + \check{R}(Y, Z)X + \check{R}(Z, X)Y &= R(X, Y)Z + R(Y, Z)X + R(Z, X)Y \\
&\quad + \{s(Z, Y) - s(Y, Z)\}X + \{s(X, Z) - s(Z, X)\}Y + \{s(Y, X) - s(X, Y)\}Z \\
&\quad - 2\{da(X, Y)Z + da(Y, Z)X + da(Z, X)Y\}.
\end{aligned} \tag{2.42}$$

Using first Bianchi identity and in particular let consider the two 1-forms η and a be closed and since s is symmetric, we get (ii).

Using equations (2.40) and (2.30), we have

$$\begin{aligned}
\check{K}(X, Y, Z, U) &= K(X, Y, Z, U) + s(Y, Z)g(X, U) \\
&\quad - s(X, Z)g(Y, U) - g(X, Z)g(BY, U) \\
&\quad + g(Y, Z)g(BX, U) - 2da(Y, X)g(U, Z).
\end{aligned} \tag{2.43}$$

Interchanging Z and U in equation (2.43) we have

$$\begin{aligned}
\check{K}(X, Y, U, Z) &= K(X, Y, U, Z) + s(Y, U)g(X, Z) \\
&\quad - s(X, U)g(Y, Z) - g(X, U)g(BY, Z) \\
&\quad + g(Y, U)g(BX, Z) - 2da(Y, X)g(Z, U).
\end{aligned} \tag{2.44}$$

Adding equations (2.43) and (2.44) we get (iii).

Using equations (2.40) and (2.30) we have

$$\begin{aligned}\check{K}(Z, U, X, Y) &= K(Z, U, X, Y) + s(Z, X)g(U, Y) \\ &\quad - s(U, X)g(Z, Y) - g(U, X)g(BZ, Y) \\ &\quad + g(Z, X)g(BU, Y) - 2da(Z, U)g(Y, X).\end{aligned}\tag{2.45}$$

Subtracting equation (2.45) from equation (2.43) and since s is symmetric, we get (iv).

Using equation (2.41) we have

$$\begin{aligned}\check{K}(X, Y, Z, U) + \check{K}(X, Z, U, Y) + \check{K}(X, U, Y, Z) \\ = g(\check{R}(X, Y)Z + \check{R}(Y, Z)X + \check{R}(Z, X)Y, U).\end{aligned}\tag{2.46}$$

Using equation (ii) in equation (2.46), we get (v). $\square$

Lemma 2.2. *In an LP-Sasakian manifold M^n with the SSNMC $\check{\nabla}$ admitting two 1-forms. We have*

$$\begin{aligned}(i) \quad \check{R}(\xi, X)Y &= g(X, Y)\xi - g(\phi X, Y)\xi + \eta(A)g(X, Y)\xi - \eta(A)\eta(Y)X - \eta(Y)X \\ &\quad - a(\xi)g(X, Y)\xi + g(X, Y)\nabla_\xi A + a(\xi)g(X, Y)A + g(X, Y)A + \eta(Y)\phi X \\ &\quad + a(X)\eta(Y) - \eta(Y)\nabla_X A - a(X)\eta(Y)A + \eta(X)\eta(Y)A - 2\eta(\nabla_X A)Y, \\ (ii) \quad \check{R}(\xi, X)\xi &= \eta(X)\xi + X + \eta(A)X - a(\xi)\eta(X) + a(\xi)\eta(X)A - \phi X + \nabla_X A \\ &\quad + a(X)A + \eta(X)\nabla_\xi A - 2\eta(\nabla_X A)\xi + \eta(A)\eta(X)\xi - a(x)\xi, \\ (iii) \quad \check{R}(X, Y)\xi &= \eta(Y)X - \eta(X)Y + \eta(A)\eta(Y)X - \eta(A)\eta(X)Y - \eta(Y)\phi X \\ &\quad - a(X)\eta(Y)\xi + \eta(Y)\nabla_X A + a(X)\eta(Y)A + \eta(X)\phi Y + a(Y)\eta(X)\xi \\ &\quad - \eta(X)\nabla_Y A - a(Y)\eta(X)A - 2da(Y, X)\xi.\end{aligned}$$

Proof. Using equations (2.31) and (2.25), we get (i). Replacing Y with ξ in (i) and using equation (2.27), we get (ii). Using equations (2.31) and (2.26), we get (iii). $\square$

Lemma 2.3. *In an LP-Sasakian manifold with SSNMC $\check{\nabla}$ admitting two 1-forms we have*

$$(i) \quad \check{S}(X, \xi) = -a(X) - a(X)\eta(A) + \frac{1}{2}n\eta(A)\eta(X) - \eta(X)\{div \xi - div A - (n-2) - \frac{1}{2}(n-4)\eta(A)\} + \eta(\nabla_X A),$$

$$(ii) \quad \check{S}(\phi X, \phi Y) = S(X, Y) + 2(n-2)\eta(X)\eta(Y) - (n-2)\Phi(X, Y) \\ + (n-3)g(X, Y)\eta(A) + (n-3)\eta(X)\eta(Y)\eta(A) + (n-3)g(X, Y) \\ - \{g(X, Y) + \eta(X)\eta(Y)\}\{div \xi - div A\} + da(\phi X, \phi Y).$$

Proof. Using equations (2.34) and (2.28), we get (i). Using equations (2.34) and (2.29), we get (ii). $\square$

2.3.1 Projective curvature tensor

Definition 2.1. *The projective curvature tensor on a Riemannian manifold is defined in equation (1.32). In an LP-Sasakian manifold M^n , Mishra (1984) gives the projective curvature tensor of the manifold M with respect to the connection $\check{\nabla}$ by*

$$\check{P}(X, Y)Z = \check{R}(X, Y)Z + \frac{1}{n+1}\{\check{S}(X, Y) - \check{S}(Y, X)\}Z \\ - \frac{1}{n^2-1}n\{\check{S}(Y, Z)X - \check{S}(X, Z)Y\} \\ - \frac{1}{n^2-1}\{\check{S}(Z, Y)X - \check{S}(Z, X)Y\}. \quad (2.47)$$

Using equations (2.31) and (2.34) in equation (2.47), we have

$$\check{P}(X, Y)Z = P(X, Y)Z - g(Y, Z)BX + g(X, Z)BY \\ + \frac{1}{n+1}\{g(BX, Y) - g(BY, X)\}Z - \frac{1}{n^2-1}\{ng(BY, Z) \\ + g(BZ, Y)\}X + \frac{1}{n^2-1}\{ng(BX, Z) + g(BZ, X)\}Y \\ + \frac{1}{n+1}g(Y, Z)\{div \xi - div A - \frac{1}{2}(n-4) - \frac{1}{2}(n-4)\eta(A)\}X \\ - \frac{1}{n-1}g(X, Z)\{div \xi - div A - \frac{1}{2}(n-4) - \frac{1}{2}(n-4)\eta(A)\}Y \\ + \frac{1}{n+1}2\{(n+3)da(X, Y)Z - da(Y, Z)X - da(X, Z)Y\}. \quad (2.48)$$

Lemma 2.4. *The projective curvature tensor $\check{P}$ satisfies the following algebraic properties:*

$$(i) \quad \check{P}(X, Y)Z + \check{P}(Y, X)Z = 0,$$

$$(ii) \quad \check{P}(X, Y)Z + \check{P}(Y, Z)X + \check{P}(Z, X)Y = 0.$$

Proof. Interchanging X and Y in equation (2.48) and adding it with equation (2.48), we get (i).

Using equation (2.48) and treating $\eta = a$ i.e. $\xi = A$ and a is closed, we get (ii). $\square$

2.3.2 v -Weyl projective tensor

Definition 2.2. *The v -Weyl projective tensor P^v is defined by Tigaeru (1998) as*

$$P^v(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}\{S(Y, Z)X - S(X, Z)Y\}. \quad (2.49)$$

Then from the definition of the above v -Weyl projective tensor, the v -Weyl projective tensor of M with respect to the SSNMC $\check{\nabla}$ admitting two 1-forms is

$$\check{P}^v(X, Y)Z = \check{R}(X, Y)Z - \frac{1}{n-1}\{\check{S}(Y, Z)X - \check{S}(X, Z)Y\} \quad (2.50)$$

Using equations (2.31) and (2.34) we get

$$\begin{aligned} \check{P}^v(X, Y)Z &= P^v(X, Y)Z - g(Y, Z)BX + g(X, Z)BY \\ &\quad - 2da(Y, X)Z - \frac{1}{n-1}\{-g(Y, Z)(\operatorname{div} \xi - \operatorname{div} A - \frac{1}{2}(n-4) \\ &\quad - \frac{1}{2}(n-4)\eta(A))X + g(BY, Z)X + 2da(Y, Z)X \\ &\quad + g(X, Z)(\operatorname{div} \xi - \operatorname{div} A - \frac{1}{2}(n-4) - \frac{1}{2}(n-4)\eta(A))Y \\ &\quad - g(BX, Z)Y - 2da(X, Z)Y\}. \end{aligned} \quad (2.51)$$

If

$$\begin{aligned}
& g(Y, Z)BX - g(X, Z)BY + 2da(Y, X)Z \\
&= -\frac{1}{n-1} \left\{ -g(Y, Z)(\operatorname{div} \xi - \operatorname{div} A - \frac{1}{2}(n-4) - \frac{1}{2}(n-4)\eta(A))X \right. \\
&+ g(BY, Z)X + 2da(Y, Z)X + g(X, Z)(\operatorname{div} \xi - \operatorname{div} A - \frac{1}{2}(n-4) \\
&\left. - \frac{1}{2}\eta(A))Y - g(BX, Z)Y - 2da(X, Z)Y \right\}, \tag{2.52}
\end{aligned}$$

then

$$\check{P}^v(X, Y)Z = P^v(X, Y)Z. \tag{2.53}$$

Hence, we have the following Lemmas:

Lemma 2.5. *In an LP-Sasakian manifold M^n , the v -Weyl projective tensor $\check{P}^v$ of M with respect to the SSNMC $\check{\nabla}$ admitting two 1-forms is equal to the v -Weyl projective tensor P^v of the manifold if the condition of equation (2.52) is satisfied.*

Lemma 2.6. *In an LP-Sasakian manifold M^n , the v -Weyl projective tensor $\check{P}^v$ of the manifold with respect to the SSNMC $\check{\nabla}$ admitting two 1-forms satisfies the following properties:*

- (i) $\check{P}^v(X, Y)Z + \check{P}^v(Y, X)Z = 0,$
- (ii) $\check{P}^v(X, Y)Z + \check{P}^v(Y, Z)X + \check{P}^v(Z, X)Y = 0.$

Proof. Interchanging X and Y in equation (2.51) and adding it with equation (2.51) we get (i).

Using equation (2.51) and treating $\eta = a$ i.e. $\xi = A$ and a is closed then we get (ii). □

2.3.3 Concircular curvature tensor

Definition 2.3. *The concircular curvature tensor of a Riemannian M is defined in equation (1.33) and the concircular curvature tensor of an LP-Sasakian*

manifold with respect to the SSNMC $\check{\nabla}$ admitting two 1-forms is given by

$$\check{V}(X, Y)Z = \check{R}(X, Y)Z - \frac{\check{r}}{n(n-1)}\{g(Y, Z)X - g(X, Z)Y\}. \quad (2.54)$$

Using equations (2.31) and (2.35) in the above equation, we have

$$\begin{aligned} \check{V}(X, Y)Z &= V(X, Y)Z - s(Y, Z)X + s(X, Z)Y - g(Y, Z)BX \\ &\quad + g(X, Z)BY - 2da(Y, X)Z + \frac{1}{n}\{trace(s) + (div \xi \\ &\quad - div A - \frac{1}{2}(n-4) - \frac{1}{2}(n-4)\eta(A))\}\{g(Y, Z)X \\ &\quad - g(X, Z)Y\}. \end{aligned} \quad (2.55)$$

Theorem 2.4. *The concircular curvature tensor V and $\check{V}$ of the LP-Sasakian manifold and of the SSNMC $\check{\nabla}$ admitting two 1-forms $\check{\nabla}$ respectively are related by equation (2.55).*

2.4 Einstein manifold

An n dimensional Riemannian M is called an Einstein manifold with respect to Riemannian connection if equation (1.24) is satisfied. Analogous to this definition, an Einstein manifold with respect SSNMC $\check{\nabla}$ admitting two 1-forms is defined as

$$\check{S}(X, Y) = \frac{\check{r}}{n}g(X, Y). \quad (2.56)$$

Using equations (2.34) and (2.35) in equation (2.56), we have

$$\begin{aligned} \check{S}(X, Y) - \frac{\check{r}}{n}g(X, Y) &= S(X, Y) - \frac{r}{n}g(X, Y) - (n-1)s(X, Y) \\ &\quad + 2da(X, Y) + \frac{1}{n}\{(n-1)trace(s) - (div \xi \\ &\quad - div A - \frac{1}{2}(n-4) - \frac{1}{2}(n-4)\eta(A))\}g(X, Y) \\ &\quad + g(BX, Y). \end{aligned} \quad (2.57)$$

If

$$\begin{aligned}
(n-1)s(X, Y) + 2da(X, Y) &= \frac{1}{n} \{ (n-1)\text{trace}(s) - (\text{div } \xi - \text{div } A \\
&\quad - \frac{1}{2}(n-4) - \frac{1}{2}(n-4)\eta(A) \} g(X, Y) \quad (2.58) \\
&\quad + g(BX, Y).
\end{aligned}$$

Then from equation (2.57), we get

$$\check{S}(X, Y) = \frac{\check{r}}{n} g(X, Y). \quad (2.59)$$

Hence we can state the following theorem:

Theorem 2.5. *In an LP-Sasakian manifold M^n with SSNMC $\check{\nabla}$ admitting two 1-forms if the relation of equation (2.58) holds, then it is an Einstein manifold for the Riemannian connection if and only if it is an Einstein manifold for the connection $\check{\nabla}$.*

2.5 Deformation Algebra

Let the two linear connections in a Riemannian manifold be $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ respectively. Lets define the product of two vector fields X and Y by

$$X \circ Y = \overset{2}{\nabla}_X Y - \overset{1}{\nabla}_X Y. \quad (2.60)$$

The module $\chi(M)$ of all differentiable vector fields becomes an algebra over the ring $F(M)$ of all real functions over M . This algebra is called the deformation algebra (Vaisman, 1966) of the ordered pair of connection $(\overset{1}{\nabla}, \overset{2}{\nabla})$ and is denoted by $\Pi(M, \overset{1}{\nabla}, \overset{2}{\nabla})$. An element $X \in \chi(M)$ is called a characteristic vector field if there exist a function $\eta \in F(M)$ such that

$$X \circ X = \eta X. \quad (2.61)$$

By using equations (2.1) and (2.60), we have,

$$X \circ X = \eta(Y)X - a(X)Y - g(X, Y)\xi + g(X, Y)A \quad (2.62)$$

Theorem 2.6. *Every element is a characteristics vector field in the deformation algebra $\Pi(M, \overset{1}{\nabla}, \overset{2}{\nabla})$.*

2.6 Applications in the general theory of relativity

The main concern of this section is the determination of the application of SSNMC $\overset{\vee}{\nabla}$ admitting two 1-forms in the general theory of relativity. We consider an LP-Sasakian manifold (M^n, g) to be a fluid spacetime and we are also treating Ricci tensor $\overset{\vee}{S}$ of SSNMC $\overset{\vee}{\nabla}$ to be vanished *i.e.* $\overset{\vee}{S} = 0$ therefore equations (2.34) and (2.35) become

$$\begin{aligned} S(Y, Z) = (n-1)s(Y, Z) + g(Y, Z) \left\{ \operatorname{div} \xi - \operatorname{div} A - \frac{1}{2}(n-4) \right. \\ \left. - \frac{1}{2}(n-4)\eta(A) \right\} - g(BY, Z) - 2da(Y, Z), \end{aligned} \quad (2.63)$$

and

$$\begin{aligned} r = (n-1)\operatorname{trace}(s) + (n-1) \left\{ \operatorname{div} \xi - \operatorname{div} A - \frac{1}{2}(n-4) \right. \\ \left. - \frac{1}{2}(n-4)\eta(A) \right\}. \end{aligned} \quad (2.64)$$

Without cosmological constant the Einstein's field equations is

$$S(X, Y) - \frac{r}{2}g(X, Y) = k T(X, Y), \quad (2.65)$$

for all $X, Y \in \chi(M)$, where T is the energy momentum tensor and $k = 8\pi G$ (in units $C = 1$) is the Einstein gravitational constant and $G \approx 6.67430 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ is the universal gravitational constant. If possible we consider that the fluid spacetime is a perfect fluid spacetime, then the energy momentum

tensor T assumes the form

$$T(X, Y) = P_0 g(X, Y) + (P_0 + \mu) \eta(X) \eta(Y), \quad (2.66)$$

where P_0 is the isotropic pressure, μ is the energy density and ξ is the fluid flow velocity $\left(g(\xi, \xi) = -1, \quad g(A, A) = -1\right)$ of the perfect fluid spacetime. Using equations (2.63), (2.64) and (2.66) in equation (2.65), we obtain

$$\begin{aligned} & (n-1)s(X, Y) + g(Y, Z) \left\{ \operatorname{div} \xi - \operatorname{div} A - \frac{1}{2}(n-4) - \frac{1}{2}(n-4)\eta(A) \right\} \\ & - g(BY, Z) - 2da(Y, Z) - \frac{1}{2} \left\{ (n-1)\operatorname{trace}(s) + (n-1)(\operatorname{div} \xi - \operatorname{div} A \right. \\ & \left. - \frac{1}{2}(n-4) - \frac{1}{2}(n-4)\eta(A)) \right\} \\ & = k \{ P_0 g(X, Y) + (P_0 + \mu) \eta(X) \eta(Y) \}. \end{aligned} \quad (2.67)$$

Putting $X = Y = \xi$ in equation (2.67) and using equations (1.57), (1.58), (1.60), (2.32) and (2.33), we get

$$\begin{aligned} & (n-1) \left\{ \frac{1}{2} \eta(A) - \frac{1}{2} \right\} - \left\{ \operatorname{div} \xi - \operatorname{div} A - \frac{1}{2}(n-4) - \frac{1}{2}(n-4)\eta(A) \right\} \\ & + \frac{1}{2} + a(\xi) + g\{\nabla_\xi A, \xi\} - \frac{1}{2} \eta(A) + a(\xi)a(\xi) + a(\xi) \\ & - \frac{1}{2} \left\{ (n-1)\operatorname{trace}(s) + (n-1)(\operatorname{div} \xi - \operatorname{div} A - \frac{1}{2}(n-4) \right. \\ & \left. - \frac{1}{2}(n-4)\eta(A)) \right\} \\ & = k\mu. \end{aligned} \quad (2.68)$$

Taking a frame field and contracting equation (2.65), we get

$$\begin{aligned} & - \frac{(n-2)}{2} \left\{ (n-1)\operatorname{trace}(s) + (n-1)(\operatorname{div} \xi - \operatorname{div} A - \frac{1}{2}(n-4) - \frac{1}{2}(n-4)\eta(A)) \right\} \\ & = k \{ P_0(n-1) - \mu \}. \end{aligned} \quad (2.69)$$

From equations (2.68) and (2.69), we get

$$\frac{D}{E(n-1)} = \frac{P_0}{\mu}, \quad (2.70)$$

where

$$D = -2(n-2)\{(n-1)\text{trace}(s)\} - (2n^2 - 5n + 5)\left(\text{div } \xi - \text{div } A - \frac{1}{2}(n-4) - \frac{1}{2}(n-4)\eta(A)\right) + (n+2)\eta(A) - (n-2), \quad (2.71)$$

and

$$E = (n+2)\eta(A) - (n-2) - (n+1)\left(\text{div } \xi - \text{div } A - \frac{1}{2}(n-4) - \frac{1}{2}(n-4)\eta(A)\right), \quad (2.72)$$

which gives the perfect fluid spacetime equation. In a particular case, we consider that $\text{trace}(s) = 0$ and two 1-form $\eta = a$ i.e. $\xi = A$, then equation (2.70) becomes

$$\frac{P_0}{\mu} = \frac{1}{(n-1)}, \quad (2.73)$$

we suppose that $n = 4$, then the perfect fluid spacetime equation becomes

$$3P_0 = \mu. \quad (2.74)$$

If $3P = \mu$, then the spacetime is isotropic and homogenous (Ellis, 1971). Hence we can state the theorem as

Theorem 2.7. *In a perfect fluid spacetime admitting a SSNMC $\check{\nabla}$ admitting two 1-forms whose Ricci tensor endowed with SSNMC $\check{\nabla}$ vanishes and if the perfect fluid spacetime satisfies the Einstein's field equations without cosmological constant, then the spacetime equation is given by equation (2.70). In particular case for $n = 4$ and $\text{trace}(s) = 0$ and treating the two 1-forms $\eta = a$ i.e. $\xi = A$, then the space matter is filled with isotropic and homogenous perfect fluid.*

Also we have

$$(\check{R}(X, Y) \cdot \check{S})(Z, U) = -\check{S}(\check{R}(X, Y)Z, U) - \check{S}(Z, \check{R}(X, Y)U). \quad (2.75)$$

Using equations (2.31) and (2.34) in equation (2.75), we get

$$\begin{aligned}
(\check{R}(X, Y) \cdot \check{S})(Z, U) &= (R(X, Y) \cdot S)(Z, U) + g(\phi Y, Z)S(X, U) \\
&\quad - \eta(Y)\eta(Z)S(X, U) - g(\phi X, Z)S(Y, U) \\
&\quad + \eta(X)\eta(Z)S(Y, U) + 2da(Y, X)S(Z, U) \\
&\quad + (n-1)(R(X, Y), s)(Z, U) - (n-1)g(\phi Y, Z)s(X, U) \\
&\quad + (n-1)\eta(Y)\eta(Z)s(X, U) + g(\phi X, Z)s(Y, U) \\
&\quad - (n-1)\eta(X)\eta(Z)s(Y, U) - (n-1)2da(Y, X)s(Z, U) \\
&\quad - (R(X, Y), 2da)(Z, U) + g(\phi Y, Z)2da(X, U) \\
&\quad - \eta(Y)\eta(Z)2da(X, U) - g(\phi X, Z)2da(Y, U) \\
&\quad + \eta(X)\eta(Z)2da(Y, U) + 2da(Y, X)2da(Z, U) \\
&\quad + g(\phi Y, U)S(Z, X) - \eta(Y)\eta(U)S(Z, X) \\
&\quad - g(\phi X, U)S(Z, Y) + \eta(X)\eta(U)S(Z, Y) \\
&\quad + 2da(Y, X)S(Z, U) + (n-1)(R(X, Y), s)(Z, U) \\
&\quad - (n-1)g(\phi Y, U)s(Z, X) + (n-1)\eta(Y)\eta(U)s(Z, X) \\
&\quad + (n-1)g(\phi X, U)s(Z, Y) + (n-1)\eta(X)\eta(U)s(Z, Y) \\
&\quad - (n-1)2da(Y, X)s(Z, U) - (R(X, Y), 2da)(Z, U) \\
&\quad + g(\phi Y, U)2da(Z, X) - \eta(Y)\eta(U)2da(Z, X) \\
&\quad - g(\phi X, U)2da(Z, Y) + \eta(X)\eta(U)2da(Z, Y) \\
&\quad + 2da(Y, X)2da(Z, U),
\end{aligned} \tag{2.76}$$

where $(R(X, Y) \cdot S)(Z, U) = -S(R(X, Y)Z, U) - S(Z, R(X, Y)U)$. We consider

that $\check{R} \cdot \check{S} = R \cdot S$ and changing U with ξ in equation (2.76), we get

$$\begin{aligned}
& g(\phi Y, Z)(n-1)\eta(X) - (n-1)\eta(Y)\eta(Z)\eta(X) - (n-1)g(\phi X, Z)\eta(Y) \\
& + (n-1)\eta(X)\eta(Y)\eta(Z) + (n-1)2da(Y, X)\eta(Z) \\
& + (n-1)2(R(X, Y), s)(Z, \xi) - (n-1)g(\phi Y, Z)s(X, \xi) \\
& + (n-1)\eta(Y)\eta(Z)s(X, \xi) + g(\phi X, Z)s(Y, \xi) \\
& - (n-1)\eta(X)\eta(Z)s(Y, \xi) - 2(n-1)2da(Y, X)s(Z, \xi) \\
& + \eta(Y)S(Z, X) - \eta(X)S(Z, Y) + (n-1)\eta(Z)2da(Y, X) \\
& - (n-1)\eta(Y)(Z, X) - (n-1)\eta(X)s(Z, Y) + \eta(Y)2da(Z, X) \\
& - \eta(X)2da(Z, Y) \\
& = 0.
\end{aligned} \tag{2.77}$$

Again putting $X = \xi$ in equation (2.77), we get

$$\begin{aligned}
& 2(n-1)g(Y, Z) - 3(n-1)\eta(Y)\eta(Z) + (n-3)g(\phi Z, Y) + S(Z, Y) \\
& = 0.
\end{aligned} \tag{2.78}$$

If

$$(n-3)g(\phi Y, Z) = 3(n-1)\eta(Y)\eta(Z), \tag{2.79}$$

then, equation (2.78) yields

$$S(Y, Z) = 2(n-1)g(Y, Z), \tag{2.80}$$

$$r = 2n(n-1). \tag{2.81}$$

which give us that the LP-Sasakian manifold under consideration is an Einstein manifold. Hence, we can state that

Theorem 2.8. *In an LP-Sasakian manifold (M^n, g) endowed with a SSNMC $\check{\nabla}$ admitting two 1-forms, if the two 1-forms are considered equal and satisfies the relations $\check{R} \cdot \check{S} = R \cdot S$ and equation (2.79), then M is an Einstein spacetime.*

Next, we consider that the two 1-forms are equal $\eta = a$ i.e. $\xi = A$ and the perfect fluid spacetimes satisfy $\check{R} \cdot \check{S} = R \cdot S$ and equation (2.79), then from equations (2.65) and (2.66), we have

$$\begin{aligned} & 4(n-1)g(X, Y) - 2n(n-1)g(X, Y) \\ &= 2k\{P_0g(X, Y) + (P_0 + \mu)\eta(X)\eta(Y)\}. \end{aligned} \quad (2.82)$$

Treating $X = Y = \xi$ in equation (2.82), we get

$$(2n-2)(n-2) = 2k\mu. \quad (2.83)$$

For $n > 2$ and taking a frame field and contracting equation (2.82), we get

$$4(n-1)n - 2n(n-1)n = 2k\{P_0(n-1) - \mu\}. \quad (2.84)$$

From equations (2.83) and (2.84), we get

$$P_0 + \mu = 0. \quad (2.85)$$

which shows that the equation of the state of perfect fluid spacetime describes the Phantom Barrier (Srinivasa, 2006) .

Theorem 2.9. *In an LP-Sasakian manifold (M^n, g) endowed with a SSNMC $\check{\nabla}$ admitting two 1-forms, if the two 1-forms are considered equal and satisfies the relations $\check{R} \cdot \check{S} = R \cdot S$ and equation (2.79) and if the perfect fluid spacetime satisfies Einstein's field equations without cosmological constant, then the model describe the evolution in the phantom barrier.*

In the development of general theory of relativity and cosmology, the confor-

mal curvature tensor plays an important role and it is defined as (Weyl, 1918)

$$\begin{aligned} C(X, Y)Z &= R(X, Y)Z + \frac{r}{(n-1)(n-2)}\{g(Y, Z)X \\ &\quad - g(X, Z)Y\} - \frac{1}{(n-2)}\{S(Y, Z)X - S(X, Z)Y \\ &\quad + g(Y, Z)QX - g(X, Z)QY\}, \end{aligned} \quad (2.86)$$

for all $X, Y, Z \in \chi(M)$.

Using equations (2.80) and (2.81) in equation (2.86), we get

$$C(X, Y) = R(X, Y)Z - 2\{g(Y, Z)X - g(X, Z)Y\}. \quad (2.87)$$

From equation (2.87), it is obvious that

$$R \cdot C = R \cdot R. \quad (2.88)$$

In 1920, the semi-symmetric spacetimes (SSST) was characterized by Eriksen and Senovilla (2010) and proved that these spacetimes are of Petrov type **D**, **N**, or **O**. Hence we can state the following theorem.

Theorem 2.10. *Suppose a conformally SSST endowed with a SNMC $\check{\nabla}$ admitting two 1-forms are equal ($\eta = a$ i.e. $\xi = A$) and satisfies the relations $\check{R} \cdot \check{S} = R \cdot S$ and equation (2.79), then the spacetime is of Petrov type **D**, **N**, or **O**.*

2.7 Example

Let M be a 4-dimensional manifold defined as

$$M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4; x_4 \neq 0\}, \quad (2.89)$$

where $\mathbb{R}$ is the set of real numbers. Let x_1, x_2, x_3, x_4 be given by

$$e_1 = \frac{x_1}{x_4} \frac{\partial}{\partial x_1}, \quad e_2 = \frac{x_2}{x_4} \frac{\partial}{\partial x_2}, \quad e_3 = \frac{x_3}{x_4} \frac{\partial}{\partial x_3}, \quad e_4 = x_4 \frac{\partial}{\partial x_4},$$

where $\{e_1, e_2, e_3, e_4\}$ are linearly independent global frame on M . The non vanishing component of the Lie-bracket are

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_3.$$

Let the 1-forms η and a be given by

$$\eta(X) = g(X, e_4), \quad a(X) = g(X, e_3)$$

The Lorentzian metric g is defined by

$$g(e_i, e_j) = \begin{cases} -1, & i = j = 3, 4 \\ 1, & i = j = 1, 2, 3 \\ 0, & \text{otherwise.} \end{cases}$$

The linear function ϕ of M is defined by

$$\phi e_i = \begin{cases} 0, & i = 3, 4 \\ e_i, & i = 1, 2. \end{cases}$$

With the help of above equation we can see that equations (1.56) to (1.58) hold for all $e_i, i = 1, 2, 3, 4$. Using the Koszul's formula

$$\begin{aligned} 2g(\check{\nabla}_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ &\quad - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

we have

$$\nabla_{e_1} e_1 = e_4, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \quad \nabla_{e_1} e_4 = e_1,$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = e_4, \quad \nabla_{e_2} e_3 = 0, \quad \nabla_{e_2} e_4 = e_2,$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = -e_4, \quad \nabla_{e_3} e_4 = e_3,$$

$$\nabla_{e_4}e_1 = 0, \quad \nabla_{e_4}e_2 = 0, \quad \nabla_{e_4}e_3 = 0, \quad \nabla_{e_4}e_4 = 0,$$

where ∇ is the Levi-Civita connection with respect to the metric g . We can observe that $\nabla_X e_4 = \phi X$ for all X of M and $\nabla_X e_3 \neq \phi X$ for all X of M . Thus (M, g) is a 4-dimensional LP-Sasakian manifold. Using the above equation and equation (2.1) we have

$$\begin{aligned} \check{\nabla}_{e_1}e_1 &= e_3, & \check{\nabla}_{e_1}e_2 &= 0, & \check{\nabla}_{e_1}e_3 &= 0, & \check{\nabla}_{e_1}e_4 &= 0, \\ \check{\nabla}_{e_2}e_1 &= 0, & \check{\nabla}_{e_2}e_2 &= e_3, & \check{\nabla}_{e_2}e_3 &= 0, & \check{\nabla}_{e_2}e_4 &= 0, \\ \check{\nabla}_{e_3}e_1 &= e_1, & \check{\nabla}_{e_3}e_2 &= e_2, & \check{\nabla}_{e_3}e_3 &= 0, & \check{\nabla}_{e_3}e_4 &= e_4, \\ \check{\nabla}_{e_4}e_1 &= 0, & \check{\nabla}_{e_4}e_2 &= 0, & \check{\nabla}_{e_4}e_3 &= 0, & \check{\nabla}_{e_4}e_4 &= -e_3. \end{aligned}$$

With the above equation and the torsion tensor given in equation (2.5), the non-zero components of torsion tensor with respect to the connection $\check{\nabla}$ are:

$$\begin{aligned} \check{T}(e_1, e_4) &= -e_1, & \check{T}(e_1, e_3) &= -e_1, & \check{T}(e_2, e_4) &= -e_2, \\ \check{T}(e_3, e_4) &= e_4 - e_3, & \check{T}(e_2, e_3) &= -e_2, & \check{T}(e_2, e_4) &= -e_2. \end{aligned}$$

Let X and Y be any two vector fields given by

$$X = X^1e_1 + X^2e_2 + X^3e_3 + X^4e_4,$$

$$Y = Y^1e_1 + Y^2e_2 + Y^3e_3 + Y^4e_4,$$

where

$$X = \sum_{i=1}^4 X^i e_i, \quad Y = \sum_{i=1}^4 Y^i e_i \quad ; i = 1, 2, 3, 4.$$

Then we have

$$\check{T}(X, Y) = \lambda(Y)X - \lambda(X)Y,$$

where

$$\begin{aligned}\lambda(Y)X - \lambda(X)Y &= (X^1Y^3 - X^3Y^1)e_1 + (X^1Y^4 - X^4Y^1)e_1 \\ &\quad + (X^2Y^3 - X^3Y^2)e_2 + (X^2Y^4 - X^4Y^2)e_2 \\ &\quad + (X^3Y^4 - X^4Y^3)e_3 + (X^4Y^3 - X^3Y^4)e_4.\end{aligned}$$

Hence the linear connection $\check{\nabla}$ defined on equation (2.1) on M is SSNMC admitting two 1-forms. We also observe that

$$(\check{\nabla}_{e_4}g)(e_4, e_3) = -1 \neq 0.$$

Therefore the SSNMC $\check{\nabla}$ is non-metric.

Also the non-vanishing components of the curvature tensor are :

$$R(e_1, e_2)e_1 = -e_2, \quad R(e_1, e_2)e_2 = e_1, \quad R(e_1, e_3)e_1 = -e_3,$$

$$R(e_1, e_3)e_3 = -e_1, \quad R(e_1, e_4)e_1 = -e_4, \quad R(e_1, e_4)e_4 = -e_1,$$

$$R(e_2, e_1)e_1 = e_2, \quad R(e_2, e_1)e_2 = -e_1, \quad R(e_2, e_3)e_2 = -e_3,$$

$$R(e_2, e_3)e_3 = -e_2, \quad R(e_2, e_4)e_2 = -e_4, \quad R(e_2, e_4)e_4 = -e_2,$$

$$R(e_3, e_1)e_1 = e_3, \quad R(e_3, e_1)e_3 = e_1, \quad R(e_3, e_2)e_2 = e_3,$$

$$R(e_3, e_2)e_3 = e_2, \quad R(e_3, e_4)e_3 = e_4, \quad R(e_3, e_4)e_4 = -e_3,$$

$$R(e_4, e_1)e_1 = e_4, \quad R(e_4, e_1)e_4 = e_1, \quad R(e_4, e_2)e_2 = e_4,$$

$$R(e_4, e_2)e_4 = e_2, \quad R(e_4, e_3)e_3 = -e_4, \quad R(e_4, e_3)e_4 = e_3,$$

and

$$\check{R}(e_1, e_3)e_1 = e_3, \quad \check{R}(e_1, e_4)e_1 = -e_3, \quad \check{R}(e_2, e_3)e_2 = e_3$$

$$\check{R}(e_2, e_4)e_2 = -e_3, \quad \check{R}(e_3, e_1)e_1 = -e_3, \quad \check{R}(e_3, e_2)e_2 = -e_3,$$

$$\check{R}(e_3, e_1)e_1 = -e_3, \quad \check{R}(e_3, e_2)e_2 = -e_3, \quad \check{R}(e_3, e_4)e_1 = -e_1,$$

$$\check{R}(e_3, e_4)e_2 = -e_2, \quad \check{R}(e_3, e_4)e_4 = e_3 - e_4, \quad \check{R}(e_4, e_1)e_1 = e_3,$$

$$\check{R}(e_4, e_2)e_2 = e_3, \quad \check{R}(e_4, e_3)e_1 = e_1, \quad \check{R}(e_4, e_3)e_2 = e_2,$$

$$\check{R}(e_4, e_3)e_4 = e_4 - e_3.$$

Again let X, Y and Z be any three vector fields given by

$$X = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4,$$

$$Y = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4,$$

$$Z = c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4,$$

where a_i, b_i, c_i for all $i = 1, 2, 3, 4$ are all non-zero real numbers.

So

$$\begin{aligned} \check{R}(X, Y)Z &= a_1b_3c_1e_3 - a_1b_4c_1e_3 + a_2b_3c_2e_3 - a_2b_4c_2e_3 \\ &\quad - a_3b_1c_1e_3 - a_3b_2c_2e_3 - a_3b_4c_1e_1 - a_3b_4c_2e_2 \\ &\quad + a_3b_4c_4e_3 - a_3b_4c_4e_4 + a_4b_1c_1e_3 + a_4b_2c_2e_3 \\ &\quad + a_4b_3c_1e_1 + a_4b_3c_2e_2 + a_4b_3c_4e_4 - a_4b_3c_4e_3 \\ &= -(a_3b_1c_1e_3 - a_4b_1c_1e_3 + a_3b_2c_2e_3 - a_4b_2c_2e_3 \\ &\quad - a_1b_3c_1e_3 - a_2b_3c_2e_3 - a_4b_3c_1e_1 - a_4b_3c_2e_2 \\ &\quad + a_4b_3c_4e_3 - a_4b_3c_4e_4 + a_1b_4c_1e_3 + a_2b_4c_2e_3 \\ &\quad + a_3b_4c_1e_1 + a_3b_4c_2e_2 + a_3b_4c_4e_4 - a_3b_4c_4e_3) \\ &= -\check{R}(Y, X)Z. \end{aligned}$$

And also

$$\check{R}(X, Y)Z + \check{R}(Y, Z)X + \check{R}(Z, X)Y = 0.$$

Hence the manifold under consideration satisfies the properties (i) and (ii) of Theorem 2.3. Again the non-vanishing componenet of the Ricci tensor are :

$$S(e_1, e_1) = 3, \quad S(e_2, e_2) = 3, \quad S(e_3, e_3) = -3 \quad S(e_4, e_4) = -3.$$

It is obvious that $R \cdot S = 0$ and $S = 2(n - 1)g$ for $n = 4$.

Also,

$$\check{S}(e_1, e_1) = -1, \quad \check{S}(e_2, e_2) = -1, \quad \check{S}(e_4, e_4) = 1,$$

which follows $\check{R} \cdot \check{S} = 0$. Thus we can say that $\check{R} \cdot \check{S} - R \cdot S = 0$ and $S = 2(n - 1)g$.

Hence the manifold under consideration is an Einstein manifold. Therefore the Theorem 2.8 is verified.

Chapter 3

Geometrical properties of LP-Sasakian Manifolds in the tangent bundle

3.1 Introduction

In an LP-Sasakian manifold (M^n, g) , the linear connection $\check{\nabla}$ on M^n is given by (Prakash and Narain, 2011)

$$\check{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X + a(X)\phi Y, \quad (3.1)$$

where η and a are 1-form associated with vector field ξ and A on M^n is given by

$$\eta(X) = g(X, \xi), \quad (3.2)$$

$$a(X) = g(X, A), \quad (3.3)$$

R. Kumar, L. Colney, S. Shenawy, N. Bin Turki (2023), Tangent Bundles Endowed with Quarter-Symmetric Non-Metric Connection (QSNMC) in a Lorentzian Para-Sasakian Manifold *Mathematics*, **11**, 4163.

for all vector fields $X, Y \in \chi(M^n)$, where $\chi(M^n)$ is the set of all differentiable vector fields on M^n and the torsion tensor is given by

$$\check{T}(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y + a(X)\phi Y - a(Y)\phi X. \quad (3.4)$$

A linear connection satisfying equation (3.4) is called a quarter-symmetric connection. Further, by using equation (3.1), we have

$$(\check{\nabla}_X g)(Y, Z) = -\eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y) - 2a(X)g(\phi Y, Z). \quad (3.5)$$

A linear connection $\check{\nabla}$ defined by equation (3.1) which satisfies equation (3.4) and equation (3.5) is called quarter symmetric non-metric connection (briefly QSNMC).

3.2 LP-Sasakian manifolds in the tangent bundle

Let the tangent bundle be denoted by TM^n in an LP-Sasakian manifold (M^n, g) . The complete lifts of the LP-Sasakian manifold in the tangent bundle are obtained by taking complete lifts using mathematical operators on equations (1.56-1.65) and equations (2.24-2.29), we get (Kumar et. al, 2023a):

$$(\phi^2(X))^c = X^c + \eta^c(X^c)\xi^v + \eta^v(X^c)\xi^c, \quad (3.6)$$

$$\eta^c(\xi^c) = \eta^v(\xi^v) = 0, \quad \eta^c(\xi^v) = \eta^v(\xi^c) = -1, \quad (3.7)$$

$$g^c((\phi X)^c, (\phi Y)^c) = g^c(X^c, Y^c) + \eta^c(X^c)\eta^v(Y^c) + \eta^v(X^c)\eta^c(Y^c), \quad (3.8)$$

$$g^c(X^c, \xi^c) = \eta^c(X^c), \quad (3.9)$$

$$\begin{aligned} (\nabla_{X^c}^c \phi^c)Y^c &= g^c(X^c, Y^c)\xi^v + g^c(X^v, Y^c)\xi^c + \eta^c(Y^c)X^v + \eta^v(Y^c)X^c \\ &+ 2\left\{\eta^c(X^c)\eta^c(Y^c)\xi^v + \eta^c(X^c)\eta^v(Y^c)\xi^c + \eta^v(X^c)\eta^c(Y^c)\xi^c\right\}, \end{aligned} \quad (3.10)$$

$$\nabla_{X^c}^c \xi^c = (\phi X)^c, \quad (3.11)$$

$$\phi^c \xi^c = \phi^v \xi^v = \phi^c \xi^v = \phi^v \xi^c = 0, \quad (3.12)$$

$$\eta^c \circ \phi^c = \eta^v \circ \phi^v = \eta^c \circ \phi^v = \eta^v \circ \phi^c = 0, \quad (3.13)$$

$$\Phi^c(X^c, Y^c) = g^c(X^c, \phi^c Y^c), \quad (3.14)$$

$$(\nabla_{X^c}^c \eta^c) Y^c = \Phi^c(X^c, Y^c), \quad (3.15)$$

$$\Phi^c(X^c, \xi^c) = 0, \quad (3.16)$$

$$g^c(R^c(X^c, Y^c)Z^c, \xi^c) = g^c(Y^c, Z^c)\eta^v(X^c) + g^c(Y^v, Z^c)\eta^c(X^c) \quad (3.17)$$

$$- g^c(X^c, Z^c)\eta^v(Y^c) - g^c(X^v, Z^c)\eta^c(Y^c),$$

$$R^c(\xi^c, X^c)Y^c = g^c(X^c, Y^c)\xi^v + g^c(X^v, Y^c)\xi^c - \eta^c(Y^c)X^v - \eta^v(Y^c)X^c, \quad (3.18)$$

$$R^c(X^c, Y^c)\xi^c = \eta^c(Y^c)X^v + \eta^v(Y^c)X^c - \eta^c(X^c)Y^v - \eta^v(X^c)Y^c, \quad (3.19)$$

$$R^c(\xi^c, X^c)\xi^c = X^c + \eta^c(X^c)\xi^v + \eta^v(X^c)\xi^c, \quad (3.20)$$

$$S^c(X^c, \xi^c) = (n-1)\eta^c(X^c), \quad (3.21)$$

$$S^c(\phi^c X^c, \phi^c Y^c) = S^c(X^c, Y^c) + (n-1)\left\{\eta^c(X^c)\eta^v(Y^c) + \eta^v(X^c)\eta^c(Y^c)\right\}, \quad (3.22)$$

where the mathematical operators 'c' and 'v' denote the complete and the vertical lifts of the curvature tensor R , Ricci tensor S , 1-form η , characteristic vector field ξ , tensor field ϕ , Levi-Civita connection ∇ and vector fields X, Y, Z of the LP-Sasakian manifold in the tangent bundle.

3.3 QSNMC of an LP-Sasakian manifolds in the tangent bundle

In an LP-Sasakian manifold (M^n, g) and its tangent bundle TM^n , let us take complete lifts by mathematical operators on equations (3.1)–(3.5), and we have

$$\begin{aligned}\check{\nabla}_{X^c}^c Y^c &= \nabla_{X^c}^c Y^c + \eta^c(Y^c)(\phi X)^v + \eta^v(Y^c)(\phi X)^c + a^c(X^c)(\phi Y)^v \\ &\quad + a^v(X^c)(\phi Y)^c,\end{aligned}\tag{3.23}$$

$$\begin{aligned}\check{T}^c(X^c, Y^c) &= \eta^c(Y^c)(\phi X)^v + \eta^v(Y^c)(\phi X)^c - \eta^c(X^c)(\phi Y)^v \\ &\quad - \eta^v(X^c)(\phi Y)^c + a^c(X^c)(\phi Y)^v + a^v(X^c)(\phi Y)^c \\ &\quad - a^c(Y^c)(\phi X)^v - a^v(Y^c)(\phi X)^c,\end{aligned}\tag{3.24}$$

$$\eta^c(X^c) = g^c(X^c, \xi^c),\tag{3.25}$$

$$a^c(X^c) = g^c(X^c, A^c),\tag{3.26}$$

$$\begin{aligned}(\check{\nabla}_{X^c}^c g^c)(Y^c, Z^c) &= -\eta^c(Y^c)g^c((\phi X)^v, Z^c) - \eta^v(Y^c)g^c((\phi X)^c, Z^c) \\ &\quad - \eta^c(Z^c)g^c((\phi X)^v, Y^c) - \eta^v(Z^c)g^c((\phi X)^c, Y^c) \\ &\quad - 2a^c(X^c)g^c((\phi Y)^v, Z^c) - 2a^v(X^c)g^c((\phi Y)^c, Z^c).\end{aligned}\tag{3.27}$$

The connection given by equation (3.23) is said to be a QSNMC on an LP-Sasakian manifold in its tangent bundle. If the torsion tensor $\check{T}^c$ of TM^n endowed with $\check{\nabla}^c$ satisfies equation (3.24), then the complete lifts of Lorentzian metric g^c fulfill the relation equation (3.27).

Theorem 3.1. *If an LP-Sasakian manifold (M^n, g) with an almost Lorentzian para-contact metric structure (ϕ, ξ, η, g) admitting a QSNMC $\check{\nabla}$ which satisfies equations (3.24) and (3.27), then the QSNMC in the tangent bundle is given by*

$$\check{\nabla}_{X^c}^c Y^c = \nabla_{X^c}^c Y + \eta^c(Y^c)(\phi X)^v + \eta^v(Y^c)(\phi X)^c + a^c(X^c)(\phi Y)^v + a^v(X^c)(\phi Y)^c.$$

Proof. Let $\check{\nabla}^c$ be the complete lift of a linear connection in M^n given by

$$\check{\nabla}_{X^c}^c Y^c = \nabla_{X^c}^c Y^c + H^c(X^c, Y^c). \quad (3.28)$$

Now, we shall determine the complete lift of the tensor field H^c such that $\check{\nabla}^c$ satisfies equations (3.24) and (3.27). From equation (3.28), we have

$$\check{T}^c(X^c, Y^c) = H^c(X^c, Y^c) - H^c(Y^c, X^c). \quad (3.29)$$

We denote

$$G^c(X^c, Y^c, Z^c) = (\check{\nabla}_{X^c}^c g^c)(Y^c, Z^c). \quad (3.30)$$

From equations (3.28) and (3.30), we have

$$g^c(H^c(X^c, Y^c), Z^c) + g^c(H^c(X^c, Z^c), Y^c) = -G^c(X^c, Y^c, Z^c). \quad (3.31)$$

Using equations (3.27), (3.28), (3.30), and (3.31) we have

$$\begin{aligned} & g^c(\check{T}^c(X^c, Y^c), Z^c) + g^c(\check{T}^c(Z^c, X^c), Y^c) + g^c(\check{T}^c(Z^c, Y^c), X^c) \\ &= g^c(H^c(X^c, Y^c), Z^c) - g^c(H^c(Y^c, X^c), Z^c) + g^c(H^c(Z^c, X^c), Y^c) \\ & \quad - g^c(H^c(X^c, Z^c), Y^c) + g^c(H^c(Z^c, Y^c), X^c) - g^c(H^c(Y^c, Z^c), X^c) \\ &= g^c(H^c(X^c, Y^c), Z^c) - g^c(H^c(X^c, Z^c), Y^c) - G^c(Z^c, X^c, Y^c) + G^c(Y^c, X^c, Z^c) \\ &= 2g^c(H^c(X^c, Y^c), Z^c) + G^c(X^c, Y^c, Z^c) + G^c(Y^c, X^c, Z^c) - G^c(Z^c, X^c, Y^c) \\ &= 2g^c(H^c(X^c, Y^c), Z^c) - 2\left\{\eta^c(Z^c)g^c((\phi X)^v, Y^c) + \eta^v(Z^c)g^c((\phi X)^c, Y^c)\right\} \\ & \quad - 2\left\{a^c(X^c)g^c((\phi Y)^v, Z^c) + a^v(X^c)g^c((\phi Y)^c, Z^c)\right\} - 2\left\{a^c(Y^c)g^c((\phi X)^v, Z^c) \right. \\ & \quad \left. + a^v(Y^c)g^c((\phi X)^c, Z^c)\right\} + 2\left\{a^c(Z^c)g^c((\phi X)^v, Y^c) + a^v(Z^c)g^c((\phi X)^c, Y^c)\right\}, \end{aligned}$$

or,

$$\begin{aligned} H^c(X^c, Y^c) &= \frac{1}{2}\left\{\check{T}^c(X^c, Y^c) + {}'\check{T}^c(X^c, Y^c) + {}'\check{T}^c(Y^c, X^c)\right\} + a^c(X^c)(\phi Y)^v \\ & \quad + a^v(X^c)(\phi Y)^c + a^c(Y^c)(\phi X)^v + a^v(Y^c)(\phi X)^c + g^c((\phi X)^c, Y^c)\xi^v \\ & \quad + g^c((\phi X)^v, Y^c)\xi^c - g^c((\phi X)^c, Y^c)a^v - g^c((\phi X)^v, Y^c)a^c, \end{aligned}$$

where $'\check{T}^c$ is a tensor field of type $(1, 2)$ defined by

$$g^c\left(' \check{T}^c(X^c, Y^c), Z^c\right) = g^c\left(\check{T}^c(Z^c, X^c), Y^c\right),$$

or,

$$H^c(X^c, Y^c) = \eta^c(Y^c)(\phi X)^v + \eta^v(Y^c)(\phi X)^c + a^c(X^c)(\phi Y)^v + a^v(X^c)(\phi Y)^c,$$

which gives

$$\check{\nabla}_{X^c}^c Y^c = \nabla_{X^c}^c Y + \eta^c(Y^c)(\phi X)^v + \eta^v(Y^c)(\phi X)^c + a^c(X^c)(\phi Y)^v + a^v(X^c)(\phi Y)^c.$$

□

3.4 Curvature tensor of LP-Sasakian manifolds endowed with the QSNMC in the tangent bundle

Let $\check{R}^c$ and R^c be the curvature tensors of the connections $\check{\nabla}^c$ and ∇^c to tangent bundle TM^n , respectively.

$$\check{R}^c(X^c, Y^c)Z^c = \check{\nabla}_{X^c}^c \check{\nabla}_{Y^c}^c Z^c - \check{\nabla}_{Y^c}^c \check{\nabla}_{X^c}^c Z^c - \check{\nabla}_{[X^c, Y^c]}^c Z^c. \quad (3.32)$$

Using equation (3.23) in equation (3.32), we have

$$\begin{aligned}
\check{R}^c(X^c, Y^c)Z^c &= R^c(X^c, Y^c)Z^c + g^c\left((\phi X)^c, Z^c\right)(\phi Y)^v + g^c\left((\phi X)^v, Z^c\right)(\phi Y)^c \\
&\quad - g^c\left((\phi Y)^c, Z^c\right)(\phi X)^v - g^c\left((\phi Y)^v, Z^c\right)(\phi X)^c + \eta^c(Y^c)\eta^c(Z^c)X^v \\
&\quad + \eta^c(Y^c)\eta^v(Z^c)X^c + \eta^v(Y^c)\eta^c(Z^c)X^c - \eta^c(X^c)\eta^c(Z^c)Y^v \\
&\quad - \eta^c(X^c)\eta^v(Z^c)Y^c - \eta^v(X^c)\eta^c(Z^c)Y^c + a^c(Y^c)g^c(X^c, Z^c)\xi^v \\
&\quad + a^c(Y^c)g^c(X^v, Z^c)\xi^c + a^v(Y^c)g^c(X^c, Z^c)\xi^c - a^c(X^c)g^c(Y^c, Z^c)\xi^v \\
&\quad - a^c(X^c)g^c(Y^v, Z^c)\xi^c - a^v(X^c)g^c(Y^c, Z^c)\xi^c + a^c(Y^c)\eta^c(X^c)\eta^c(Z^c)\xi^v \\
&\quad + a^c(Y^c)\eta^c(X^c)\eta^v(Z^c)\xi^c + a^c(Y^c)\eta^v(X^c)\eta^c(Z^c)\xi^c + a^v(Y^c)\eta^c(X^c)\eta^c(Z^c)\xi^c \\
&\quad - a^c(X^c)\eta^c(Y^c)\eta^c(Z^c)\xi^v - a^c(X^c)\eta^c(Y^c)\eta^v(Z^c)\xi^c - a^c(X^c)\eta^v(Y^c)\eta^c(Z^c)\xi^c \\
&\quad - a^v(X^c)\eta^c(Y^c)\eta^c(Z^c)\xi^c + da^c(X^c, Y^c)(\phi Z)^v + da^v(X^c, Y^c)(\phi Z)^c,
\end{aligned} \tag{3.33}$$

where

$$R^c(X^c, Y^c)Z^c = \nabla_{X^c}^c \nabla_{Y^c}^c Z^c - \nabla_{Y^c}^c \nabla_{X^c}^c Z^c - \nabla_{[X^c, Y^c]}^c Z^c, \tag{3.34}$$

is the curvature tensor of ∇^c with respect to the Riemannian connection. Contracting equation (3.33), we obtain

$$\begin{aligned}
\check{S}^c(Y^c, Z^c) &= S^c(Y^c, Z^c) - \gamma^c g^c\left((\phi Y)^c, Z^c\right) + \left[1 - a^c(\xi^c)\right] g^c(Y^c, Z^c) \\
&\quad + \left[\eta^c - a^c(\xi^c)\right] \left[\eta^c(Y^c)\eta^v(Z^c) + \eta^v(Y^c)\eta^c(Z^c)\right] \\
&\quad + da^c\left((\phi Z)^c, Y^c\right),
\end{aligned} \tag{3.35}$$

and

$$\check{r}^c = r^c - (n-1)a^c(\xi^c) + \lambda^c - \gamma^{c^2}, \tag{3.36}$$

where $\check{S}^c$ and $\check{r}^c$ are the Ricci tensor and scalar curvature with respect to $\check{\nabla}^c$.

$$\lambda^c = \text{trace } da^c\left((\phi Z)^c, Y^c\right) \quad \text{and} \quad \gamma^c = \text{trace } \phi^c. \tag{3.37}$$

Hence, we have the following theorem:

Theorem 3.2. *In an LP-Sasakian manifold (M^n, g) with tangent bundle TM^n admitting QSNMC, we have the following:*

- (i) *The complete lift of curvature tensor $\check{R}^c$ is given by equation (3.33).*
- (ii) *The complete lift of Ricci tensor $\check{S}^c$ is given by equation (3.35).*
- (iii) *The complete lift of scalar curvature $\check{r}^c$ is given by equation (3.36).*

Let us consider that $\check{R}^c(X^c, Y^c) = 0$ in equation (3.33), and by contracting it we also obtain

$$\begin{aligned} S^c(Y^c, Z^c) &= \gamma^c g^c((\phi Y)^c, Z^c) - [1 - a^c(\xi^c)]g^c(Y^c, Z^c) \\ &\quad - [\eta^c - a^c(\xi^c)] [\eta^c(Y^c)\eta^v(Z^c) + \eta^v(Y^c)\eta^c(Z^c)] \\ &\quad - da^c((\phi Z)^c, Y^c), \end{aligned} \quad (3.38)$$

which gives

$$r^c = (n-1)a^c(\xi^c) - \lambda^c + \gamma^{c^2}. \quad (3.39)$$

Hence, we have the following theorems:

Theorem 3.3. *In an LP-Sasakian manifold, (M^n, g) , with tangent bundle TM^n endowed with QSNMC whose curvature tensor vanishes, then the complete lift of r^c is given by equation (3.39).*

From equation (3.33), it follows that

$${}'\check{R}^c(X^c, Y^c, Z^c, W^c) + {}'\check{R}^c(Y^c, X^c, Z^c, W^c) = 0, \quad (3.40)$$

$$\begin{aligned}
& {}'\check{R}^c(X^c, Y^c, Z^c, W^c) + {}'\check{R}^c(X^c, Y^c, W^c, Z^c) \\
&= \eta^c(Y^c)\eta^v(Z^c)g^c(X^c, W^c) + \eta^v(Y^c)\eta^c(Z^c)g^c(X^c, W^c) \\
&\quad - \eta^c(X^c)\eta^v(Z^c)g^c(Y^c, W^c) - \eta^v(X^c)\eta^c(Z^c)g^c(Y^c, W^c) \\
&\quad + \eta^c(Y^c)\eta^v(W^c)g^c(X^c, Z^c) + \eta^v(Y^c)\eta^c(W^c)g^c(X^c, Z^c) \\
&\quad - \eta^c(X^c)\eta^v(W^c)g^c(Y^c, Z^c) - \eta^v(X^c)\eta^c(W^c)g^c(Y^c, Z^c) \\
&\quad + a^c(Y^c)\eta^v(W^c)g^c(X^c, Z^c) + a^v(Y^c)\eta^c(W^c)g^c(X^c, Z^c) \\
&\quad - a^c(X^c)\eta^v(W^c)g^c(Y^c, Z^c) - a^v(X^c)\eta^c(W^c)g^c(Y^c, Z^c) \\
&\quad + a^c(Y^c)\eta^v(Z^c)g^c(X^c, W^c) + a^v(Y^c)\eta^c(Z^c)g^c(X^c, W^c) \\
&\quad - a^c(X^c)\eta^v(Z^c)g^c(Y^c, W^c) - a^v(X^c)\eta^c(Z^c)g^c(Y^c, W^c) \\
&\quad + 2\left[a^c(Y^c)\eta^c(X^c)\eta^c(Z^c)\eta^v(W^c) + a^c(Y^c)\eta^c(X^c)\eta^v(Z^c)\eta^c(W^c)\right. \\
&\quad \left.+ a^c(Y^c)\eta^v(X^c)\eta^c(Z^c)\eta^c(W^c) + a^v(Y^c)\eta^c(X^c)\eta^c(Z^c)\eta^c(W^c)\right] \\
&\quad - 2\left[a^c(X^c)\eta^c(Y^c)\eta^c(Z^c)\eta^v(W^c) + a^c(X^c)\eta^c(Y^c)\eta^v(Z^c)\eta^c(W^c)\right. \\
&\quad \left.+ a^c(X^c)\eta^v(Y^c)\eta^c(Z^c)\eta^c(W^c) + a^v(X^c)\eta^c(Y^c)\eta^c(Z^c)\eta^c(W^c)\right] \\
&\quad + 2da^c(X^c, Y^c)g^c((\phi Z)^c, W^c).
\end{aligned} \tag{3.41}$$

$$\begin{aligned}
& {}'\check{R}^c(X^c, Y^c, Z^c, W^c) - {}'\check{R}^c(Z^c, W^c, X^c, Y^c) \\
&= \eta^c(Y^c)\eta^v(Z^c)g^c(X^c, W^c) + \eta^v(Y^c)\eta^c(Z^c)g^c(X^c, W^c) \\
&\quad - \eta^c(X^c)\eta^v(W^c)g^c(Y^c, Z^c) - \eta^v(X^c)\eta^c(W^c)g^c(Y^c, Z^c) \\
&\quad + a^c(Y^c)\eta^v(W^c)g^c(X^c, Z^c) + a^v(Y^c)\eta^c(W^c)g^c(X^c, Z^c) \\
&\quad - a^c(X^c)\eta^v(W^c)g^c(Y^c, Z^c) - a^v(X^c)\eta^c(W^c)g^c(Y^c, Z^c) \\
&\quad + a^c(Z^c)\eta^v(Y^c)g^c(X^c, W^c) + a^v(Z^c)\eta^c(Y^c)g^c(X^c, W^c) \\
&\quad - a^c(W^c)\eta^v(Y^c)g^c(X^c, Z^c) - a^v(W^c)\eta^c(Y^c)g^c(X^c, Z^c) \\
&\quad + a^c(Y^c)\eta^c(X^c)\eta^c(Z^c)\eta^v(W^c) + a^c(Y^c)\eta^c(X^c)\eta^v(Z^c)\eta^c(W^c) \\
&\quad + a^c(Y^c)\eta^v(X^c)\eta^c(Z^c)\eta^c(W^c) + a^v(Y^c)\eta^c(X^c)\eta^c(Z^c)\eta^c(W^c) \\
&\quad - a^c(X^c)\eta^c(Y^c)\eta^c(Z^c)\eta^v(W^c) - a^c(X^c)\eta^c(Y^c)\eta^v(Z^c)\eta^c(W^c) \\
&\quad - a^c(X^c)\eta^v(Y^c)\eta^c(Z^c)\eta^c(W^c) - a^v(X^c)\eta^c(Y^c)\eta^c(Z^c)\eta^c(W^c) \\
&\quad + a^c(Z^c)\eta^c(X^c)\eta^c(Y^c)\eta^v(W^c) + a^c(Z^c)\eta^c(X^c)\eta^v(Y^c)\eta^c(W^c) \\
&\quad + a^c(Z^c)\eta^v(X^c)\eta^c(Y^c)\eta^c(W^c) + a^v(Z^c)\eta^c(X^c)\eta^c(Y^c)\eta^c(W^c) \\
&\quad - a^c(W^c)\eta^c(X^c)\eta^c(Y^c)\eta^v(Z^c) - a^c(W^c)\eta^c(X^c)\eta^v(Y^c)\eta^c(Z^c) \\
&\quad - a^c(W^c)\eta^v(X^c)\eta^c(Y^c)\eta^c(Z^c) - a^v(W^c)\eta^c(X^c)\eta^c(Y^c)\eta^c(Z^c) \\
&\quad + da^c(X^c, Y^c)g^c((\phi Z)^c, W^c) - da^c(Z^c, W^c)g^c((\phi X)^c, Y^c),
\end{aligned} \tag{3.42}$$

and

$$\begin{aligned}
& {}'\check{R}^c(X^c, Y^c, Z^c, W^c) + {}'\check{R}^c(Y^c, Z^c, X^c, W^c) + {}'\check{R}^c(Z^c, X^c, Y^c, W^c) \\
&= da^c(X^c, Y^c)g^c((\phi Z)^c, W^c) + da^c(Y^c, Z^c)g^c((\phi X)^c, W^c) \\
&\quad + da^c(Z^c, X^c)g^c((\phi Y)^c, W^c).
\end{aligned} \tag{3.43}$$

If the 1-form a^c is closed, then from equation (3.43) we have

$${}'\check{R}^c(X^c, Y^c, Z^c, W^c) + {}'\check{R}^c(Y^c, Z^c, X^c, W^c) + {}'\check{R}^c(Z^c, X^c, Y^c, W^c) = 0, \tag{3.44}$$

where

$$\check{R}^c(X^c, Y^c, Z^c, W^c) = g^c(\check{R}^c(X^c, Y^c)Z^c, W^c)$$

and

$$'R^c(X^c, Y^c, Z^c, W^c) = g^c\left(R^c(X^c, Y^c)Z^c, W^c\right).$$

Hence, we have the following theorem:

Theorem 3.4. *In an LP-Sasakian manifold, (M^n, g) with tangent bundle TM^n endowed with a QSNMC, the curvature tensor satisfies the relations of equations (3.40)–(3.43). In particular, if the complete lift of 1-form a^c is closed, then*

$$' \check{R}^c(X^c, Y^c, Z^c, W^c) + ' \check{R}^c(Y^c, Z^c, X^c, W^c) + ' \check{R}^c(Z^c, X^c, Y^c, W^c) = 0.$$

3.5 Symmetric and skew-symmetric condition of the Ricci tensor of an LP-Sasakian Manifolds endowed with the QSNMC in the tangent bundle

From equation (3.35), we have

$$\begin{aligned} \check{S}^c(Z^c, Y^c) &= S^c(Z^c, Y^c) - \gamma^c g^c\left((\phi Z)^c, Y^c\right) + \left[1 - a^c(\xi^c)\right] g^c(Y^c, Z^c) \\ &\quad + \left[\eta^c - a^c(\xi^c)\right] \left[\eta^c(Y^c)\eta^v(Z^c) + \eta^v(Y^c)\eta^c(Z^c)\right] + da^c\left((\phi Y)^c, Z^c\right). \end{aligned} \quad (3.45)$$

From equations (3.35) and (3.45), we have

$$\check{S}^c(Y^c, Z^c) - \check{S}^c(Z^c, Y^c) = da^c\left((\phi Z)^c, Y^c\right) - da^c\left((\phi Y)^c, Z^c\right). \quad (3.46)$$

If $\check{S}^c(Y^c, Z^c)$ is symmetric, then the left-hand side of equation (3.46) vanishes, and then

$$da^c\left((\phi Z)^c, Y^c\right) = da^c\left((\phi Y)^c, Z^c\right). \quad (3.47)$$

Moreover, if equation (3.47) holds, then from equation (3.46), $\check{S}^c(Y^c, Z^c)$ is symmetric. Hence, we have the following theorem:

Theorem 3.5. *In an LP-Sasakian manifold (M^n, g) with tangent bundle TM^n endowed with QSNMC $\check{\nabla}^c$, the Ricci tensor $\check{S}^c(Y^c, Z^c)$ with respect to QSNMC is symmetric if and only if the relation of equation (3.47) holds.*

From equations (3.35) and (3.45), we have

$$\begin{aligned} \check{S}^c(Y^c, Z^c) + \check{S}^c(Z^c, Y^c) &= 2S^c(Y^c, Z^c) - 2\gamma^c g^c((\phi Y)^c, Z^c) \\ &\quad + 2[1 - a^c(\xi^c)]g^c(Y^c, Z^c) + 2[n - a^c(\xi^c)]\left[\eta^c(Y^c)\eta^v(Z^c) \right. \\ &\quad \left. + \eta^v(Y^c)\eta^c(Z^c)\right] + da^c((\phi Y)^c, Z^c) + da^c((\phi Z)^c, Y^c). \end{aligned} \quad (3.48)$$

By taking the skew-symmetry of $\check{S}^c(Y^c, Z^c)$, the left-hand side of equation (3.48) will vanish and we have

$$\begin{aligned} S^c(Y^c, Z^c) &= \gamma^c g^c((\phi Y)^c, Z^c) - [1 - a^c(\xi^c)]g^c(Y^c, Z^c) \\ &\quad - [n - a^c(\xi^c)]\left[\eta^c(Y^c)\eta^v(Z^c) - \eta^v(Y^c)\eta^c(Z^c)\right] \\ &\quad - \frac{1}{2}\left[da^c((\phi Y)^c, Z^c) + da^c((\phi Z)^c, Y^c)\right]. \end{aligned} \quad (3.49)$$

Moreover, if $S^c(Y^c, Z^c)$ is given by equations (3.49), then from equation (3.48), we have

$$S^c(Y^c, Z^c) + S^c(Z^c, Y^c) = 0.$$

Hence, we have the following theorem:

Theorem 3.6. *The necessary and sufficient condition for the Ricci tensor of the QSNMC connection $\check{\nabla}^c$ on an LP-Sasakian manifold (M^n, g) to be skew-symmetric in the tangent bundle TM^n is that the Ricci tensor of the Levi-Civita connection ∇^c satisfies equation (3.49).*

3.6 Skew-symmetric properties of the projective Ricci tensor in an LP-Sasakian manifolds endowed with the QSNMC in the tangent bundle

Chaki and Saha (1994) defined the projective Ricci tensor in a Riemannian manifold as

$$P(X, Y) = \frac{n}{n-1} \left[S(X, Y) - \frac{r}{n} g(X, Y) \right]. \quad (3.50)$$

So, the projective Ricci tensor with respect to QSNMC $\check{\nabla}$ is defined as

$$\check{P}(X, Y) = \frac{n}{n-1} \left[\check{S}(X, Y) - \frac{\check{r}}{n} g(X, Y) \right]. \quad (3.51)$$

Taking a complete lift by mathematical operators on equation (3.51), we have

$$\check{P}^c(X^c, Y^c) = \frac{n}{n-1} \left[\check{S}^c(X^c, Y^c) - \frac{\check{r}^c}{n} g^c(X^c, Y^c) \right]. \quad (3.52)$$

Using equations (3.35) and (3.36) in equation (3.52), we have

$$\begin{aligned} \check{P}^c(X^c, Y^c) = & \frac{n}{n-1} \left[S^c(X^c, Y^c) - \gamma^c g^c((\phi X)^c, Y^c) + (1 - a^c(\xi^c)) g^c(X^c, Y^c) \right. \\ & + (n - a^c(\xi^c)) (\eta^c(X^c) \eta^v(Y^c) + \eta^v(X^c) \eta^c(Y^c)) + da^c((\phi Y)^c, X^c) \\ & \left. - \frac{1}{n} (r^c - (n-1)a^c(\xi^c) + \lambda^c - \gamma^{c^2}) g^c(X^c, Y^c) \right]. \end{aligned} \quad (3.53)$$

Similarly, we have

$$\begin{aligned} \check{P}^c(Y^c, X^c) = & \frac{n}{n-1} \left[S^c(Y^c, X^c) - \gamma^c g^c((\phi Y)^c, X^c) + (1 - a^c(\xi^c)) g^c(Y^c, X^c) \right. \\ & + (n - a^c(\xi^c)) (\eta^c(X^c) \eta^v(Y^c) + \eta^v(X^c) \eta^c(Y^c)) + da^c((\phi X)^c, Y^c) \\ & \left. - \frac{1}{n} (r^c - (n-1)a^c(\xi^c) + \lambda^c - \gamma^{c^2}) g^c(Y^c, X^c) \right]. \end{aligned} \quad (3.54)$$

From equations (3.53) and (3.54), we have

$$\begin{aligned}
\check{P}^c(X^c, Y^c) + \check{P}^c(Y^c, X^c) = & \frac{n}{n-1} \left[2S^c(X^c, Y^c) - 2\gamma^c g^c((\phi X)^c, Y^c) \right. \\
& + 2(1 - a^c(\xi^c))g^c(X^c, Y^c) + 2(n - a^c(\xi^c)) \\
& \left(\eta^c(X^c)\eta^v(Y^c) + \eta^v(X^c)\eta^c(Y^c) \right) - \frac{2}{n} \left(r^c \right. \\
& \left. - (n-1)a^c(\xi^c) + \lambda^c - \gamma^{c^2} \right) g^c(X^c, Y^c) \\
& \left. + da^c((\phi X)^c, Y^c) + da^c((\phi Y)^c, X^c) \right]. \quad (3.55)
\end{aligned}$$

If $\check{P}^c(X^c, Y^c)$ is skew-symmetric, then the left-hand side of equation (3.55) vanishes and we have

$$\begin{aligned}
S^c(X^c, Y^c) = & \left[\gamma^c g^c((\phi X)^c, Y^c) - (1 - a^c(\xi^c))g^c(X^c, Y^c) \right. \\
& - (n - a^c(\xi^c)) \left(\eta^c(X^c)\eta^v(Y^c) + \eta^v(X^c)\eta^c(Y^c) \right) \\
& + \frac{1}{n} \left(r^c - (n-1)a^c(\xi^c) + \lambda^c - \gamma^{c^2} \right) g^c(X^c, Y^c) \\
& \left. - \frac{1}{2} \left(da^c((\phi X)^c, Y^c) + da^c((\phi Y)^c, X^c) \right) \right]. \quad (3.56)
\end{aligned}$$

Moreover, if $S^c(X^c, Y^c)$ is given by equation (3.56), then from equation (3.55) we obtain

$$\check{P}^c(X^c, Y^c) + \check{P}^c(Y^c, X^c) = 0, \quad (3.57)$$

which gives a skew-symmetric condition of the projective Ricci tensor of $\check{\nabla}^c$.

Hence, we have the following theorem:

Theorem 3.7. *The necessary and sufficient condition for the projective Ricci tensor of the QSNMC connection $\check{\nabla}^c$ on an LP-Sasakian manifold (M^n, g) to be skew-symmetric in the tangent bundle TM^n is that the Ricci tensor of the Levi-Civita connection ∇^c satisfies equation (3.56).*

3.7 Einstein manifold endowed with the QSNMC in an LP-Sasakian manifold to the tangent bundle

A Riemannian manifold (M^n, g) is called an Einstein manifold with respect to Riemannian connection if

$$S^c(X^c, Y^c) = \frac{r^c}{n} g^c(X^c, Y^c). \quad (3.58)$$

Then, the Einstein manifold with respect to QSNMC $\check{\nabla}^c$ is given by

$$\check{S}^c(X^c, Y^c) = \frac{\check{r}^c}{n} g^c(X^c, Y^c). \quad (3.59)$$

Using equations (3.35) and (3.36) in equation (3.59), we have

$$\begin{aligned} & \check{S}^c(X^c, Y^c) - \frac{\check{r}^c}{n} g^c(X^c, Y^c) \\ &= S^c(X^c, Y^c) - \frac{r^c}{n} g^c(X^c, Y^c) - \gamma^c g^c((\phi X)^c, Y^c) \\ &+ da^c((\phi Y)^c, X^c) + \frac{1}{n} [n + \gamma^{c^2} - \lambda^c - a^c(\xi^c)] g^c(X^c, Y^c) \\ &+ (n - a^c(\xi^c)) [\eta^c(X^c) \eta^v(Y^c) + \eta^v(X^c) \eta^c(Y^c)]. \end{aligned} \quad (3.60)$$

If

$$\begin{aligned} & \gamma^c g^c((\phi X)^c, Y^c) + da^c(X^c, (\phi Y)^c) \\ &= \frac{1}{n} [n + \gamma^{c^2} - \lambda^c - a^c(\xi^c)] g^c(X^c, Y^c) \\ &+ (n - a^c(\xi^c)) [\eta^c(X^c) \eta^v(Y^c) + \eta^v(X^c) \eta^c(Y^c)], \end{aligned} \quad (3.61)$$

then from equation (3.60), we have

$$\check{S}^c(X^c, Y^c) - \frac{\check{r}^c}{n} g^c(X^c, Y^c) = S^c(X^c, Y^c) - \frac{r^c}{n} g^c(X^c, Y^c). \quad (3.62)$$

Hence, we have the following theorem:

Theorem 3.8. *In an LP-Sasakian manifold (M^n, g) whose tangent bundle TM^n admits a QSNMC connection, the manifold is Einstein with respect to the Riemannian (Levi-Civita) connection if and only if it is Einstein with respect to the connection $\check{\nabla}^c$, provided that equation (3.61) holds.*

3.8 Example

Let M be a four-dimensional manifold defined as

$$M = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4; x_4 \neq 0 \right\}, \quad (3.63)$$

where $\mathbb{R}$ is the set of real numbers. Let x_1, x_2, x_3, x_4 be given by

$$e_1 = \frac{x_1}{x_4} \frac{\partial}{\partial x_1}, \quad e_2 = \frac{x_2}{x_4} \frac{\partial}{\partial x_2}, \quad e_3 = \frac{x_3}{x_4} \frac{\partial}{\partial x_3}, \quad e_4 = x_4 \frac{\partial}{\partial x_4},$$

where $\{e_1, e_2, e_3, e_4\}$ are a linearly independent global frame on M . Let the 1-form η be given by

$$\eta(X) = g(X, e_4).$$

The Lorentzian metric g is defined by

$$g(e_i, e_j) = \begin{cases} -1, & i = j = 4 \\ 1, & i = j = 1, 2, 3 \\ 0, & \text{otherwise.} \end{cases}$$

Let ϕ be the tensor field defined by

$$\phi e_i = \begin{cases} 0, & i = 4 \\ e_i, & i = 1, 2, 3. \end{cases}$$

Using the linearity of ϕ and g , we acquire $\eta(e_4) = -1$, $\phi^2 X = -X + \eta(X)e_4$ and $g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$. Thus, for $e_4 = \xi$, the structure (ϕ, ξ, η, g) is

an almost para-contact metric structure on M and M is called an almost para-contact metric manifold. In addition, M satisfies

$$(\nabla_X \phi)Y = g(X, Y)e_4 + \eta(Y)X + 2\eta(X)\eta(Y)e_4.$$

Here, for $e_4 = \xi$, M is an LP-Sasakian manifold. In tangent bundle TM , let the complete and vertical lifts of e_1, e_2, e_3, e_4 be $e_1^c, e_2^c, e_3^c, e_4^c$ and $e_1^v, e_2^v, e_3^v, e_4^v$ on M and let g^c be the complete lift of the Lorentzian metric g on TM such that

$$g^c(X^v, e_4^c) = \left(g^c(X, e_4)\right)^v = \left(\eta(X)\right)^v, \quad (3.64)$$

$$g^c(X^c, e_4^c) = \left(g^c(X, e_4)\right)^c = \left(\eta(X)\right)^c, \quad (3.65)$$

$$g^c(e_4^c, e_4^c) = -1, \quad g^v(X^v, e_4^c) = 0, \quad g^v(e_4^v, e_4^c) = 0, \quad (3.66)$$

and so on. Let ϕ^c and ϕ^v be the complete and vertical lifts of the $(1, 1)$ tensor field ϕ defined by

$$\phi^v(e_4^v) = \phi^c(e_4^c) = 0, \quad (3.67)$$

$$\phi^v(e_1^v) = e_1^v, \quad \phi^c(e_1^c) = e_1^c, \quad (3.68)$$

$$\phi^v(e_2^v) = e_2^v, \quad \phi^c(e_2^c) = e_2^c, \quad (3.69)$$

$$\phi^v(e_3^v) = e_3^v, \quad \phi^c(e_3^c) = e_3^c. \quad (3.70)$$

Using the linearity of ϕ and g , we infer that

$$(\phi^2 X)^c = X^c + \eta^c(X)e_4^v + \eta^v(X)e_4^c, \quad (3.71)$$

$$g^c\left((\phi e_4)^c, (\phi e_3)^c\right) = g^c(e_4^c, e_3^c) + \eta^c(e_4^c)\eta^v(e_3^c) + \eta^v(e_4^c)\eta^c(e_3^c). \quad (3.72)$$

Thus, for $e_4 = \xi$ in equations (3.64)–(3.66) and (3.71), the structure $(\phi^c, \xi^c, \eta^c, g^c)$ is an almost para-contact metric structure on TM and satisfies the relation

$$\begin{aligned} (\nabla_{e_4^c}^c \phi^c)e_3^c &= g^c(e_4^c, e_3^c)\xi^v + g^c(e_4^v, e_3^c)\xi^c + \eta^c(e_3^c)e_4^v + \eta^v(e_3^c)e_4^c \\ &\quad + 2\left\{\eta^c(e_4^c)\eta^c(e_3^c)\xi^v + \eta^c(e_4^c)\eta^v(e_3^c)\xi^c + \eta^v(e_4^c)\eta^c(e_3^c)\xi^c\right\}. \end{aligned}$$

Thus, $(\phi^c, \xi^c, \eta^c, g^c, TM)$ is an LP-Sasakian manifold.

Chapter 4

Some properties on statistical manifolds and Sasakian statistical manifolds in the tangent bundle

4.1 Statistical Manifolds

The statistical manifold $(M, \ddot{\nabla}, g)$ and submanifold are defined in equations (1.68-1.75, 1.79-1.83), the linear connection $\check{\nabla}$ on M is given by (Yildirim, 2022)

$$\check{\nabla}_X Y = \ddot{\nabla}_X Y + \eta(Y)X - K_X Y, \quad (4.1)$$

where η is a 1-form associated with vector field ξ

$$\eta(X) = g(X, \xi), \quad (4.2)$$

for all vector fields $X, Y \in \chi(M)$. By using equation (1.71) in equation (4.1), we get

$$\check{\nabla}_X Y = \ddot{\nabla}_X^* Y + \eta(Y)X + K_X Y, \quad (4.3)$$

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where K is a tensor field of type $(1, 2)$. The torsion tensor $\check{T}$ with respect to $\check{\nabla}$ is given by

$$\begin{aligned}\check{T}(X, Y) &= \check{\nabla}_X Y + \eta(Y)X - K_X Y + \check{\nabla}_X^* Y + \eta(Y)X + K_X Y - [X, Y] \\ &= \eta(Y)X - \eta(X)Y.\end{aligned}\tag{4.4}$$

A linear connection satisfying equation (4.4) is called semi symmetric connection (SSC). For any vector fields X, Y, Z on M , we have

$$(\check{\nabla}_X g)(Y, Z) = Xg(Y, Z) - g(\check{\nabla}_X Y, Z) - g(Y, \check{\nabla}_X Z).\tag{4.5}$$

Using equations (1.69), (1.72) and (4.1), we get

$$\begin{aligned}(\check{\nabla}_X g)(Y, Z) &= g(\check{\nabla}_X Y, Z) + g(Y, \check{\nabla}_X^* Z) + g(Y, \check{\nabla}_X Z - 2K_X Z) \\ &\quad - g(\check{\nabla}_X Y + \eta(Y)X - K_X Y, Z) - g(Y, \check{\nabla}_X Z + \eta(Z)X \\ &\quad - K_X Z) \\ &= -\eta(Y)g(X, Z) - \eta(Z)g(X, Y).\end{aligned}\tag{4.6}$$

So, $\check{\nabla}g \neq 0$. Hence a linear connection $\check{\nabla}$ defined by equation (4.1) satisfies equations (4.4) and (4.6), and it is called semi-symmetric non-metric connection (briefly SSNMC).

4.1.1 Statistical manifolds in the tangent bundle

Let us denote TM to be the tangent bundle of a statistical manifold M . The properties of the statistical manifold and its submanifold in the tangent bundle are obtained by taking complete lifts using mathematical operators on equations (1.68-1.75) and (1.79-1.83), we get (Kumar et al., 2023b):

$$(\check{\nabla}_{X^c}^c g^c)(Y^c, Z^c) = (\check{\nabla}_{Y^c}^c g^c)(X^c, Z^c),\tag{4.7}$$

$$X^c g^c(Y^c, Z^c) = g^c(\check{\nabla}_{X^c}^c Y^c, Z^c) + g^c(Y^c, \check{\nabla}_{X^c}^{*c} Z^c),\tag{4.8}$$

$$\nabla^c = \frac{1}{2}(\ddot{\nabla}^c + \nabla^{*c}), \quad (4.9)$$

$$K_{X^c}^c Y^c = \ddot{\nabla}_{X^c}^c Y^c - \nabla_{X^c}^c Y^c, \quad K_{X^c}^c Y^c = \frac{1}{2}(\ddot{\nabla}_{X^c}^c Y^c - \ddot{\nabla}_{X^{*c}}^c Y^c), \quad (4.10)$$

$$g^c(K_{X^c}^c Y^c, Z^c) = g^c(K_{X^c}^c Z^c, Y^c), \quad K_{X^c}^c Y^c = K_{Y^c}^c X^c, \quad (4.11)$$

$$\ddot{R}^c(X^c, Y^c)Z^c = \ddot{\nabla}_{X^c}^c \ddot{\nabla}_{Y^c}^c Z^c - \ddot{\nabla}_{Y^c}^c \ddot{\nabla}_{X^c}^c Z^c - \ddot{\nabla}_{[X^c, Y^c]}^c Z^c, \quad (4.12)$$

$$\ddot{R}^c(X^c, Y^c)Z^c = -\ddot{R}^c(Y^c, X^c)Z^c, \quad \ddot{R}^{*c}(X^c, Y^c)Z^c = -\ddot{R}^{*c}(Y^c, X^c)Z^c, \quad (4.13)$$

$$g^c(\ddot{R}^c(X^c, Y^c)Z^c, W^c) = -g^c(\ddot{R}^{*c}(X^c, Y^c)W^c, Z^c), \quad (4.14)$$

$$\ddot{R}^c(X^c, Y^c)Z^c + \ddot{R}^c(Y^c, Z^c)X^c + \ddot{R}^c(Z^c, X^c)Y^c = 0, \quad (4.15)$$

$$\nabla_{X^c}^c Y^c = \nabla_{X^c}^{\odot c} Y^c + h^{\odot c}(X^c, Y^c), \quad \nabla_{X^c}^c Q^c = -A_{Q^c}^{\odot c} X^c + D_{X^c}^{\odot c} Q^c, \quad (4.16)$$

$$\ddot{\nabla}_{X^c}^c Y^c = \dot{\nabla}_{X^c}^c Y^c + h^c(X^c, Y^c), \quad \ddot{\nabla}_{X^c}^c Q^c = -A_{Q^c}^c X^c + D_{X^c}^c Q^c, \quad (4.17)$$

$$\ddot{\nabla}_{X^c}^{*c} Y^c = \dot{\nabla}_{X^c}^{*c} Y^c + h^{*c}(X^c, Y^c), \quad \ddot{\nabla}_{X^c}^{*c} Q^c = -A_{Q^c}^{*c} X^c + D_{X^c}^{*c} Q^c, \quad (4.18)$$

$$g^c(A_{Q^c}^c X^c, Y^c) = g^c(h^c(X^c, Y^c), Q^c), \quad (4.19)$$

$$g^c(A_{Q^c}^{*c} X^c, Y^c) = g^c(h^{*c}(X^c, Y^c), Q^c), \quad (4.20)$$

$$h^c(X^c, Y^c) = H^c g^c(X^c, Y^c), \quad h^{*c}(X^c, Y^c) = H^{*c} g^c(X^c, Y^c), \quad (4.21)$$

where the mathematical operators 'c' and 'v' denote the complete and vertical lifts respectively, X, Y, Z, W are the vector fields on M , g is a Riemannian metric, K is the tensor field of type $(1, 2)$, $\ddot{R}$ is the curvature tensor of the affine connection $\ddot{\nabla}$, $\ddot{R}^*$ is the curvature tensor of the dual affine connection $\ddot{\nabla}^*$, ∇ is the Levi-Civita connection, $\nabla^\odot$ and $h^\odot$ are the induced connection and the second form on statistical submanifold N respectively, $A^\odot$ is the shape operator and $D^\odot$ is the normal connection on $N^\perp$. $\dot{\nabla}, \dot{\nabla}^*$ are statistical and dual connections on N , A and A^* denotes the shape operators, D and D^* are the normal connections on $N^\perp$. h and h^* are second fundamental forms on N . H and H^* are the mean curvature vectors admitting $\ddot{\nabla}$ and $\ddot{\nabla}^*$ respectively.

4.1.2 SSNMC of a statistical manifolds in the tangent bundle

In an n -dimensional statistical manifold and its tangent bundle TM , we are taking complete lifts by using mathematical operators on equations (4.1-4.6), we get

$$\check{\nabla}_{X^c}^c Y^c = \ddot{\nabla}_{X^c}^c Y^c + \eta^c(Y^c)X^v + \eta^v(Y^c)X^c - K_{X^c}^c Y^c, \quad (4.22)$$

$$\eta^c(X^c) = g^c(X^c, \xi^c), \quad (4.23)$$

$$\check{\nabla}_{X^c}^c Y^c = \ddot{\nabla}_{X^c}^{*c} Y^c + \eta^c(Y^c)X^v + \eta^v(Y^c)X^c + K_{X^c}^c Y^c, \quad (4.24)$$

$$\check{T}^c(X^c, Y^c) = \eta^c(Y^c)X^v + \eta^v(Y^c)X^c - \eta^c(X^c)Y^v - \eta^v(X^c)Y^c, \quad (4.25)$$

$$\begin{aligned} (\check{\nabla}_{X^c}^c g^c)(Y^c, Z^c) &= X^c g^c(Y^v, Z^c) + X^v g^c(Y^c, Z^c) \\ &\quad - g^c(\check{\nabla}_{X^c}^c Y^c, Z^c) - g^c(Y^c, \check{\nabla}_{X^c}^c Z^c), \end{aligned} \quad (4.26)$$

$$\begin{aligned} (\check{\nabla}_{X^c}^c g^c)(Y^c, Z^c) &= -\eta^c(Y^c)g^c(X^v, Z^c) - \eta^v(Y^c)g^c(X^c, Z^c) \\ &\quad - \eta^c(Z^c)g^c(X^v, Y^c) - \eta^v(Z^c)g^c(X^c, Y^c). \end{aligned} \quad (4.27)$$

Equation (4.22) is said to be a SSNMC if the torsion tensor $\check{T}^c$ of TM with respect to $\check{\nabla}^c$ satisfies equation (4.25) and the Riemannian metric g^c holds the relation of equation (4.27). From now on, we will denote the statistical manifold $(M, \ddot{\nabla}, g)$ in the tangent bundle TM as $M^{\ddot{\nabla}^c}$.

Theorem 4.1. *Let $M^{\ddot{\nabla}^c}$ be a statistical manifold admitting a SSNMC in the tangent bundle TM which satisfies equations (4.25) and (4.27), then the SSNMC of the statistical manifold in the tangent bundle is given by equation (4.22) and satisfies equation (4.24).*

Proof. Let $\check{\nabla}^c$ be the complete lift of a linear connection defined on M by

$$\check{\nabla}_{X^c}^c Y^c = \ddot{\nabla}_{X^c}^c Y^c + W^c(X^c, Y^c), \quad (4.28)$$

where $\ddot{\nabla}^c$ is the complete lift of the statistical connection and W^c is complete lift

of a tensor field of type $(0, 2)$ defined on M and we have $\check{\nabla}^c$ satisfies equations (4.25) and (4.27) and from equations (4.27) and (4.28), we have

$$\begin{aligned}
& -\eta^c(Y^c)g^c(X^v, Z^c) - \eta^v(Y^c)g^c(X^c, Z^c) - \eta^c(Z^c)g^c(X^v, Y^c) \\
& - \eta^v(Z^c)g^c(X^c, Y^c) \\
& = (\check{\nabla}_{X^c}^c g^c)(Y^c, Z^c) \\
& = X^c g^c(Y^v, Z^c) + X^v g^c(Y^c, Z^c) - g^c(\check{\nabla}_{X^c}^c Y^c, Z^c) - g^c(Y^c, \check{\nabla}_{X^c}^c Z^c) \\
& = X^c g^c(Y^v, Z^c) + X^v g^c(Y^c, Z^c) - g^c(\check{\nabla}_{X^c}^c Y^c + \omega^c(X^c, Y^c), Z^c) \\
& - g^c(Y^c, \check{\nabla}_{X^c}^c Z^c + \omega^c(X^c, Z^c)).
\end{aligned} \tag{4.29}$$

Then by considering equation (4.8), we have

$$\begin{aligned}
& -\eta^c(Y^c)g^c(X^c, Z^c) - \eta^v(Y^c)g^c(X^c, Z^c) - \eta^c(Z^c)g^c(X^c, Y^c) \\
& - \eta^v(Z^c)g^c(X^c, Y^c) \\
& = g^c(\check{\nabla}_{X^c}^c Y^c, Z^c) + g^c(Y^c, \check{\nabla}_{X^c}^{*c} Z^c) - g^c(\check{\nabla}_{X^c}^c Y^c, Z^c) \\
& - g^c(W^c(X^c, Y^c), Z^c) - g^c(Y^c, \check{\nabla}_{X^c}^c Z^c) - g^c(Y^c, W^c(X^c, Z^c)) \\
& = -2g^c(Y^c, K_{X^c}^c Z^c) - g^c(W^c(X^c, Y^c), Z^c) - g^c(Y^c, W^c(X^c, Z^c)).
\end{aligned} \tag{4.30}$$

Using equation (4.10) in equation (4.30), we have

$$\begin{aligned}
& g^c(W^c(X^c, Y^c), Z^c) + g^c(Y^c, W^c(X^c, Z^c)) = \eta^c(Y^c)g^c(X^v, Z^c) \\
& + \eta^v(Y^c)g^c(X^c, Z^c) + \eta^c(Z^c)g^c(X^v, Y^c) \\
& + \eta^v(Z^c)g^c(X^c, Y^c) - 2g^c(Y^c, K_{X^c}^c Z^c).
\end{aligned} \tag{4.31}$$

Beside that from equation (4.28), we have

$$\check{T}^c(X^c, Y^c) = W^c(X^c, Y^c) - W^c(Y^c, X^c), \tag{4.32}$$

From equations (4.31) and (4.32), we have

$$\begin{aligned}
& g^c(\check{T}^c(X^c, Y^c)Z^c) + g^c(\check{T}^c(Z^c, X^c)Y^c) + g^c(\check{T}^c(Z^c, Y^c)X^c) \\
& = 2g^c(W^c(X^c, Y^c), Z^c) - 2\eta^c(Z^c)g^c(X^v, Y^c) \\
& \quad - 2\eta^v(Z^c)g^c(X^c, Y^c) + 2g^c(Y^c, K_{X^c}^c Z^c).
\end{aligned} \tag{4.33}$$

Also using equation (4.25) and the inner product properties, we obtain

$$\begin{aligned}
& g^c(\check{T}^c(X^c, Y^c)Z^c) + g^c(\check{T}^c(Z^c, X^c)Y^c) + g^c(\check{T}^c(Z^c, Y^c)X^c) \\
& = 2\eta^c(Y^c)g^c(X^v, Z^c) + 2\eta^v(Y^c)g^c(X^c, Z^c) \\
& \quad - 2\eta^c(Z^c)g^c(X^v, Y^c) - 2\eta^v(Z^c)g^c(X^c, Y^c),
\end{aligned} \tag{4.34}$$

in which $g^c(\check{T}^c(X^c, Y^c), Z^c) = \check{T}^c(X^c, Y^c, Z^c)$. Then by equations (4.33) and (4.34), we get the result

$$W^c(X^c, Y^c) = \eta^c(Y^c)X^v + \eta^v(Y^c)X^c - K_{X^c}^c Y^c. \tag{4.35}$$

Further, if we consider equations (4.10) and (4.22), we can obtain equation (4.24). $\square$

4.1.3 Curvature tensor of statistical manifolds in the tangent bundle

Let us denote the curvature tensor associated with the SSNMC $\check{\nabla}$ by $\check{R}$. Similar to the definition of curvature tensor of a Riemannian manifold M according to the Riemannian connection ∇ , the curvature tensor of M is defined according to SSNMC $\check{\nabla}$ by

$$\check{R}(X, Y)Z = \check{\nabla}_X \check{\nabla}_Y Z - \check{\nabla}_Y \check{\nabla}_X Z - \check{\nabla}_{[X, Y]} Z, \tag{4.36}$$

for all vector fields $X, Y, Z \in \chi(M)$. Taking a complete lifts of equation (4.36), we get

$$\check{R}^c(X^c, Y^c)Z^c = \check{\nabla}_{X^c}^c \check{\nabla}_{Y^c}^c Z^c - \check{\nabla}_{Y^c}^c \check{\nabla}_{X^c}^c Z^c - \check{\nabla}_{[X^c, Y^c]}^c Z^c. \quad (4.37)$$

Now, we can obtain the relations between the complete lifts of the curvature tensor of $\check{R}^c$, $\check{R}^c$ and $\check{R}^c$, $\check{R}^{*c}$ with respect to the statistical connection $\check{\nabla}^c, \check{\nabla}^{*c}$ respectively.

Theorem 4.2. *In a statistical manifold $M^{\check{\nabla}^c}$. The complete lifts of the curvature tensor $\check{R}^c$ of a SSNMC $\check{\nabla}^c$ in the tangent bundle TM satisfies the conditions.*

$$\begin{aligned} \check{R}^c(X^c, Y^c)Z^c &= \check{R}^c(X^c, Y^c)Z^c - (\check{\nabla}_{X^c}^c K^c)(Y^c, Z^c) + (\check{\nabla}_{Y^c}^c K^c)(X^c, Z^c) \\ &\quad + K_{X^c}^c K^c(Y^c, Z^c) - K_{Y^c}^c K^c(X^c, Z^c) - g^c(\eta^c(X^c)\xi^v + \eta^v(X^c)\xi^c \\ &\quad - \check{\nabla}_{X^c}^c \xi^c + K_{X^c}^c \xi^c, Z^c)Y^c + g^c(\eta^c(Y^c)\xi^v + \eta^v(Y^c)\xi^c \\ &\quad - \check{\nabla}_{Y^c}^c \xi^c + K_{Y^c}^c \xi^c, Z^c)X^c, \end{aligned} \quad (4.38)$$

and

$$\begin{aligned} \check{R}^c(X^c, Y^c)Z^c &= \check{R}^{*c}(X^c, Y^c)Z^c + (\check{\nabla}_{X^c}^{*c} K^c)(Y^c, Z^c) - (\check{\nabla}_{Y^c}^{*c} K^c)(X^c, Z^c) \\ &\quad + K_{X^c}^c K^c(Y^c, Z^c) - K_{Y^c}^c K^c(X^c, Z^c) - g^c(\eta^c(X^c)\xi^v + \eta^v(X^c)\xi^c \\ &\quad - \check{\nabla}_{X^c}^{*c} \xi^c + K_{X^c}^c \xi^c, Z^c)Y^c + g^c(\eta^c(Y^c)\xi^v + \eta^v(Y^c)\xi^c \\ &\quad - \check{\nabla}_{Y^c}^{*c} \xi^c + K_{Y^c}^c \xi^c, Z^c)X^c. \end{aligned} \quad (4.39)$$

Proof. Using equations (4.8), (4.10), (4.12), (4.22) and (4.24), we will obtain the result. $\square$

Proposition 4.1. *In a statistical manifold $M^{\check{\nabla}^c}$. The following relations hold:*

- (i) $-g^c(\check{\nabla}_{X^c}^c \xi^c, W^c) + g^c(K_{X^c}^c \xi^c, W^c) = -g^c(\check{\nabla}_{X^c}^{*c} \xi^c, W^c) + g^c(K_{X^c}^c \xi^c, W^c),$
- (ii) $g^c((\check{\nabla}_{Y^c}^c K^c)(X^c, Z^c), W^c) - g^c((\check{\nabla}_{X^c}^c K^c)(Y^c, Z^c), W^c)$
 $= g^c((\check{\nabla}_{Y^c}^{*c} K^c)(X^c, W^c), Z^c) - g^c((\check{\nabla}_{X^c}^{*c} K^c)(Y^c, W^c), Z^c),$
- (iii) $g^c(K_{X^c}^c K_{Y^c}^c Z^c, W^c) = g^c(K_{Y^c}^c K_{X^c}^c W^c, Z^c).$

Proof. Using equation (4.10), we get (i). For (ii), we use equation (4.11) and prove that

$$\begin{aligned}
g^c(K_{\ddot{\nabla}_{Y^c}^c X^c}^c Z^c, W^c) - g^c(K_{\ddot{\nabla}_{X^c}^c Y^c}^c Z^c, W^c) &= g^c(K_{[Y^c, X^c]}^c Z^c, W^c) \\
&= g^c(K_{[Y^c, X^c]}^c W^c, Z^c) \\
&= g^c(K_{\ddot{\nabla}_{Y^c}^{*c} X^c}^c W^c, Z^c) - g^c(K_{\ddot{\nabla}_{X^c}^{*c} Y^c}^c W^c, Z^c).
\end{aligned} \tag{4.40}$$

Now, using equations (4.8) and (4.11), we get

$$\begin{aligned}
&g^c((\ddot{\nabla}_{Y^c}^c K^c)(X^c, Z^c), W^c) \\
&= g^c(\ddot{\nabla}_{Y^c}^c K_{X^c}^c Z^c, W^c) - g^c(K_{X^c}^c \ddot{\nabla}_{Y^c}^c Z^c, W^c) - g^c(K_{\ddot{\nabla}_{Y^c}^c X^c}^c Z^c, W^c) \\
&= Y^c g^c(W^v, K_{X^c}^c Z^c) + Y^v g^c(W^c, K_{X^c}^c Z^c) - g^c(K_{X^c}^c Z^c, \ddot{\nabla}_{Y^c}^{*c} W^c) \\
&\quad - g^c(K_{X^c}^c W^c, \ddot{\nabla}_{Y^c}^c Z^c) - g^c(K_{\ddot{\nabla}_{Y^c}^c X^c}^c Z^c, W^c) \\
&= Y^c g^c(W^v, K_{X^c}^c Z^c) + Y^v g^c(W^c, K_{X^c}^c Z^c) - g^c(K_{X^c}^c \ddot{\nabla}_{Y^c}^{*c} W^c, Z^c) \\
&\quad - Y^c g^c(Z^v, K_{X^c}^c W^c) - Y^v g^c(Z^c, K_{X^c}^c W^c) + g^c(Z^c, \ddot{\nabla}_{X^c}^{*c} K_{X^c}^c W^c) \\
&\quad - g^c(K_{\ddot{\nabla}_{Y^c}^c X^c}^c Z^c, W^c) \\
&= g^c(Z^c, \ddot{\nabla}_{Y^c}^{*c} K_{X^c}^c W^c) - g^c(K_{X^c}^c \ddot{\nabla}_{Y^c}^{*c} W^c, Z^c) - g^c(K_{\ddot{\nabla}_{Y^c}^c X^c}^c Z^c, W^c).
\end{aligned} \tag{4.41}$$

From equations (4.40) and (4.41), we obtain

$$\begin{aligned}
&g^c((\ddot{\nabla}_{Y^c}^c K^c)(X^c, Z^c), W^c) - g^c((\ddot{\nabla}_{X^c}^c K^c)(Y^c, Z^c), W^c) \\
&= g^c(Z^c, \ddot{\nabla}_{Y^c}^{*c} K_{X^c}^c W^c) - g^c(K_{X^c}^c \ddot{\nabla}_{Y^c}^{*c} W^c, Z^c) - g^c(K_{\ddot{\nabla}_{Y^c}^c X^c}^c W^c, Z^c) \\
&\quad - g^c(Z^c, \ddot{\nabla}_{X^c}^{*c} K_{Y^c}^c W^c) + g^c(K_{Y^c}^c \ddot{\nabla}_{X^c}^{*c} W^c, Z^c) + g^c(K_{\ddot{\nabla}_{X^c}^{*c} Y^c}^c W^c, Z^c) \\
&= g^c((\ddot{\nabla}_{Y^c}^{*c} K^c)(X^c, W^c), Z^c) - g^c((\ddot{\nabla}_{X^c}^{*c} K^c)(Y^c, W^c), Z^c).
\end{aligned} \tag{4.42}$$

For (iii), from symmetry property of K^c , we deduce

$$g^c(K_{X^c}^c K_{Y^c}^c Z^c, W^c) = g^c(K_{X^c}^c W^c, K_{Y^c}^c Z^c) = g^c(K_{Y^c}^c K_{X^c}^c W^c, Z^c). \tag{4.43}$$

□

Proposition 4.2. *In a statistical manifold $M^{\ddot{\nabla}^c}$ admitting a SSNMC $\ddot{\nabla}^c$ in the*

tangent bundle TM . The complete lift of curvature tensor $\check{R}^c$ with respect to $\check{\nabla}^c$ in the tangent bundle satisfy

$$\begin{aligned}
(i) \quad & \check{R}^c(X^c, Y^c)Z^c = -\check{R}^c(Y^c, X^c)Z^c, \\
(ii) \quad & \check{R}^c(X^c, Y^c, Z^c, W^c) + \check{R}^c(X^c, Y^c, W^c, Z^c) \\
&= \eta^c(Y^c)\eta^c(Z^c)g^c(X^v, W^c) + \eta^c(Y^c)\eta^v(Z^c)g^c(X^c, W^c) + \eta^v(Y^c)\eta^c(Z^c)g^c(X^c, W^c) \\
&\quad - g^c(Z^c, \check{\nabla}_{Y^c}^c \xi^c)g^c(X^v, W^c) - g^c(Z^v, \check{\nabla}_{Y^c}^c \xi^c)g^c(X^c, W^c) + g^c(Z^c, K_{Y^c}^c \xi^c)g^c(X^v, W^c) \\
&\quad + g^c(Z^v, K_{Y^c}^c \xi^c)g^c(X^c, W^c) + \eta^c(Y^c)\eta^c(W^c)g^c(X^v, Z^c) + \eta^c(Y^c)\eta^v(W^c)g^c(X^c, Z^c) \\
&\quad + \eta^v(Y^c)\eta^c(W^c)g^c(X^c, Z^c) - g^c(W^c, \check{\nabla}_{Y^c}^c \xi^c)g^c(X^v, Z^c) - g^c(W^v, \check{\nabla}_{Y^c}^c \xi^c)g^c(X^c, Z^c) \\
&\quad + g^c(W^c, K_{Y^c}^c \xi^c)g^c(X^v, Z^c) + g^c(W^v, K_{Y^c}^c \xi^c)g^c(X^c, Z^c) - \eta^c(X^c)\eta^c(Z^c)g^c(Y^v, W^c) \\
&\quad - \eta^c(X^c)\eta^v(Z^c)g^c(Y^c, W^c) - \eta^v(X^c)\eta^c(Z^c)g^c(Y^c, W^c) + g^c(Z^c, \check{\nabla}_{X^c}^c \xi^c)g^c(Y^v, W^c) \\
&\quad + g^c(Z^v, \check{\nabla}_{X^c}^c \xi^c)g^c(Y^c, W^c) - g^c(Z^c, K_{X^c}^c \xi^c)g^c(Y^v, W^c) - g^c(Z^v, K_{X^c}^c \xi^c)g^c(Y^c, W^c) \\
&\quad + \eta^c(X^c)\eta^c(W^c)g^c(Y^v, Z^c) + \eta^c(X^c)\eta^v(W^c)g^c(Y^c, Z^c) + \eta^v(X^c)\eta^c(W^c)g^c(Y^c, Z^c) \\
&\quad - g^c(W^c, \check{\nabla}_{X^c}^c \xi^c)g^c(Y^v, Z^c) - g^c(W^v, \check{\nabla}_{X^c}^c \xi^c)g^c(Y^c, Z^c) + g^c(W^c, K_{X^c}^c \xi^c)g^c(Y^v, Z^c) \\
&\quad + g^c(W^v, K_{X^c}^c \xi^c)g^c(Y^c, Z^c), \\
(iii) \quad & \check{R}^c(X^c, Y^c)Z^c + \check{R}^c(Y^c, Z^c)X^c + \check{R}^c(Z^c, X^c)Y^c \\
&= g^c(Z^c, \check{\nabla}_{X^c}^c \xi^c)Y^v + g^c(Z^v, \check{\nabla}_{X^c}^c \xi^c)Y^c - g^c(Z^c, \check{\nabla}_{Y^c}^c \xi^c)X^v - g^c(Z^v, \check{\nabla}_{Y^c}^c \xi^c)X^c \\
&\quad + g^c(X^c, \check{\nabla}_{Y^c}^c \xi^c)Z^v + g^c(X^v, \check{\nabla}_{Y^c}^c \xi^c)Z^c - g^c(X^c, \check{\nabla}_{Z^c}^c \xi^c)Y^v - g^c(X^v, \check{\nabla}_{Z^c}^c \xi^c)Y^c \\
&\quad + g^c(Y^c, \check{\nabla}_{Z^c}^c \xi^c)X^v + g^c(Y^v, \check{\nabla}_{Z^c}^c \xi^c)X^c - g^c(Y^c, \check{\nabla}_{X^c}^c \xi^c)Z^v - g^c(Y^v, \check{\nabla}_{X^c}^c \xi^c)Z^c.
\end{aligned}$$

Proof. By considering equation (4.8), we get (i). Using equations (4.38), (4.39) and Proposition 4.1, we get (ii). By cycling $\check{R}^c$ on X^c, Y^c, Z^c and a direct computation we get (iii). $\square$

Corollary 4.1. *In a statistical manifold $(M^{\check{\nabla}^c})$ admitting a SSNMC $\check{\nabla}^c$ in the tangent bundle TM , if η^c and η^v are closed, then*

$$\check{R}^c(X^c, Y^c)Z^c + \check{R}^c(Y^c, Z^c)X^c + \check{R}^c(Z^c, X^c)Y^c = 0.$$

Proof. By considering equation (4.8), we get

$$\begin{aligned}
& g^c(Z^c, \ddot{\nabla}_{X^c}^c \xi^c) Y^v + g^c(Z^v, \ddot{\nabla}_{X^c}^c \xi^c) Y^c - g^c(X^c, \ddot{\nabla}_{Z^c}^c \xi^c) Y^v \\
& - g^c(X^v, \ddot{\nabla}_{Z^c}^c \xi^c) Y^c \\
& = X^c g^c(\xi^c, Z^c) Y^v + X^c g^c(\xi^v, Z^c) Y^c + X^v g^c(\xi^c, Z^c) Y^c \\
& - g^c(\xi^c, \ddot{\nabla}_{X^c}^{*c} Z^c) Y^v - g^c(\xi^v, \ddot{\nabla}_{X^c}^{*c} Z^c) Y^c - Z^c g^c(\xi^c, X^c) Y^v \\
& - Z^c g^c(\xi^v, X^c) Y^c - Z^v g^c(\xi^c, X^c) Y^c + g^c(\xi^c, \ddot{\nabla}_{Z^c}^{*c} X^c) Y^v \\
& + g^c(\xi^v, \ddot{\nabla}_{Z^c}^{*c} X^c) Y^c, \\
& = X^c \eta^v(Z^c) + X^v \eta^c(Z^c) - Z^c \eta^v(X^c) - Z^v \eta^c(X^c) - \eta^c[X^c, Z^c], \\
& = d\eta^c(X^c, Z^c) Y^v + d\eta^v(X^c, Z^c) Y^c.
\end{aligned} \tag{4.44}$$

Using (iii) of Proposition 4.2, we have

$$\begin{aligned}
& \check{R}^c(X^c, Y^c) Z^c + \check{R}^c(Y^c, Z^c) X^c + \check{R}^c(Z^c, X^c) Y^c \\
& = d\eta^c(X^c, Z^c) Y^v + d\eta^v(X^c, Z^c) Y^c + d\eta^c(Y^c, X^c) Z^v \\
& + d\eta^v(Y^c, X^c) Z^c + d\eta^c(Z^c, Y^c) X^v + d\eta^v(Z^c, Y^c) X^c \\
& = 0,
\end{aligned}$$

If η^c and η^v are closed, then the above equation yield and this complete the proof. $\square$

4.1.4 SSNMC on statistical submanifolds in the tangent bundle

The Gauss formula admitting SSNMC $\check{\nabla}$ is expressed as

$$\check{\nabla}_X Y = \nabla_X^\bullet Y + h^\bullet(X, Y), \tag{4.45}$$

which $\nabla^\bullet$ and $h^\bullet$ denote the induced connection and second fundamental form on the submanifold N that admits a SSNMC $\check{\nabla}$. If the second fundamental form $h^\bullet$ satisfies the condition $h^\bullet(X, Y) = H^\bullet g(X, Y)$. Then the submanifold N is said to

be totally umbilical which admits a SSNMC $\check{\nabla}$, where $H^\bullet$ is the mean curvature tensor with respect to $\check{\nabla}$. Also if $h^\bullet$ vanishes, then N is called totally geodesic.

Let $\nabla^{\bullet c}$ and $h^{\bullet c}$ be the complete lift of the induced connection and complete lifts of second fundamental form in the submanifold TN of the tangent bundle TM . Now taking a complete lift of equation (4.45), we get

$$\check{\nabla}_{X^c}^c Y^c = \nabla_{X^c}^{\bullet c} Y^c + h^{\bullet c}(X^c, Y^c), \quad (4.46)$$

Now we will examine the relationship between $\nabla^{\bullet c}$ and ∇^c in the tangent bundle TM , assuming $X^c, Y^c \in \chi(TN)$ and the subscript c denote the complete lift.

Theorem 4.3. *In a statistical manifold $(M^{\check{\nabla}^c})$ with respect to SSNMC $\check{\nabla}^c$ in the tangent bundle TM , TN be a submanifold of $(M^{\check{\nabla}^c})$ and $X \in \chi(N)$. Then the following relations hold:*

$$\nabla_{X^c}^{\bullet c} Y^c = \check{\nabla}_{X^c}^c Y^c + \eta^c(Y^c)X^v + \eta^v(Y^c)X^c - K_{X^c}^c Y^c, \quad (4.47)$$

$$h^{\bullet c}(X^c, Y^c) = \frac{1}{2} \left(h^c(X^c, Y^c) + h^{*c}(X^c, Y^c) \right), \quad (4.48)$$

in which $K_{X^c}^c Y^c = \frac{1}{2}(\check{\nabla}^c - \check{\nabla}^{*c}), \forall X, Y \in \chi(N)$.

Proof. Using the Gauss formula and equation (4.22) in equation (4.18), we obtain the following

$$\begin{aligned} \check{\nabla}_{X^c}^c Y^c &= \check{\nabla}_{X^c}^c Y^c + \eta^c(Y^c)X^v + \eta^v(Y^c)X^c - K_{X^c}^c Y^c \\ &= \check{\nabla}_{X^c}^c Y^c + h^c(X^c, Y^c) + \eta^c(Y^c)X^v + \eta^v(Y^c)X^c \\ &\quad - \frac{1}{2}(\check{\nabla}_{X^c}^c Y^c + h^c(X^c, Y^c) - \check{\nabla}_{X^c}^{*c} Y^c - h^{*c}(X^c, Y^c)) \\ &= \check{\nabla}_{X^c}^c Y^c + \eta^c(Y^c)X^v + \eta^v(Y^c)X^c - K_{X^c}^c Y^c \\ &\quad + \frac{1}{2}(h^c(X^c, Y^c) + h^{*c}(X^c, Y^c)). \end{aligned} \quad (4.49)$$

If we separate the tangential and normal part we get our assertion. $\square$

Remark 4.1. *From the above proof, the relation of the complete lifts of $\nabla^{\bullet c}$ and*

∇^{*c} in the tangent bundle is

$$\nabla_{X^c}^{\bullet c} Y^c = \dot{\nabla}_{X^c}^{*c} Y^c + \eta^c(Y^c)X^v + \eta^v(Y^c)X^c + K_{X^c}^c Y^c, \quad (4.50)$$

for all $X, Y \in \chi(N)$.

Now we can obtain the following corollaries by considering equations (4.9), (4.46), (4.47) and (4.48).

Corollary 4.2. *In a statistical manifold $(M^{\ddot{\nabla}^c})$ with respect to SSNMC $\ddot{\nabla}^c$ and TN be a submanifold of $(M^{\ddot{\nabla}^c})$ in the tangent bundle TM . Then the second fundamental form $h^{\bullet c}$ coincides with $h^\odot$ in the tangent bundle.*

Corollary 4.3. *In a statistical manifold $(M^{\ddot{\nabla}^c})$ with respect to SSNMC $\ddot{\nabla}^c$ and TN be a submanifold of $(M^{\ddot{\nabla}^c})$ in the tangent bundle TM . Then the induced connection $\nabla^{\bullet c}$ of the SSNMC $\ddot{\nabla}^c$ is also SSNMC in the tangent bundle.*

Proposition 4.3. *Let $(M^{\ddot{\nabla}^c})$ be a statistical manifold with respect to SSNMC $\ddot{\nabla}^c$ and TN be a submanifold of $(M^{\ddot{\nabla}^c})$ in the tangent bundle TM . If TN is totally umbilical with respect to the statistical connections then TN is totally umbilical with respect to SSNMC in the tangent bundle.*

Proof. In view of equation (4.21), we use equations (4.9) and (4.47), and we obtain

$$\begin{aligned} h^{\bullet c}(X^c, Y^c) &= \frac{1}{2}(h^c(X^c, Y^c) + h^{*c}(X^c, Y^c)) \\ &= \frac{1}{2}(H^c + H^{*c})g^c(X^c, Y^c). \end{aligned} \quad (4.51)$$

So, TN is totally umbilical. Also, the mean curvature tensor that admits $\ddot{\nabla}^c$ is

$$H^{\bullet c}(X^c, Y^c) = \frac{1}{2}(H^c + H^{*c}). \quad (4.52)$$

□

Theorem 4.4. *Let $(M^{\ddot{\nabla}^c})$ be a statistical manifold with respect to SSNMC $\check{\nabla}^c$ and TN be a submanifold of $(M^{\ddot{\nabla}^c})$ in the tangent bundle TM and $\xi^c \in \chi(TN)$, for all $X, Y \in \chi(M)$ and $Q^c \in \chi(TN^\perp)$*

$$\check{\nabla}_{X^c}^c Q^c = -\frac{1}{2}(A_{Q^c}^c X^c + A_{Q^c}^{*c} X^c) + \frac{1}{2}(D_{X^c}^c Q^c + D_{X^c}^{*c} Q^c) \quad (4.53)$$

Proof. Since $Q^c \in \chi(TN^\perp)$, and $\eta^c(Q^c) = 0$. Then equations (4.17) and (4.22) imply that

$$\begin{aligned} \check{\nabla}_{X^c}^c Q^c &= \ddot{\nabla}_{X^c}^c Q^c + \eta^c(Q^c)X^v + \eta^v(Q^c)X^c - K_{X^c}^c Q^c \\ &= -A_{Q^c}^c X^c + D_{X^c}^c Q^c - \frac{1}{2}(\ddot{\nabla}_{X^c}^c Q^c - \ddot{\nabla}_{X^c}^{*c} Q^c) \\ &= -\frac{1}{2}(A_{Q^c}^c X^c + A_{Q^c}^{*c} X^c) + \frac{1}{2}(D_{X^c}^c Q^c + D_{X^c}^{*c} Q^c). \end{aligned} \quad (4.54)$$

□

Proposition 4.4. *Let $(M^{\ddot{\nabla}^c})$ be a statistical manifold with respect to SSNMC $\check{\nabla}^c$ and TN be a submanifold of $(M^{\ddot{\nabla}^c})$ in the tangent bundle TM and $\xi^c \in \chi(TN)$. Then $A^{\bullet c}$ and $D^{\bullet c}$ coincides with $A^{\odot c}$ and $D^{\odot c}$ in the tangent bundle respectively.*

Proof. The Weingarten formula admitting SSNMC is expressed as

$$\check{\nabla}_{X^c}^c Q^c = -A_{Q^c}^{\bullet c} X^c + D_{X^c}^{\bullet c} Q^c. \quad (4.55)$$

If we consider the above equation with equation (4.49) and separating the tangential and normal parts, we have

$$A_{Q^c}^{\bullet c} X^c = \frac{1}{2}(A_{Q^c}^c X^c + A_{Q^c}^{*c} X^c), \quad D_{X^c}^{\bullet c} Q^c = \frac{1}{2}(D_{X^c}^c Q^c + D_{X^c}^{*c} Q^c). \quad (4.56)$$

By considering equations (1.83) and (4.56), we obtain the results. In the light of

equations (4.20) , (4.48) and (4.56), we get

$$\begin{aligned}
 g^c(A_{Q^c}^{\bullet c} X^c, Y^c) &= \frac{1}{2} g^c \left((A_{Q^c}^c X^c + A_{Q^c}^{*c} X^c), Y^c \right) \\
 &= \frac{1}{2} g^c \left((h^c(X^c, Y^c) + h^{*c}(X^c, Y^c)), Q^c \right) \\
 &= g^c \left(h^{\bullet c}(X^c, Y^c), Q^c \right).
 \end{aligned} \tag{4.57}$$

□

Theorem 4.5. *Let $(M^{\ddot{\nabla}^c})$ be a statistical manifolds with respect to SSNMC $\ddot{\nabla}^c$ and TN be a submanifold of $(M^{\ddot{\nabla}^c})$ in the tangent bundle TM and $\xi^c \in (TN^\perp)$. then $\ddot{\nabla}^c$ and $h^{\bullet c}$ coincides with ∇^c and $h^{\odot c}$ in the tangent bundle respectively.*

Proof. Let $\xi^c \in \chi(TN^\perp)$, then similar to the proof in Theorem 4.3. If we obtain the Gauss formula as follows.

$$\ddot{\nabla}_{X^c}^c Y^c = \dot{\nabla}_{X^c}^c Y^c - K_{X^c}^c Y^c + \frac{1}{2} \left(h^c(X^c, Y^c) + h^{*c}(X^c, Y^c) \right). \tag{4.58}$$

So, if we taking tangential part of above equation, we have

$$\nabla_{Y^c}^{\bullet c} Y^c = \dot{\nabla}_{X^c}^c Y^c - K_{X^c}^c Y^c. \tag{4.59}$$

Using equations (4.10) and (4.16) in equation (4.59), we get

$$\nabla_{X^c}^{\odot c} Y^c = \dot{\nabla}_{X^c}^c Y^c - K_{X^c}^c Y^c. \tag{4.60}$$

In that case, from the last equation we have $\nabla^{\bullet c} = \dot{\nabla}^c$. then to show the coincides with of $h^{\bullet c}$ and $h^{\odot c}$, we take the normal part of equation (4.58), so

$$h^{\bullet c}(X^c, Y^c) = \frac{1}{2} \left(h^c(X^c, Y^c) + h^{*c}(X^c, Y^c) \right), \tag{4.61}$$

for any $X^c, Y^c \in \chi(TN)$. From equation (4.16),

$$h^{\bullet c}(X^c, Y^c) = h^{\odot c}(X^c, Y^c). \tag{4.62}$$

Thus, the result is clear. □

Lemma 4.1. *Let $(M^{\ddot{\nabla}^c})$ be a statistical manifolds $(M^{\ddot{\nabla}^c})$ with respect to SSNMC $\ddot{\nabla}^c$ and TN be a submanifold of $(M^{\ddot{\nabla}^c})$ in the tangent bundle TM . Then $\dot{\nabla}^c$ is a non-metric connection in the tangent bundle.*

Proof. For any vector fields X, Y, Z on M , we have

$$\begin{aligned} (\ddot{\nabla}_{X^c}^c g^c)(Y^c, Z^c) &= X^c g^c(Y^v, Z^c) + X^v g^c(Y^c, Z^c) - g^c(\ddot{\nabla}_{X^c}^c Y^c, Z^c) \\ &\quad - g^c(Y^c, \ddot{\nabla}_{X^c}^c Z^c). \end{aligned} \quad (4.63)$$

Using equations (4.8), (4.11) and (4.46) in equation (4.63), we get

$$\begin{aligned} (\ddot{\nabla}_{X^c}^c g^c)(Y^c, Z^c) &= X^c g^c(Y^v, Z^c) + X^v g^c(Y^c, Z^c) - g^c(\dot{\nabla}_{X^c}^c Y^c, Z^c) \\ &\quad - \eta^c(Y^c) g^c(X^v, Z^c) - \eta^v(Y^c) g^c(X^c, Z^c) + g^c(K_{X^c}^c Y^c, Z^c) \\ &\quad - g^c(Y^c, \dot{\nabla}_{X^c}^c Z^c) - \eta^c(Z^c) g^c(X^v, Y^c) - \eta^v(Z^c) g^c(X^c, Y^c) \\ &\quad + g^c(Y^c, K_{X^c}^c Z^c). \end{aligned} \quad (4.64)$$

Using equation (4.27) in equation (4.64), we have

$$\begin{aligned} &(\dot{\nabla}_{X^c}^c g^c)(Y^c, Z^c) - \eta^c(Y^c) g^c(X^v, Z^c) - \eta^v(Y^c) g^c(X^c, Z^c) + g^c(K_{X^c}^c Y^c, Z^c) \\ &\quad - \eta^c(Z^c) g^c(X^v, Y^c) - \eta^v(Z^c) g^c(X^c, Y^c) + g^c(Y^c, K_{X^c}^c Z^c) \\ &= -\eta^c(Y^c) g^c(X^v, Z^c) - \eta^v(Y^c) g^c(X^c, Z^c) - \eta^c(Z^c) g^c(X^v, Y^c) \\ &\quad - \eta^v(Z^c) g^c(X^c, Y^c), \\ &\Rightarrow (\dot{\nabla}_{X^c}^c g^c)(Y^c, Z^c) = -2g^c(K_{X^c}^c Y^c, Z^c). \end{aligned} \quad (4.65)$$

Consequently, we obtain $\dot{\nabla}^c g^c \neq 0$, so $\dot{\nabla}^c$ is a non-metric connection. $\square$

4.1.5 The Gauss, Codazzi and Ricci equation in the tangent bundle

In this section, the relationship between the lifts of Riemannian curvature tensors with respect to statistical connections and the SSNMC to the tangent

bundle are examined. We denote the tangential and normal parts of the Riemannian curvature tensors $\check{R}^c$ corresponding to the statistical connections by $R^{\bullet c}$ and $R^{\perp c}$ respectively. Using equations (4.46) and (4.55), we have

$$\begin{aligned} \check{\nabla}_{X^c}^c \check{\nabla}_{Y^c}^c Z^c &= \nabla_{X^c}^{\bullet c} \nabla_{Y^c}^{\bullet c} Z^c - A_{h^{\bullet c}(Y^c, Z^c)}^{\bullet c} X^c + D_{X^c}^{\bullet c} h^{\bullet c}(Y^c, Z^c) \\ &\quad + h^{\bullet c}(X^c, \nabla_{Y^c}^{\bullet c} Z^c). \end{aligned} \quad (4.66)$$

Interchanging X and Y in equation (4.66), we get $\check{\nabla}_{Y^c}^c \check{\nabla}_{X^c}^c Z^c$. Then

$$\begin{aligned} \check{R}^c(X^c, Y^c)Z^c &= R^{\bullet c}(X^c, Y^c)Z^c + A_{h^{\bullet c}(X^c, Z^c)}^c Y^c - A_{h^{\bullet c}(Y^c, Z^c)}^c X^c \\ &\quad + D_{X^c}^{\bullet c} h^{\bullet c}(Y^c, Z^c) - D_{Y^c}^{\bullet c} h^{\bullet c}(X^c, Z^c) + h^{\bullet c}(X^c, \nabla_{Y^c}^{\bullet c} Z^c) \\ &\quad - h^{\bullet c}(Y^c, \nabla_{X^c}^{\bullet c} Z^c) - h^{\bullet c}([X^c, Y^c], Z^c). \end{aligned} \quad (4.67)$$

In the view of equation (4.67), taking $W \in \chi(N)$, we get the Gauss equation admitting SSNMC expressed as

$$\begin{aligned} g^c(\check{R}^c(X^c, Y^c)Z^c, W^c) &= g^c(R^c(X^c, Y^c)Z^c, W^c) + g^c(h^{\bullet c}(X^c, Z^c), h^{\bullet c}(Y^c, W^c)) \\ &\quad - g^c(h^{\bullet c}(X^c, W^c), h^{\bullet c}(Y^c, Z^c)) \end{aligned} \quad (4.68)$$

Proposition 4.5. *Let $(M^{\check{\nabla}^c})$ be a statistical manifold $(M^{\check{\nabla}^c})$ with respect to SSNMC $\check{\nabla}^c$ and TN a submanifold of $(M^{\check{\nabla}^c})$ in the tangent bundle TM and*

$\xi^c \in \chi(TN)$. Then, we have

$$\begin{aligned}
g^c(\check{R}^c(X^c, Y^c)Z^c, W^c) &= g^c(R^c(X^c, Y^c)Z^c, W^c) + g^c((\dot{\nabla}_{Y^c}^c K^c)(X^c, Z^c)W^c) \\
&+ g^c(Z^c, \dot{\nabla}_{X^c}^c \xi^c)g^c(Y^v, W^c) + g^c(Z^v, \dot{\nabla}_{X^c}^c \xi^c)g^c(Y^c, W^c) \\
&- \eta^c(K_{X^c}^c Z^c)g^c(Y^v, W^c) - \eta^v(K_{X^c}^c Z^c)g^c(Y^c, W^c) \\
&+ \eta^c(Y^c)\eta^c(Z^c)g^c(X^v, W^c) + \eta^c(Y^c)\eta^v(Z^c)g^c(X^c, W^c) \\
&+ \eta^v(Y^c)\eta^c(Z^c)g^c(X^c, W^c) - g^c((\dot{\nabla}_{X^c}^c K^c)(Y^c, Z^c), W^c) \\
&- \eta^c(K_{Y^c}^c Z^c)g^c(X^v, W^c) - \eta^v(K_{Y^c}^c Z^c)g^c(X^c, W^c) \\
&+ g^c(K^c(X^c, (Y^c, Z^c), W^c)) - g^c(Z^c, \dot{\nabla}_{Y^c}^c \xi^c)g^c(X^v, W^c) \\
&- g^c(Z^v, \nabla_{Y^c}^c \xi^c)g^c(X^c, W^c) - \eta^c(X^c)\eta^c(Z^c)g^c(Y^v, W^c) \\
&- \eta^c(X^c)\eta^v(Z^c)g^c(Y^c, W^c) - \eta^v(X^c)\eta^c(Z^c)g^c(Y^c, W^c) \\
&- g^c(K^c(Y^c, K^c(X^c, Z^c), W^c)) + \frac{1}{4}g^c(h^c(X^c, Z^c) + h^{*c}(X^c, Z^c) \\
&, h^c(Y^c, W^c) + h^{*c}(Y^c, W^c)) - \frac{1}{4}g^c(h^c(X^c, W^c) + h^{*c}(X^c, W^c) \\
&, h^c(Y^c, Z^c) + h^{*c}(Y^c, Z^c)),
\end{aligned}$$

in which R^c is the curvature tensor of the induced statistical connection ∇^c on TN in the tangent bundle.

Proof. The proof can be obtained from equations (4.47), (4.48) and (4.68). $\square$

Theorem 4.6. Let $(M^{\check{\nabla}^c})$ be a statistical manifold with respect to SSNMC $\check{\nabla}^c$, TN a submanifold of $(M^{\check{\nabla}^c})$ in the tangent bundle TM and $\xi^c \in \chi(TN)$. Then,

the Codazzi equation with respect to SSNMC $\check{\nabla}^c$ is given by

$$\begin{aligned}
g^c(\check{R}^c(X^c, Y^c)Z^c, \lambda^c) &= g^c((\nabla_{X^c}^{\bullet c} h^{\bullet c})(Y^c, Z^c), \lambda^c) \\
&\quad - g^c((\nabla_{Y^c}^{\bullet c} h^{\bullet c})(X^c, Z^c), \lambda^c) \\
&\quad + \eta^c(Y^c)g^c(\lambda^v, h^{\bullet c}(X^c, Z^c)) \\
&\quad + \eta^v(Y^c)g^c(\lambda^c, h^{\bullet c}(X^c, Z^c)) \\
&\quad - \eta^c(X^c)g^c(\lambda^v, h^{\bullet c}(Y^c, Z^c)) \\
&\quad - \eta^v(X^c)g^c(\lambda^c, h^{\bullet c}(Y^c, Z^c)),
\end{aligned} \tag{4.69}$$

in which

$$(\nabla_{X^c}^{\bullet c} h^{\bullet c})(Y^c, Z^c) = D_{X^c}^{\bullet c} h^{\bullet c}(Y^c, Z^c) - h^{\bullet c}(\nabla_{X^c}^{\bullet c} Y^c, Z^c) - h^{\bullet c}(Y^c, \nabla_{X^c}^{\bullet c} Z^c).$$

Proof. Since $\lambda^c \in \chi(TN^\perp)$ and the inner product of equation (4.67) implies

$$\begin{aligned}
g^c(\check{R}^c(X^c, Y^c)Z^c, \lambda^c) &= g^c(h^{\bullet c}(X^c, \nabla_{Y^c}^{\bullet c} Z^c), \lambda^c) \\
&\quad + g^c(D_{X^c}^{\bullet c} h^{\bullet c}(Y^c, Z^c), \lambda^c) \\
&\quad - g^c(h^{\bullet c}(Y^c, \nabla_{X^c}^{\bullet c} Z^c), \lambda^c) \\
&\quad + g^c(D_{Y^c}^{\bullet c} h^{\bullet c}(X^c, Z^c), \lambda^c) \\
&\quad - g^c(h^{\bullet c}([X^c, Y^c], Z^c), \lambda^c).
\end{aligned} \tag{4.70}$$

If the lie bracket of X and Y are consider as equal in equation (4.70), then the proof is completed. $\square$

Theorem 4.7. *In a statistical manifold $(M^{\check{\nabla}^c})$ with respect to SSNMC $\check{\nabla}^c$, TN a submanifold of M in the tangent bundle TM and $\xi^c \in \chi(TN)$. Then the Ricci equation with respect to SSNMC $\check{\nabla}^c$ in the tangent bundle is given by*

$$g^c(\check{R}^c(X^c, Y^c)Q^c, P^c) = g^c(R^{\perp c}(X^c, Y^c)Q^c, P^c) + g^c([A_{Q^c}^{\bullet c}, A_{P^c}^{\bullet c}]X^c, Y^c),$$

where $X^c, Y^c \in \chi(TN)$, $Q^c, P^c \in \chi(TN^\perp)$ and $[A_{Q^c}^{\bullet c}, A_{P^c}^{\bullet c}] = A_{Q^c}^{\bullet c} A_{P^c}^{\bullet c} - A_{P^c}^{\bullet c} A_{Q^c}^{\bullet c}$.

Proof. In view of equations (4.46) and (4.55), we obtain

$$\begin{aligned}
g^c(\check{R}^c(X^c, Y^c)Q^c, P^c) &= g^c(\check{\nabla}_{X^c}^c \check{\nabla}_{Y^c}^c Q^c, P^c) - g^c(\check{\nabla}_{Y^c}^c \check{\nabla}_{X^c}^c Q^c, P^c) \\
&\quad - g^c(\check{\nabla}_{[X^c, Y^c]}^c Q^c, P^c) \\
&= g^c(R^{\perp c}(X^c, Y^c)Q^c, P^c) \\
&\quad - g^c(h^{\bullet c}(X^c, A_{Q^c}^{\bullet c} Y^c), P^c) \\
&\quad + g^c(Y^c, A_{Q^c}^{\bullet c} X^c, P^c),
\end{aligned} \tag{4.71}$$

for any $X^c, Y^c \in \chi(TN)$ and $Q^c, P^c \in \chi(TN^\perp)$. Then by considering equation (4.57) in equation (4.71), we obtain

$$\begin{aligned}
g^c(\check{R}^c(X^c, Y^c)Q^c, P^c) &= g^c(R^{\perp c}(X^c, Y^c)Q^c, P^c) + g^c(A_{Q^c}^{\bullet c} A_{P^c}^{\bullet c} X^c, Y^c) \\
&\quad - g^c(A_{P^c}^{\bullet c} A_{Q^c}^{\bullet c} Y^c, X^c) \\
&= g^c(R^{\perp c}(X^c, Y^c)Q^c, P^c) + g^c([A_{Q^c}^{\bullet c}, A_{P^c}^{\bullet c}]X^c, Y^c).
\end{aligned}$$

□

4.2 Sasakian Statistical Manifolds

In a n -dimensional Sasakian statistical manifold defined in equations (1.76-1.78), the linear connection $\check{\nabla}$ on Riemannian manifold M is given by (Yano, 1970)

$$\check{\nabla}_X Y = \ddot{\nabla}_X Y + \eta(Y)X - g(X, Y)\xi - K(X, Y), \tag{4.72}$$

for all vector fields X and $Y \in \chi(M^n)$ and η is a 1-form associated with vector field ξ and defined by

$$\eta(X) = g(X, \xi). \tag{4.73}$$

R. Kumar, S. Shenawy, N. Bin Turki, L. Colney, U. C. De (2023), Lifts of a Semi-Symmetric Metric Connection (SSMC) from Sasakian Statistical Manifolds to Tangent Bundle *Mathematics*, **12**, 226.

By using equation (1.71) in equation (4.72), we get

$$\check{\nabla}_X Y = \check{\nabla}_X^* Y + \eta(Y)X - g(X, Y)\xi + K(X, Y), \quad (4.74)$$

The torsion tensor is given by

$$\check{T}(X, Y) = \eta(Y)X - \eta(X)Y. \quad (4.75)$$

A linear connection satisfying equation (4.75) is called semi symmetric connection (SSC). For any vector fields X, Y, Z on M , we have

$$(\check{\nabla}_X g)(Y, Z) = Xg(Y, Z) - g(\check{\nabla}_X Y, Z) - g(Y, \check{\nabla}_X Z). \quad (4.76)$$

Using equations (1.69), (1.72) and (4.72), we get

$$(\check{\nabla}_X g)(Y, Z) = 0. \quad (4.77)$$

Hence a linear connection $\check{\nabla}$ defined by equation (4.72) satisfies equations (4.75) and (4.77) and is called semi-symmetric metric connection (briefly SSMC).

4.2.1 Statistical manifolds and Sasakian statistical manifolds in the tangent bundle

The properties of the statistical and Sasakian statistical manifolds in the tangent bundle are obtained by taking complete lifts using mathematical operators on equations (1.68-1.75), (1.40-1.44) and (1.76-1.78), we get (Kumar et. al, 2024):

$$(\phi\xi)^c = 0, \quad \eta^c(\xi^c) = 1, \quad g^c(\xi^c, X^c) = \eta^c(X^c), \quad (4.78)$$

$$(\phi^2 X)^c = -X^c + \eta^c(X^c)\xi^c + \eta(X^c)\xi^c, \quad (4.79)$$

$$g^c((\phi X)^c, (\phi Y)^c) = g^c(X^c, Y^c) - \eta^c(X^c)\eta^v(Y^c) - \eta^v(X^c)\eta^c(Y^c), \quad (4.80)$$

$$d\eta^c = 0, \quad \nabla_{X^c}^c \xi^c = (\phi X)^c, \quad (4.81)$$

$$(\nabla_{X^c}^c \phi^c)Y^c = \eta^c(Y^c)X^v + \eta^v(Y^c)X^c - g^c(X^c, Y^c)\xi^v - g^v(X^c, Y^c)\xi^c, \quad (4.82)$$

$$(\ddot{\nabla}_{X^c}^c g^c)(Y^c, Z^c) = (\ddot{\nabla}_{Y^c}^c g^c)(X^c, Z^c), \quad (4.83)$$

$$X^c g^c(Y^c, Z^c) = g^c(\ddot{\nabla}_{X^c}^c Y^c, Z^c) + g^c(Y^c, \ddot{\nabla}_{X^c}^{*c} Z^c), \quad (4.84)$$

$$\nabla^c = \frac{1}{2}(\ddot{\nabla}^c + \acute{\nabla}^{*c}), \quad (4.85)$$

$$K^c(X^c, Y^c) = \ddot{\nabla}_{X^c}^c Y^c - \nabla_{X^c}^c Y^c, \quad (4.86)$$

$$K^c(X^c, Y^c) = \frac{1}{2}(\ddot{\nabla}_{X^c}^c Y^c - \ddot{\nabla}_{X^c}^{*c} Y^c), \quad (4.87)$$

$$g^c(K^c(X^c, Y^c), Z^c) = g^c(K^c(X^c, Z^c), Y^c), \quad K^c(X^c, Y^c) = K^c(Y^c, X^c), \quad (4.88)$$

$$\ddot{R}^c(X^c, Y^c)Z^c = \ddot{\nabla}_{X^c}^c \ddot{\nabla}_{Y^c}^c Z^c - \ddot{\nabla}_{Y^c}^c \ddot{\nabla}_{X^c}^c Z^c - \ddot{\nabla}_{[X^c, Y^c]}^c, \quad (4.89)$$

$$\ddot{R}^c(X^c, Y^c)Z^c = -\ddot{R}^c(Y^c, X^c)Z^c, \quad \ddot{R}^{*c}(X^c, Y^c)Z^c = -\ddot{R}^{*c}(Y^c, X^c)Z^c, \quad (4.90)$$

$$g^c(\ddot{R}^c(X^c, Y^c)Z^c, W^c) = -g^c(\ddot{R}^{*c}(X^c, Y^c)W^c, Z^c), \quad (4.91)$$

$$\ddot{R}^c(X^c, Y^c)Z^c + \ddot{R}^c(Y^c, Z^c)X^c + \ddot{R}^c(Z^c, X^c)Y^c = 0. \quad (4.92)$$

$$K^c(X^c, (\phi Y)^c) + \phi^c K^c(X^c, Y^c) = 0, \quad (4.93)$$

$$\begin{aligned} \ddot{\nabla}_{X^c}^c \phi^c Y^c - \phi^c \ddot{\nabla}_{X^c}^{*c} Y^c &= g^c(Y^c, \xi^c)X^v + g^c(Y^v, \xi^c)X^c \\ &\quad - g^c(Y^c, X^c)\xi^v - g^c(Y^v, X^c)\xi^c, \end{aligned} \quad (4.94)$$

$$\ddot{\nabla}_{X^c}^c \xi^c = \phi^c X^c + g^c(\xi^c, \ddot{\nabla}_{X^c}^c \xi^c)\xi^v + g^c(\xi^v, \ddot{\nabla}_{X^c}^c \xi^c)\xi^c, \quad (4.95)$$

where, the mathematical operators 'c' and 'v' denote the complete and vertical lifts respectively, X, Y, Z, W are the vector fields, K is a tensor field of type $(1, 2)$, η is a 1-form, ϕ is a tensor field, g is a Riemannian metric, $\ddot{R}$ is the curvature tensor of the affine connection $\ddot{\nabla}$, $\ddot{R}^*$ is the curvature tensor of the dual affine connection $\ddot{\nabla}^*$, ∇ is the Levi-Civita connection, $\acute{\nabla}, \acute{\nabla}^*$ are statistical and dual connections respectively.

4.2.2 SSMC of a Sasakian statistical manifold in the tangent bundle

In a Sasakian statistical manifold, let the tangent bundle be denoted by TM and we obtain the complete lifts of equations (4.72-4.77) by using mathematical operators, we get

$$\begin{aligned}\check{\nabla}_{X^c}^c Y^c &= \check{\nabla}_{X^c}^c Y^c + \eta^c(Y^c)X^v + \eta^v(Y^c)X^c - g^c(X^c, Y^c)\xi^v \\ &\quad - g^c(X^v, Y^c)\xi^c - K^c(X^c, Y^c),\end{aligned}\tag{4.96}$$

$$\eta^c(X^c) = g^c(X^c, \xi^c),\tag{4.97}$$

$$\begin{aligned}\check{\nabla}_{X^c}^c Y^c &= \check{\nabla}_{X^c}^{*c} Y^c + \eta^c(Y^c)X^v + \eta^v(Y^c)X^c - g^c(X^c, Y^c)\xi^v \\ &\quad - g^c(X^v, Y^c)\xi^c + K^c(X^c, Y^c),\end{aligned}\tag{4.98}$$

$$\check{T}^c(X^c, Y^c) = \eta^c(Y^c)X^v + \eta^v(Y^c)X^c - \eta^c(X^c)Y^v - \eta^v(X^c)Y^c,\tag{4.99}$$

$$\begin{aligned}(\check{\nabla}_{X^c}^c g^c)(Y^c, Z^c) &= X^c g^c(Y^v, Z^c) + X^v g^c(Y^c, Z^c) - g^c(\check{\nabla}_{X^c}^c Y^c, Z^c) \\ &\quad - g^c(Y^c, \check{\nabla}_{X^c}^c Z^c),\end{aligned}\tag{4.100}$$

$$(\check{\nabla}_{X^c}^c g^c)(Y^c, Z^c) = 0.\tag{4.101}$$

Theorem 4.8. *Let M be a Sasakian statistical manifold admitting a SSMC in the tangent bundle TM which satisfies equations (4.99) and (4.101), then the SSMC of Sasakian statistical manifold in the tangent bundle is given by equation (4.96) and satisfies equation (4.98).*

Proof. Let $\check{\nabla}^c$ be a metric connection in the tangent bundle satisfying equation (4.99) on a Sasakian statistical manifold M defined by

$$\check{\nabla}_{X^c}^c Y^c = \check{\nabla}_{X^c}^c Y^c + H^c(X^c, Y^c),\tag{4.102}$$

where $\check{\nabla}^c$ is a torsion free connection and H^c is the tensor field of type $(1, 2)$ in

the tangent bundle. Using equations (4.84) and (4.102), we get

$$\begin{aligned}
& (\check{\nabla}_{X^c}^c g^c)(Y^c, Z^c) = 0, \\
& \implies X^c g^c(Y^c, Z^c) - g^c(\check{\nabla}_{X^c}^c Y^c, Z^c) - g^c(Y^c, \check{\nabla}_{X^c}^c Z^c) = 0, \\
& \implies X^c g^c(Y^c, Z^c) - g^c(\check{\nabla}_{X^c}^c Y^c + H^c(X^c, Y^c), Z^c) - g^c(Y^c, \check{\nabla}_{X^c}^c Z^c + H^c(X^c, Z^c)) = 0, \\
& \implies -2g^c(K^c(X^c, Z^c), Y^c) - g^c(H^c(X^c, Y^c), Z^c) - g^c(H^c(X^c, Z^c), Y^c) = 0.
\end{aligned} \tag{4.103}$$

So

$$g^c(H^c(X^c, Y^c), Z^c) + g^c(H^c(X^c, Z^c), Y^c) = -2g^c(K^c(X^c, Z^c), Y^c). \tag{4.104}$$

Now, using equation (4.102), we get

$$\check{T}^c(X^c, Y^c) = H^c(X^c, Y^c) - H^c(Y^c, X^c). \tag{4.105}$$

By using equation (4.99), we have

$$\begin{aligned}
& g^c(\check{T}^c(X^c, Y^c), Z^c) + g^c(\check{T}^c(Z^c, X^c), Y^c) + g^c(\check{T}^c(Z^c, Y^c), X^c) \\
& = g^c(H^c(X^c, Y^c) - H^c(Y^c, X^c), Z^c) + g^c(H^c(Z^c, X^c) - H^c(X^c, Z^c), Y^c) \\
& + g^c(H^c(Z^c, Y^c) - H^c(Y^c, Z^c), X^c) \\
& = 2(g^c(H^c(X^c, Y^c), Z^c) + g^c(K^c(X^c, Z^c), Y^c)).
\end{aligned} \tag{4.106}$$

Using equation (4.99) in equation (4.106), we get

$$\begin{aligned}
g^c(H^c(X^c, Y^c), Z^c) &= \frac{1}{2} \left[g^c(\eta^c(Y^c)X^v + \eta^v(Y^c)X^c - \eta^c(X^c)Y^v \right. \\
& \quad \left. - \eta^v(X^c)Y^c, Z^c) + g^c(\eta^c(X^c)Z^v + \eta^v(X^c)Z^c \right. \\
& \quad \left. - \eta^c(Z^c)X^v - \eta^v(Z^c)X^c, Y^c) + g^c(\eta^c(Y^c)Z^v \right. \\
& \quad \left. + \eta^v(Y^c)Z^c - \eta^c(Z^c)Y^v - \eta^v(Z^c)Y^c, X^c) \right] \\
& \quad - g^c(K^c(X^c, Z^c), Y^c).
\end{aligned} \tag{4.107}$$

Hence, we get

$$\begin{aligned} H^c(X^c, Y^c) &= \eta^c(Y^c)X^v + \eta^v(Y^c)X^c - g^c(X^c, Y^c)\xi^v - g^c(X^v, Y^c)\xi^c \\ &\quad - K^c(X^c, Y^c). \end{aligned} \quad (4.108)$$

Further by using equations (4.86), (4.87) and (4.96), we get equation (4.98) $\square$

4.2.3 Curvature tensor of SSMC on statistical manifolds in the tangent bundle

In an n -dimensional Riemannian manifold M and the statistical structure $(\ddot{\nabla}^c, g^c)$ on M in the tangent bundle TM , we have the relation between the complete lifts of the SSMC $\check{\nabla}^c$ and torsion free connection $\ddot{\nabla}^c$ in the tangent bundle from equations (4.86) and (4.96) by

$$\begin{aligned} \check{\nabla}_{X^c}^c Y^c &= \ddot{\nabla}_{X^c}^c Y^c + \eta^c(Y^c)X^v + \eta^v(Y^c)X^c - g^c(X^c, Y^c)\xi^v \\ &\quad - g^c(X^v, Y^c)\xi^c - K^c(X^c, Y^c). \end{aligned} \quad (4.109)$$

The complete lift of the Riemannian curvature tensor $\check{R}^c$ of M associated with the SSMC $\check{\nabla}^c$ in the tangent bundle is given by

$$\check{R}^c(X^c, Y^c)Z^c = \check{\nabla}_{X^c}^c \check{\nabla}_{Y^c}^c Z^c - \check{\nabla}_{Y^c}^c \check{\nabla}_{X^c}^c Z^c - \check{\nabla}_{[X^c, Y^c]}^c Z^c, \quad (4.110)$$

Using equation (4.109) in equation (4.110), we get

$$\begin{aligned}
\check{R}^c(X^c, Y^c)Z^c &= \ddot{R}^c(X^c, Y^c)Z^c + \left(\eta^c(X^c)\xi^v + \eta^v(X^c)\xi^c - \eta^c(\xi^c)X^v \right. \\
&\quad \left. - \eta^v(\xi^c)X^c - \ddot{\nabla}_{X^c}^c \xi^c + K^c(X^c, \xi^c) \right) g^c(Y^c, Z^c) \\
&\quad - \left(\eta^c(Y^c)\xi^v + \eta^v(Y^c)\xi^c - \eta^c(\xi^c)Y^v - \eta^v(\xi^c)Y^c \right. \\
&\quad \left. - \ddot{\nabla}_{Y^c}^c \xi^c + K^c(Y^c, \xi^c) \right) g^c(X^c, Z^c) - g^c \left(\eta^c(X^c)\xi^v \right. \\
&\quad \left. + \eta^v(X^c)\xi^c - \ddot{\nabla}_{X^c}^c \xi^c + K^c(X^c, \xi^c), Z^c \right) Y^c \\
&\quad + g^c \left(\eta^c(Y^c)\xi^v + \eta^v(Y^c)\xi^c - \ddot{\nabla}_{Y^c}^c \xi^c \right. \\
&\quad \left. + K^c(Y^c, \xi^c), Z^c \right) X^c - (\ddot{\nabla}_{X^c}^c K^c)(Y^c, Z^c) \\
&\quad + (\ddot{\nabla}_{Y^c}^c K^c)(X^c, Z^c) + K^c(X^c, K^c(Y^c, Z^c)) \\
&\quad - K^c(Y^c, K^c(X^c, Z^c)),
\end{aligned}$$

where $\check{R}^c$ is the complete lifts of the curvature tensor of M associated with torsion-free connection $\ddot{\nabla}^c$ in the tangent bundle TM and is define as

$$\check{R}^c(X^c, Y^c)Z^c = \ddot{\nabla}_{X^c}^c \ddot{\nabla}_{Y^c}^c Z^c - \ddot{\nabla}_{Y^c}^c \ddot{\nabla}_{X^c}^c Z^c - \ddot{\nabla}_{[X^c, Y^c]}^c Z^c, \quad (4.111)$$

Similarly, we obtain the complete lifts of the relation between the SSMC $\check{\nabla}^c$ and dual connection $\ddot{\nabla}^{*c}$ in the tangent bundle using equations (4.87) and (4.109) as

$$\begin{aligned}
\check{\nabla}_{X^c}^c Y^c &= \ddot{\nabla}_{X^c}^{*c} Y^c + \eta^c(Y^c)X^v + \eta^v(Y^c)X^c - g^c(X^c, Y^c)\xi^v \\
&\quad - g^c(X^v, Y^c)\xi^c + K^c(X^c, Y^c).
\end{aligned} \quad (4.112)$$

The relation between the complete lifts of the Riemannian curvature tensor $\check{R}^c$ associated with the SSMC $\check{\nabla}^c$ and Riemannian curvature tensor $\check{R}^{*c}$ associated with the dual connection $\ddot{\nabla}^{*c}$ in the tangent bundle TM is obtain by using equation

(4.112) in equation (4.110) by

$$\begin{aligned}
\check{R}^c(X^c, Y^c)Z^c &= \check{R}^{*c}(X^c, Y^c)Z^c + \left(\eta^c(X^c)\xi^v + \eta^v(X^c)\xi^c - \eta^c(\xi^c)X^v \right. \\
&\quad \left. - \eta^v(\xi^c)X^c - \check{\nabla}_{X^c}^{*c}\xi^c - K^c(X^c, \xi^c) \right) g^c(Y^c, Z^c) \\
&\quad - \left(\eta^c(Y^c)\xi^v + \eta^v(Y^c)\xi^c - \eta^c(\xi^c)Y^v - \eta^v(\xi^c)Y^c \right. \\
&\quad \left. - \check{\nabla}_{Y^c}^{*c}\xi^c - K^c(Y^c, \xi^c) \right) g^c(X^c, Z^c) - g^c \left(\eta^c(X^c)\xi^v \right. \\
&\quad \left. + \eta^v(X^c)\xi^c - \check{\nabla}_{X^c}^{*c}\xi^c - K^c(X^c, \xi^c), Z^c \right) Y^c \\
&\quad + g^c \left(\eta^c(Y^c)\xi^v + \eta^v(Y^c)\xi^c - \check{\nabla}_{Y^c}^{*c}\xi^c \right. \\
&\quad \left. - K^c(Y^c, \xi^c), Z^c \right) X^c + (\check{\nabla}_{X^c}^{*c}K^c)(Y^c, Z^c) \\
&\quad - (\check{\nabla}_{Y^c}^{*c}K^c)(X^c, Z^c) + K^c(X^c, K^c(Y^c, Z^c)) \\
&\quad - K^c(Y^c, K^c(X^c, Z^c)),
\end{aligned} \tag{4.113}$$

where, $\check{R}^{*c}$ is the complete lifts of the Riemannian curvature tensor of M associated with the lift of the dual connection $\check{\nabla}^{*c}$ in the tangent bundle TM and is defined as

$$\check{R}^{*c}(X^c, Y^c)Z^c = \check{\nabla}_{X^c}^{*c}\check{\nabla}_{Y^c}^{*c}Z^c - \check{\nabla}_{Y^c}^{*c}\check{\nabla}_{X^c}^{*c}Z^c - \check{\nabla}_{[X^c, Y^c]}^{*c}Z^c. \tag{4.114}$$

Theorem 4.9. *In a Riemannian manifold M , let the complete lifts of the statistical structure be $(\check{\nabla}^c, g^c)$ in the tangent bundle TM . Then,*

- (i) *The relation between the lifts of the Riemannian curvature tensor $\check{R}^c$ of SSMC $\check{\nabla}^c$ and the Riemannian curvature tensor $\check{R}^c$ of torsion-free connection $\check{\nabla}^c$ in the tangent bundle TM is given by equation (4.111).*
- (ii) *The relation between the lifts of the Riemannian curvature tensor $\check{R}^c$ of SSMC $\check{\nabla}^c$ and the Riemannian curvature tensor $\check{R}^{*c}$ of dual connection $\check{\nabla}^{*c}$ in the tangent bundle TM is given by equation (4.113).*

Proposition 4.6. *For a statistical manifold in the tangent bundle, the following relation hold:*

$$\begin{aligned}
(i) \quad & -g^c(\ddot{\nabla}_{X^c}^c \xi^c, W^c) + g^c(K^c(X^c, \xi^c), W^c) \\
& = -g^c(\ddot{\nabla}_{X^c}^{*c} \xi^c, W^c) + g^c(K^c(X^c, \xi^c), W^c), \\
(ii) \quad & g^c((\ddot{\nabla}_{Y^c}^c K^c)(X^c, Z^c), W^c) - g^c((\ddot{\nabla}_{X^c}^c K^c)(Y^c, Z^c), W^c) \\
& = g^c((\ddot{\nabla}_{Y^c}^{*c} K^c)(X^c, W^c), Z^c) - g^c((\ddot{\nabla}_{X^c}^{*c} K^c)(Y^c, W^c), Z^c), \\
(iii) \quad & g^c(K^c(X^c, K^c(Y^c, Z^c)), W^c) = g^c(K^c(Y^c, K^c(X^c, W^c)), Z^c).
\end{aligned}$$

Proof. Using equations (4.86) and (4.87), we get (i). For (ii), we use equation (4.88) and prove that

$$\begin{aligned}
& g^c(K^c(\ddot{\nabla}_{Y^c}^c X^c, Z^c), W^c) - g^c(K^c(\ddot{\nabla}_{X^c}^c Y^c, Z^c), W^c) \\
& = g^c(K^c([Y^c, X^c], Z^c), W^c) \\
& = g^c(K^c([Y^c, X^c], W^c), Z^c) \\
& = g^c(K^c(\ddot{\nabla}_{Y^c}^{*c} X^c, W^c), Z^c) - g^c(K^c(\ddot{\nabla}_{X^c}^{*c} Y^c, W^c), Z^c).
\end{aligned} \tag{4.115}$$

Now, using equations (4.84) and (4.88), we get

$$\begin{aligned}
& g^c((\ddot{\nabla}_{Y^c}^c K^c)(X^c, Z^c), W^c) \\
& = g^c(\ddot{\nabla}_{Y^c}^c K^c(X^c, Z^c), W^c) - g^c(K^c(X^c, \ddot{\nabla}_{Y^c}^c Z^c), W^c) \\
& \quad - g^c(K^c(\ddot{\nabla}_{Y^c}^c X^c, Z^c), W^c) \\
& = Y^c g^c(W^v, K^c(X^c, Z^c)) + Y^v g^c(W^c, K^c(X^c, Z^c)) \\
& \quad - g^c(K^c(X^c, Z^c), \ddot{\nabla}_{Y^c}^{*c} W^c) - g^c(K^c(X^c, W^c), \ddot{\nabla}_{Y^c}^c Z^c) \\
& \quad - g^c(K^c(\ddot{\nabla}_{Y^c}^c X^c, Z^c), W^c) \\
& = Y^c g^c(W^v, K^c(X^c, Z^c)) + Y^v g^c(W^c, K^c(X^c, Z^c)) \\
& \quad - g^c(K^c(X^c, \ddot{\nabla}_{Y^c}^{*c} W^c), Z^c) - Y^c g^c(Z^v, K^c(X^c, W^c)) \\
& \quad - Y^v g^c(Z^c, K^c(X^c, W^c)) + g^c(Z^c, \ddot{\nabla}^{*c} K^c(X^c, W^c)) \\
& \quad - g^c(K^c(\ddot{\nabla}_{Y^c}^c X^c, Z^c), W^c) \\
& = g^c(Z^c, \ddot{\nabla}_{Y^c}^{*c} K^c(X^c, W^c)) - g^c(K^c(X^c, \ddot{\nabla}_{Y^c}^{*c} W^c), Z^c) \\
& \quad - g^c(K^c(\ddot{\nabla}_{Y^c}^c X^c, Z^c), W^c).
\end{aligned} \tag{4.116}$$

From equations (4.115) and (4.116), we obtain

$$\begin{aligned}
& g^c((\ddot{\nabla}_{Y^c}^c K^c)(X^c, Z^c), W^c) - g^c((\ddot{\nabla}_{X^c}^c K^c)(Y^c, Z^c), W^c) \\
&= g^c(Z^c, \ddot{\nabla}_{Y^c}^{*c} K^c(X^c, W^c)) - g^c(K^c(X^c, \ddot{\nabla}_{Y^c}^{*c} W^c), Z^c) \\
&- g^c(K^c(\ddot{\nabla}_{Y^c}^{*c} X^c, W^c), Z^c) - g^c(Z^c, \ddot{\nabla}_{X^c}^{*c} K^c(Y^c, W^c)) \\
&+ g^c(K^c(Y^c, \ddot{\nabla}_{X^c}^{*c} W^c), Z^c) + g^c(K^c(\ddot{\nabla}_{X^c}^{*c} Y^c, W^c), Z^c) \\
&= g^c((\ddot{\nabla}_{Y^c}^{*c} K^c)(X^c, W^c), Z^c) - g^c((\ddot{\nabla}_{X^c}^{*c} K^c)(Y^c, W^c), Z^c).
\end{aligned} \tag{4.117}$$

For (iii), from symmetry property of K^c , we deduce

$$\begin{aligned}
g^c(K^c(X^c, K^c(Y^c, Z^c)), W^c) &= g^c(K^c(X^c, W^c), K^c(Y^c, Z^c)) \\
&= g^c(K^c(Y^c, K^c(X^c, W^c)), Z^c).
\end{aligned} \tag{4.118}$$

□

4.2.4 Curvature tensor of SSMC on Sasakian statistical manifolds in the tangent bundle

Let $(M, \ddot{\nabla}^c, g^c, \phi^c, \eta^c, \xi^c)$ be the complete lifts of $(2n+1)$ dimensional Sasakian statistical manifold in the tangent bundle TM . Then, the complete lifts of the curvature tensors $\check{R}^c$ associated with SSMC $\check{\nabla}^c$ in the tangent bundle are given by

$$\check{R}^c(X^c, Y^c)Z^c = \check{\nabla}_{X^c}^c \check{\nabla}_{Y^c}^c Z^c - \check{\nabla}_{Y^c}^c \check{\nabla}_{X^c}^c Z^c - \check{\nabla}_{[X^c, Y^c]}^c Z^c. \tag{4.119}$$

Using equations (4.109) and (4.78-4.95) in equation (4.119), we can obtain the relation of the complete lifts of the curvature tensor $\check{R}^c$ of the SSMC $\check{\nabla}^c$ and the curvature tensor $\ddot{R}^c$ of the torsion-free connection $\ddot{\nabla}^c$ in the tangent bundle TM

as

$$\begin{aligned}
\check{R}^c(X^c, Y^c)Z^c &= \ddot{R}^c(X^c, Y^c)Z^c + \left((\phi^2 X)^c - (\phi X)^c\right)g^c(Y^c, Z^c) \\
&\quad - \left((\phi^2 Y)^c - (\phi Y)^c\right)g^c(X^c, Z^c) + g^c\left((\phi X)^c, Z^c\right)Y^v \\
&\quad + g^c\left((\phi X)^v, Z^c\right)Y^c - g^c\left((\phi Y)^c, Z^c\right)X^v - g^c\left((\phi Y)^v, Z^c\right)X^c \\
&\quad - \eta^c(X^c)\eta^c(Z^c)Y^v - \eta^c(X^c)\eta^v(Z^c)Y^c - \eta^v(X^c)\eta^c(Z^c)Y^c \\
&\quad + \eta^c(Y^c)\eta^c(Z^c)X^v + \eta^c(Y^c)\eta^v(Z^c)X^c + \eta^v(Y^c)\eta^c(Z^c)X^c \\
&\quad - (\ddot{\nabla}_{X^c}^c K^c)(Y^c, Z^c) + (\ddot{\nabla}_{Y^c}^c K^c)(X^c, Z^c) \\
&\quad + K^c\left(X^c, K^c(Y^c, Z^c)\right) - K^c\left(Y^c, K^c(X^c, Z^c)\right).
\end{aligned} \tag{4.120}$$

Similarly, we obtain the relation of the complete lifts of the curvature tensor $\check{R}^c$ of the SSMC $\check{\nabla}^c$ and the curvature tensor $\ddot{R}^{*c}$ of the dual connection $\ddot{\nabla}^{*c}$ in the tangent bundle TM by using equations (4.112) and (4.78-4.95) in equation (4.119) as

$$\begin{aligned}
\check{R}^c(X^c, Y^c)Z^c &= \ddot{R}^{*c}(X^c, Y^c)Z^c + \left((\phi^2 X)^c - (\phi_0 X)^c\right)g^c(Y^c, Z^c) \\
&\quad - \left((\phi^2 Y)^c - (\phi Y)^c\right)g^c(X^c, Z^c) + g^c\left((\phi X)^c, Z^c\right)Y^v \\
&\quad + g^c\left((\phi X)^v, Z^c\right)Y^c - g^c\left((\phi Y)^c, Z^c\right)X^v - g^c\left((\phi Y)^v, Z^c\right)X^c \\
&\quad - \eta^c(X^c)\eta^c(Z^c)Y^v - \eta^c(X^c)\eta^v(Z^c)Y^c - \eta^v(X^c)\eta^c(Z^c)Y^c \\
&\quad + \eta^c(Y^c)\eta^c(Z^c)X^v + \eta^c(Y^c)\eta^v(Z^c)X^c + \eta^v(Y^c)\eta^c(Z^c)X^c \\
&\quad + (\ddot{\nabla}_{X^c}^{*c} K^c)(Y^c, Z^c) - (\ddot{\nabla}_{Y^c}^{*c} K^c)(X^c, Z^c) \\
&\quad + K^c\left(X^c, K^c(Y^c, Z^c)\right) - K^c\left(Y^c, K^c(X^c, Z^c)\right).
\end{aligned} \tag{4.121}$$

Proposition 4.7. *Let $(M, \ddot{\nabla}^c, g^c, \phi^c, \eta^c, \xi^c)$ be the complete lifts of a $(2n+1)$ dimensional Sasakian statistical manifold in the tangent bundle TM . Then, we have*

$$\begin{aligned}
(i) \quad \check{R}^c(X^c, Y^c)\xi^c &= \ddot{R}^c(X^c, Y^c)\xi^c + \eta^v(X^c)(\phi Y)^c + \eta^c(X^c)(\phi Y)^v \\
&\quad - \eta^v(Y^c)(\phi X)^c - \eta^c(Y^c)(\phi X)^v - (\ddot{\nabla}_{X^c}^c K^c)(Y^c, \xi^c)
\end{aligned}$$

$$+ (\ddot{\nabla}_{Y^c}^c K^c)(X^c, \xi^c),$$

$$\begin{aligned} (ii) \quad \check{R}^c(X^c, Y^c)\xi^c &= \check{R}^{*c}(X^c, Y^c)\xi^c + \eta^v(X^c)(\phi Y)^c + \eta^c(X^c)(\phi Y)^v \\ &\quad - \eta^v(Y^c)(\phi X)^c - \eta^c(Y^c)(\phi X)^v + (\ddot{\nabla}_{X^c}^{*c} K^c)(Y^c, \xi^c) \\ &\quad - (\ddot{\nabla}_{Y^c}^{*c} K^c)(X^c, \xi^c), \end{aligned}$$

$$\begin{aligned} (iii) \quad \check{R}^c(\xi^c, X^c)Y^c &= \check{R}^c(\xi^c, X^c)Y^c + \eta^c(Y^c)(\phi X)^v + \eta^v(Y^c)(\phi X)^c \\ &\quad - g^c((\phi X)^c, Y^c)\xi^v - g^c((\phi X)^v, Y^c)\xi^c - (\ddot{\nabla}_{\xi^c}^c K^c)(X^c, Y^c) \\ &\quad + (\ddot{\nabla}_{X^c}^c K^c)(\xi^c, Y^c) + K^c(\xi^c, K^c(X^c, Y^c)) - K^c(X^c, K^c(\xi^c, Y^c)), \end{aligned}$$

$$\begin{aligned} (iv) \quad \check{R}^c(\xi^c, X^c)Y^c &= \check{R}^{*c}(\xi^c, X^c)Y^c + \eta^c(Y^c)(\phi X)^v + \eta^v(Y^c)(\phi X)^c \\ &\quad - g^c((\phi X)^c, Y^c)\xi^v - g^c((\phi X)^v, Y^c)\xi^c + (\ddot{\nabla}_{\xi^c}^{*c} K^c)(X^c, Y^c) \\ &\quad - (\ddot{\nabla}_{X^c}^{*c} K^c)(\xi^c, Y^c) + K^c(\xi^c, K^c(X^c, Y^c)) - K^c(X^c, K^c(\xi^c, Y^c)) \end{aligned}$$

Proof. In a Sasakian manifold, we have from (Furuhata et. al, 2017) as

$$\ddot{\nabla}_X \xi = \phi X + \eta(\ddot{\nabla}_X \xi)\xi. \quad (4.122)$$

Obtaining the complete lifts of equation (4.122), we have

$$\ddot{\nabla}_{X^c}^c \xi^c = (\phi X)^c + \eta^c(\ddot{\nabla}_{X^c}^c \xi^c)\xi^v + \eta^v(\ddot{\nabla}_{X^c}^c \xi^c)\xi^c. \quad (4.123)$$

From equation (4.86), we have

$$K^c(X^c, \xi^c) = \eta^c(\ddot{\nabla}_{X^c}^c \xi^c)\xi^v + \eta^v(\ddot{\nabla}_{X^c}^c \xi^c)\xi^c. \quad (4.124)$$

So,

$$\begin{aligned} K^c(X^c, K^c(Y^c, \xi^c)) &= \eta^c(\ddot{\nabla}_{X^c}^c \xi^c)\eta^c(\ddot{\nabla}_{Y^c}^c \xi^c)\xi^v + \eta^c(\ddot{\nabla}_{X^c}^c \xi^c)\eta^v(\ddot{\nabla}_{Y^c}^c \xi^c)\xi^c \\ &\quad + \eta^v(\ddot{\nabla}_{X^c}^c \xi^c)\eta^c(\ddot{\nabla}_{Y^c}^c \xi^c)\xi^c. \end{aligned} \quad (4.125)$$

Hence, we get

$$K^c(X^c, K^c(Y^c, \xi^c)) = K^c(Y^c, K^c(X^c, \xi^c)). \quad (4.126)$$

Using equations (4.78-4.80) and (4.126) in equations (4.120) and (4.121), we get (i)-(iv). $\square$

4.2.5 Ricci tensor associated with SSMC of Sasakian statistical manifolds in the tangent bundle

The complete lifts of the Ricci tensor $\check{S}^c$ associated with the connection $\check{\nabla}^c$ in the tangent bundle TM is given as

$$\check{S}^c(X^c, Y^c) = \text{Trace of the map: } Z^c \rightarrow \check{R}^c(X^c, Y^c)Z^c. \quad (4.127)$$

Then from equations (4.120) and (4.121), we have

$$\begin{aligned} \check{S}^c(X^c, Y^c) &= \check{S}^c(X^c, Y^c) - (2n+1)g^c\left((\phi X)^c, (\phi Y)^c + Y^c\right) \\ &\quad - \sum_{i=1}^{2n+1} g^c\left((\check{\nabla}_{X^c}^c K^c)(e_i^c, e_i^c) - (\check{\nabla}_{e_i^c}^c K^c)(X^c, e_i^c)\right) \\ &\quad - K^c(X^c, K^c(e_i^c, e_i^c)) + K^c(e_i^c, K^c(X^c, e_i^c)), Y^c, \end{aligned} \quad (4.128)$$

and

$$\begin{aligned} \check{S}^c(X^c, Y^c) &= \check{S}^{*c}(X^c, Y^c) - (2n+1)g^c\left((\phi X)^c, (\phi Y)^c + Y^c\right) \\ &\quad - \sum_{i=1}^{2n+1} g^c\left((\check{\nabla}_{X^c}^{*c} K^c)(e_i^c, e_i^c) - (\check{\nabla}_{e_i^c}^{*c} K^c)(X^c, e_i^c)\right) \\ &\quad + K^c(X^c, K^c(e_i^c, e_i^c)) - K^c(e_i^c, K^c(X^c, e_i^c)), Y^c. \end{aligned} \quad (4.129)$$

Theorem 4.10. *Let $(M, \check{\nabla}^c, g^c, \phi^c, \eta^c, \xi^c)$ be the complete lifts of a $(2n+1)$ dimensional Sasakian statistical manifold in the tangent bundle TM . Then*

(i) *The relation between the Ricci tensor $\check{S}^c$ of the SSMC $\check{\nabla}^c$ and the Ricci tensor $\check{S}^c$ of the torsion free connection $\check{\nabla}^c$ in the tangent bundle TM is given by equation (4.128).*

(ii) *The relation between the Ricci tensor $\check{S}^c$ of the SSMC $\check{\nabla}^c$ and the Ricci tensor $\check{S}^{*c}$ of the dual connection $\check{\nabla}^{*c}$ in the tangent bundle TM is given by*

equation (4.129).

4.2.6 Scalar curvature associated with SSMC of Sasakian statistical manifolds in the tangent bundle

From equations (4.128) and (4.129), it follows that

$$\begin{aligned} \check{r}^c = \ddot{r}^c + 2n - 4n^2 - \sum_{i,j=1}^{2n+1} g^c \Big((\ddot{\nabla}_{e_j^c}^c K^c)(e_i^c, e_i^c) - (\ddot{\nabla}_{e_i^c}^c K^c)(e_j^c, e_i^c) \\ - K^c(e_j^c, K^c(e_i^c, e_i^c)) + K^c(e_i^c, K^c(e_j^c, e_i^c)), e_j^c \Big), \end{aligned} \quad (4.130)$$

and

$$\begin{aligned} \check{r}^c = \ddot{r}^{*c} + 2n - 4n^2 + \sum_{i,j=1}^{2n+1} g^c \Big((\ddot{\nabla}_{e_j^c}^{*c} K^c)(e_i^c, e_i^c) - (\ddot{\nabla}_{e_i^c}^{*c} K^c)(e_j^c, e_i^c) \\ + K^c(e_j^c, K^c(e_i^c, e_i^c)) - K^c(e_i^c, K^c(e_j^c, e_i^c)), e_j^c \Big). \end{aligned} \quad (4.131)$$

where $\check{r}^c$, $\ddot{r}^c$, $\ddot{r}^{*c}$ are the complete lifts of the scalar curvatures associated with the SSMC $\check{\nabla}^c$, torsion free connection $\ddot{\nabla}^c$ and dual connection $\ddot{\nabla}^{*c}$ in the tangent bundle TM respectively.

Theorem 4.11. *Let $(M, \check{\nabla}^c, g^c, \phi^c, \eta^c, \xi^c)$ be the complete lifts of a $(2n+1)$ dimensional Sasakian statistical manifold in the tangent bundle TM . Then*

- (i) *The relation between the scalar curvature $\check{r}^c$ of the SSMC $\check{\nabla}^c$ and the scalar curvature $\ddot{r}^c$ of the torsion free connection $\ddot{\nabla}^c$ in the tangent bundle TM is given by equation (4.130).*
- (ii) *The relation between the scalar curvature $\check{r}^c$ of the SSMC $\check{\nabla}^c$ and the scalar curvature $\ddot{r}^{*c}$ of the dual connection $\ddot{\nabla}^{*c}$ in the tangent bundle TM is given by equation (4.131).*

Theorem 4.12. *In a $(2n+1)$ dimensional Sasakian statistical manifold in the tangent bundle TM , the complete lift of the Ricci tensor of the Sasakian statistical*

manifold associated with SSMC in the tangent bundle TM is said to be Ricci flat if the complete lift of the scalar curvature with respect to torsion free connection $\ddot{\nabla}^c$ and dual connection $\ddot{\nabla}^{*c}$ in the tangent bundle TM satisfies

$$\begin{aligned} \ddot{r}^c = -2n + 4n^2 + \sum_{i,j=1}^{2n+1} g^c \Big((\ddot{\nabla}_{e_j^c}^c K^c)(e_i^c, e_i^c) - (\ddot{\nabla}_{e_i^c}^c K^c)(e_j^c, e_i^c) \\ - K^c(e_j^c, K^c(e_i^c, e_i^c)) + K^c(e_i^c, K^c(e_j^c, e_i^c)), e_j^c \Big), \end{aligned} \quad (4.132)$$

and

$$\begin{aligned} \ddot{r}^{*c} = -2n + 4n^2 - \sum_{i,j=1}^{2n+1} g^c \Big((\ddot{\nabla}_{e_j^c}^{*c} K^c)(e_i^c, e_i^c) - (\ddot{\nabla}_{e_i^c}^{*c} K^c)(e_j^c, e_i^c) \\ + K^c(e_j^c, K^c(e_i^c, e_i^c)) - K^c(e_i^c, K^c(e_j^c, e_i^c)), e_j^c \Big). \end{aligned} \quad (4.133)$$

Proof. Treating $\check{S}^c(X^c, Y^c) = 0$ in equations (4.128) and (4.129), then contracting it with respect to X^c and Y^c in the tangent bundle TM , we get the results. $\square$

4.2.7 Example

In this section, we shall show the example of the lifts of a Sasakian statistical manifold in the tangent bundle TM . Let us consider M to be a 3-dimensional manifold which is defined as

$$M = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3; x_3 \neq 0 \right\}, \quad (4.134)$$

where $\mathbb{R}$ is the set of real numbers. Let x_1, x_2, x_3 be given by

$$e_1 = \frac{\partial}{\partial x_1}, \quad e_2 = -x_1 \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3} \right) + \frac{\partial}{\partial x_2}, \quad e_3 = \frac{1}{2} \frac{\partial}{\partial x_3},$$

where $\{e_1, e_2, e_3\}$ are linearly independent global frame on M . Let TM be the tangent bundle and let the 1-form η be given by

$$\eta(X) = g(X, e_3).$$

The Riemannian metric g is defined by

$$g(e_i, e_j) = \begin{cases} 1, & i = j = 1, 2, 3 \\ 0, & \text{otherwise.} \end{cases}$$

Let ϕ be the tensor field defined by

$$\phi e_i = \begin{cases} -e_2, & i = 1 \\ e_1, & i = 2 \\ 0, & i = 3. \end{cases}$$

Using the linearity of ϕ and g , we acquire $\eta(e_3) = 1$, $\phi^2 X = -X + \eta(X)e_3$ and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$. Thus, for $e_3 = \xi$, the structure (ϕ, ξ, η, g) is an almost contact metric structure on M . In addition, M satisfies,

$$(\nabla_X \phi)Y = \eta(Y)X - g(X, Y)e_3.$$

Thus for $e_3 = \xi$, M is a Sasakian manifold.

Also M satisfies

$$K(X, \nabla_Y e_3) + \phi K(X, Y) = 0, \quad (4.135)$$

$$\ddot{\nabla}_X(\nabla_Y e_3) - \phi \ddot{\nabla}_X^* Y = g(Y, e_3)X - g(Y, X)e_3, \quad (4.136)$$

$$\ddot{\nabla}_X e_3 = \nabla_X e_3 + g(\ddot{\nabla}_X e_3, e_3)e_3. \quad (4.137)$$

Then M is called a Sasakian statistical manifold.

In tangent bundle TM let the complete and vertical lifts of e_1, e_2, e_3 be e_1^c, e_2^c, e_3^c and e_1^v, e_2^v, e_3^v on M and let g^c be the complete lift of the Riemannian metric g on TM such that

$$g^c(X^v, e_3^c) = \left(g^c(X, e_3)\right)^v = \left(\eta(X)\right)^v \quad (4.138)$$

$$g^c(X^c, e_3^c) = \left(g^c(X, e_3)\right)^c = \left(\eta(X)\right)^c \quad (4.139)$$

$$g^c(e_3^c, e_3^c) = 1, \quad g^v(X^v, e_3^c) = 0, \quad g^v(e_3^v, e_3^c) = 0, \quad (4.140)$$

and so on. Let ϕ^c and ϕ^v be the complete and vertical lifts of the $(1, 1)$ tensor field ϕ defined by

$$\phi^v(e_3^v) = \phi^c(e_3^c) = 0, \quad (4.141)$$

$$\phi^v(e_1^v) = e_1^v, \quad \phi^c(e_1^c) = e_1^c, \quad (4.142)$$

$$\phi^v(e_2^v) = e_2^v, \quad \phi^c(e_2^c) = e_2^c, \quad (4.143)$$

$$\phi^v(e_3^v) = e_3^v, \quad \phi^c(e_3^c) = e_3^c. \quad (4.144)$$

Using the linearity of ϕ and g , we infer that

$$(\phi^2 Y)^c = -Y^c + \eta^c(Y)e_3^v + \eta^v(Y)e_3^c, \quad (4.145)$$

$$g^c((\phi e_3)^c, (\phi e_2)^c) = g^c(e_3^c, e_2^c) - \eta^c(e_3^c)\eta^v(e_2^c) - \eta^v(e_3^c)\eta^c(e_2^c). \quad (4.146)$$

$$(\nabla_{e_3^c}^c \phi^c)e_2^c = \eta^c(e_2^c)e_3^v + \eta^v(e_2^c)e_3^c - g^c(e_3^c, e_2^c)\xi^v - g^c(e_3^v, e_2^c)\xi^c. \quad (4.147)$$

Also

$$K^c(e_3^c, \tilde{\nabla}_{e_2^c}^c \xi^c) + \phi^c K^c(e_3^c, e_2^c) = 0, \quad (4.148)$$

$$\begin{aligned} \ddot{\nabla}_{e_3^c}^c (\nabla_{e_2^c}^c \xi^c) - \phi^c \ddot{\nabla}_{e_3^c}^{*c} e_2^c &= g^c(e_2^c, \xi^c)e_3^v + g^c(e_2^v, \xi^c)e_3^c - g^c(e_2^c, e_3^c)\xi^c \\ &\quad - g^c(e_2^v, e_3^c)\xi^c, \end{aligned} \quad (4.149)$$

$$\ddot{\nabla}_{e_3^c}^c \xi^c = \nabla_{e_3^c}^c \xi^c + g^c(\xi^c, \ddot{\nabla}_{e_3^c}^c \xi^c)\xi^v + g^c(\xi^v, \ddot{\nabla}_{e_3^c}^c \xi^c)\xi^v. \quad (4.150)$$

Then for $e_3 = \xi$ in (4.138-4.145), the structure $(\phi^c, \xi^c, \eta^c, g^c)$ is an almost contact metric structure on TM and satisfies the relation (4.147). Then, $(\phi^c, \xi^c, \eta^c, g^c, TM)$ is a Sasakian manifold. Also, this structure satisfies the relation (4.148-4.150) and is called a Sasakian statistical manifold.

Chapter 5

Some properties on Kenmotsu manifolds admitting certain connections in the tangent bundle

5.1 Kenmotsu Manifolds with a Non-Symmetric Non-Metric Connection (NSNMC)

In a $(2n + 1)$ dimensional Kenmotsu manifold M defined in (1.45-1.48), the following relations also hold (Kumar et. al, 2023, 2024a):

$$\nabla_X \xi = X - \eta(X)\xi, \quad (5.1)$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y) = g(\phi X, \phi Y), \quad (5.2)$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (5.3)$$

R. Kumar, L. Colney, M. N. I. Khan (2023), Proposed Theorems on the Lifts of Kenmotsu Manifolds Admitting a Non-Symmetric Non-Metric Connection (NSNMC) in the Tangent Bundle *Symmetry*, **15**, 2037.

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \quad (5.4)$$

$$R(\xi, X)\xi = X - \eta(X)\xi, \quad (5.5)$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \quad (5.6)$$

$$S(\phi X, \phi Y) = S(X, Y) + 2n\eta(X)\eta(Y), \quad (5.7)$$

$$S(X, \xi) = -2n\eta(X), \quad (5.8)$$

$$S(X, Y) = g(Q, Y), \quad (5.9)$$

where R , S , and Q refer to the curvature tensor, the Ricci tensor, and the Ricci operator, respectively. These entities are derived from the Levi-Civita connection, which is the unique connection that preserves the metric and is torsion-free. The linear connection is given by (Singh et. al, 2022).

$$\check{\nabla}_X Y = \nabla_X Y + g(\phi X, Y)\xi, \quad (5.10)$$

satisfying the torsion tensor

$$\check{T}(X, Y) = 2g(\phi X, Y)\xi, \quad (5.11)$$

and

$$(\check{\nabla}_X g)(Y, Z) = -\eta(Z)g(\phi X, Y) - \eta(Y)g(\phi X, Z), \quad (5.12)$$

for arbitrary vector fields $X, Y, Z \in \chi(M^{(2n+1)})$; this is called a non-symmetric non-metric connection (NSNMC) .

5.1.1 Kenmotsu manifolds in the tangent bundle

Let TM be the tangent bundle on the Kenmotsu manifold M , the complete lift of the Kenmotsu manifold in the tangent bundle are obtained by taking complete lifts using mathematical operators on equations (1.45-1.48) and (5.1-5.12), we get

(Kumar et. al, 2023, 2024a):

$$\eta^c(\xi^c) = 1, \quad (\phi\xi)^c = 0, \quad \eta^c\left((\phi(X))^c\right) = 0, \quad g^c(X^c, \xi^c) = \eta^c(X^c), \quad (5.13)$$

$$(\phi^2(X))^c = -X^c + \eta^c(X^c)\xi^v + \eta^v(X^c)\xi^c, \quad (5.14)$$

$$g^c\left(X^c, (\phi Y)^c\right) = -g^c\left((\phi X)^c, Y^c\right), \quad (5.15)$$

$$g^c\left((\phi X)^c, (\phi Y)^c\right) = g^c(X^c, Y^c) - \eta^c(X^c)\eta^v(Y^c) - \eta^v(X^c)\eta^c(Y^c), \quad (5.16)$$

$$\begin{aligned} (\nabla_{X^c}^c \phi^c)Y^c &= g^c\left((\phi X)^c, Y^c\right)\xi^v + g^c\left((\phi X)^v, Y^c\right)\xi^c - \eta^c(Y^c)(\phi X)^v \\ &\quad - \eta^v(Y^c)(\phi X)^c, \end{aligned} \quad (5.17)$$

$$\nabla_{X^c}^c \xi^c = X^c - \eta^c(X^c)\xi^v - \eta^v(X^c)\xi^c, \quad (5.18)$$

$$\begin{aligned} (\nabla_{X^c}^c \eta^c)Y^c &= g^c(X^c, Y^c) - \eta^c(X^c)\eta^v(Y^c) - \eta^v(X^c)\eta^c(Y^c) \\ &= g^c\left((\phi X)^c, (\phi Y)^c\right), \end{aligned} \quad (5.19)$$

$$\begin{aligned} R^c(X^c, Y^c)\xi^c &= \eta^c(X^c)Y^v + \eta^v(X^c)Y^c - \eta^c(Y^c)X^v \\ &\quad - \eta^v(Y^c)X^c, \end{aligned} \quad (5.20)$$

$$\begin{aligned} R^c(\xi^c, X^c)Y^c &= \eta^c(Y^c)X^v + \eta^v(Y^c)X^c - g^c(X^c, Y^c)\xi^v \\ &\quad - g^c(X^v, Y^c)\xi^c, \end{aligned} \quad (5.21)$$

$$R^c(\xi^c, X^c)\xi^c = X^c - \eta^c(X^c)\xi^v - \eta^v(X^c)\xi^c, \quad (5.22)$$

$$\begin{aligned} \eta^c(R^c(X^c, Y^c)Z^c) &= g^c(X^c, Z^c)\eta^v(Y^c) + g^c(X^v, Z^c)\eta^c(Y^c) \\ &\quad - g^c(Y^c, Z^c)\eta^v(X) - g^c(Y^v, Z^c)\eta^c(X), \end{aligned} \quad (5.23)$$

$$\begin{aligned} S^c\left((\phi X)^c, (\phi Y)^c\right) &= S^c(X^c, Y^c) + 2n\eta^c(X^c)\eta^v(Y^c) \\ &\quad + 2n\eta^v(X^c)\eta^c(Y^c), \end{aligned} \quad (5.24)$$

$$S^c(X^c, \xi^c) = -2n\eta^c(X^c), \quad (5.25)$$

$$S^c(X^c, Y^c) = g^c\left((QX)^c, Y^c\right), \quad (5.26)$$

$$S^c(X^c, Y^c) = \lambda_1 g^c(X^c, Y^c) + \lambda_2 \eta^c(X^c)\eta^v(Y^c) + \lambda_2 \eta^v(X^c)\eta^c(Y^c), \quad (5.27)$$

$$(L_V g)^c + 2S^c + 2\lambda g^c = 0, \quad (5.28)$$

where $\eta, \xi, \phi, R, S, Q, \lambda$ are the 1-form, vector field, tensor field, curvature tensor, Ricci tensor, Ricci operator and constant respectively. And the mathematical operators ' c ' and ' v ' denote the complete and vertical lifts, and X, Y are vector fields on manifold M^{2n+1} .

5.1.2 NSNMC on Kenmotsu Manifolds in the Tangent Bundle

In a Kenmotsu manifold M , Now, let us take the complete lifts of equations (5.10-5.12) by using mathematical operators, we have

$$\check{\nabla}_{X^c}^c Y^c = \nabla_{X^c}^c Y^c + g^c((\phi X)^c, Y^c)\xi^v + g^c((\phi X)^v, Y^c)\xi^c, \quad (5.29)$$

$$\check{T}^c(X^c, Y^c) = 2g^c((\phi X)^c, Y^c)\xi^v + 2g^c((\phi X)^v, Y^c)\xi^c, \quad (5.30)$$

$$\begin{aligned} (\check{\nabla}_{X^c}^c g^c)(Y^c, Z^c) &= -\eta^c(Z^c)g^c((\phi X)^v, Y^c) - \eta^v(Z^c)g^c((\phi X)^c, Y^c) \\ &\quad - \eta^c(Y^c)g^c((\phi X)^v, Z^c) - \eta^v(Y^c)g^c((\phi X)^c, Z^c). \end{aligned} \quad (5.31)$$

We also have

$$\begin{aligned} (\check{\nabla}_{X^c}^c \phi^c)(Y^c) &= (\nabla_{X^c}^c \phi^c)(Y^c) + g^c((\phi X)^c, (\phi Y)^c)\xi^v \\ &\quad + g^c((\phi X)^v, (\phi Y)^c)\xi^c, \end{aligned} \quad (5.32)$$

$$(\check{\nabla}_{X^c}^c \eta^c)(Y^c) = (\nabla_{X^c}^c \eta^c)(Y^c) - g^c((\phi X)^c, (Y)^c), \quad (5.33)$$

$$\begin{aligned} (\check{\nabla}_{X^c}^c g^c)((\phi Y)^c, Z^c) &= -\eta^c(Z^c)g^c((\phi X)^v, (\phi Y)^c) \\ &\quad - \eta^v(Z^c)g^c((\phi X)^c, (\phi Y)^c). \end{aligned} \quad (5.34)$$

Replacing Y with ξ in equation (5.29), we obtain

$$(\check{\nabla}_{X^c}^c \xi^c) = \nabla_{X^c} \xi^c. \quad (5.35)$$

Again, putting $X = \xi$ in equation (5.12), we obtain

$$(\check{\nabla}_{\xi^c}^c g^c)(Y^c, Z^c) = 0. \quad (5.36)$$

Hence, we can propose the following proposition.

Proposition 5.1. *The complete lift of vector field ξ^c in the tangent bundle TM is invariant with respect to the lift of the Levi-Civita connection ∇^c and the lift of NSNMC $\check{\nabla}^c$ in the tangent bundle TM .*

Proposition 5.2. *The complete lift of the co-variant differentiation of Riemannian metric g^c associated with the lift of contra-variant vector field ξ^c vanishes identically in a contact metric manifold admitting the lift of NSNMC $\check{\nabla}^c$ in the tangent bundle TM .*

5.1.3 Curvature tensor on Kenmotsu manifolds endowed with the NSNMC in the tangent bundle

The curvature tensor $\check{R}$ of NSNMC $\check{\nabla}$ is given by (Singh et. al, 2022)

$$\check{R}(X, Y)Z = \check{\nabla}_X \check{\nabla}_Y Z - \check{\nabla}_Y \check{\nabla}_X Z - \check{\nabla}_{[X, Y]} Z. \quad (5.37)$$

Taking a complete lift by mathematical operators on equation (5.37), we obtain

$$\check{R}^c(X^c, Y^c)Z^c = \check{\nabla}_{X^c}^c \check{\nabla}_{Y^c}^c Z^c - \check{\nabla}_{Y^c}^c \check{\nabla}_{X^c}^c Z^c - \check{\nabla}_{[X^c, Y^c]}^c Z^c \quad (5.38)$$

for all $X^c, Y^c, Z^c \in \chi(TM)$. By using equation (5.29), we obtain

$$\begin{aligned} \check{R}^c(X^c, Y^c)Z^c &= R^c(X^c, Y^c)Z^c + g^c\left((\nabla_X \phi)Y\right)^c, Z^c \xi^v \\ &\quad + g^c\left((\nabla_X \phi)Y\right)^v, Z^c \xi^c - g^c\left((\nabla_Y \phi)X\right)^c, Z^c \xi^v \\ &\quad - g^c\left((\nabla_Y \phi)X\right)^v, Z^c \xi^c + g^c\left((\phi Y)^c, Z^c\right)(\nabla_X \xi)^v \\ &\quad + g^c\left((\phi Y)^v, Z^c\right)(\nabla_X \xi)^c - g^c\left((\phi X)^c, Z^c\right)(\nabla_Y \xi)^v \\ &\quad - g^c\left((\phi X)^v, Z^c\right)(\nabla_Y \xi)^c, \end{aligned} \quad (5.39)$$

where $R^c(X^c, Y^c)Z^c$ is the complete lift of the Riemannian curvature tensor of the Levi-Civita connection in the tangent bundle such that

$$R^c(X^c, Y^c)Z^c = \nabla_{X^c}^c \nabla_{Y^c}^c Z^c - \nabla_{Y^c}^c \nabla_{X^c}^c Z^c - \nabla_{[X^c, Y^c]}^c Z^c. \quad (5.40)$$

Proposition 5.3. *The relation between the complete lifts of $\check{R}^c$ associated with the lift of NSNMC $\check{\nabla}^c$ and R^c associated with the lift of the Levi-Civita connection ∇ is given by equation (5.39).*

Using equations (5.14), (5.15), (5.17), and (5.18) in equation (5.39), we obtain

$$\begin{aligned} \check{R}^c(X^c, Y^c)Z^c &= R^c(X^c, Y^c)Z^c + 2\eta^c(Z^c)g^c((\phi X)^c, Y^c)\xi^v \\ &\quad + 2\eta^c(Z^c)g^c((\phi X)^v, Y^c)\xi^c + 2\eta^v(Z^c)g^c((\phi X)^c, Y^c)\xi^c \\ &\quad - g^c((\phi X)^c, Z^c)Y^v - g^c((\phi X)^v, Z^c)Y^c \\ &\quad + g^c((\phi Y)^c, Z^c)X^v + g^c((\phi Y)^v, Z^c)X^c. \end{aligned} \quad (5.41)$$

Taking the contraction of equation (5.41) with respect to X , we obtain

$$\check{S}^c(Y^c, Z^c) = S^c(Y^c, Z^c) + 2ng^c((\phi Y)^c, Z^c). \quad (5.42)$$

Putting equation (5.26) into equation (5.42), we obtain

$$\check{Q}^c(Y^c) = Q^c(Y^c) + 2n(\phi Y)^c. \quad (5.43)$$

Again, from equation (5.42), by taking the contraction, it follows that

$$\check{r}^c = r^c, \quad (5.44)$$

where $\check{S}^c, \check{Q}^c, \check{r}^c, S^c, Q^c$ and r^c are the complete lifts of the Ricci tensors, Ricci operators, and scalar curvatures of the lifts of NSNMC $\check{\nabla}^c$ and Levi-Civita connection ∇^c , respectively.

By interchanging Z with ξ in equation (5.41), and putting equations (5.14)

and (5.15), we obtain

$$\check{R}^c(\xi^c, Y^c)Z^c = R^c(\xi^c, Y^c)Z^c + g^c((\phi Y)^c, Z^c)\xi^v + g^c((\phi Y)^v, Z^c)\xi^c. \quad (5.45)$$

From equations (5.21) and (5.45), we obtain

$$\begin{aligned} \check{R}^c(\xi^c, Y^c)Z^c &= \eta^c(Z^c)Y^v + \eta^v(Z^c)Y^c - g^c(Y^c, Z^c)\xi^v - g^c(Y^v, Z^c)\xi^c \\ &\quad + g^c((\phi Y)^c, Z^c)\xi^v + g^c((\phi Y)^v, Z^c)\xi^c. \end{aligned} \quad (5.46)$$

Again, interchanging Z with ξ in equation (5.41) and putting equations (5.14), (5.16), and (5.20), we obtain

$$\begin{aligned} \check{R}^c(X^c, Y^c)\xi^c &= R^c(X^c, Y^c)\xi^c + 2g^c((\phi Y)^c, Z^c)\xi^v + 2g^c((\phi Y)^v, Z^c)\xi^c \\ &= \eta^c(X^c)Y^v + \eta^v(X^c)Y^c - \eta^c(Y^c)X^v - \eta^v(Y^c)X^c \\ &\quad + 2g^c((\phi Y)^c, Z^c)\xi^v + 2g^c((\phi Y)^v, Z^c)\xi^c \\ &\neq 0. \end{aligned} \quad (5.47)$$

Thus, we state the following theorem.

Theorem 5.1. *Every $(2n + 1)$ -dimensional Kenmotsu manifold associated with the lift of the NSNMC $\check{\nabla}^c$ in the tangent bundle is regular with respect to $\check{\nabla}^c$.*

Let us operate η on both sides of equation (5.41) and putting equation (5.14), we obtain

$$\begin{aligned} \eta^c(\check{R}^c(X^c, Y^c)Z^c) &= 2g^c((\phi X)^c, Y^c)\eta^v(Z^c) + 2g^c((\phi X)^v, Y^c)\eta^c(Z^c) \\ &\quad + \eta^c(Y^c)g^c(X^v, Z^c) + \eta^v(Y^c)g^c(X^c, Z^c) \\ &\quad - \eta^c(X^c)g^c(Y^v, Z^c) - \eta^v(X^c)g^c(Y^c, Z^c) \\ &\quad + g^c((\phi Y)^c, Z^c)\eta^v(X^c) + g^c((\phi Y)^v, Z^c)\eta^c(X^c) \\ &\quad - g^c((\phi X)^c, Z^c)\eta^v(Y^c) - g^c((\phi X)^v, Z^c)\eta^c(Y^c). \end{aligned} \quad (5.48)$$

Now, taking the contraction of equation (5.47) with respect to X , we obtain

$$\check{S}^c(Y^c, \xi^c) = -2n\eta^c(Y^c). \quad (5.49)$$

Putting $\check{R}^c(X^c, Y^c)Z^c = 0$ into equation (5.41), we obtain

$$\begin{aligned}
R^c(X^c, Y^c)Z^c &= g^c((\phi X)^c, Z^c)Y^v + g^c((\phi X)^v, Z^c)Y^c \\
&\quad - g^c((\phi Y)^c, Z^c)X^v - g^c((\phi Y)^v, Z^c)X^c \\
&\quad - 2g^c((\phi X)^c, Y^c)\eta^c(Z^c)\xi^v - 2g^c((\phi X)^c, Y^c)\eta^v(Z^c)\xi^c \\
&\quad - 2g^c((\phi X)^v, Y^c)\eta^c(Z^c)\xi^c.
\end{aligned} \tag{5.50}$$

In view of $'R^c(X^c, Y^c, Z^c, W^c) = g^c(R^c(X^c, Y^c)Z^c, W^c)$ and equation (5.50), we get

$$\begin{aligned}
'R^c(X^c, Y^c, Z^c, W) &= g^c((\phi X)^c, Z^c)g^c(Y^v, W^c) + g^c((\phi X)^v, Z^c)g^c(Y^c, W^c) \\
&\quad - g^c((\phi Y)^c, Z^c)g^c(X^v, W^c) - g^c((\phi Y)^v, Z^c)g^c(X^c, W^c) \\
&\quad - 2g^c((\phi X)^c, Y^c)\eta^c(Z^c)\eta^v(W^c) - 2g^c((\phi X)^c, Y^c)\eta^v(Z^c)\eta^c(W^c) \\
&\quad - 2g^c((\phi X)^v, Y^c)\eta^c(Z^c)\eta^c(W^c).
\end{aligned} \tag{5.51}$$

Taking the contraction of equation (5.51) with respect to X , we obtain

$$S^c(Y^c, Z^c) = -2ng^c((\phi Y)^c, Z^c). \tag{5.52}$$

Putting equation (5.26) into equation (5.52), we obtain

$$Q^c(Y^c) = -2n(\phi Y)^c. \tag{5.53}$$

From equation (5.52), it follows that

$$r^c = 0. \tag{5.54}$$

The conformal curvature tensor in Riemannian manifold is defined in equation

(1.35). For the NSNMC $\check{\nabla}$, it can be written as (Singh et. al, 2022)

$$\begin{aligned}\check{C}(X, Y)Z &= \check{R}(X, Y)Z - \frac{1}{2n-1} \left[\check{S}(Y, Z)X - \check{S}(X, Z)Y \right. \\ &\quad \left. + g(Y, Z)\check{Q}X - g(X, Z)\check{Q}Y \right] \\ &\quad + \frac{\check{r}}{2n(2n-1)} \left[g(Y, Z)X - g(X, Z)Y \right].\end{aligned}\tag{5.55}$$

Taking a complete lift by mathematical operators on equation (5.55), we obtain

$$\begin{aligned}\check{C}^c(X^c, Y^c)Z^c &= \check{R}^c(X^c, Y^c)Z^c - \frac{1}{2n-1} \left[\check{S}^c(Y^c, Z^c)X^v + \check{S}^v(Y^c, Z^c)X^c \right. \\ &\quad \left. - \check{S}^c(X, Z)Y^v - \check{S}^v(X, Z)Y^c + g^c(Y^c, Z^c)(\check{Q}X)^v \right. \\ &\quad \left. + g^c(Y^v, Z^c)(\check{Q}X)^c - g^c(X^c, Z^c)(\check{Q}Y)^v - g^c(X^v, Z^c)(\check{Q}Y)^c \right] \\ &\quad + \frac{\check{r}}{2n(2n-1)} \left[g^c(Y^c, Z^c)X^v + g^c(Y^v, Z^c)X^c - g^c(X^c, Z^c)Y^v \right. \\ &\quad \left. - g^c(X^v, Z^c)Y^c \right].\end{aligned}\tag{5.56}$$

Putting equations (5.41-5.43) into equation (5.56), we obtain

$$\begin{aligned}\check{C}^c(X^c, Y^c)Z^c - C^c(X^c, Y^c)Z^c &= 2g^c((\phi X)^c, Y^c)\eta^c(Z^c)\xi^v \\ &\quad + 2g^c((\phi X)^c, Y^c)\eta^v(Z^c)\xi^c + 2g^c((\phi X)^v, Y^c)\eta^c(Z^c)\xi^c \\ &\quad + g^c((\phi Y)^c, Z^c)X^v + g^c((\phi Y)^v, Z^c)X^c - g^c((\phi X)^c, Z^c)Y^v \\ &\quad - g^c((\phi X)^v, Z^c)Y^c - \frac{2n}{(2n-1)} \left[g^c((\phi Y)^c, Z^c)X^v \right. \\ &\quad \left. + g^c((\phi Y)^v, Z^c)X^c - g^c((\phi X)^c, Z^c)Y^v - g^c((\phi X)^v, Z^c)Y^c \right. \\ &\quad \left. + g^c(Y^c, Z^c)(\phi X)^v + g^c(Y^v, Z^c)(\phi X)^c - g^c(X^c, Z^c)(\phi Y)^v \right. \\ &\quad \left. - g^c(X^v, Z^c)(\phi Y)^c \right],\end{aligned}\tag{5.57}$$

where C^c is the complete lift of the conformal curvature tensor associated with

the lift of the Levi-Civita connection ∇^c in the tangent bundle as

$$\begin{aligned}
C^c(X^c, Y^c)Z^c &= R^c(X^c, Y^c)Z^c - \frac{1}{2n-1} \left[S^c(Y^c, Z^c)X^v + S^v(Y^c, Z^c)X^c \right. \\
&\quad - S^c(X, Z)Y^v - S^v(X, Z)Y^c + g^c(Y^c, Z^c)(QX)^v + g^c(Y^v, Z^c)(QX)^c \\
&\quad \left. - g^c(X^c, Z^c)(QY)^v - g^c(X^v, Z^c)(QY)^c \right] + \frac{r}{2n(2n-1)} \left[g^c(Y^c, Z^c)X^v \right. \\
&\quad \left. + g^c(Y^v, Z^c)X^c - g^c(X^c, Z^c)Y^v - g^c(X^v, Z^c)Y^c \right].
\end{aligned} \tag{5.58}$$

The conharmonic curvature tensor in Riemmanian manifold is defined in equation (1.34). For the NSNMC $\check{\nabla}$ it can be written as (Singh et. al, 2022)

$$\begin{aligned}
\check{L}(X, Y)Z &= \check{R}(X, Y)Z - \frac{1}{(2n-1)} \left[\check{S}(Y, Z)X \right. \\
&\quad - \check{S}(X, Z)Y + g(Y, Z)\check{Q}X \\
&\quad \left. - g(X, Z)\check{Q}Y \right].
\end{aligned} \tag{5.59}$$

Now, taking a complete lift by mathematical operators on equation (5.59), we obtain

$$\begin{aligned}
\check{L}^c(X^c, Y^c)Z^c &= \check{R}^c(X^c, Y^c)Z^c - \frac{1}{(2n-1)} \left[\check{S}^c(Y^c, Z^c)X^v \right. \\
&\quad + \check{S}^v(Y^c, Z^c)X^c - \check{S}^c(X^c, Z^c)Y^v \\
&\quad - \check{S}^v(X^c, Z^c)Y^c + g^c(Y^c, Z^c)(\check{Q}X)^v \\
&\quad + g^c(Y^v, Z^c)(\check{Q}X)^c - g^c(X^c, Z^c)(\check{Q}Y)^v \\
&\quad \left. - g^c(X^v, Z^c)(\check{Q}Y)^c \right].
\end{aligned} \tag{5.60}$$

Putting equations (5.41-5.44) into equation (5.60), we obtain

$$\begin{aligned}
\check{L}^c(X^c, Y^c)Z^c - L^c(X^c, Y^c)Z^c &= 2g^c((\phi X)^c, Y^c)\eta^c(Z^c)\xi^v \\
&+ 2g^c((\phi X)^c, Y^c)\eta^v(Z^c)\xi^c + 2g^c((\phi X)^v, Y^c)\eta^c(Z^c)\xi^c \\
&+ g^c((\phi Y)^c, Z^c)X^v + g^c((\phi Y)^v, Z^c)X^c - g^c((\phi X)^c, Z^c)Y^v \\
&- g^c((\phi X)^v, Z^c)Y^c - \frac{2n}{(2n-1)} \left[g^c((\phi Y)^c, Z^c)X^v \right. \\
&+ g^c((\phi Y)^v, Z^c)X^c - g^c((\phi X)^c, Z^c)Y^v \\
&- g^c((\phi X)^v, Z^c)Y^c + g^c(Y^c, Z^c)(\phi X)^v \\
&+ g^c(Y^v, Z^c)(\phi X)^c - g^c(X^c, Z^c)(\phi Y)^v \\
&\left. - g^c(X^v, Z^c)(\phi Y)^c \right], \tag{5.61}
\end{aligned}$$

where L^c is the complete lift of the conharmonic curvature tensor associated with the lift of the Levi-Civita connection ∇^c in the tangent bundle as

$$\begin{aligned}
L^c(X^c, Y^c)Z^c &= R^c(X^c, Y^c)Z^c - \frac{1}{(2n-1)} \left[S^c(Y^c, Z^c)X^v \right. \\
&+ S^v(Y^c, Z^c)X^c - S^c(X^c, Z^c)Y^v \\
&- S^v(X^c, Z^c)Y^c + g^c(Y^c, Z^c)(QX)^v \\
&+ g^c(Y^v, Z^c)(QX)^c - g^c(X^c, Z^c)(QY)^v \\
&\left. - g^c(X^v, Z^c)(QY)^c \right]. \tag{5.62}
\end{aligned}$$

Hence, we can proposed the following:

Proposition 5.4. *The relationship between the complete lift of the conformal curvature tensor $\check{C}^c$ associated with the lift of NSNMC $\check{\nabla}^c$ and the complete lift of the conformal curvature tensor C^c associated with the lift of the Levi-Civita connection ∇^c is given by equation (5.57).*

Proposition 5.5. *The relationship between the complete lift of the conharmonic curvature tensor $\check{L}^c$ associated with the lift of NSNMC $\check{\nabla}^c$ and the complete lift of the conharmonic curvature tensor L^c associated with the lift of the Levi-Civita*

connection ∇^c is given by equation (5.61).

Theorem 5.2. *In a Kenmotsu manifold admitting a NSNMC $\check{\nabla}^c$, the conformal curvature tensor lifted from $\check{\nabla}^c$ coincides with that of ∇^c if and only if their corresponding conharmonic curvature tensor lifts are equal in the tangent bundle.*

The concircular curvature tensor in Riemannian manifold is defined in equation (1.33). For the NSNMC $\check{\nabla}$ it can be written as (Singh et. al, 2022)

$$V(X, Y)Z = R(X, Y)Z + \frac{r}{2n(2n-1)} \left[g(Y, Z)X - g(X, Z)Y \right]. \quad (5.63)$$

Taking the complete lift of concircular curvature tensor defined in equation (5.63) by mathematical operators, we obtain

$$\begin{aligned} V^c(X^c, Y^c)Z^c &= R^c(X^c, Y^c)Z^c + \frac{r^c}{2n(2n-1)} \left[g^c(Y^c, Z^c)X^v \right. \\ &\quad \left. + g^c(Y^v, Z^c)X^c - g^c(X^c, Z^c)Y^v \right. \\ &\quad \left. - g^c(X^v, Z^c)Y^c \right]. \end{aligned} \quad (5.64)$$

Using equations (5.44), (5.50), (5.54), and (5.64), we can state the following theorem:

Theorem 5.3. *In a Kenmotsu manifold admitting an NSNMC $\check{\nabla}^c$, the lift of the concircular curvature tensor coincides with the lift of the curvature tensor if and only if the scalar curvature of $\check{\nabla}^c$ vanishes upon lifting to the tangent bundle.*

5.1.4 Ricci semi-symmetric Kenmotsu manifolds endowed with the NSNMC in the tangent bundle

A $(2n+1)$ -dimensional contact metric manifold with an NSNMC is said to be Ricci semi-symmetric if (Singh et. al, 2022)

$$\left(\check{R}(X, Y) \cdot \check{S} \right)(Z, U) = 0. \quad (5.65)$$

Taking a complete lift of equation (5.65), we obtain

$$\left(\check{R}^c(X^c, Y^c) \cdot \check{S}^c\right)(Z^c, U^c) = 0, \quad (5.66)$$

i.e.,

$$\check{S}^c\left(\check{R}^c(X^c, Y^c)Z^c, U^c\right) + \check{S}^c\left(Z^c, \check{R}^c(X^c, Y^c)U^c\right) = 0. \quad (5.67)$$

Interchanging X with ξ , and using equation (5.46) in equation (5.67), we obtain

$$\check{S}^c\left(\check{R}^c(\xi^c, Y^c)Z^c, U^c\right) + \check{S}^c\left(Z^c, \check{R}^c(\xi^c, Y^c)U^c\right) = 0, \quad (5.68)$$

$$\begin{aligned} \implies & \eta^c(Z^c)\check{S}^v(Y^c, U^c) + \eta^v(Z^c)\check{S}^c(Y^c, U^c) - g^c(Y^c, Z^c)\check{S}^v(\xi^c, U^c) \\ & - g^c(Y^v, Z^c)\check{S}^c(\xi^c, U^c) + g^c\left((\phi Y)^c, Z^c\right)\check{S}^v(\xi^c, U^c) + g^c\left((\phi Y)^v, Z^c\right)\check{S}^c(\xi^c, U^c) \\ & + \eta^c(U^c)\check{S}^v(Z^c, Y^c) + \eta^v(U^c)\check{S}^c(Z^c, Y^c) - g^c(Y^c, U^c)\check{S}^v(Z^c, \xi^c) \\ & - g^c(Y^v, U^c)\check{S}^c(Z^c, \xi^c) + g^c\left((\phi Y)^c, U^c\right)\check{S}^v(Z^c, \xi^c) + g^c\left((\phi Y)^v, U^c\right)\check{S}^c(Z^c, \xi^c) \\ & = 0. \end{aligned} \quad (5.69)$$

Using equation (5.49), the above equation (5.69) reduces to

$$\begin{aligned} & \eta^c(Z^c)\check{S}^v(Y^c, U^c) + \eta^v(Z^c)\check{S}^c(Y^c, U^c) + 2ng^c(Y^c, Z^c)\eta^v(U^c) \\ & + 2ng^c(Y^v, Z^c)\eta^c(U^c) - 2ng^c\left((\phi Y)^c, Z^c\right)\eta^v(U^c) \\ & - 2ng^c\left((\phi Y)^v, Z^c\right)\eta^c(U^c) + \eta^c(U^c)\check{S}^v(Z^c, Y^c) \\ & + \eta^v(U^c)\check{S}^c(Z^c, Y^c) + 2ng^c(Y^c, U^c)\eta^v(Z^c) \\ & + 2ng^c(Y^v, U^c)\eta^c(Z^c) - 2ng^c\left((\phi Y)^c, U^c\right)\eta^v(Z^c) \\ & - 2ng^c\left((\phi Y)^v, U^c\right)\eta^c(Z^c) \\ & = 0. \end{aligned} \quad (5.70)$$

Putting $U = \xi$ in equation (5.70) and using equation (5.49), we obtain

$$\check{S}^c(Y^c, Z^c) = 2ng^c\left((\phi Y)^c, Z^c\right) - 2ng^c(Y^c, Z^c). \quad (5.71)$$

Putting equation (5.42) into equation (5.71), we obtain

$$S^c(Y^c, Z^c) = -2ng^c(Y^c, Z^c). \quad (5.72)$$

Taking the contraction of equation (5.72), we obtain

$$r^c = -2n(2n + 1). \quad (5.73)$$

Using equation (5.73) in equation (5.44), we obtain

$$\check{r}^c = -2n(2n + 1). \quad (5.74)$$

By using equation (5.72), we can propose the following theorem.

Theorem 5.4. *A Ricci semi-symmetric Kenmotsu manifold associated with the lift of an NSNMC $\check{\nabla}^c$ is an Einstein manifold in the tangent bundle.*

The lift of the Ricci soliton (g^c, V^c, λ) in the tangent bundle is defined in equation (5.28). Here we obtain two conditions with respect to $V^c : V^c \in \text{span}(\xi^c)$ and $V^c \perp \text{span}(\xi^c)$. Let us concentrate on the first condition, which is $V^c : V^c \in \text{span}(\xi^c)$. The data of the lift of the Ricci soliton (g^c, V^c, λ) of the Kenmotsu manifold associated with the lift of NSNMC $\check{\nabla}^c$ in the tangent bundle are defined as

$$(\check{L}_\xi g)^c(X^c, Y^c) + 2\check{S}^c(X^c, Y^c) + 2\lambda g^c(X^c, Y^c) = 0 \quad (5.75)$$

for all $X^c, Y^c \in \chi(TM)$, where $(\check{L}_\xi g)^c$ is the lift of the Lie-derivative of Riemannian metric g^c with respect to ξ^c containing a lift of NSNMC $\check{\nabla}^c$ in the tangent

bundle and is defined by

$$\begin{aligned}
(\check{L}_\xi g)^c(X^c, Y^c) &= \xi^c g^c(X^v, Y^c) + \xi^v g^c(X^c, Y^c) \\
&\quad + g^c((\check{L}_\xi X)^c, Y^c) + g^c(X^c, (\check{L}_\xi Y)^c) \\
&= \xi^c g^c(X^v, Y^c) + g^c((\check{\nabla}_\xi X)^c, Y^c) \\
&\quad + g^c(X^c, (\check{\nabla}_\xi Y)^c) - g^c((\check{\nabla}_X \xi)^c, Y^c) \\
&\quad - g^c(X^c, (\check{\nabla}_Y \xi)^c).
\end{aligned} \tag{5.76}$$

Hence,

$$\begin{aligned}
(\check{L}_\xi g)^c(X^c, Y^c) &= (\check{\nabla}_\xi g)^c(X^c, Y^c) - g^c((\check{\nabla}_X \xi)^c, Y^c) \\
&\quad - g^c(X^c, (\check{\nabla}_Y \xi)^c).
\end{aligned} \tag{5.77}$$

Using equations (5.14), (5.16), (5.18), (5.31), (5.35) in equation (5.77), we obtain

$$(\check{L}_\xi g)^c(X^c, Y^c) = -2g^c((\phi X)^c, (\phi Y)^c). \tag{5.78}$$

Putting equation (5.72) and (5.78) into equation (5.75), we obtain

$$\begin{aligned}
&-g^c((\phi X)^c, (\phi Y)^c) - 2ng^c(X^c, Y^c) + 2ng^c((\phi X)^c, Y^c) \\
&\quad + \lambda g^c(X^c, Y^c) \\
&= 0.
\end{aligned} \tag{5.79}$$

Again putting $X = Y = \xi$ in equation (5.79) and using equation (5.14), we obtain

$$\lambda = 2n > 0. \tag{5.80}$$

Hence, we can state the following theorem.

Theorem 5.5. *On a Ricci semi-symmetric Kenmotsu manifold associated with the lift of an NSNMC $\check{\nabla}^c$, the data of the Ricci soliton (g^c, ξ^c, λ) are expanding in the tangent bundle.*

5.2 Kenmotsu manifolds with generalized Tanaka Webster connection (GTWC)

In a $(2n + 1)$ dimensional Kenmotsu manifold M defined in equations (1.45-1.48) and (5.1-5.9), we used the generalized Tanaka Webster connection (GTWC) $\check{\nabla}$ given by (Ghosh and De ,2017; Kumar et. al, 2022)

$$\check{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi - \eta(X)\phi(Y), \quad (5.81)$$

where ∇ is the Levi-Civita connection and X, Y are vector fields on $M^{(2n+1)}$.

5.2.1 Curvature properties of the Kenmotsu manifolds endowed with the GTWC in the tangent bundle

Taking the complete lift of equation (5.81) by using mathematical operators, we get

$$\begin{aligned} \check{\nabla}_{X^c}^c Y^c &= \nabla_{X^c}^c Y^c - \eta^c(Y^c)(\nabla_X \xi)^v - \eta^v(Y^c)(\nabla_X \xi)^c + (\nabla_X \eta)^c Y^c \xi^v \\ &\quad + (\nabla_X \eta)^c Y^v \xi^c + (\nabla_X \eta)^v Y^c \xi^c - \eta^c(X^c)(\phi Y)^v - \eta^v(X^c)(\phi Y)^c. \end{aligned} \quad (5.82)$$

Employing equations (5.18) and (5.19) in the above equation, we get

$$\begin{aligned} \check{\nabla}_{X^c}^c Y^c &= \nabla_{X^c}^c Y^c + g^c(X^c, Y^c)\xi^v + g^c(X^v, Y^c)\xi^c - \eta^c(Y^c)X^v \\ &\quad - \eta^v(Y^c)X^c - \eta^c(X^c)(\phi Y)^v - \eta^v(X^c)(\phi Y)^c. \end{aligned} \quad (5.83)$$

By setting $Y = \xi$ in the above equation and employing equation (5.18), we get

$$\check{\nabla}_X \xi = 0. \quad (5.84)$$

R. Kumar, S. Shenawy, L. Colney, N. Bin Turki (2024), Certain results on tangent bundle endowed with generalized Tanaka Webster connection on Kenmotsu manifolds *AIMS Mathematics*, **9(11)**, 30364-30383.

The complete lift of the Riemannian curvature tensor in the tangent bundle are given in equation (5.38). By employing equation (5.83) in equation (5.38), we get

$$\begin{aligned}\check{R}^c(X^c, Y^c)Z^c &= R^c(X^c, Y^c)Z^c + g^c(Y^c, Z^c)X^v + g^c(Y^v, Z^c)X^c \\ &\quad - g^c(X^c, Z^c)Y^v - g^c(X^v, Z^c)Y^c.\end{aligned}\tag{5.85}$$

By setting $Z = \xi$ in the above equation and using equation (5.20), we get

$$\check{R}^c(X^c, Y^c)\xi^c = 0.\tag{5.86}$$

Contracting equation (5.85), we get

$$\check{S}^c(Y^c, Z^c) = S^c(Y^c, Z^c) + 2ng^c(Y^c, Z^c),\tag{5.87}$$

where $\check{S}^c$ and S^c are the complete lifts of the Ricci tensor associated with the GTWC and Levi-Civita connections, respectively. The complete lift of the Ricci operator $\check{Q}^c$ associated with the GTWC is obtained by

$$\check{Q}^c Y^c = Q^c Y^c + 2nY^c,\tag{5.88}$$

where Q^c is the complete lift of the Ricci operator associated with the Levi-Civita connection. Again, contracting equation (5.87), we get

$$\check{r}^c = r^c + 2n(2n + 1).\tag{5.89}$$

where $\check{r}^c$ and r^c are the complete lifts of the scalar curvature associated with the GTWC and Levi-Civita connections, respectively.

Theorem 5.6. *The complete lifts of the GTWC of Kenmotsu manifold in the tangent bundle are the only affine connection, which are metric and its complete*

lift of the torsion are given by

$$\begin{aligned}\check{T}^c(X^c, Y^c) &= \eta^c(X^c)Y^v + \eta^v(X^c)Y^c - \eta^c(Y^c)X^v - \eta^v(Y^c)X^c \\ &\quad - \eta^c(X^c)(\phi Y)^v - \eta^v(X^c)(\phi Y)^c + \eta^c(Y^c)(\phi X)^v \\ &\quad + \eta^v(Y^c)(\phi X)^c.\end{aligned}\tag{5.90}$$

Proof. We have

$$\left((\check{\nabla}_X \eta)Y\right)^c = \left(\check{\nabla}_X \eta(Y)\right)^c - \left(\eta(\check{\nabla}_X Y)\right)^c.\tag{5.91}$$

Employing equation (5.83) in the above equation, we get

$$\left((\check{\nabla}_X \eta)Y\right)^c = \left(\nabla_X \eta(Y)\right)^c - g^c(X^c, Y^c) + \eta^c(Y^c)\eta^v(X^c) + \eta^v(Y^c)\eta^c(X^c).\tag{5.92}$$

Employing equation (5.19) in the above equation, we get

$$\left((\check{\nabla}_X \eta)Y\right)^c = 0.\tag{5.93}$$

Again,

$$\left((\check{\nabla}_X g)(Y, Z)\right)^c = \left(\check{\nabla}_X g(Y, Z)\right)^c - \left(g(\check{\nabla}_X Y, Z)\right)^c - \left(g(Y, \check{\nabla}_X Z)\right)^c.\tag{5.94}$$

Employing equation (5.83) in the above equation, we get

$$\left((\check{\nabla}_X g)(Y, Z)\right)^c = 0.\tag{5.95}$$

This shows that the GTWC is a metric connection. The expression of the complete lift of the torsion tensor in the tangent bundle are given by

$$\check{T}^c(X^c, Y^c) = \check{\nabla}_{X^c}^c Y^c - \check{\nabla}_{Y^c}^c X^c.\tag{5.96}$$

Employing equation (5.83) in the above equation, we get

$$\begin{aligned}\check{T}^c(X^c, Y^c) &= \eta^c(X^c)Y^v + \eta^v(X^c)Y^c - \eta^c(Y^c)X^v - \eta^v(Y^c)X^c \\ &\quad - \eta^c(X^c)(\phi Y)^v - \eta^v(X^c)(\phi Y)^c + \eta^c(Y^c)(\phi X)^v \\ &\quad + \eta^v(Y^c)(\phi X)^c.\end{aligned}\tag{5.97}$$

Any complete lifts of metric connection can be put in the expression with the help of $\check{T}^c$ as

$$\begin{aligned}\left(g(\check{\nabla}_X Y, Z)\right)^c &= \left(g(\nabla_X Y, Z)\right)^c + \frac{1}{2} \left[\left(g(\check{T}(X, Y), Z)\right)^c \right. \\ &\quad \left. - \left(g(\check{T}(X, Z), Y)\right)^c - \left(g(\check{T}(Y, Z), X)\right)^c \right].\end{aligned}\tag{5.98}$$

Employing equation (5.97) in the above equation, we get

$$\begin{aligned}\left(g(\check{\nabla}_X Y, Z)\right)^c &= \left(g(\nabla_X Y, Z)\right)^c + \eta^c(Z^c)g^c(X^v, Y^c) \\ &\quad + \eta^v(Z^c)g^c(X^c, Y^c) - \eta^c(Y^c)g^c(X^v, Z^c) \\ &\quad - \eta^v(Y^c)g^c(X^c, Z^c) - \eta^c(X^c)g^c((\phi Y)^v, Z^c) \\ &\quad - \eta^v(X^c)g^c((\phi Y)^c, Z^c).\end{aligned}\tag{5.99}$$

Contracting the above equation with Z , we get

$$\begin{aligned}\check{\nabla}_{X^c}^c Y^c &= \nabla_{X^c}^c Y^c + g^c(X^c, Y^c)\xi^v + g^c(X^v, Y^c)\xi^c - \eta^c(Y^c)X^v \\ &\quad - \eta^v(Y^c)X^c - \eta^c(X^c)(\phi Y)^v - \eta^v(X^c)(\phi Y)^c.\end{aligned}\tag{5.100}$$

□

Proposition 5.6. *In the complete lifts of Kenmotsu manifold associated with the GTWC, ξ^c, η^c , and g^c are parallel in the tangent bundle.*

Proposition 5.7. *The complete lifts of the Kenmotsu manifold associated with GTWC in the tangent bundle are a metric connections.*

Proposition 5.8. *In the complete lifts of Kenmotsu manifold associated with the GTWC, the complete lift of the integral curves of the vector field ξ^c are geodesic in the tangent bundle.*

5.2.2 Ricci semi-symmetry Kenmotsu manifolds endowed with the GTWC in the tangent bundle

The Kenmotsu manifold associated with the GTWC are said to be Ricci semi-symmetry if (Ghosh and De, 2017)

$$(\breve{R}(X, Y) \cdot \breve{S})(V, U) = -\breve{S}(\breve{R}(X, Y)V, U) - \breve{S}(V, \breve{R}(X, Y)U). \quad (5.101)$$

Obtaining the complete lifts of the above equation by using mathematical operators, we get

$$\left((\breve{R}(X, Y) \cdot \breve{S})(V, U) \right)^c = -\left(\breve{S}(\breve{R}(X, Y)V, U) \right)^c - \left(\breve{S}(V, \breve{R}(X, Y)U) \right)^c. \quad (5.102)$$

Theorem 5.7. *The complete lifts of the Ricci semi-symmetric Kenmotsu manifold associated with the GTWC and Levi-Civita connections are equal if and only if the manifold is an Einstein manifold, which is associated with the Levi-Civita connection in the tangent bundle.*

Proof. Employing equation (5.85) in the above equation, we get

$$\begin{aligned} \left((\breve{R}(X, Y) \cdot \breve{S})(V, U) \right)^c &= \left((R(X, Y) \cdot S)(V, U) \right)^c \\ &\quad + g^c(X^c, V^c)S^v(Y^c, U^c) \\ &\quad + g^c(X^v, V^c)S^c(Y^c, U^c) \\ &\quad + g^c(X^c, U^c)S^v(V^c, Y^c) \\ &\quad + g^c(X^v, U^c)S^c(V^c, Y^c) \\ &\quad - g^c(Y^c, V^c)S^v(X^c, U^c) \\ &\quad - g^c(Y^v, V^c)S^c(X^c, U^c) \\ &\quad - g^c(Y^c, U^c)S^v(V^c, X^c) \\ &\quad - g^c(Y^v, U^c)S^c(V^c, X^c). \end{aligned} \quad (5.103)$$

Suppose

$$\left((\check{R}(X, Y) \cdot \check{S})(V, U) \right)^c = \left((R(X, Y) \cdot S)(V, U) \right)^c. \quad (5.104)$$

Then, equation (5.103) becomes

$$\begin{aligned} & g^c(X^c, V^c)S^v(Y^c, U^c) + g^c(X^v, V^c)S^c(Y^c, U^c) + g^c(X^c, U^c)S^v(V^c, Y^c) \\ & + g^c(X^v, U^c)S^c(V^c, Y^c) - g^c(Y^c, V^c)S^v(X^c, U^c) - g^c(Y^v, V^c)S^c(X^c, U^c) \\ & - g^c(Y^c, U^c)S^v(V^c, X^c) - g^c(Y^v, U^c)S^c(V^c, X^c) \\ & = 0. \end{aligned} \quad (5.105)$$

By setting $X = V = e_i$, $1 \leq i \leq (2n + 1)$ in the above equation, we get

$$S^c(Y^c, U^c) = \frac{r^c}{(2n + 1)}g^c(Y^c, U^c), \quad (5.106)$$

which shows that the manifold is Einstein with the Levi-Civita connection in the tangent bundle. Again if

$$S^c(Y^c, U^c) = \frac{r^c}{(2n + 1)}g^c(Y^c, U^c). \quad (5.107)$$

Then, from (5.103), we get

$$\left((\check{R}(X, Y) \cdot \check{S})(V, U) \right)^c = \left((R(X, Y) \cdot S)(V, U) \right)^c, \quad (5.108)$$

which shows that the complete lifts of the Ricci semi-symmetric Kenmotsu manifold associated with the GTWC and Levi-Civita connection are equal. $\square$

5.2.3 Projective curvature tensor of a Kenmotsu manifolds endowed with the GTWC in the tangent bundle

The projective curvature tensor of Riemannian manifold is defined in equation (1.32), then the complete lift of the Kenmotsu manifold associated with the

GTWC in the tangent bundle is given by

$$\begin{aligned} \check{P}^c(X^c, Y^c)Z^c &= \check{R}^c(X^c, Y^c)Z^c - \frac{1}{2n} \left[\check{S}^c(Y^c, Z^c)X^v + \check{S}^v(Y^c, Z^c)X^c \right. \\ &\quad \left. - \check{S}^c(X^c, Z^c)Y^v - \check{S}^v(X^c, Z^c)Y^c \right]. \end{aligned} \quad (5.109)$$

Theorem 5.8. *If the complete lifts of the projective curvature tensor of the Kenmotsu manifold $M^{(2n+1)}$ associated with the GTWC to its tangent bundle vanish, then the complete lift of the curvature tensor of Kenmotsu manifold associated with the GTWC is also found to vanish to its tangent bundle.*

Proof. By setting $\check{P}^c = 0$ in equation (5.109), we get

$$\begin{aligned} \check{R}^c(X^c, Y^c)Z^c &= \frac{1}{2n} \left[\check{S}^c(Y^c, Z^c)X^v + \check{S}^v(Y^c, Z^c)X^c - \check{S}^c(X^c, Z^c)Y^v \right. \\ &\quad \left. - \check{S}^v(X^c, Z^c)Y^c \right]. \end{aligned} \quad (5.110)$$

Employing equations (5.85) and (5.87) in the above equation, we get

$$\begin{aligned} &g^c \left(R^c(X^c, Y^c)Z^c, U^c \right) + g^c(Y^c, Z^c)g^c(X^v, U^c) + g^c(Y^v, Z^c)g^c(X^c, U^c) \\ &- g^c(X^c, Z^c)g^c(Y^v, U^c) - g^c(X^v, Z^c)g^c(Y^c, U^c) \\ &= \frac{1}{2n} \left[S^c(Y^c, Z^c)g^c(X^v, U^c) + S^v(Y^c, Z^c)g^c(X^c, U^c) \right. \\ &\quad + 2ng^c(Y^c, Z^c)g^c(X^v, U^c) + 2ng^c(Y^v, Z^c)g^c(X^c, U^c) \\ &\quad \left. - S^c(X^c, Z^c)g^c(Y^v, U^c) - S^v(X^c, Z^c)g^c(Y^c, U^c) \right]. \end{aligned} \quad (5.111)$$

By setting $U = \xi$ in the above equation, we get

$$\begin{aligned} &S^v(Y^c, Z^c)\eta^c(X^c) + S^c(Y^c, Z^c)\eta^v(X^c) - S^v(X^c, Z^c)\eta^c(Y^c) \\ &- S^c(X^c, Z^c)\eta^v(Y^c) \\ &= 2n \left[g^c(X^c, Z^c)\eta^v(Y^c) + g^c(X^v, Z^c)\eta^c(Y^c) - g^c(Y^c, Z^c)\eta^v(X^c) \right. \\ &\quad \left. - g^c(Y^v, Z^c)\eta^c(X^c) \right]. \end{aligned} \quad (5.112)$$

Also, by setting $X = \xi$ in the above equation, we get

$$S^c(Y^c, Z^c) = -2ng^c(Y^c, Z^c). \quad (5.113)$$

which gives

$$r^c = -2n(2n + 1). \quad (5.114)$$

Employing equation (5.113) in equation (5.110), we get

$$\check{R}^c = 0, \quad (5.115)$$

which shows that the complete lifts of the curvature tensor of Kenmotsu manifold associated with the GTWC vanish. $\square$

Theorem 5.9. *In the complete lifts of the Kenmotsu manifold $M^{(2n+1)}$ with GTWC to its tangent bundle, the following properties hold:*

(i) *The complete lift of the projective curvature tensor is skew-symmetric.*

(ii) *The complete lift of the projective curvature tensor is cyclic.*

Proof. For (i), we interchanged X and Y in equation (5.109) as

$$\begin{aligned} \check{P}^c(Y^c, X^c)Z^c &= \check{R}^c(Y^c, X^c)Z^c - \frac{1}{2n} \left[\check{S}^c(X^c, Z^c)Y^v + \check{S}^v(X^c, Z^c)Y^c \right. \\ &\quad \left. - \check{S}^c(Y^c, Z^c)X^v - \check{S}^v(Y^c, Z^c)X^c \right]. \end{aligned} \quad (5.116)$$

On adding the above equation to equation (5.109) and employing $R^c(X^c, Y^c)Z^c + R^c(Y^c, X^c)Z^c = 0$, we get

$$\check{P}^c(X^c, Y^c)Z^c + \check{P}^c(Y^c, X^c)Z^c = 0. \quad (5.117)$$

For (ii), employing equations (5.85) and (5.109) and the complete lifts of the first Bianchi identity $R^c(X^c, Y^c)Z^c + R^c(Y^c, Z^c)X^c + R^c(Z^c, X^c)Y^c = 0$, we get

$$\check{P}^c(X^c, Y^c)Z^c + \check{P}^c(Y^c, Z^c)X^c + \check{P}^c(Z^c, X^c)Y^c = 0. \quad (5.118)$$

$\square$

Definition 5.1. *The Kenmotsu manifold $M^{(2n+1)}$ associated with the GTWC is*

said to be ξ -projectively flat if (Kumar et. al, 2022)

$$\check{P}(X, Y)\xi = 0,$$

for any vector field X, Y on $M^{(2n+1)}$.

Theorem 5.10. *The complete lift of the Kenmotsu manifold $M^{(2n+1)}$ associated with GTWC to its tangent bundle is ξ -projectively flat.*

Proof. By setting $Z = \xi$ in equation (5.109) and employing equations (5.86) and (5.87), we get

$$\check{P}^c(X^c, Y^c)\xi^c = 0. \quad (5.119)$$

Which proves the theorem □

5.2.4 ϕ -projectively semi-symmetric Kenmotsu manifolds endowed with the GTWC in the tangent bundle

Definition 5.2. *A Kenmotsu manifold $M^{(2n+1)}$ associated with the GTWC is said to be ϕ -projectively semi-symmetric if (Kumar et. al, 2022)*

$$\check{P}(X, Y) \cdot \phi = 0. \quad (5.120)$$

for any vector field X, Y on $M^{(2n+1)}$.

Taking a complete lift of equation (5.120) by using mathematical operators,

$$\left((\check{P}(X, Y) \cdot \phi) Z \right)^c = 0. \quad (5.121)$$

The above equation becomes

$$\left(\check{P}(X, Y) \phi Z \right)^c - \left(\phi \check{P}(X, Y) Z \right)^c = 0. \quad (5.122)$$

Employing equations (5.109), (5.85), and (5.87) in the above equation, we get

$$\begin{aligned}
& \left(R(X, Y)\phi Z \right)^c - \left(\phi R(X, Y)Z \right)^c - \frac{1}{2n} \left[S^c(Y^c, (\phi Z)^c)X^v \right. \\
& + S^v(Y^c, (\phi Z)^c)X^c - S^c(X^c, (\phi Z)^c)Y^v - S^v(X^c, (\phi Z)^c)Y^c \\
& + S^c(X^c, Z^c)(\phi Y)^v + S^v(X^c, Z^c)(\phi Y)^c - S^c(Y^c, Z^c)(\phi X)^v \\
& \left. - S^v(Y^c, Z^c)(\phi X)^c \right] \\
& = 0.
\end{aligned} \tag{5.123}$$

Interchanging Y by ξ in the above equation and employing equations (5.20) and (5.25), we get

$$\begin{aligned}
& S^c(X^c, (\phi Z)^c)\xi^v + S^v(X^c, (\phi Z)^c)\xi^c \\
& = -2n \left[g^c(X^c, (\phi Z)^c)\xi^v + g^c(X^v, (\phi Z)^c)\xi^c \right].
\end{aligned} \tag{5.124}$$

Operating an inner product with ξ , interchanging X by ϕX , and employing equations (5.16) and (5.24) in the above equation, we get

$$S^c(X^c, Z^c) = -2ng^c(X^c, Z^c), \tag{5.125}$$

$$\text{and } r^c = -2n(2n+1). \tag{5.126}$$

Thus, by substituting equation (5.125) in equation (5.109), we get

$$\begin{aligned}
\check{P}^c(X^c, Y^c)Z^c &= R^c(X^c, Y^c)Z^c + \left[g^c(Y^c, Z^c)X^v + g^c(Y^v, Z^c)X^c \right. \\
&\quad \left. - g^c(X^c, Z^c)Y^v - g^c(X^v, Z^c)Y^c \right].
\end{aligned} \tag{5.127}$$

Hence, we can claim the following theorem.

Theorem 5.11. *The complete lift of the Kenmotsu manifold $M^{(2n+1)}$ associated with the GTWC are said to be ϕ -projectively semi-symmetric if and only if $S^c(X^c, Z^c) = -2ng^c(X^c, Z^c)$ in the tangent bundle. Further, if $\check{P}^c = 0$, then the complete lifts of the manifold are said to be isomorphic to the hyperbolic space $H^{(2n+1)}(-1)$ in the tangent bundle.*

By setting $\left[(\check{P}(X, Y) \cdot \check{S})(Z, U) \right]^c = 0$ on a Kenmotsu manifold $M^{(2n+1)}$, we get

$$\check{S}^c(\check{P}(X, Y)Z, U)^c + \check{S}^c(Z, \check{P}(X, Y)U)^c = 0. \quad (5.128)$$

By setting $X = \xi$ in the above equation, we get

$$\check{S}^c(\check{P}(\xi, Y)Z, U)^c + \check{S}^c(Z, \check{P}(\xi, Y)U)^c = 0. \quad (5.129)$$

Employing equation (5.109) in the above equation, we get

$$\begin{aligned} & \check{S}^c(Y^c, Z^c)\eta^v(U^c) + \check{S}^v(Y^c, Z^c)\eta^c(U^c) + \check{S}^c(Y^c, U^c)\eta^v(Z^c) \\ & + \check{S}^v(Y^c, U^c)\eta^c(Z^c) \\ & = 0. \end{aligned} \quad (5.130)$$

Again employing equation (5.87) in the above equation, we get

$$\begin{aligned} & S^c(Y^c, Z^c)\eta^v(U^c) + S^v(Y^c, Z^c)\eta^c(U^c) + S^c(Y^c, U^c)\eta^v(Z^c) \\ & + S^v(Y^c, U^c)\eta^c(Z^c) + 2n \left[g^c(Y^c, Z^c)\eta^v(U^c) + g^c(Y^v, Z^c)\eta^c(U^c) \right. \\ & \left. + g^c(Y^c, U^c)\eta^v(Z^c) + g^c(Y^v, U^c)\eta^c(Z^c) \right] \\ & = 0. \end{aligned} \quad (5.131)$$

By setting $U = \xi$ in the above equation and contracting it with Y and Z , we get

we get

$$S^c(Y^c, Z^c) = -2ng^c(Y^c, Z^c), \quad (5.132)$$

$$\text{and } r^c = -2n(2n + 1). \quad (5.133)$$

Thus, by substituting equation (5.132) in equation (5.109), we get

$$\begin{aligned} \check{P}^c(X^c, Y^c)Z^c &= R^c(X^c, Y^c)Z^c + \left[g^c(Y^c, Z^c)X^v + g^c(Y^v, Z^c)X^c \right. \\ & \left. - g^c(X^c, Z^c)Y^v - g^c(X^v, Z^c)Y^c \right]. \end{aligned} \quad (5.134)$$

Hence, we can claim the following theorem.

Theorem 5.12. *The complete lifts of the Kenmotsu manifold $M^{(2n+1)}$ associated with the GTWC satisfy $\check{P}^c \cdot \check{S}^c = 0$ if and only if $S^c(Y^c, Z^c) = -2ng^c(Y^c, Z^c)$ in the tangent bundle. Further, if $\check{P}^c = 0$, then the complete lifts of the manifold are said to be isomorphic to the hyperbolic space $H^{(2n+1)}(-1)$ in the tangent bundle.*

5.2.5 Conharmonic curvature tensor of a Kenmotsu manifolds endowed with the GTWC in the tangent bundle

The conharmonic curvature tensor of Riemannian manifold is defined in equation (1.34), then the complete lift of the Kenmotsu manifold associated with the GTWC in the tangent bundle are given by (Kumar et. al, 2024b)

$$\begin{aligned} \check{K}^c(X^c, Y^c)Z^c &= \check{R}^c(X^c, Y^c)Z^c - \frac{1}{2n-1} \left[\check{S}^c(Y^c, Z^c)X^v + \check{S}^v(Y^c, Z^c)X^c \right. \\ &\quad - \check{S}^c(X^c, Z^c)Y^v - \check{S}^v(X^c, Z^c)Y^c + g^c(Y^c, Z^c)(\check{Q}X)^v \\ &\quad \left. + g^c(Y^v, Z^c)(\check{Q}X)^c - g^c(X^c, Z^c)(\check{Q}Y)^v - g^c(X^v, Z^c)(\check{Q}Y)^c \right]. \end{aligned} \quad (5.135)$$

Theorem 5.13. *If the complete lift of the conharmonic curvature tensor of a Kenmotsu manifold $M^{(2n+1)}$ with GTWC to its tangent bundle vanishes, then the complete lift of the curvature tensor of a Kenmotsu manifold with the GTWC is also found to vanish to its tangent bundle.*

Proof. By setting $\check{K}(X, Y)Z = 0$, equation (5.135) becomes

$$\begin{aligned} \check{R}^c(X^c, Y^c)Z^c &= \frac{1}{2n-1} \left[\check{S}^c(Y^c, Z^c)X^v + \check{S}^v(Y^c, Z^c)X^c - \check{S}^c(X^c, Z^c)Y^v \right. \\ &\quad - \check{S}^v(X^c, Z^c)Y^c + g^c(Y^c, Z^c)(\check{Q}X)^v + g^c(Y^v, Z^c)(\check{Q}X)^c \\ &\quad \left. - g^c(X^c, Z^c)(\check{Q}Y)^v - g^c(X^v, Z^c)(\check{Q}Y)^c \right]. \end{aligned} \quad (5.136)$$

Employing equations (5.85), (5.87), and (5.88) in the above equation, we get

$$\begin{aligned}
& \left(g(R(X, Y)Z, U) \right)^c + g^c(Y^c, Z^c)g^c(X^v, U^c) + g^c(Y^v, Z^c)g^c(X^c, U^c) \\
& - g^c(X^c, Z^c)g^c(Y^v, U^c) - g^c(X^v, Z^c)g^c(Y^c, U^c) \\
& = \frac{1}{2n-1} \left[S^c(Y^c, Z^c)g^c(X^v, U^c)S^c(Y^c, Z^c)g^c(X^v, U^c) \right. \\
& + S^v(Y^c, Z^c)g^c(X^c, U^c) + 4ng^c(Y^c, Z^c)g^c(X^v, U^c) \\
& + 4ng^c(Y^v, Z^c)g^c(X^c, U^c) - S^c(X^c, Z^c)g^c(Y^v, U^c) \\
& - S^v(X^c, Z^c)g^c(Y^c, U^c) - 4ng^c(X^c, Z^c)g^c(Y^v, U^c) \\
& - 4ng^c(X^v, Z^c)g^c(Y^c, U^c) + S^c(X^c, U^c)g^c(Y^v, Z^c) \\
& + S^v(X^c, U^c)g^c(Y^c, Z^c) - S^c(Y^c, U^c)g^c(X^v, Z^c) \\
& \left. - S^v(Y^c, U^c)g^c(X^c, Z^c) \right]. \tag{5.137}
\end{aligned}$$

By setting $U = \xi$ in the above equation, we get

$$\begin{aligned}
& S^c(Y^c, Z^c)\eta^v(X^c) + S^v(Y^c, Z^c)\eta^c(X^c) - S^c(X^c, Z^c)\eta^v(Y^c) \\
& - S^v(X^c, Z^c)\eta^c(Y^c) - 2n \left[g^c(X^c, Z^c)\eta^v(Y^c) + g^c(X^v, Z^c)\eta^c(Y^c) \right. \\
& \left. - g^c(Y^c, Z^c)\eta^v(X^c) - g^c(Y^v, Z^c)\eta^c(X^c) \right] \\
& = 0. \tag{5.138}
\end{aligned}$$

By setting $X = \xi$ in the above equation, we get

$$S^c(Y^c, Z^c) = -2ng^c(Y^c, Z^c). \tag{5.139}$$

Contracting the above equation gives

$$r^c = -2n(2n+1). \tag{5.140}$$

Employing equation (5.139) in equation (5.136), we get

$$\check{R}^c(X^c, Y^c)Z^c = 0. \tag{5.141}$$

□

5.2.6 Concircular curvature tensor of a Kenmotsu manifold endowed with the GTWC in the tangent bundle

The concircular curvature tensor of Riemannian manifold is defined in equation (1.33), then the complete lift of the Riemannian manifold in the tangent bundle is given by (Kumar et. al, 2024b)

$$\begin{aligned} K^c(X^c, Y^c)Z^c &= R^c(X^c, Y^c)Z^c - \frac{r^c}{2n(2n+1)} \left(g^c(Y^c, Z^c)X^v \right. \\ &\quad \left. + g^c(Y^v, Z^c)X^c - g^c(X^c, Z^c)Y^v - g^c(X^v, Z^c)Y^c \right), \end{aligned} \quad (5.142)$$

where X, Y, Z are vector fields on the Riemannian manifold, R is the Riemannian curvature tensor, and r is the scalar curvature associated with the Levi-Civita connection.

Similarly, the complete lift of the concircular curvature tensor of Kenmotsu manifold in the tangent bundle is given as

$$\begin{aligned} \check{K}^c(X^c, Y^c)Z^c &= \check{R}^c(X^c, Y^c)Z^c - \frac{\check{r}^c}{2n(2n+1)} \left(g^c(Y^c, Z^c)X^v \right. \\ &\quad \left. + g^c(Y^v, Z^c)X^c - g^c(X^c, Z^c)Y^v - g^c(X^v, Z^c)Y^c \right), \end{aligned} \quad (5.143)$$

where X, Y, Z are vector fields on Kenmotsu manifold, $\check{R}$ is the Riemannian curvature tensor, and $\check{r}$ is the scalar curvature associated with the GTWC.

Theorem 5.14. *The complete lifts of concircular curvature tensors of Kenmotsu manifold associated with the GTWC and Levi-Civita connections are equal in the tangent bundle.*

Proof. Employing (5.85) and (5.89) in the above equation, we get

$$\check{K}^c(X^c, Y^c)Z^c = K^c(X^c, Y^c)Z^c. \quad (5.144)$$

□

Theorem 5.15. *If the complete lifts of the Kenmotsu manifold endowed with GTWC to its tangent bundle satisfy the condition $\check{K}^c(X^c, Y^c) \cdot \check{S}^c = 0$, then the manifold is classified as an Einstein manifold with respect to the Levi-Civita connection in the tangent bundle.*

Proof. By setting $\check{K}^c(X^c, Y^c) \cdot \check{S}^c = 0$, we have

$$\left(\check{S}(\check{K}(X, Y)V, U) \right)^c + \left(\check{S}(V, \check{K}(X, Y)U) \right)^c = 0, \quad (5.145)$$

for all $X, Y, V, U \in \chi(M^{(2n+1)})$. Replacing X with ξ in the above equation, we get

$$\left(\check{S}(\check{K}(\xi, Y)V, U) \right)^c + \left(\check{S}(V, \check{K}(\xi, Y)U) \right)^c = 0. \quad (5.146)$$

Employing equations (5.85), (5.89), and (5.143) in the above equation, we get

$$\begin{aligned} & \eta^c(V^c)\check{S}^v(Y^c, U^c) + \eta^v(V^c)\check{S}^c(Y^c, U^c) - \eta^c(U^c)\check{S}^v(V^c, Y^c) \\ & - \eta^v(U^c)\check{S}^c(V^c, Y^c) \\ & = 0. \end{aligned} \quad (5.147)$$

By setting $V = \xi$ in the above equation, we get $\check{S}^c(Y^c, U^c) = 0$. Then, from equation (5.87) it follows that

$$S^c(Y^c, U^c) = -2ng^c(Y^c, U^c). \quad (5.148)$$

Which shows that the manifold is an Einstein manifold with respect to the Levi-Civita connection in the tangent bundle. $\square$

5.2.7 Weyl conformal curvature tensor of a Kenmotsu manifolds endowed with the GTWC in the tangent bundle

In a Riemannian manifold, the complete lift of the Weyl conformal curvature tensor associated with the GTWC in the tangent bundle is given by (Kumar et. al, 2024b)

$$\begin{aligned}
\check{C}^c(X^c, Y^c)Z^c &= \check{R}^c(X^c, Y^c)Z^c - \frac{1}{2n-1} \left[\check{S}^c(Y^c, Z^c)X^v \right. \\
&\quad + \check{S}^v(Y^c, Z^c)X^c + g^c(Y^c, Z^c)(\check{Q}X)^v + g^c(Y^v, Z^c)(\check{Q}X)^c \\
&\quad \left. - g^c(X^c, Z^c)(\check{Q}Y)^v - g^c(X^v, Z^c)(\check{Q}Y)^c \right] \\
&\quad + \frac{\check{r}^c}{2n(2n-1)} \left[g^c(Y^c, Z^c)X^v + g^c(Y^v, Z^c)X^c \right. \\
&\quad \left. - g^c(X^c, Z^c)Y^v - g^c(X^v, Z^c)Y^c \right].
\end{aligned} \tag{5.149}$$

Theorem 5.16. *In a Kenmotsu manifold $M^{(2n+1)}$ with the GTWC, the complete lifts of the Weyl conformal curvature tensor with the Levi-Civita connection and the complete lifts of the Weyl conformal curvature tensor with the GTWC are equivalent in the tangent bundle.*

Proof. Employing equations (5.85), (5.87)–(5.89) in (5.149), we get

$$\check{C}^c(X^c, Y^c)Z^c = C^c(X^c, Y^c)Z^c. \tag{5.150}$$

for all X, Y, Z on $M^{(2n+1)}$. □

Definition 5.3. *A Kenmotsu manifold associated with the GTWC is called recurrent if its curvature tensor $\check{R}$ satisfies the condition (Kumar et. al, 2022b)*

$$(\check{\nabla}_U \check{R})(X, Y)Z = D(U)\check{R}(X, Y)Z, \tag{5.151}$$

where $\check{R}$ is the curvature tensor associated with the GTWC and D is a 1-form

associated with vector field ρ such that $D(X) = g(X, \rho)$.

Taking the complete lift of equation (5.151) by mathematical operators, we get

$$\left((\check{\nabla}_U \check{R})(X, Y)Z \right)^c = \left(D(U) \check{R}(X, Y)Z \right)^c, \quad (5.152)$$

We can write the above equation as

$$\begin{aligned} & \left(\check{\nabla}_U \check{R}(X, Y)Z \right)^c - \left(\check{R}(\check{\nabla}_U X, Y)Z \right)^c - \left(\check{R}(X, \check{\nabla}_U Y)Z \right)^c \\ & - \left(\check{R}(X, Y) \check{\nabla}_U Z \right)^c \\ & = \left(D(U) \check{R}(X, Y)Z \right)^c. \end{aligned} \quad (5.153)$$

Employing equations (5.83), (5.85), and (5.87) in the above equation, we get

$$\begin{aligned} & g^c(U^c, R^c(X^c, Y^c)Z^c)\xi^v + g^c(U^v, R^c(X^c, Y^c)Z^c)\xi^c \\ & - \left(g(U^c, X^c)R(\xi, Y)Z \right)^c - \left(g(U, Y)R(X, \xi)Z \right)^c \\ & - \left(g(U, Z)R(X, Y)\xi \right)^c - \left(\eta(R(X, Y)Z)U \right)^c \\ & + \left(\eta(X)R(U, Y)Z \right)^c + \left(\eta(Y)R(X, U)Z \right)^c \\ & + \left(\eta(Z)R(X, Y)U \right)^c - \left[\eta(U) \left(\phi R(X, Y)Z \right. \right. \\ & \left. \left. - R(\phi X, Y)Z - R(X, \phi Y)Z - R(X, Y)\phi Z \right) \right]^c \\ & = D^c(U^c) \left(g^c(Y^c, Z^c)X^v + g^c(Y^v, Z^c)X^c - g^c(X^c, Z^c)Y^v \right. \\ & \left. - g^c(X^v, Z^c)Y^c \right). \end{aligned} \quad (5.154)$$

Interchanging Z by ξ in the above equation and employing equations (5.13), (5.14), (5.20), (5.21), and (5.23) in the above equation, we get

$$\begin{aligned} & D^c \left(\eta^c(Y^c)X^v + \eta^v(Y^c)X^c - \eta^c(X^c)Y^v - \eta^v(X^c)Y^c \right) \\ & = g^c(U^c, Y^c)X^v + g^c(U^v, Y^c)X^c - g^c(U^c, X^c)Y^v \\ & - g^c(U^v, X^c)Y^c + R^c(X^c, Y^c)U^c. \end{aligned} \quad (5.155)$$

Obtaining an inner product with V in the above equation, we get

$$\begin{aligned}
 & D^c \left(\eta^c(Y^c)g^c(X^v, V^c) + \eta^v(Y^c)g^c(X^c, V^c) - \eta^c(X^c)g^c(Y^v, V^c) \right. \\
 & \quad \left. - \eta^v(X^c)g^c(Y^c, V^c) \right) \\
 & = g^c(U^c, Y^c)g^c(X^v, V^c) + g^c(U^v, Y^c)g^c(X^c, V^c) - g^c(U^c, X^c)g^c(Y^v, V^c) \\
 & \quad - g^c(U^v, X^c)g^c(Y^c, V^c) + R^c(X^c, Y^c, U^c, V^c).
 \end{aligned} \tag{5.156}$$

Suppose $\{e_1, e_2, \dots, e_{2n+1}\}$ is a local orthonormal basis of vector fields in M . Then, by setting $X = V = e_i$ in equation (5.156) and summing up with respect to i , $1 \leq i \leq 2n+1$, we get

$$S^c(Y^c, U^c) = -(n-1) \left[g^c(Y^c, U^c) - \eta^c(Y^c)D^v(U^c) - \eta^v(Y^c)D^c(U^c) \right]. \tag{5.157}$$

We suppose that the vector field ρ is in co-direction with vector field ξ . Then their associated 1-forms will be equal, *i.e.* $D = \eta$, and so the above equation becomes

$$S^c(Y^c, U^c) = -(n-1) \left[g^c(Y^c, U^c) - \eta^c(Y^c)\eta^v(U^c) - \eta^v(Y^c)\eta^c(U^c) \right]. \tag{5.158}$$

Hence, we can claim the following theorem.

Theorem 5.17. *If the complete lift of the Kenmotsu manifold associated with the GTWC is recurrent and the associated 1-form D is equal to the associated 1-form η , then the manifold is an η -Einstein manifold in the tangent bundle.*

5.2.8 Ricci soliton on Kenmotsu manifolds endowed with the GTWC in the tangent bundle

The Ricci soliton of Kenmotsu manifold associated with the GTWC is given by (Kumar et. al, 2022)

$$(\check{L}_V g)(X, Y) + 2\check{S}(X, Y) + 2\gamma g(X, Y) = 0. \tag{5.159}$$

Obtaining the complete lift of equation (5.159) by using mathematical operators, we get

$$(\breve{L}_{V^c}^c g^c)(X^c, Y^c) + 2\breve{S}^c(X^c, Y^c) + 2\gamma g^c(X^c, Y^c) = 0. \quad (5.160)$$

If the complete lift of the potential vector field V^c is the structure of the vector field ξ^c in the tangent bundle, and since the complete lift of ξ^c is a parallel vector field with respect to the GTWC (from equation (5.84)) in the tangent bundle causing the first term of equation (5.160) to vanish, then the manifold reduces to an Einstein manifold. If the complete lift of V^c is pointwise collinear with the structure vector field ξ^c in the tangent bundle such that $V^c = a\xi^c$, where a is a function on $M^{(2n+1)}$, then equation (5.160) implies that

$$\begin{aligned} & ag^c(\breve{\nabla}_{X^c}^c \xi^c, Y^c) + a(X^c \eta^v(Y^c) + X^v \eta^c(Y^c)) + ag^c(X^c, \breve{\nabla}_{Y^c}^c \xi^c) \\ & + a(Y^c \eta^v(X^c) + Y^v \eta^c(X^c)) + 2\breve{S}^c(X^c, Y^c) + 2\gamma g^c(X^c, Y^c) \\ & = 0. \end{aligned} \quad (5.161)$$

Employing equations (5.84) and (5.87) in the above equation, we get

$$\begin{aligned} & a(\eta^c(Y^c)X^v + \eta^v(Y^c)X^c) + a(\eta^c(X^c)Y^v + \eta^v(X^c)Y^c) \\ & + 2S^c(X^c, Y^c) + 2(2n + \gamma)g^c(X^c, Y^c) \\ & = 0. \end{aligned} \quad (5.162)$$

By setting $Y = \xi$ in the above equation and employing equation (5.25), we get

$$a(X^c) = -(2\gamma + a\xi)\eta^c(X^c). \quad (5.163)$$

Interchanging X with ξ in the above equation, we get

$$a(\xi^c) = -\gamma. \quad (5.164)$$

Employing the above equation in equation (5.163), we get

$$a(X^c) = -\gamma\eta^c(X^c). \quad (5.165)$$

By differentiating the above equation, we get

$$\gamma d\eta^c = 0. \quad (5.166)$$

Since, $d\eta \neq 0$ from the above equation, we get

$$\gamma = 0. \quad (5.167)$$

Employing the above equation in equation (5.165), we can say that a is a constant. Thus, it is verified from equation (5.162) that

$$S^c(X^c, Y^c) = -(2n + \gamma)g^c(X^c, Y^c) + \gamma\left(\eta^v(X^c)\eta^c(Y^c)\right). \quad (5.168)$$

Thus, we state the following theorem.

Theorem 5.18. *If the complete lift of the vector field X^c on the Ricci soliton of Kenmotsu manifold associated with the GTWC is pointwise collinear with the complete lift of the vector field ξ^c in the tangent bundle, then the manifold is an η -Einstein manifold and the data provided by the complete lift of the Ricci soliton is found to be steady in the tangent bundle.*

5.3 Example

This section is devoted to a comprehensive examination of a particular example. It will provide a detailed analysis of the manifold's structural properties and the relevant mathematical framework. Let M be a five-dimensional manifold defined as

$$M = \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \right\},$$

where $\mathbb{R}$ represents the set of real numbers. Consider $\{E_1, E_2, E_3, E_4, E_5\}$ as a linearly independent global frame on M where

$$\begin{aligned} E_1 &= e^{-x_5} \frac{\partial}{\partial x_1}, & E_2 &= e^{-x_5} \frac{\partial}{\partial x_2}, & E_3 &= e^{-x_5} \frac{\partial}{\partial x_3}, \\ E_4 &= e^{-x_5} \frac{\partial}{\partial x_4}, & E_5 &= e^{-x_5} \frac{\partial}{\partial x_5}. \end{aligned} \quad (5.169)$$

Let the 1-form η be given by

$$\eta(Y) = g(Y, \xi). \quad (5.170)$$

The Riemannian metric g is defined by

$$g(E_i, E_j) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise.} \end{cases} \quad (5.171)$$

Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi E_i = \begin{cases} E_3, & i = 1 \\ E_4, & i = 2 \\ -E_1, & i = 3 \\ -E_2, & i = 4 \\ 0, & i = 5. \end{cases} \quad (5.172)$$

Using the linearity of ϕ and g , we acquire $\eta(E_5) = 1$, $\phi^2 X = -X + \eta(X)E_5$ and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$. Thus, for $E_5 = \xi$, the structure (ϕ, ξ, η, g) is then an almost contact metric structure on M , and M is called an almost contact metric manifold. In addition, M satisfies

$$(\nabla_X \phi)Y = g(\phi X, Y)E_5 - \eta(Y)\phi X. \quad (5.173)$$

Here, for $E_5 = \xi$, M is a Kenmotsu manifold. Let the complete and vertical lifts of E_1, E_2, E_3, E_4, E_5 be $E_1^c, E_2^c, E_3^c, E_4^c, E_5^c$ and $E_1^v, E_2^v, E_3^v, E_4^v, E_5^v$, respectively, in the tangent bundle TM of manifold M , and let the complete lift of the Riemannian metric g be g^c on TM such that

$$g^c(X^v, E_5^c) = \left(g^c(X, E_5)\right)^v = \left(\eta(X)\right)^v \quad (5.174)$$

$$g^c(X^c, E_5^c) = \left(g^c(X, E_5)\right)^c = \left(\eta(X)\right)^c \quad (5.175)$$

$$g^c(E_5^c, E_5^c) = 1, \quad g^v(X^v, E_5^c) = 0, \quad g^v(E_5^v, E_5^c) = 0, \quad (5.176)$$

and so on. Let the complete and vertical lifts of the $(1, 1)$ tensor field ϕ_0 be ϕ_0^c and ϕ_0^v , respectively, and be defined by

$$\phi^v(E_5^v) = \phi^c(E_5^c) = 0, \quad (5.177)$$

$$\phi^v(E_1^v) = E_3^v, \quad \phi^c(E_1^c) = E_3^c, \quad (5.178)$$

$$\phi^v(E_2^v) = E_4^v, \quad \phi^c(E_2^c) = E_4^c, \quad (5.179)$$

$$\phi^v(E_3^v) = -E_1^v, \quad \phi^c(E_3^c) = -E_1^c, \quad (5.180)$$

$$\phi^v(E_4^v) = -E_2^v, \quad \phi^c(E_4^c) = -E_2^c. \quad (5.181)$$

Using the linearity of ϕ and g , we infer that

$$(\phi^2 X)^c = -X^c + \eta^c(X^c)E_5^v + \eta^v(X^c)E_5^c, \quad (5.182)$$

$$g^c\left((\phi E_5)^c, (\phi E_4)^c\right) = g^c(E_5^c, E_4^c) - \eta^c(E_5^c)\eta^v(E_4^c) - \eta^v(E_5^c)\eta^c(E_4^c). \quad (5.183)$$

Thus, for $E_5 = \xi$ in equations (5.174)–(5.176) and (5.182)–(5.183), the structure $(\phi^c, \xi^c, \eta^c, g^c)$ is an almost contact metric structure on TM and satisfies the relation

$$\begin{aligned} (\nabla_{E_5^c}^c \phi^c)E_4^c &= g^c\left((\phi E_5)^c, E_4^c\right)\xi^v + g^c\left((\phi E_5)^v, E_4^c\right)\xi^c \\ &\quad - \eta^c(E_4^c)(\phi E_5)^v - \eta^v(E_4^c)(\phi E_5)^c. \end{aligned} \quad (5.184)$$

Thus, $(\phi^c, \xi^c, \eta^c, g^c, TM)$ is a Kenmotsu manifold.

Chapter 6

CL-Transformation on LP-Kenmotsu manifolds and 3-dimensional quasi Sasakian manifolds

6.1 LP-Kenmotsu manifolds

An LP-Kenmotsu manifold defined in equations (1.56-1.61) and (1.66-1.67), the following relations also hold (Haseeb and Prasad, 2018)

$$\nabla_X \xi = -\phi^2 X = -X - \eta(X)\xi, \quad (6.1)$$

$$(\nabla_X \eta)Y = -g(X, Y) - \eta(X)\eta(Y), \quad (6.2)$$

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \quad (6.3)$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \quad (6.4)$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad (6.5)$$

$$S(X, \xi) = (n - 1)\eta(X), \quad (6.6)$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y). \quad (6.7)$$

6.1.1 Infinitesimal CL -transformation on LP-Kenmotsu manifolds

Definition 6.1. A vector field V on an LP-Kenmotsu manifold M is said to be an infinitesimal CL -transformation if it satisfies

$$L_V \begin{pmatrix} h \\ j & i \end{pmatrix} = \rho_j \delta_i^h + \rho_i \delta_j^h + \alpha(\eta_j \phi_i^h + \eta_i \phi_j^h) \quad (6.8)$$

for a certain constant α , where ρ_i are the components of an 1-form ρ , L_V denotes the Lie derivative with respect to V and $\begin{pmatrix} h \\ j & i \end{pmatrix}$ is the Christoffel symbol of the Riemannian metric g .

Theorem 6.1. If V is an infinitesimal CL -transformation on an LP-Kenmotsu manifold, then the relation

$$(L_V g)(Y, Z) = -(\nabla_Y \rho)(Z) - \alpha g(\phi Y, Z) \quad (6.9)$$

holds for any vector fields Y and Z on M .

Proof. It is known from (Yano and Kon, 1984) that

$$L_V R_{kji}^h = \nabla_k L_V \begin{pmatrix} h \\ j & i \end{pmatrix} - \nabla_j L_V \begin{pmatrix} h \\ k & i \end{pmatrix} \quad (6.10)$$

Employing equations (6.8), (1.67) and (6.2) in the above equation, we get

$$\begin{aligned} (L_V R)(X, Y)Z &= (\nabla_X \rho)(Z)Y - (\nabla_Y \rho)(Z)X + \alpha \left[\{g(Y, Z)\phi X \right. \\ &\quad \left. - g(X, Z)\phi Y\} - \{\eta(Y)\phi X - \eta(X)\phi Y\}\eta(Z) \right. \\ &\quad \left. + \{g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y) + 2g(\phi X, Y)\eta(Z)\}\xi \right], \end{aligned} \quad (6.11)$$

for any vector fields X, Y and Z on M . Operating η in the above equation, we get

$$\begin{aligned}\eta\left((L_V R)(X, Y)Z\right) &= (\nabla_X \rho)(Z)\eta(Y) - (\nabla_Y \rho)(Z)\eta(X) \\ &\quad + \alpha\{-g(\phi Y, Z)\eta(X) + g(\phi X, Z)\eta(Y) \\ &\quad - 2g(\phi X, Y)\eta(Z)\}.\end{aligned}\tag{6.12}$$

Taking Lie derivative of equation (6.3) with respect to V and using the above equation and then interchanging X and Y to Y and ξ respectively, we get

$$\begin{aligned}(L_V g)(Z, Y) &= -\{(\nabla_\xi \rho)(Y) - (L_V g)(\xi, Y)\}\eta(Z) - (\nabla_Z \rho)(Y) \\ &\quad - \alpha g(\phi Z, Y).\end{aligned}\tag{6.13}$$

Interchanging Y and Z in the above equation and then subtracting it from equation (6.13), we get

$$\begin{aligned}\{(\nabla_\xi \rho)(Z) - (L_V g)(\xi, Z)\}\eta(Y) &= \{(\nabla_\xi \rho)(Y) - (L_V g)(\xi, Y)\}\eta(Z) \\ &\quad + 2\alpha g(\phi Z, Y).\end{aligned}\tag{6.14}$$

Setting $Y = \xi$ in the above equation, we get

$$(L_V g)(\xi, Z)(\xi, Z) - (\nabla_\xi \rho)(Z) = \{(\nabla_\xi \rho)(\xi) - (L_V g)(\xi, \xi)\}.\tag{6.15}$$

From equations (6.13) and (6.15), we get

$$\begin{aligned}(L_V g)(Y, Z) &= -(\nabla_Y \rho)(Z) + \{(\nabla_\xi \rho)(\xi) - (L_V g)(\xi, \xi)\}\eta(Y)\eta(Z) \\ &\quad - \alpha g(\phi Y, Z).\end{aligned}\tag{6.16}$$

Now, we take the inner product of equation (6.11) with a vector field W on M and then contracting X and W , we get

$$(L_V S)(Y, Z) = (n - 1)(\nabla_Y \rho)(Z).\tag{6.17}$$

Setting $Y = \xi$ in the above equation, we have

$$(L_V S)(\xi, Z) = (n - 1)(\nabla_\xi \rho)(Z). \quad (6.18)$$

Taking Lie derivative of equation (6.6) with respect V and using the above equation and then setting $Z = \xi$, we get

$$(\nabla_\xi \rho)(\xi) = (L_V g)(\xi, \xi). \quad (6.19)$$

Employing the above equation in equation (6.16), we obtain equation (6.9). This completes the proof. $\square$

From equation (6.9), we can state the following:

Theorem 6.2. *An infinitesimal CL-transformation V on an LP-Kenmotsu manifold M is not a projective killing vector field unless $\alpha = 0$.*

Corollary 6.1. *Any infinitesimal CL-transformation V on an Einstein LP-Kenmotsu manifold M is not necessarily a projective killing vector field.*

Proof. Let M be an Einstein LP-Kenmotsu manifold. Then,

$$S(Y, Z) = (n - 1)g(Y, Z). \quad (6.20)$$

Taking Lie derivative of the above equation with respect to V and employing equations (6.9) and (6.17), we get

$$\alpha g(\phi Y, Z) = -2(\nabla_Y \rho)(Z), \quad (6.21)$$

from which we get $\alpha \neq 0$. $\square$

6.1.2 CL-transformation on LP-Kenmotsu manifold

Definition 6.2. *A transformation f on an n -dimensional LP-Kenmotsu manifold M with structure (ϕ, ξ, η, g) is said to be a CL-transformation if the Levi-Civita*

connection ∇ and a symmetric affine connection ∇^f induced from ∇ by f is related by

$$\nabla_X^f Y = \nabla_X Y + \rho(X)Y + \rho(Y)X + \alpha\{\eta(X)\phi Y + \eta(Y)\phi X\} \quad (6.22)$$

where ρ is a 1-form and α is a constant, R^f is the curvature tensor of the connection ∇^f , S^f is the Ricci tensor of the connection ∇^f and both R^f and S^f are the geometric objects with respect to affine connection ∇^f .

If f is a CL -transformation on an LP-Kenmotsu manifold M , then by virtue of equations (1.56), (1.57), (1.58), (1.60), (1.67), (6.2) and (6.22), we get

$$\begin{aligned} R^f(X, Y)Z &= R(X, Y)Z + \{B(X, Y) - B(Y, X)\}Z + B(X, Z)Y \\ &\quad - B(Y, Z)X - \alpha\left[\{\eta(Y)\phi X - \eta(X)\phi Y\}\eta(Z) \right. \\ &\quad + \{g(X, Z)\phi Y - g(Y, Z)\phi X\} - \{g(\phi Y, Z)\eta(X) \\ &\quad \left. - g(\phi X, Z)\eta(Y) - 2g(X, \phi Y)\eta(Z)\}\xi\right], \end{aligned} \quad (6.23)$$

for any vector field X, Y and Z on M , where the tensor field $B(X, Y)$ is defined by

$$\begin{aligned} B(X, Y) &= (\nabla_X \rho)(Y) - \rho(X)\rho(Y) - \alpha^2\eta(X)\eta(Y) - \alpha\{\eta(X)\rho(\phi X) \\ &\quad + \eta(Y)\rho(\phi X)\}. \end{aligned} \quad (6.24)$$

From equation (6.23), we get

$$\begin{aligned} g(R^f(X, Y)Z, U) &= g(R(X, Y)Z, U) + \{B(X, Y) - B(Y, X)\}g(Z, U) \\ &\quad + B(X, Z)g(Y, U) - B(Y, Z)g(X, U) - \alpha\left[\{\eta(Y)g(\phi X, U) \right. \\ &\quad - \eta(X)g(\phi Y, U)\}\eta(Z) + \{g(X, Z)g(\phi Y, U) - g(Y, Z)g(\phi X, U)\} \\ &\quad \left. - \{g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y) - 2g(X, \phi Y)\eta(Z)\}g(\xi, U)\right], \end{aligned} \quad (6.25)$$

where U is any vector field on M . Taking contraction of equation (6.25) over X

and U and also over Z and U and then adding the results and proceeding the same manner as in (Koto and Nagao, 1966), it is easy to check that $B(X, Y)$ is symmetric and hence from equation (6.24), we see that the 1-form ρ is closed.

Theorem 6.3. *Let A be the tensor field of type $(1, 3)$ given by*

$$\begin{aligned}
 A(X, Y)Z = & R(X, Y)Z - \frac{1}{n-1} \left[\left(S(Y, Z)X - S(X, Z)Y \right) \right. \\
 & - \left(\eta(Y)\eta(Z) - g(Y, Z) \right) QX + \left(\eta(X)\eta(Z) - g(X, Z) \right) QY \\
 & + \left[\left(S(X, Z) - (n-1)g(X, Z) \right) \eta(Y) - \left(S(Y, Z) \right. \right. \\
 & \left. \left. - (n-1)g(Y, Z) \right) \eta(X) + 2 \left(S(X, Y) - (n-1)g(X, Y) \right) \eta(Z) \right] \xi \\
 & \left. + \left(g(Y, Z) - \eta(Y)\eta(Z) \right) X - \left(g(X, Z) - \eta(X)\eta(Z) \right) Y \right]
 \end{aligned} \tag{6.26}$$

where Q is the tensor field of type $(1, 1)$ given by $g(QX, Y) = S(X, Y)$, where X and Y are vector fields on M . This tensor field A is invariant under CL-transformation of an LP-Kenmotsu manifold M .

Proof. Assume that f is a CL-transformation on an LP-Kenmotsu manifold M . Then the relations of equations (6.22-6.25) hold. Since the tensor $B(X, Y)$ is symmetric, equation (6.23) can be rewritten as

$$\begin{aligned}
 R^f(X, Y)Z = & R(X, Y)Z + B(X, Z)Y - B(Y, Z)X \\
 & - \alpha \left[\{ \eta(Y)\phi X - \eta(X)\phi Y \} \eta(Z) + \{ g(X, Z)\phi Y \right. \\
 & \left. - g(Y, Z)\phi X \} - \{ g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y) \right. \\
 & \left. - 2g(X, \phi Y)\eta(Z) \} \xi \right],
 \end{aligned} \tag{6.27}$$

which yields

$$S^f(Y, Z) = S(Y, Z) + B(Y, Z) - nB(Y, Z), \tag{6.28}$$

$$S^f(Y, Z) = S(Y, Z) - (n-1)B(Y, Z), \tag{6.29}$$

$$(n-1)B(Y, Z) = S(Y, Z) - S^f(Y, Z). \tag{6.30}$$

Substituting equation (6.30) into equation (6.27), we get

$$\begin{aligned}
R^f(X, Y)Z &= R(X, Y)Z + \frac{1}{n-1}S(X, Z)Y - \frac{1}{n-1}S^f(X, Z)Y \\
&\quad - \frac{1}{n-1}S(Y, Z)X + \frac{1}{n-1}S^f(Y, Z)X - \alpha \left[\{\eta(Y)\phi X \right. \\
&\quad \left. - \eta(X)\phi Y\}\eta(Z) + \{g(X, Z)\phi Y - g(Y, Z)\phi X\} \right. \\
&\quad \left. - \{g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y) \right. \\
&\quad \left. - 2g(X, \phi Y)\eta(Z)\}\xi \right].
\end{aligned} \tag{6.31}$$

and

$$\begin{aligned}
P^f(X, Y)Z &= P(X, Y)Z - \alpha \left[\{\eta(Y)\phi X - \eta(X)\phi Y\}\eta(Z) \right. \\
&\quad \left. + \{g(X, Z - 0)\phi Y - g(Y, Z)\phi X\} - \{g(\phi Y, Z)\eta(X) \right. \\
&\quad \left. - g(\phi X, Z)\eta(Y) - 2g(X, \phi Y)\eta(Z)\}\xi \right],
\end{aligned} \tag{6.32}$$

where P is the projective curvature tensor given by (Weyl, 1918)

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}\{S(Y, Z)X - S(X, Z)Y\}. \tag{6.33}$$

Replacing X to ξ in equation (6.32), we have

$$P^f(\xi, Y)Z = P(\xi, Y)Z - \alpha\{2\eta(Z)\phi Y + g(\phi Y, Z)\xi\}. \tag{6.34}$$

Operating η on both sides of the above equation, we get

$$\eta(P^f(\xi, Y)Z) = \eta(P(\xi, Y)Z) + \alpha g(\phi Y, Z), \tag{6.35}$$

$$\eta(P^f(\xi, Y)Z) - \eta(P(\xi, Y)Z) = \alpha g(\phi Y, Z). \tag{6.36}$$

Using equation (6.36) in equation (6.32), we get □

$$\begin{aligned}
P^f(X, Y)Z &= P(X, Y)Z - \alpha \left[\{ \eta(Y)\phi X - \eta(X)\phi Y \} \eta(Z) \right. \\
&\quad \left. + \{ g(X, Z)\phi Y - g(Y, Z)\phi X \} \right] - \eta\{P^f(\xi, Y)Z\}\eta(X)\xi \\
&\quad + \eta\{P(\xi, Y)Z\}\eta(X)\xi + \eta\{P^f(\xi, X)Z\}\eta(Y)\xi \\
&\quad - \eta\{P(\xi, X)Z\}\eta(Y)\xi + 2\eta\{P^f(\xi, X)Y\}\eta(Z)\xi \\
&\quad - 2\eta\{P(\xi, X)Y\}\eta(Z)\xi,
\end{aligned} \tag{6.37}$$

$$\begin{aligned}
H^f(X, Y)Z &= H(X, Y)Z - \alpha \left[\{ g(X, Z)\phi Y - g(Y, Z)\phi X \} \right. \\
&\quad \left. + \{ \eta(Y)\phi X - \eta(X)\phi Y \} \eta(Z) \right],
\end{aligned} \tag{6.38}$$

where

$$\begin{aligned}
H(X, Y)Z &= P(X, Y)Z - \left[\eta\{P(\xi, X)Z\}\eta(Y) - \eta\{P(\xi, Y)Z\}\eta(X) \right. \\
&\quad \left. + 2\eta\{P(\xi, X)Y\}\eta(Z) \right] \xi.
\end{aligned} \tag{6.39}$$

Taking the inner product of equation (6.38) with W , we get

$$\begin{aligned}
g\left(H^f(X, Y)Z, W\right) &= g\left(H(X, Y)Z, W\right) - \alpha \left[\{ g(X, Z)g(\phi Y, W) \right. \\
&\quad \left. - g(Y, Z)g(\phi X, W) \} + \eta(Y)g(\phi X, W)\eta(Z) \right. \\
&\quad \left. - \eta(X)g(\phi Y, W)\eta(Z) \right] \\
&= g\left(H(X, Y)Z, W\right) - g(\alpha X, Z)\alpha - (\phi Y \alpha, W) \\
&\quad + g(Y, Z)\alpha g(\phi X, W) - \eta(Y)\eta(Z)\alpha g(\phi X, W) \\
&\quad + \eta(X)\eta(Z)\alpha g(\phi Y, W).
\end{aligned} \tag{6.40}$$

Using equation (6.95) in the above equation, we get

$$\begin{aligned}
g\left(H^f(X, Y)Z, W\right) &= g\left(H(X, Y)Z, W\right) - g(X, Z)\left[\eta\{P^f(\xi, Y)W\}\right. \\
&\quad \left.- \eta\{P(\xi, Y)W\}\right] + g(Y, Z)\left[\eta\{P^f(\xi, X)W\}\right. \\
&\quad \left.- \eta\{P(\xi, X)W\}\right] - \eta(Y)\eta(Z)\left[\eta\{P^f(\xi, X)W\}\right. \\
&\quad \left.- \eta\{P(\xi, X)W\}\right] + \eta(X)\eta(Z)\left[\eta\{P^f(\xi, Y)W\}\right. \\
&\quad \left.- \eta\{P(\xi, Y)W\}\right],
\end{aligned} \tag{6.41}$$

$$g\left(L^f(X, Y)Z, W\right) = g\left(L(X, Y)Z, W\right), \tag{6.42}$$

where the tensor field L is defined by

$$\begin{aligned}
g\left(L(X, Y)Z, W\right) &= g\left(H(X, Y)Z, W\right) + \{g(X, Z) \\
&\quad - \eta(X)\eta(Z)\}\eta\{P(\xi, Y)W\} - \{g(Y, Z) \\
&\quad - \eta(Y)\eta(Z)\}\eta\{P(\xi, X)W\},
\end{aligned} \tag{6.43}$$

and L^f also is defined similarly. Using equations (6.3), (6.6), (6.33) and (6.43), we have

$$g\left(L(X, Y)Z, W\right) = g\left(A(X, Y)Z, W\right), \tag{6.44}$$

that is $L = A$, similarly $L^f = A^f$. Hence, we obtain

$$g\left(A^f(X, Y)Z, W\right) = g\left(A(X, Y)Z, W\right). \tag{6.45}$$

This completes the proof.

This tensor field A on an LP-Kenmotsu manifold M is invariant under a CL -transformation is said to be the CL -curvature tensor field on M .

6.1.3 CL -flat and CL -semi symmetric on LP-Kenmotsu manifolds

Definition 6.3. An LP-Kenmotsu manifold M is said to be CL -flat if the curvature (CL) tensor field A of type $(1, 3)$ vanishes identically on M .

The CL -flat manifold was introduced by Kota and Nagao (1966) for a Sasakian manifold.

From equation (6.26), we have

$$\begin{aligned}
 R(X, Y)Z = & \frac{1}{n-1} \left[\left(S(Y, Z)X - S(X, Z)Y \right) - \left(\eta(Y)\eta(Z) \right. \right. \\
 & \left. \left. - g(Y, Z) \right) QX + \left(\eta(X)\eta(Z) - g(X, Z) \right) QY \right. \\
 & + \left[\left(S(X, Z) - (n-1)g(X, Z) \right) \eta(Y) - \left(S(Y, Z) \right. \right. \\
 & \left. \left. - (n-1)g(Y, Z) \right) \eta(X) + 2 \left(S(X, Y) \right. \right. \\
 & \left. \left. - (n-1)g(X, Y) \right) \eta(Z) \right] \xi - \left(g(Y, Z) - \eta(Y)\eta(Z) \right) X \\
 & + \left(g(X, Z) - \eta(X)\eta(Z) \right) Y.
 \end{aligned} \tag{6.46}$$

Taking inner product with U and contracting over Y and Z , we get

$$r = n(n-1). \tag{6.47}$$

Theorem 6.4. An LP-Kenmotsu manifold M is said to be CL -flat if the CL -scalar curvature of the manifold satisfied equation (6.47).

Definition 6.4. An LP-Kenmotsu manifold M is said to be CL -semisymmetric if $R(U, W)A = 0$.

From equation (6.26), we have

$$\begin{aligned}
& (n-1)(R(U, W)A)(X, Y)Z \\
&= (n-1)(R(U, W) \cdot R)(X, Y)Z - \{(R(U, W) \cdot S)(Y, Z)X \\
&\quad - (R(U, W) \cdot S)(X, Z)Y\} + [\{\eta(Y)\eta(Z) \\
&\quad - g(Y, Z)\}(R(U, W) \cdot \ddot{Q})(X) - \{\eta(X)\eta(Z) \\
&\quad - g(X, Z)\}(R(U, W) \cdot \ddot{Q})(Y)] - \{(R(U, W) \cdot S)(X, Z)\eta(Y) \\
&\quad - (R(U, W) \cdot S)(Y, Z)\eta(X) + 2(R(U, W) \cdot S)(X, Y)\eta(Z)\}\xi \quad (6.48) \\
&\quad - \{E(X, Z)\eta(Y) - E(Y, Z)\eta(X) + 2E(X, Y)\eta(Z)\}\{\eta(U)W \\
&\quad - \eta(W)U\} + \{g(U, Y)\eta(W) - g(W, Y)\eta(U)\}\{E(X, Z)\xi \\
&\quad - \eta(Z)\xi X\} - \{g(U, X)\eta(W) - g(W, X)\eta(U)\}\{E(Y, Z)\xi \\
&\quad - \eta(Z)\xi Y\} + \{g(U, Z)\eta(W) - g(W, Z)\eta(U)\}\left[2E(X, Y)\xi \right. \\
&\quad \left. - \eta(Z)\{\eta(Y)\xi X - \eta(X)\xi Y\}\right],
\end{aligned}$$

where we put

$$E(X, Y) = S(X, Y) - (n-1)g(X, Y), \quad (6.49)$$

and ξ is given by

$$g(\xi X, Y) = E(X, Y), \quad (6.50)$$

Theorem 6.5. *An LP-Kenmotsu manifold M is said to be CL-semisymmetric if*

and only if

$$\begin{aligned}
(n-1)(R(U, W) \cdot R)(X, Y)Z &= \{(R(U, W) \cdot S)(Y, Z)X \\
&\quad - (R(U, W) \cdot S)(X, Z)Y\} - \left[\{\eta(Y)\eta(Z) \right. \\
&\quad \left. - g(Y, Z)\}(R(U, W) \cdot \ddot{Q})(X) - \{\eta(X)\eta(Z) \right. \\
&\quad \left. - g(X, Z)\}(R(U, W) \cdot \ddot{Q})(Y) \right] + \{(R(U, W) \cdot S)(X, Z)\eta(Y) \\
&\quad - (R(U, W) \cdot S)(Y, Z)\eta(X) + 2(R(U, W) \cdot S)(X, Y)\eta(Z)\}\xi \\
&\quad + \{E(X, Z)\eta(Y) - E(Y, Z)\eta(X) + 2E(X, Y)\eta(Z)\}\{\eta(U)W \\
&\quad - \eta(W)U\} - \{g(U, Y)\eta(W) - g(W, Y)\eta(U)\}\{E(X, Z)\xi \\
&\quad - \eta(Z)\xi X\} + \{g(U, X)\eta(W) - g(W, X)\eta(U)\}\{E(Y, Z)\xi \\
&\quad - \eta(Z)\xi Y\} - \{g(U, Z)\eta(W) - g(W, Z)\eta(U)\} \left[2E(X, Y)\xi \right. \\
&\quad \left. - \eta(Z)\{\eta(Y)\xi X - \eta(X)\xi Y\} \right]
\end{aligned}$$

Proof. Setting $(n-1)(R(U, W) \cdot A)(X, Y)Z = 0$ in equation (6.48), we get the result. $\square$

6.2 3-dimensional quasi Sasakian manifolds

A 3-dimensional quasi Sasakian manifold with structure (ϕ, ξ, η, g) defined in (1.49-1.52) is cosymplectic if and only if $\beta = 0$. If β is a constant, the manifold is referred to as a β -Sasakian manifold, and $\beta = 1$ gives the Sasakian structure. Throughout this section, it is assumed that β is constant. For a 3-dimensional quasi Sasakian manifold, the following properties hold (Olszak, 1986, Akgün and Acet, 2023; Barman and De, 2020):

$$(\nabla_X \phi)Y = -\beta(g(X, Y)\xi - \eta(Y)X), \quad (6.51)$$

R. Kumar, L. Colney, D. Alhwikem (2025), Investigation on the CL -Transformation of 3-dimensional quasi Sasakian manifolds and their Ricci solitons *Mathematics*, **13**, 1543.

$$(\nabla_X \eta)Y = -\beta g(\phi X, Y), \quad (6.52)$$

$$\begin{aligned} R(X, Y)Z &= \left(\frac{r}{2} - 2\beta^2\right) (g(Y, Z)X - g(X, Z)Y) \\ &+ \left(3\beta^2 - \frac{r}{2}\right) \left(g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \right. \\ &\left. + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\right), \end{aligned} \quad (6.53)$$

$$S(X, Y) = \left(\frac{r}{2} - \beta^2\right) g(X, Y) - \left(3\beta^2 - \frac{r}{2}\right) \eta(X)\eta(Y), \quad (6.54)$$

$$\eta(R(X, Y)Z) = \beta^2 (g(Y, Z)\eta(X) - g(X, Z)\eta(Y)), \quad (6.55)$$

$$R(\xi, Y)Z = \beta^2 (g(Y, Z)\xi - \eta(Z)Y), \quad (6.56)$$

$$S(X, \xi) = (r - 4\beta^2)\eta(X), \quad (6.57)$$

$$(L_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi), \quad (6.58)$$

for all vector fields X, Y, Z on M .

Definition 6.5. A quasi Sasakian manifold is termed an η -Einstein manifold if its non-vanishing Ricci tensor S is expressed as:

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

where a and b are smooth functions on the manifold M , and if a and b are constants then it was called an η -Einstein manifold. Also if $b = 0$, then the manifold is called an Einstein manifold (Akgün and Acet, 2023).

From now on, we will denote a 3-dimensional Quasi Sasakian manifold as 3-D Quasi-Sasakian manifold.

6.2.1 Infinitesimal CL -transformation on 3-D Quasi-Sasakian manifolds

Definition 6.6. A vector field V on a 3-D quasi Sasakian manifold M is said to be an infinitesimal CL -transformation (Kumar et al., 2025a) if it satisfies:

$$L_V \begin{pmatrix} h \\ j \ i \end{pmatrix} = \rho_j \delta_i^h + \rho_i \delta_j^h + \alpha(\eta_j \phi_i^h + \eta_i \phi_j^h) \quad (6.59)$$

for a certain constant α , where ρ_i is the components of an 1-form ρ , L_V denotes the Lie derivative with respect to vector field V and $\begin{pmatrix} h \\ j \ i \end{pmatrix}$ is the Christoffel symbol of the Riemannian metric g .

Proposition 6.1. If V is an infinitesimal CL -transformation on a 3-D quasi Sasakian manifold, then the 1-form ρ is closed (Kumar et al., 2025a).

Proof. Contracting equation (6.59) with h and j , we have

$$L_V \begin{pmatrix} h \\ j \ i \end{pmatrix} = 4\rho_i. \quad (6.60)$$

The Lie-derivative of the Christoffel symbols can be expressed as

$$L_V \begin{pmatrix} h \\ j \ i \end{pmatrix} = \frac{\partial}{\partial d_i} \phi, \quad (6.61)$$

we can write as

$$\rho = d\left(\frac{\phi}{4}\right). \quad (6.62)$$

Since $d(df) = 0$ for any smooth function f , so for equation (6.62), we have

$$d\rho = 0$$

Hence the 1-form ρ is closed. □

Theorem 6.6. *If V is an infinitesimal CL-transformation on a 3-D quasi Sasakian manifold M and the scalar curvature $r = 6\beta^2$, then the relation*

$$\beta^2(L_V g)(Y, Z) = -(\nabla_Y \rho)(Z) + \alpha\{\beta(g(Y, Z) - \eta(Y)\eta(Z))\} \quad (6.63)$$

holds for any vector fields Y and Z on M , where α is constant.

Proof. It is known from (Yano and Kon, 1984) that

$$L_V R_{kji}^h = \nabla_k L_V \begin{pmatrix} h \\ j & i \end{pmatrix} - \nabla_j L_V \begin{pmatrix} h \\ k & i \end{pmatrix} \quad (6.64)$$

Employing equations (6.51), (6.52) and (6.59) in the above equation, we get

$$\begin{aligned} (L_V R)(X, Y)Z &= (\nabla_X \rho)(Z)Y - (\nabla_Y \rho)(Z)X + \alpha \left[\beta \left(g(\phi Y, Z)\phi X \right. \right. \\ &\quad \left. \left. - g(\phi X, Z)\phi Y \right) - 2\beta g(\phi X, Y)\phi Z - \beta \left(\eta(Y)g(X, Z) \right. \right. \\ &\quad \left. \left. - \eta(X)g(Y, Z) \right) \xi \right], \end{aligned} \quad (6.65)$$

for any vector fields X, Y and Z on M . Operating η in the above equation, we get

$$\begin{aligned} \eta \left((L_V R)(X, Y)Z \right) &= (\nabla_X \rho)(Z)\eta(Y) - (\nabla_Y \rho)(Z)\eta(X) \\ &\quad - \alpha\beta \left[g(X, Z)\eta(Y) - g(Y, Z)\eta(X) \right]. \end{aligned} \quad (6.66)$$

First we take the Lie derivative of equation (6.55) with respect to vector field V and using the above equation and then interchanging X and Y to Y and ξ respectively, we get

$$\begin{aligned} \beta^2(L_V g)(Y, Z) &= \left(\beta^2(L_V g)(\xi, Z) + (\nabla_\xi \rho)(Z) \right) \eta(Y) - (\nabla_Y \rho)(Z) \\ &\quad + \alpha\beta \left[g(Y, Z) - \eta(Y)\eta(Z) \right]. \end{aligned} \quad (6.67)$$

Again, interchanging Y and Z in the above equation, we have

$$\begin{aligned} \beta^2(L_V g)(Z, Y) &= \left(\beta^2(L_V g)(\xi, Y) + (\nabla_\xi \rho)(Y) \right) \eta(Z) - (\nabla_Z \rho)(Y) \\ &\quad + \alpha\beta \left[g(Z, Y) - \eta(Z)\eta(Y) \right]. \end{aligned} \quad (6.68)$$

Subtracting equation (6.68) from equation (6.67), we get

$$\left(\beta^2(L_V g)(\xi, Z) + (\nabla_\xi \rho)(Z)\right)\eta(Y) = \left(\beta^2(L_V g)(\xi, Y) + (\nabla_\xi \rho)(Y)\right)\eta(Z). \quad (6.69)$$

Setting $Y = \xi$ in the above equation, we get

$$\beta^2(L_V g)(\xi, Z) + (\nabla_\xi \rho)(Z) = \left(\beta^2(L_V g)(\xi, \xi) + (\nabla_\xi \rho)(\xi)\right)\eta(Z). \quad (6.70)$$

From equations (6.70) and (6.67), we get

$$\begin{aligned} \beta^2(L_V g)(Y, Z) &= \left(\beta^2(L_V g)(\xi, \xi) + (\nabla_\xi \rho)(\xi)\right)\eta(Y)\eta(Z) - (\nabla_Y \rho)(Z) \\ &\quad + \alpha\beta \left[g(Y, Z) - \eta(Y)\eta(Z)\right]. \end{aligned} \quad (6.71)$$

Now, we take the inner product of equation (6.65) with a vector field W on M .

Then, by contracting along vector fields X and W , we get

$$(L_V S)(Y, Z) = -2(\nabla_Y \rho)(Z) + \alpha\beta \left[g(Y, Z) - \eta(Y)\eta(Z)\right]. \quad (6.72)$$

Setting $Y = \xi$ in the above equation, we have

$$(L_V S)(\xi, Z) = -2(\nabla_\xi \rho)(Z). \quad (6.73)$$

Taking Lie derivative of equation (6.57) with respect to the vector field V and using the above equation we get

$$-2(\nabla_\xi \rho)(Z) = (r - 4\beta^2)(L_V g)(Z, \xi). \quad (6.74)$$

Replacing $Z = \xi$ in the above equation, we get

$$-2(\nabla_\xi \rho)(\xi) = (r - 4\beta^2)(L_V g)(\xi, \xi). \quad (6.75)$$

Employing the above equation in equation (6.71), we get

$$\begin{aligned} \beta^2(L_V g)(Y, Z) &= \left[-\frac{(r - 4\beta^2)}{2}(L_V g)(\xi, \xi) + \beta^2(L_V g)(\xi, \xi)\right]\eta(Y)\eta(Z) \\ &\quad - (\nabla_Y \rho)(Z) + \alpha\beta \left[g(Y, Z) - \eta(Y)\eta(Z)\right]. \end{aligned} \quad (6.76)$$

Again, setting $r = 6\beta^2$, then we get

$$\beta^2(L_V g)(Y, Z) = -(\nabla_Y \rho)(Z) + \alpha\beta[g(Y, Z) - \eta(Y)\eta(Z)]. \quad (6.77)$$

This completes the proof. $\square$

From equation (6.77), we can state the following:

Theorem 6.7. *An infinitesimal CL-transformation V on a 3-D quasi Sasakian manifold M is not a projective killing vector field unless $\alpha = 0$ and $r = 6\beta^2$.*

Corollary 6.2. *Any infinitesimal CL-transformation V on a 3-D quasi Sasakian manifold M is not necessarily a projective killing vector field.*

6.2.2 CL-transformation on a 3-D quasi Sasakian manifolds

Definition 6.7. *A transformation f on a 3-D quasi Sasakian manifold M with structure (ϕ, ξ, η, g) is said to be a CL-transformation if the Levi-Civita connection ∇ and a symmetric affine connection ∇^f induced from ∇ by f are related by (Kumar et al., 2025a)*

$$\nabla_X^f Y = \nabla_X Y + \rho(X)Y + \rho(Y)X + \alpha\{\eta(X)\phi Y + \eta(Y)\phi X\}, \quad (6.78)$$

where ρ is a 1-form and α is a constant.

We represent the geometric objects R^f and S^f as curvature tensor and Ricci tensor respectively with respect to the affine connection ∇^f .

If f is a CL-transformation on a 3-D quasi Sasakian manifold M , then by virtue of equations (6.78) (1.49), (6.51) and (6.52), the curvature tensor $R^f(X, Y)Z$ of the

connection ∇^f is given by

$$\begin{aligned} R^f(X, Y)Z &= R(X, Y)Z + \{B(X, Y) - B(Y, X)\}Z + B(X, Z)Y \\ &\quad - B(Y, Z)X - \alpha \left[\beta \{g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y\} \right. \\ &\quad \left. - \{\beta g(Y, Z)\eta(X) - \beta g(X, Z)\eta(Y)\}\xi \right], \end{aligned} \quad (6.79)$$

for any vector field X, Y and Z on M , where the tensor field $B(X, Y)$ is defined by

$$\begin{aligned} B(X, Y) &= (\nabla_X \rho)(Y) - \rho(X)\rho(Y) - \alpha^2 \eta(X)\eta(Y) - \alpha \{\eta(X)\rho(\phi Y) \\ &\quad + \eta(Y)\rho(\phi X)\}. \end{aligned} \quad (6.80)$$

From equation (6.79), we get

$$\begin{aligned} g(R^f(X, Y)Z, U) &= g(R(X, Y)Z, U) + \{B(X, Y) \\ &\quad - B(Y, X)\}g(Z, U) + B(X, Z)g(Y, U) \\ &\quad - B(Y, Z)g(X, U) - \alpha \left[\beta \{g(\phi Y, Z)g(\phi X, U) \right. \\ &\quad - g(\phi X, Z)g(\phi Y, U)\} - \beta \{g(Y, Z)\eta(X) \\ &\quad \left. - g(X, Z)\eta(Y)\}g(\xi, U) \right], \end{aligned} \quad (6.81)$$

where U is any vector field on M . By contraction equation (6.81) over X and U , and subsequently adding the resulting expressions, we follow the method outlined in (Koto and Nagao, 1966). This allow us to verify that $B(X, Y)$ is symmetric. Hence, equation (6.80) implies that the 1-form ρ is closed.

Theorem 6.8. *Let A be the tensor field of $(1, 3)$ -type given by*

$$\begin{aligned}
 A(X, Y)Z &= R(X, Y)Z - \frac{1}{2} \left[\left(S(Y, Z)X - S(X, Z)Y \right) \right. \\
 &\quad - \left(\frac{r}{2} - \beta^2 \right) g(Y, Z) \eta(X) \xi + \left(\frac{r}{2} - \beta^2 \right) g(X, Z) \eta(Y) \xi \\
 &\quad \left. + \left(\frac{r}{2} - \beta^2 \right) g(\phi Y, Z) \phi X - \left(\frac{r}{2} - \beta^2 \right) g(\phi X, Z) \phi Y \right] \\
 &\quad - \beta^2 \left[g(Y, Z) \eta(X) \xi - g(X, Z) \eta(Y) \xi \right] + \beta^2 \left[g(\phi Y, Z) \phi X \right. \\
 &\quad \left. - g(\phi X, Z) \phi Y \right].
 \end{aligned} \tag{6.82}$$

where X and Y are vector fields on M . This tensor field A is invariant under CL -transformation and is said to be the CL -curvature tensor field on a 3-D quasi Sasakian manifold M .

Proof. Assume that f is a CL -transformation on a 3-D quasi Sasakian manifold M . Then the relations of equations (6.78-6.81) hold. Since the tensor $B(X, Y)$ is symmetric, equation (6.79) can be rewritten as

$$\begin{aligned}
 R^f(X, Y)Z &= R(X, Y)Z + B(X, Z)Y - B(Y, Z)X \\
 &\quad - \alpha \left[\beta \{ g(\phi Y, Z) \phi X - g(\phi X, Z) \phi Y \} - \beta \{ g(Y, Z) \eta(X) \right. \\
 &\quad \left. - g(X, Z) \eta(Y) \} \xi \right],
 \end{aligned} \tag{6.83}$$

which yields

$$2B(Y, Z) = S(Y, Z) - S^f(Y, Z). \tag{6.84}$$

Substituting equation (6.84) into equation (6.83), we get

$$\begin{aligned}
 R^f(X, Y)Z &= R(X, Y)Z + \frac{1}{2} S(X, Z)Y - \frac{1}{2} S^f(X, Z)Y \\
 &\quad - \frac{1}{2} S(Y, Z)X + \frac{1}{2} S^f(Y, Z)X - \alpha \left[\beta \{ g(\phi Y, Z) \phi X \right. \\
 &\quad \left. - g(\phi X, Z) \phi Y \} - \beta \{ g(Y, Z) \eta(X) - g(X, Z) \eta(Y) \} \xi \right].
 \end{aligned} \tag{6.85}$$

Also from equation (1.32), we have

$$P^f(X, Y)Z = R^f(X, Y)Z - \frac{1}{2} \{ S^f(Y, Z)X - S^f(X, Z)Y \}. \tag{6.86}$$

Substituting equation (6.84) into equation (6.86), we get

$$\begin{aligned} P^f(X, Y)Z &= P(X, Y)Z - \alpha\beta \left[g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \right. \\ &\quad \left. - \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi \right], \end{aligned} \quad (6.87)$$

where P is the projective curvature tensor given by (Weyl, 1918)

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2}\{S(Y, Z)X - S(X, Z)Y\}. \quad (6.88)$$

Replacing X to ξ in equation (6.87), we have

$$P^f(\xi, Y)Z = P(\xi, Y)Z + \alpha\beta \left[g(Y, Z) - \eta(Y)\eta(Z) \right] \xi. \quad (6.89)$$

Operating η on both sides of the above equation, we get

$$\eta(P^f(\xi, Y)Z) - \eta(P(\xi, Y)Z) = \alpha\beta \left[g(Y, Z) - \eta(Y)\eta(Z) \right]. \quad (6.90)$$

Using equation (6.90) in equation (6.87), we get

$$\begin{aligned} P^f(X, Y)Z &= P(X, Y)Z - \alpha\beta \left[g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \right] \\ &\quad + \eta\{P^f(\xi, Y)Z\}\eta(X)\xi - \eta\{P(\xi, Y)Z\}\eta(X)\xi \\ &\quad - \eta\{P^f(\xi, X)Z\}\eta(Y)\xi + \eta\{P(\xi, X)Z\}\eta(Y)\xi, \end{aligned} \quad (6.91)$$

and

$$H^f(X, Y)Z = H(X, Y)Z - \alpha\beta \left[g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \right], \quad (6.92)$$

where

$$H(X, Y)Z = P(X, Y)Z - \eta\{P(\xi, Y)Z\}\eta(X)\xi + \eta\{P(\xi, X)Z\}\eta(Y)\xi. \quad (6.93)$$

$H^f(X, Y)Z$ also define in similar way.

Taking the inner product of equation (6.92) with vector field W , we get

$$g\left(H^f(X, Y)Z, W\right) = g\left(H(X, Y)Z, W\right) - \alpha\beta\left[g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W)\right]. \quad (6.94)$$

Using equation (6.90) in the above equation, we get

$$\begin{aligned} g\left(H^f(X, Y)Z, W\right) &= g\left(H(X, Y)Z, W\right) - \eta\{P^f(\xi, \phi Y)Z\}g(\phi X, W) \\ &\quad + \eta\{P(\xi, \phi Y)Z\}g(\phi X, W) + \eta\{P^f(\xi, \phi X)Z\}g(\phi Y, W) \\ &\quad - \eta\{P(\xi, \phi X)Z\}g(\phi Y, W). \end{aligned} \quad (6.95)$$

It can be written as

$$g\left(L^f(X, Y)Z, W\right) = g\left(L(X, Y)Z, W\right), \quad (6.96)$$

where the tensor field L is defined by

$$\begin{aligned} g\left(L(X, Y)Z, W\right) &= g\left(H(X, Y)Z, W\right) + \eta\{P(\xi, \phi Y)Z\}g(\phi X, W) \\ &\quad - \eta\{P(\xi, \phi X)Z\}g(\phi Y, W), \end{aligned} \quad (6.97)$$

and L^f also is defined similarly. Using equations (6.55), (6.57), (6.88) and (6.97), we have

$$g\left(L(X, Y)Z, W\right) = g\left(A(X, Y)Z, W\right), \quad (6.98)$$

that is $L = A$, similarly $L^f = A^f$. Hence, we obtain

$$g\left(A^f(X, Y)Z, W\right) = g\left(A(X, Y)Z, W\right). \quad (6.99)$$

This completes the proof. □

This tensor field A on a 3-D quasi Sasakian manifold M is invariant under a CL -transformation and is said to be the CL -curvature tensor field on M .

Corollary 6.3. *The CL -curvature tensor field on a 3-D cosymplectic quasi Sasakian*

manifold M is given by

$$\begin{aligned}
 A(X, Y)Z &= R(X, Y)Z - \frac{1}{2} \left[\left(S(Y, Z)X - S(X, Z)Y \right) \right. \\
 &\quad \left. - \frac{r}{2}g(Y, Z)\eta(X)\xi + \frac{r}{2}g(X, Z)\eta(Y)\xi \right. \\
 &\quad \left. + \frac{r}{2}g(\phi Y, Z)\phi X - \frac{r}{2}g(\phi X, Z)\phi Y \right].
 \end{aligned} \tag{6.100}$$

Proof. By setting $\beta = 0$ in equation (6.82), we get the result. $\square$

6.2.3 CL -flat and CL -symmetric on a 3-D Quasi-Sasakian manifolds

Definition 6.8. A 3-D quasi Sasakian manifold M is said to be CL -flat if the CL -curvature tensor field A of type $(1, 3)$ vanishes identically on M .

The CL -flat manifold was introduced by Kota and Nagao (1966) for a Sasakian manifold.

From equation (6.82), we have

$$\begin{aligned}
 R(X, Y)Z &= \frac{1}{2} \left[\left(S(Y, Z)X - S(X, Z)Y \right) - \left(\frac{r}{2} - \beta^2 \right)g(Y, Z)\eta(X)\xi \right. \\
 &\quad \left. + \left(\frac{r}{2} - \beta^2 \right)g(X, Z)\eta(Y)\xi + \left(\frac{r}{2} - \beta^2 \right)g(\phi Y, Z)\phi X \right. \\
 &\quad \left. - \left(\frac{r}{2} - \beta^2 \right)g(\phi X, Z)\phi Y \right] + \beta^2 \left[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \right] \\
 &\quad - \beta^2 \left[g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \right].
 \end{aligned} \tag{6.101}$$

Taking inner product of the above equation with U and contracting over Y and Z , we get

$$S(X, U) = \frac{1}{3} \left[3\beta^2 - 2r \right] g(X, U) + \frac{1}{3} \left[9\beta^2 - 2r \right] \eta(X)\eta(U). \tag{6.102}$$

Hence, from equation (6.102), we have the following theorem:

Theorem 6.9. A 3-D quasi Sasakian manifold M is said to be CL -flat if the Ricci tensor of M satisfied equation (6.102).

With the help of Theorem 6.9, we have the following corollaries:

Corollary 6.4. *A CL-flat 3-D quasi Sasakian manifold M is said to be an η -Einstein manifold if $r \neq \frac{9}{2}\beta^2$.*

Corollary 6.5. *If $r = \frac{9}{2}\beta^2$, then a CL-flat 3-D Quasi-Sasakian manifold M is an Einstein manifold.*

Theorem 6.10. *A 3-D cosymplectic quasi Sasakian manifold M is said to be CL-flat if the Ricci tensor of M satisfied*

$$S(X, U) = -\frac{2}{3}rg(X, U) - \frac{2}{3}r\eta(X)\eta(U). \quad (6.103)$$

Proof. By setting $\beta = 0$ in equation (6.102), we get the result. $\square$

With the help of Theorem 6.10, we have the following corollary:

Corollary 6.6. *A CL-flat 3-D cosymplectic quasi Sasakian manifold is said to be η -Einstein manifold.*

Theorem 6.11. *In a CL-flat 3-D quasi Sasakian manifold M , the scalar curvature of the manifold is constant if and only if $r = \frac{3}{2}\beta^2$.*

Proof.

$$(\nabla_B S)(X, U) = \nabla_B S(X, U) - S(\nabla_B X, U) - S(X, \nabla_B U), \quad (6.104)$$

where B, U and X are vector fields on M . So, taking an orthonormal frame field on (6.102) and contracting along X and U , we get

$$dr(B) = 3\beta^2 - 2r. \quad (6.105)$$

$\square$

Theorem 6.12. *In a CL-flat 3-D cosymplectic quasi Sasakian manifold M , the scalar curvature of the manifold is constant if and only if $r = 0$.*

Proof. Setting $\beta = 0$ in equation (6.105), we get the result. $\square$

Definition 6.9. A 3-D quasi Sasakian manifold M is said to be a CL -symmetric if

$$(\nabla_U A)(X, Y)Z = 0,$$

for all vector fields U, X, Y and Z on M .

Differentiating equation (6.82) covariantly with respect to vector field U and using equations (1.52), (6.51) and (6.52), we get

$$\begin{aligned} (\nabla_U A)(X, Y)Z &= (\nabla_U R)(X, Y)Z - \frac{1}{2} \left[(\nabla_U S)(Y, Z)X - (\nabla_U S)(X, Z)Y \right. \\ &\quad - \frac{1}{2} \left(dr(U)g(Y, Z)\eta(X) - dr(U)g(X, Z)\eta(Y) \right) \xi \\ &\quad + \beta \left(\frac{1}{2} - \beta^2 \right) g(Y, Z)g(\phi U, X)\xi + \beta \left(\frac{r}{2} - \beta^2 \right) g(Y, Z)\eta(X)\phi U \\ &\quad - \beta \left(\frac{1}{2} - \beta^2 \right) g(X, Z)g(\phi U, Y)\xi - \beta \left(\frac{r}{2} - \beta^2 \right) g(X, Z)\eta(Y)\phi U \\ &\quad + \frac{1}{2} dr(U) \left(g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \right) - \beta \left(\frac{r}{2} - \beta^2 \right) g(\phi Y, Z)g(U, X)\xi \\ &\quad + \beta \left(\frac{r}{2} - \beta^2 \right) g(\phi Y, Z)\eta(X)U + \beta \left(\frac{r}{2} - \beta^2 \right) g(\phi X, Z)g(U, Y)\xi \\ &\quad - \beta \left(\frac{r}{2} - \beta^2 \right) g(\phi X, Z)\eta(Y)U \left. \right] + \beta^3 \left[\left(g(Y, Z)g(\phi U, X) - g(X, Z)g(\phi U, Y) \right. \right. \\ &\quad - g(\phi Y, Z)g(U, X) + g(\phi X, Z)g(U, Y) \left. \right) \xi + \left(g(Y, Z)\eta(X) \right. \\ &\quad \left. \left. - g(X, Z)\eta(Y) \right) \phi U + \left(g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y) \right) U \right]. \end{aligned} \tag{6.106}$$

We consider the condition that our manifold is a CL -symmetric i.e. $(\nabla_U A)(X, Y)Z =$

0, then equation (6.106) becomes

$$\begin{aligned}
(\nabla_U R)(X, Y, Z, W) = & \frac{1}{2} \left[(\nabla_U S)(Y, Z)g(X, W) - (\nabla_U S)(X, Z)g(Y, W) \right. \\
& - \frac{1}{2} \left(dr(U)g(Y, Z)\eta(X) - dr(U)g(X, Z)\eta(Y) \right) g(\xi, W) \\
& + \beta \left(\frac{1}{2} - \beta^2 \right) g(Y, Z)g(\phi U, X)g(\xi, W) \\
& + \beta \left(\frac{r}{2} - \beta^2 \right) g(Y, Z)\eta(X)g(\phi U, W) \\
& - \beta \left(\frac{1}{2} - \beta^2 \right) g(X, Z)g(\phi U, Y)g(\xi, W) \\
& - \beta \left(\frac{r}{2} - \beta^2 \right) g(X, Z)\eta(Y)g(\phi U, W) \\
& + \frac{1}{2} dr(U) \left(g(\phi Y, Z)g(\phi X, W) \right. \\
& \left. - g(\phi X, Z)g(\phi Y, W) \right) \\
& - \beta \left(\frac{r}{2} - \beta^2 \right) g(\phi Y, Z)g(U, X)g(\xi, W) \\
& + \beta \left(\frac{r}{2} - \beta^2 \right) g(\phi Y, Z)\eta(X)g(U, W) \\
& + \beta \left(\frac{r}{2} - \beta^2 \right) g(\phi X, Z)g(U, Y)g(\xi, W) \\
& \left. - \beta \left(\frac{r}{2} - \beta^2 \right) g(\phi X, Z)\eta(Y)g(U, W) \right] \\
& - \beta^3 \left[\left(g(Y, Z)g(\phi U, X) - g(X, Z)g(\phi U, Y) \right. \right. \\
& \left. \left. - g(\phi Y, Z)g(U, X) + g(\phi X, Z)g(U, Y) \right) g(\xi, W) \right. \\
& + \left(g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \right) g(\phi U, W) \\
& \left. + \left(g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y) \right) g(U, W) \right].
\end{aligned} \tag{6.107}$$

Contracting the above equation with Y and Z , we get

$$\begin{aligned}
(\nabla_U S)(X, W) = & \frac{1}{2} \left[dr(U)g(X, W) - (\nabla_U S)(X, W) \right. \\
& - \frac{1}{2} \left(3dr(U)\eta(X)\eta(W) - dr(U)\eta(X)\eta(W) \right) \\
& + 3\beta\left(\frac{r}{2} - \beta^2\right)g(\phi U, X)\eta(W) + 3\beta\left(\frac{r}{2} - \beta^2\right)g(\phi U, W)\eta(X) \\
& - \beta\left(\frac{r}{2} - \beta^2\right)g(X, \phi U)\eta(W) - \beta\left(\frac{r}{2} - \beta^2\right)\eta(X)g(\phi U, W) \\
& + \frac{1}{2}dr(U)(-g(\phi^2 X, W)) + \beta\left(\frac{r}{2} - \beta^2\right)g(\phi X, U)\eta(W) \Big] \\
& - \beta^3 \left[3g(\phi U, X)\eta(W) - g(X, \phi U)\eta(W) + g(\phi X, U)\eta(W) \right. \\
& \left. + 3\eta(X)g(\phi U, W) - \eta(X)g(\phi U, W) \right],
\end{aligned} \tag{6.108}$$

which yields

$$dr(U) = 0. \tag{6.109}$$

Hence, we can state the following theorem

Theorem 6.13. *In a CL -symmetric 3-D quasi Sasakian manifold, the scalar curvature is found to be constant.*

6.2.4 CL - ϕ symmetric and CL - ϕ Recurrent on a 3-D quasi Sasakian manifolds

Definition 6.10. *A 3-dimensional Quasi-Sasakian manifold M is said to be a CL - ϕ symmetric if*

$$\phi^2 \left((\nabla_U A)(X, Y)Z \right) = 0, \tag{6.110}$$

for all vector fields U, X, Y and Z on M .

If the vector fields U, X, Y and Z are orthonormal with vector field ξ or U, X, Y and Z are horizontal vector fields, then equation (6.110) is called locally CL - ϕ symmetric. So, using equations (1.49) and (6.110), we have

$$-(\nabla_U A)(X, Y)Z + \eta \left((\nabla_U A)(X, Y)Z \right) \xi = 0,$$

which gives

$$\begin{aligned}
(\nabla_U A)(X, Y)Z &= g\left((\nabla_U A)(X, Y)Z, \xi\right)\xi \\
&= \frac{r}{2}\beta g(Y, Z)g(\phi U, X)\xi - \frac{r}{2}\beta g(X, Z)g(\phi U, Y)\xi \\
&\quad - \frac{1}{2}dr(U)\eta(X)\eta(Y)\eta(Z)\xi + \frac{r}{2}\beta g(\phi U, Y)\eta(X)\eta(Z)\xi \\
&\quad - \frac{1}{2}dr(U)\beta g(\phi U, X)\eta(Y)\eta(Z)\xi - \frac{1}{2}\left[(\nabla_U S)(Y, Z)\eta(X)\xi \right. \\
&\quad \left. - (\nabla_U S)(X, Z)\eta(Y)\xi - \frac{1}{2}\left(dr(U)g(Y, Z)\eta(X) \right. \right. \\
&\quad \left. \left. - dr(U)g(X, Z)\eta(Y)\right)\xi + \beta\left(\frac{r}{2} - \beta^2\right)g(Y, Z)g(\phi U, X)\xi \right. \\
&\quad \left. - \beta\left(\frac{r}{2} - \beta^2\right)g(X, Z)g(\phi U, Y)\xi - \beta\left(\frac{r}{2} - \beta^2\right)g(\phi Y, Z)g(U, X)\xi \right. \\
&\quad \left. + \beta\left(\frac{r}{2} - \beta^2\right)g(\phi Y, Z)\eta(X)\xi + \beta\left(\frac{r}{2} - \beta^2\right)g(\phi X, Z)g(U, Y)\xi \right. \\
&\quad \left. - \beta\left(\frac{r}{2} - \beta^2\right)g(\phi X, Z)\eta(Y)\eta(U)\xi\right] + \beta^3\left[\left(g(Y, Z)g(\phi U, X) \right. \right. \\
&\quad \left. \left. - g(X, Z)g(\phi U, Y) - g(\phi Y, Z)g(U, X) + g(\phi X, Z)g(U, Y)\right)\xi \right. \\
&\quad \left. + g(\phi Y, Z)\eta(X)\eta(U)\xi - g(\phi X, Z)\eta(Y)\eta(U)\xi\right]. \tag{6.111}
\end{aligned}$$

Hence, we can conclude the following theorems:

Theorem 6.14. *A 3-D quasi Sasakian manifold is said to be CL- ϕ symmetric if and only if equation (6.111) is a satisfied.*

Also, we can obtain the condition of locally-symmetric condition by considering the vector fields U, X, Y and Z to be orthogonal to vector field ξ . So equation (6.111) becomes

$$\begin{aligned}
(\nabla_U A)(X, Y)Z &= \frac{r}{2}\beta\left(g(Y, Z)g(\phi U, X)\xi - g(X, Z)g(\phi U, Y)\xi\right) \\
&\quad - \frac{1}{2}\left[\beta\left(\frac{r}{2} - \beta^2\right)\left(g(Y, Z)g(\phi U, X)\xi - g(X, Z)g(\phi U, Y)\xi\right) \right. \\
&\quad \left. - \beta\left(\frac{r}{2} - \beta^2\right)\left(g(\phi Y, Z)g(U, X)\xi - g(\phi X, Z)g(U, Y)\xi\right)\right] \\
&\quad + \beta^3\left[g(Y, Z)g(\phi U, X) - g(X, Z)g(\phi U, Y) \right. \\
&\quad \left. - g(\phi Y, Z)g(U, X) + g(\phi X, Z)g(U, Y)\right]\xi. \tag{6.112}
\end{aligned}$$

Hence, we can have

Theorem 6.15. *A 3-D quasi Sasakian manifold M is said to be locally CL - ϕ symmetric if and only if the equation (6.112) is satisfied.*

Theorem 6.16. *A 3-D cosymplectic quasi Sasakian manifold is said to be CL - ϕ symmetric if and only if*

$$\begin{aligned} (\nabla_U A)(X, Y)Z = & -\frac{1}{2}dr(U)\eta(X)\eta(Y)\eta(Z)\xi - \frac{1}{2}\left[(\nabla_U S)(Y, Z)\eta(X)\xi \right. \\ & - (\nabla_U S)(X, Z)\eta(Y)\xi - \frac{1}{2}\left(dr(U)g(Y, Z)\eta(X) \right. \\ & \left. \left. - dr(U)g(X, Z)\eta(Y)\right)\xi\right]. \end{aligned} \quad (6.113)$$

Proof. Setting $\beta = 0$ in equation (6.111), we get the result. $\square$

With the help of Theorem 6.16, we can have the following corollary:

Corollary 6.7. *A locally CL - ϕ symmetric 3-D cosymplectic quasi Sasakian manifold M is a CL -symmetric manifold*

Proof. Setting $\beta = 0$ in equation (6.112), we get

$$(\nabla_U A)(X, Y)Z = 0. \quad (6.114)$$

$\square$

Definition 6.11. *A 3-D quasi Sasakian manifold M is said to be CL - ϕ recurrent if*

$$\phi^2\left((\nabla_U A)(X, Y)Z\right) = D(U)A(X, Y)Z, \quad (6.115)$$

where D is a 1-form.

By the use of equation (1.49) in equation (6.115), we have

$$-(\nabla_U A)(X, Y)Z + g\left((\nabla_U A)(X, Y)Z, \xi\right)\xi = D(U)A(X, Y)Z. \quad (6.116)$$

Using equations (6.82) and (6.106) and contracting along with vector field X , we get

$$\begin{aligned}
-(\nabla_U A^s)(Y, Z) &= -dr(U)\eta(Y)\eta(Z) - \frac{1}{2} \left[(\nabla_U S)(Y, Z) \right. \\
&\quad - \frac{1}{2} dr(U)\eta(Y)\eta(Z) + \beta r \eta(Y)g(\phi U, Z) \\
&\quad - \frac{1}{2} dr(U)g(Y, Z) - \beta \left(\frac{r}{2} - \beta^2 \right) \eta(Z)g(\phi U, Y) \\
&\quad \left. - \beta \left(\frac{r}{2} - \beta^2 \right) \eta(U)g(\phi Y, Z) + \beta \left(\frac{r}{2} - \beta^2 \right) g(\phi Y, Z) \right] \\
&\quad + \beta^3 \eta(Z)g(\phi U, Y).
\end{aligned} \tag{6.117}$$

Again contracting along the vector fields Y and Z , we get

$$dA^r(U) = 0, \tag{6.118}$$

where A^s and A^r are the CL -Ricci tensor and CL -scalar curvature respectively.

So, we can conclude the following theorem

Theorem 6.17. *A 3-D Quasi Sasakian manifold is said to be CL - ϕ recurrent if and only if the CL -scalar curvature is constant.*

6.2.5 Ricci soliton on the CL transformation of a 3-D quasi Sasakian manifolds

In this section, in order to find the condition of the Ricci soliton, we consider the curvature of the 3-D quasi Sasakian manifold is flat $R(X, Y)Z = 0$. So, equation (6.82) becomes

$$\begin{aligned}
A(X, Y)Z &= -\frac{1}{2} \left[\beta^2 g(Y, Z)\eta(X)\xi - \beta^2 g(X, Z)\eta(Y)\xi \right. \\
&\quad \left. - \beta^2 g(\phi Y, Z)\phi X + \beta^2 g(\phi X, Z)\phi Y \right] \\
&\quad - \beta^2 \left[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \right] + \beta^2 \left[g(\phi Y, Z)\phi X \right. \\
&\quad \left. - g(\phi X, Z)\phi Y \right].
\end{aligned} \tag{6.119}$$

By contraction along the vector field X , we get

$$A^s(Y, Z) = -\frac{1}{2}\beta^2 g(Y, Z), \quad (6.120)$$

where A^s is the CL -Ricci tensor. From (Hamilton, 1988), we know the expression of the Ricci soliton as

$$L_\xi g(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \quad (6.121)$$

So, in our case we can write it as

$$L_\xi g(X, Y) + 2A^s(X, Y) + 2\lambda g(X, Y) = 0. \quad (6.122)$$

Using equations (1.50) and (1.52), we get

$$\begin{aligned} L_\xi g(X, Y) &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \\ &= 0. \end{aligned} \quad (6.123)$$

Using equation (6.123) in equation (6.122), we get

$$A^s(Y, Z) = -\lambda g(Y, Z). \quad (6.124)$$

Theorem 6.18. *If (g, ξ, λ) is a Ricci soliton on the CL -transformation of 3-D quasi Sasakian manifold with the condition that the curvature of the 3-D quasi Sasakian manifold is flat, then the CL -transformation on the 3-D quasi Sasakian manifold is an Einstein manifold.*

Also, by the use of equation (6.120) in equation (6.124), we get

$$\lambda = \frac{1}{2}\beta^2. \quad (6.125)$$

Hence, we have the following corollaries:

Corollary 6.8. *In the CL -transformation of 3-D quasi Sasakian manifold with the condition that the curvature of the 3-D quasi Sasakian manifold is flat, the*

Ricci soliton (g, ξ, λ) of CL-transformation on 3-D quasi Sasakian manifold is found to be shrinking, steady or expanding when $\beta^2 < 0$, $\beta^2 = 0$ or $\beta^2 > 0$ respectively.

Corollary 6.9. *In the CL-transformation of 3-D cosymplectic quasi Sasakian manifold with the condition that the curvature of the 3-D cosymplectic quasi Sasakian manifold is flat, the Ricci soliton (g, ξ, λ) of CL-transformation on 3-D cosymplectic quasi Sasakian manifold is found to be steady.*

Chapter 7

Summary and Conclusion

In the present thesis, we study the certain structures on Para-contact and Lorentzian Para-contact manifolds. The following objectives are used in the study:

1. To study the LP-Sasakian manifold with some curvature tensor.
2. To introduce new connection on LP-Sasakian manifold and normal contact Lorentzian manifold.
3. To introduce some new structures on Lorentzian Para-contact manifold and discuss its geometrical properties.
4. To study the CL -transformation on Lorentzian Para-contact manifold and discuss its symmetric and recurrent properties.

In chapter 1, we provide a comprehensive introduction, which includes a review of relevant literature and an overview of fundamental concepts. These concepts encompass definitions such as topological manifold, smooth manifold, vector field, tangent vector, tangent space, covariant derivative, Lie brackets, Lie derivative, exterior derivative, Riemannian manifold, connection on Riemannian manifold, tangent bundle, vertical and complete lifts, curvature tensor, Ricci ten-

tor, important curvature tensors on Riemannian manifold, Ricci soliton, Einstein manifold and η -Einstein manifold, submanifold, almost contact metric manifold, Sasakian manifold, Kenmotsu manifold, 3-dimensional Quasi-Sasakian manifold, Almost Para-contact metric manifold, Lorentzian manifold, Lorentzian Para-contact metric manifold, LP-Sasakian manifold, LP-Kenmotsu manifold, statistical manifold, Sasakian statistical manifold, statistical submanifold.

In chapter 2, we introduced and analyzed a novel class of semi-symmetric non-metric connection (SSNMC) on LP-Sasakian manifolds, characterized by the inclusion of two distinct 1-forms. We began by establishing the definition of such a connection, denoted by $\check{\nabla}$, and derived its torsion tensor and non-metricity condition, confirming its classification as a SSNMC. We proved the existence and uniqueness of this connection in Theorem 2.1 by verifying that it satisfies the essential structural equations. We explored the implications of SSNMC in LP-Sasakian manifolds, particularly focusing on the curvature tensor, Ricci tensor, and scalar curvature derived from this connection. We demonstrated in Theorem 2.2 that the presence of this connection modifies curvature behaviors, including the Ricci and scalar curvature, while preserving key algebraic identities such as the Bianchi identity. Further, we analyzed the projective curvature tensor and the v -Weyl projective tensor, proving that they maintain important symmetries and coincide with their Riemannian counterparts under specific conditions. Additionally, we constructed the concircular curvature tensor in Theorem 2.4. A significant result of this chapter was the characterization of Einstein manifolds under the SSNMC. In Theorem 2.5, we derived the condition under which the manifold remains Einstein with respect to both the Levi-Civita connection and the SSNMC. We also developed the deformation algebra framework between two linear connections and proved in Theorem 2.6 that every element is a characteristic vector field in the associated deformation algebra. The physical applications of

our geometric framework were explored through the lens of the general theory of relativity. We treated the LP-Sasakian manifold as a fluid spacetime and studied Einstein's field equations without a cosmological constant. Assuming a perfect fluid form of the energy-momentum tensor, we derived the relation $3P_0 = \mu$ in the case of $n = 4$, isotropy, and homogeneity, and established Theorem 2.7 confirming this physical interpretation. Furthermore, we proved in Theorem 2.8 that under specific geometric conditions, the LP-Sasakian manifold endowed with SS-NMC becomes an Einstein spacetime. We extended this to show in Theorem 2.9 that if the perfect fluid spacetime satisfies the Einstein field equations and the 1-forms coincide, the resulting state equation $P_0 + \mu = 0$ represents a phantom barrier model in cosmology. Finally, in Theorem 2.10, we demonstrated that if the manifold satisfies $\check{R} \cdot \check{S} = R \cdot S$ and a structural curvature relation, then the spacetime belongs to the Petrov classification type **D**, **N**, or **O**, confirming its status as a conformally semi-symmetric spacetime. All the theoretical findings were substantiated with a detailed example of a 4-dimensional LP-Sasakian manifold, where the explicit forms of torsion, curvature tensors, and structure tensors were computed, confirming the validity of our results.

In chapter 3, we investigated the geometry of the tangent bundle of a Lorentzian para-Sasakian (LP-Sasakian) manifold endowed with a quarter-symmetric non-metric connection (QSNMC). Our goal was to understand how the key geometric objects—such as connections, curvature tensors, Ricci tensor and scalar curvature, behave when lifted from the base manifold to its tangent bundle. We began by reviewing the structures of LP-Sasakian manifolds and defined the QSNMC using an adapted torsion tensor and non-metric condition. We then employed the complete lift method to transfer various geometric entities—including vector fields, metrics, connections, and tensor fields—from the base manifold to the tangent bundle. Our main focus was on the complete lift of QSNMC and its influence

on geometric quantities. We derived explicit expressions for the lifted torsion tensor, curvature tensor, Ricci tensor, and scalar curvature associated with the lifted QSNMC. Moreover, we established conditions under which the lifted Ricci tensor is symmetric or skew-symmetric. We also proved that the projective Ricci tensor becomes skew-symmetric under specific conditions involving the exterior derivative of a related 1-form. One of our significant results was the characterization of Einstein manifolds in the tangent bundle context. We demonstrated that the lifted structure satisfies the Einstein condition if and only if certain algebraic and differential relations among the geometric objects hold. Finally, we constructed an explicit example of a four-dimensional LP-Sasakian manifold, by calculating its lifted structures.

In chapter 4, we focus on examining the properties of statistical manifolds and Sasakian statistical manifolds within the tangent bundle. This chapter is divided into two sections. The first section explores the lifts of statistical manifolds and their submanifolds in the tangent bundle, equipped with a semi-symmetric non-metric connection (abbreviated as SSNMC) introduced by Yildirim (2022). Our main objective was to derive and prove key theorems concerning the behavior of such lifts and to investigate their curvature properties, including the Gauss, Codazzi, and Ricci equations in the context of SSNMC. We began by revisiting the theory of vertical and complete lifts of fundamental geometric objects, such as vector fields, 1-forms, tensor fields, and affine connections. We then examined the interplay between the statistical connection and the SSNMC under these lift operations, constructing the explicit form of the SSNMC in the tangent bundle and analyzing its compatibility with the metric structure. We proposed and proved several results concerning the curvature tensor of the lifted SSNMC, showing how the complete lifts of the statistical curvature tensor and its dual evolve in the tangent bundle. We also formulated generalized expressions for

the curvature, Ricci, and scalar curvature tensors, characterizing the geometric behavior of lifted statistical manifolds with SSNMC. Moreover, we extended our investigation to statistical submanifolds, proving that the induced connection of the SSNMC in the tangent bundle remains a SSNMC. We derived lifted versions of the Gauss and Weingarten formulas, and established conditions under which the lifted submanifold is totally geodesic or totally umbilical. In particular, we showed that if the second fundamental form vanishes or satisfies a proportionality condition involving the metric, then the submanifold retains its respective property in the lifted geometry. Finally, we demonstrated that the Gauss, Codazzi, and Ricci equations hold in the tangent bundle for both the statistical and the SSNMC structures. We provided a rigorous framework for understanding how these classic geometric relations persist and transform under the influence of SSNMC in lifted statistical manifolds. Our findings offer a comprehensive extension of statistical and differential geometry in tangent bundles, enriching the theoretical foundation for future studies in lifted geometrical structures, especially in contexts involving non-metric connections and information geometry.

The second section investigates the lifts of Sasakian statistical manifolds in the tangent bundle with a semi-symmetric metric connection (abbreviated as SSMC) introduced by Yano (1970). Our objective was to understand the geometric behavior of these lifted structures, particularly focusing on the properties of the connection, curvature tensors, and Ricci tensors under the influence of this connection. We began by reviewing the necessary preliminaries of semi-symmetric metric connections, statistical manifolds, and Sasakian statistical structures. We provided explicit formulas for vertical and complete lifts of vector fields, 1-forms, tensor fields, and affine connections from the base manifold to its tangent bundle. We then constructed the complete lift of the SSMC and showed that it preserves key geometric identities, including torsion and metric compatibility. Following this, we examined the lifted curvature tensors. We established the relationships

between the curvature tensor of the lifted SSMC and those of the lifted torsion-free and dual connections. Moreover, we expressed the lifted Ricci tensor and scalar curvature in terms of their counterparts on the base manifold, augmented by correction terms involving the torsion tensor and statistical structure tensor K . We derived several important theorems characterizing the behavior of curvature and Ricci tensors under the complete lift. In particular, we proved conditions under which the lifted manifold is Ricci-flat and presented explicit formulae for the scalar curvatures r^c and r^{*c} associated with the torsion-free and dual connections, respectively. To validate our theoretical framework, we constructed an explicit example of a three-dimensional Sasakian statistical manifold. Our work contributes to the deeper understanding of differential geometry in tangent bundles, particularly under non-trivial affine connections, and opens avenues for future study in geometric statistics and higher-dimensional contact geometry.

In chapter 5, we explore various properties of the Kenmotsu manifolds within the tangent bundle. This chapter is divided into two sections. The first section focuses on the study of the lifts of the Kenmotsu manifolds in the tangent bundle with a non-symmetric non-metric connection (abbreviated as NSNMC) introduced by Singh et al. (2022). Our primary goal was to analyze the behavior of curvature tensors and Ricci solitons in this context and to formulate new theorems concerning their lifted counterparts. We examined the vertical and complete lifts of fundamental geometrical objects, including vector fields, 1-forms, and affine connections. By applying these lifting techniques to Kenmotsu manifolds, we derived the explicit forms of lifted structures and demonstrated that these lifts preserve essential compatibility conditions with the Kenmotsu framework. A major contribution of our work was to investigate how the NSNMC behaves under lifting and to derive the relationships between the lifted Levi-Civita connection and the lifted NSNMC in the tangent bundle. We found that the complete lift of

the vector field ξ^c remains invariant under both connections, and that the lifted covariant derivative of the Riemannian metric with respect to ξ^c vanishes. Furthermore, we computed and analyzed the lifted forms of several curvature tensors: the Riemannian curvature tensor, the Ricci tensor, and scalar curvature, as well as the conformal, conharmonic, and concircular curvature tensors. We showed that the lifted scalar curvature remains zero and provided conditions under which the lifted conformal and conharmonic tensors coincide. We also explored the properties of Ricci semi-symmetric Kenmotsu manifolds in the tangent bundle, proving that such manifolds become Einstein manifolds under the lift of NSNMC. Specifically, we established the result: $S(X, Y) = -2n\tilde{g}(X, Y)$, $\check{r} = -2n(2n + 1)$, which confirms the Einstein nature of the lifted structure. Lastly, we studied the existence of Ricci solitons on the lifted Kenmotsu manifold. Assuming the potential vector field is aligned with ξ , we showed that the soliton constant is positive: $\lambda = 2n > 0$, implying that the Ricci soliton is of expanding type.

The second section examines the lifts of the Kenmotsu manifolds in the tangent bundle with a generalized Tanaka-Webster connection (abbreviated as GTWC) introduced by Ghosh and De (2017). Our primary objective was to explore the behavior of curvature tensors and geometric structures under the complete lift procedure, particularly in the context of Ricci solitons and various curvature symmetries. We began by recalling the essential background on Kenmotsu manifolds and the construction of the GTWC. We then defined the vertical and complete lifts of vector fields, metric tensors, and affine connections, and extended these definitions to the GTWC setting in the tangent bundle. This allowed us to compute the complete lifts of curvature tensors such as the Riemannian, Ricci, projective, conharmonic, concircular, and Weyl conformal curvature tensors associated with the GTWC. A central result of our work was the derivation of conditions under which the lifted manifold becomes Ricci semi-symmetric, ϕ -projectively semi-symmetric, ξ -projectively flat, or Einstein. We established that

if the lifted projective curvature tensor vanishes, the manifold becomes a hyperbolic space and all other curvature tensors also vanish in the tangent bundle. We also studied the recurrence of curvature tensors and the implications for η -Einstein structures. In particular, we proved that if the lifted Kenmotsu manifold is recurrent with respect to a 1-form D that coincides with the structure 1-form η , then the manifold is necessarily an η -Einstein manifold. Additionally, we investigated the complete lifts of Ricci solitons in the tangent bundle. By analyzing the Ricci soliton equation under GTWC and assuming that the potential vector field is collinear with the lifted structure vector field ξ^c , we demonstrated that the soliton is of steady type and the manifold becomes η -Einstein in the tangent bundle. To illustrate our theoretical findings, we provided a concrete example of a five-dimensional Kenmotsu manifold. Our findings contribute significantly to the understanding of Kenmotsu geometry in tangent bundles and offer a foundation for future studies involving generalized connections, Ricci flows, and higher-dimensional curvature symmetries in differential geometry.

In chapter 6, we investigate the CL -transformation on LP-Kenmotsu manifolds and 3-dimensional quasi Sasakian manifolds. This chapter is structured into two Sections. The first section addresses the CL -transformation on LP-Kenmotsu manifolds, Our main goal was to define and analyze infinitesimal CL -transformations and CL -transformations in this setting, and to construct an invariant tensor field that characterizes the transformed geometry. We began by establishing the conditions under which a vector field on an LP-Kenmotsu manifold generates an infinitesimal CL -transformation. We proved that such a transformation leads to a closed 1-form and derived the associated Lie derivative formulas for the metric and Ricci tensors. These calculations revealed that infinitesimal CL -transformations on LP-Kenmotsu manifolds are not necessarily projective Killing vector fields unless the parameter α vanishes. Furthermore,

we proved that even on Einstein LP-Kenmotsu manifolds, such transformations need not preserve projective properties, unless additional constraints are met. We then defined the notion of a CL -transformation as a mapping that generates a symmetric affine connection from the Levi-Civita connection via a specific formula involving a 1-form and the structure tensor φ . Using this, we derived new expressions for the curvature and Ricci tensors under CL -transformations and constructed a new tensor field A of type $(1, 3)$, which we called the CL -curvature tensor. We demonstrated that this tensor remains invariant under CL -transformations, serving as a fundamental geometric invariant in this context. Our study continued with the derivation of geometric conditions under which the LP-Kenmotsu manifold becomes CL -flat or CL -semisymmetric. We defined CL -flatness as the vanishing of the CL -curvature tensor A and showed that this condition is equivalent to the scalar curvature satisfying $r = n(n - 1)$. We also established a detailed criterion for CL -semisymmetry in terms of the curvature, Ricci, and structure tensors, proving that these conditions yield a rich set of identities connecting different curvature operators. In conclusion, we provided a comprehensive theoretical framework for understanding CL -transformations on LP-Kenmotsu manifolds. Our results generalize earlier work on Sasakian and Kenmotsu manifolds and open the door for future investigations into geometric flows, curvature symmetries, and invariant structures in the setting of Lorentzian paracontact geometry.

In the second section, we investigated the geometric behavior of 3-dimensional quasi Sasakian manifolds under CL -transformations, with a particular focus on the existence of Ricci solitons and the development of new curvature invariants. We introduced a novel tensor, the CL -curvature tensor, which remains invariant under CL -transformations and serves as a powerful tool for analyzing the structure of such manifolds. Through this tensor, we characterized several curvature-related conditions, including CL -flatness, CL -symmetry, CL - ϕ symmetry, and CL - ϕ re-

currence, each expressed through explicit differential equations. A significant part of our investigation was devoted to examining infinitesimal CL -transformations, where we demonstrated that these transformations do not generally yield projective Killing vector fields unless specific curvature constraints are satisfied, namely $\alpha = 0$ and $r = 6\beta^2$. Furthermore, we explored Ricci solitons on CL -transformed quasi Sasakian manifolds and established that, under the assumption of flat curvature, such structures reduce to Einstein manifolds. In this case, the Ricci soliton constant λ is given by $\lambda = \frac{1}{2}\beta^2$. Accordingly, the Ricci soliton is classified as shrinking, steady, or expanding depending on whether β^2 is negative, zero, or positive, respectively. Our findings contribute to the broader understanding of contact metric geometry and pave the way for future research. This includes studying the interaction of CL -transformations with other geometric flows such as Ricci and Yamabe flows, extending the analysis to higher-dimensional quasi Sasakian manifolds, and exploring applications in modified gravity theories such as $f(R)$ and $f(R, T)$ gravity models.

In conclusion, the majority of our work extends the contributions of numerous geometers, including U. C. De, M. N. I. Khan, S. Shenawy, N. Bin Turki, S. K. Chaubey, and A. A. Sheikh. We investigated significant geometrical structures such as complete and vertical lifts, CL -transformations, and the tangent bundle. Additionally, we derived results that may have potential applications in theoretical physics, particularly in the study of general relativity. Several examples were also constructed to illustrate and validate the obtained results. Given the abstract nature of the subject, most of our findings are purely theoretical and lack immediate practical applications. However, we are optimistic that these results will enhance our understanding and support future research in differential geometry.

Appendices

(A) LIST OF RESEARCH PUBLICATIONS/ ACCEPTED/ COMMUNICATIONS

- (1) R. Kumar, **L. Colney** and M. N. I. Khan (2023). Lifts of a semi-symmetric non-metric connection (SSNMC) from statistical manifolds to the tangent Bundle, *Results in Nonlinear Analysis*, **6(3)**, 50-65.
- (2) Rajesh Kumar, **L. Colney**, S. Shenawy and N. Bin Turki (2023). Tangent Bundles Endowed with Quarter-Symmetric Non-Metric Connection (QSNMC) in a Lorentzian Para-Sasakian Manifold, *Mathematics*, **11**, 4163.
- (3) R. Kumar, **L. Colney** and M. N. I. Khan (2023). Proposed Theorems on the Lifts of Kenmotsu Manifolds Admitting a Non-Symmetric Non-Metric Connection (NSNMC) in the Tangent Bundle, *Symmetry*, **15**, 2037.
- (4) R. Kumar, S. Shenawy, N. Bin Turki, **L. Colney** and U. C. De (2024). Lifts of a Semi-Symmetric Metric Connection (SSMC) from Sasakian Statistical Manifolds to Tangent Bundle, *Mathematics*, **12**, 226.
- (5) R. Kumar, S. Shenawy, **L. Colney** and N. Bin Turki (2024). Certain results on tangent bundle endowed with generalized Tanaka Webster connection (GTWC) on Kenmotsu manifolds, *AIMS Mathematics*, **9(11)**, 2024.
- (6) R. Kumar, **L. Colney** and D. Alhwikem (2025). CL-Transformation on 3-Dimensional quasi Sasakian manifolds and their Ricci solitons, *Mathematics*, **13**, 2025.
- (7) R. Kumar and **L. Colney** (2024). Applications of LP-Sasakian manifold admitting a new type of semi-symmetric non-metric connection (SSNMC) to general theory of relativity, Accepted to Southeast Asian Bulletin of Mathematics.

- (8) **L. Colney** and R. Kumar (2025). Soliton on Sasakian Manifold endowed with Quarter-Symmetric Non-Metric Connection (QSNMC) on the tangent bundle, Accepted to Arab Journal of Mathematical Sciences.
- (9) R. Kumar and **L. Colney** (2022). Riemannian manifold containing a new type of semi-symmetric non-metric connection (SSNMC) admitting two 1-forms, (Communicated)
- (10) R. Kumar and **L. Colney** (2022). LP-Sasakian manifold admitting a new type of semi-symmetric non-metric connection (SSNMC) containing two 1-forms,, (Communicated).
- (11) R. Kumar and **L. Colney** (2022). On a Generalized Projective Curvature Tensor of a Semi-Symmetric Non-Metric Connection Admitting Two 1-forms on a Riemannian Manifold, (Communicated).
- (12) **L. Colney** and R. Kumar (2025). Curvature and Solitonic structures of para-Sasakian Manifolds with Shouten van-Kampen connection (SVKC) in the tangernt bundle, (Communicated).
- (13) Y. Li, R. Kumar, **L. Colney**, S. Shenawy and N. Bin Turki (2025). Solitons on the tangent bundle of hyperbolic Kenmotsu manifold endowed with semi-symmetric non-metric connection (SSNMC), (Communicated).
- (14) R. Kumar and **L. Colney** (2025). Some transformation on LP-contact metric manifolds, (Communicated).

(B) CONFERENCES/ SEMINARS/ WORKSHOPS

- (1) Participated in the Short-Term Course (Online) on Research and Publication Ethics held from 1st – 7th October, 2021, organized by University Grants Commission and Human Resource Development Centre, Mizoram University.

- (2) Presented a research paper titled, "A note on some properties on tensor and generalised projective curvature tensor of a new class of semi-symmetric non-metric connection on a Riemannian manifold" at the Physical Science discipline technical session in the 4th Mizoram Science Congress held during 24-25 November 2022, organized by Mizoram Science, Technology, and Innovation Council (MISTIC) in Collaboration with Mizo Academy of Sciences (MAS), Mizoram Science Society (MSS), Science Teachers' Association, Mizoram (STAM), Geological Society of Mizoram (GAM), Mizoram Mathematics Society (MMS), Biodiversity and Nature Conservation Network (BIOCONe) and Mizoram Information & Technology Society (MITS).
- (3) Participated in the International Faculty Development Programme (FDP) on Data Analysis and Machine Learning held during 21st – 25th March 2022, organized by Mizoram University and North-Eastern Hill University, India.
- (4) Participated as a trainee in the offline One Week Training Program on Mathematical Modeling and Computing held during 26th April to 2nd May 2022, organized by Department of Mathematics and Computer Science, Mizoram University.
- (5) Participated in Teacher's Enrichment Workshop on Groups, Rings and Number Theory (TEW 2022) organized by Department of Mathematics and Computer Science, Mizoram University, from 12th – 17th December 2022.
- (6) Participated in the Webinar on "Celebration of International Women's Day", organized by UGC-Women's Studies Centre, Mizoram University, initiated by Ministry of Women & Child Development on 8th March 2022.
- (7) Presented a paper in an International Workshop on Research and Project Development, supported by DBT-BUILDER, Department of Biotechnology,

Gov't of India, held at Pachhunga University College, from 24th – 25th January 2023, organized by Research and Development Cell, Pachhunga University College.

- (8) Presented a research paper titled "On a type of semi symmetric non-metric connection on a Riemannian manifold" in the National Conference on Recent Developments in Mathematics and Computer Science (NCRDMCS23), organized by Mizoram Mathematical Research Association (MMRA) and Department of Mathematics, Gov't Champhai College (GCC) held during 13th to 14th March 2023.
- (9) Presented a paper titled "Introduction to Algebra" in Seminar on High School level Mathematics Curricula 2023, organized by Mizoram Mathematics Society, Mizoram Science, Technology & Innovation Council (MISTIC) and Department of Mathematics, Pachhunga University College, catalyzed and supported by National Council for Science and Technology Communication (NCSTC), DST, Gov't of India, New Delhi during 20th – 22nd March 2023.
- (10) Presented a research paper titled "Applications of LP-Sasakian manifold admitting a new type of SSNMC to general theory of relativity" at the National Seminar on Role of Number Theory and Graph Theory in Mathematical Science, hosted by Pachhunga University College, held during 12th – 13th June 2023.
- (11) Attending the National Workshop on Role of Mathematics Modeling in Environment (NWRMME-2023) during 18th and 19th May 2023, organized by Department of Mathematics and Computer Science, Mizoram University.
- (12) Presented a research paper titled "A study of certain Transformation on Lorentzian Paracontact Metric Manifold" at the International Conference

on Recent Advances in Mathematical, Physical, and Chemical Sciences, Mizoram University, during 21st – 23rd February 2024.

- (13) Presented a paper titled "Geometry" in Seminar on High School level Mathematics Curricula 2024, organized by Mizoram Mathematics Society, Mizoram Science, Technology & Innovation Council (MISTIC) and Department of Mathematics, Pachhunga University College, catalyzed and supported by National Council for Science and Technology Communication (NCSTC), DST, Gov't of India, New Delhi during 4th – 8th March 2024.
- (14) Presented a paper titled "Coordinate Geometry" in Seminar on High School level Mathematics Curricula 2024, organized by Mizoram Mathematics Society, Mizoram Science, Technology & Innovation Council (MISTIC) and Department of Mathematics, Pachhunga University College, catalyzed and supported by National Council for Science and Technology Communication (NCSTC), DST, Gov't of India, New Delhi during 4th – 8th March 2024.
- (15) Presented a research paper titled "Applications of LP-Sasakian manifold to general theory of relativity with semi-symmetric non-metric connection (SSNMC)" in the International Conference on Mathematics and Applications In Science, Technology and Society-2024, held during 9th – 10th May 2024, organized by the Department of Mathematics, Assam Don Bosco University, India.
- (16) Acted as a mentor in the MTTS Mathematics Overture held at Department of Mathematics, Pachhunga University College, during 5th – 6th November 2024, organized by Pachhunga University College and MTTS-Trust.
- (17) Acted as olympiad facilitator in the Mathematics Olympiad Exposure Camp held at Department of Mathematics, Pachhunga University College, from

9th – 13th December 2024, organized by Department of Mathematics, Pachhunga University College, funded by Homi Bhabha Centre for Science Education (HBCSE), TIFR, Mumbai with support from National Board for Higher mathematics (NBHM), Government of India.

- (18) Acted as mentor in the Mathematics Olympiad Exposure Camp held at Department of Mathematics, Pachhunga University College, from 9th – 13th December 2024, organized by Department of Mathematics, Pachhunga University College, funded by Homi Bhabha Centre for Science Education (HBCSE), TIFR, Mumbai with support from National Board for Higher Mathematics (NBHM), Government of India.

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ABSTRACT

A STUDY OF CERTAIN STRUCTURES ON PARA-CONTACT AND LORENTZIAN PARA-CONTACT MANIFOLDS

**AN ABSTRACT SUBMITTED IN PARTIAL FULFILLMENT OF
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**A STUDY OF CERTAIN STRUCTURES ON
PARA-CONTACT AND LORENTZIAN PARA-CONTACT
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ABSTRACT

In 1828, Gauss introduced the concept of Gaussian curvature, a fundamental property of surfaces, in his renowned work “*Theorema Egregium*”. Riemann later extended Gauss’s ideas to higher-dimensional spaces, known as manifolds, laying the foundation for the mathematical structures used in various branches of science and mathematics. This development played a pivotal role in Einstein’s formulation of general relativity, where he utilized Riemannian geometry and its generalization, Finsler geometry, to model spacetime. Beyond relativity, Riemannian geometry has found broad applications in information geometry, probability, statistics, group theory, representation theory, and mathematical analysis.

A notable extension of Riemannian geometry is the study of Lorentzian manifolds, a special class of pseudo-Riemannian manifolds characterized by a metric with signature $(1, n - 1)$. Lorentzian manifolds are central to the mathematical framework of general relativity, where spacetime is modeled as a 4-dimensional Lorentzian manifold with a signature $(3, 1)$ or equivalently $(1, 3)$. Unlike Riemannian manifolds, which have positive-definite metrics, Lorentzian metrics enable the classification of tangent vectors into timelike, null, or spacelike, thus facilitating the study of temporal and spatial phenomena. Such manifolds with signatures $(p, 1)$ or $(1, q)$ are often time-orientable, both locally and globally, making them essential in understanding relativistic and quantum mechanical systems. Moreover, the study of Lorentzian Para-Sasakian manifolds has proven to be highly relevant in scientific and engineering fields, particularly in relativity and quantum mechanics.

Differentiable manifolds, introduced to generalize curves and surfaces to higher-dimensional objects, have been instrumental in advancing numerous disciplines. These manifolds form the backbone of many areas of mathematics and physics, appearing naturally in Lie group theory, homogeneous spaces, classical mechanics, algebraic geometry, relativity theory, and particle physics. Contact geometry, which emerged from the mathematical formalism of classical mechanics, has matured into a robust field with diverse applications, including geometrical optics, integrable systems, thermodynamics, and control theory.

Furthermore, differential geometry, encompassing the study of both Riemannian and Lorentzian manifolds, continues to play a crucial role in a wide array of fields such as chemistry, biophysics, structural geology, engineering, and physics. Its ability to model complex geometrical and physical phenomena has made it an indispensable tool for understanding and solving real-world problems. In this thesis, we attempt to further understand the properties of Riemannian manifold, and para-contact manifold along with Lorentzian para- contact manifolds by considering the following objectives.

1. To study the LP-Sasakian manifold with some curvature tensor.
2. To introduce new connection on LP-Sasakian manifold and normal contact Lorentzian manifold.
3. To introduce some new structures on Lorentzian Para-contact manifold and discuss its geometrical properties.
4. To study the CL-transformation on Lorentzian Para-contact manifold and discuss its symmetric and recurrent properties.

In chapter 1, we give the general introduction which includes the literature reviews and the basic definitions such as topological manifold, smooth manifold, Vector field, tangent vector, tangent space, covariant derivative, Lie brackets, Lie derivative, exterior derivative, Riemannian manifold, connection on Riemannian manifold, tangent bundle, vertical and complete lifts, curvature tensor, Ricci tensor, important curvature tensors on Riemannian manifold, Ricci soliton, submanifold, almost contact metric manifold, Einstein manifold and η -Einstein manifold, Sasakian manifold, Kenmotsu manifold, 3-dimensional Quasi-Sasakian manifold, Almost Para-contact metric manifold, Lorentzian manifold, Lorentzian Para-contact metric manifold, LP-Kenmotsu manifolds, LP-Sasakian manifold, statistical manifold, Sasakian statistical manifold, statistical submanifold.

In chapter 2, we focus on exploring a new class of semi-symmetric non-metric connections on an LP-Sasakian manifold. It is structured into seven sections. The first section provides an introduction to Riemannian manifold and a new class of semi-symmetric non-metric connections admitting two 1-forms. Section 2 provides the existence of the connection. In section 3, the curvature tensors of LP-Sasakian manifolds are examined with respect to the new connection. Section 4 investigates the conditions under which the manifold becomes an Einstein manifold in relation to the new connection. Section 5 delves into the study of deformation algebra. In section 6, we explore the application of LP-Sasakian manifolds in the general theory of relativity with the newly introduced connection. Lastly, section 7 provides an example to validate our findings.

In chapter 3, we study the geometrical properties of LP-Sasakian manifold with quarter symmetric non-metric connection (briefly QSNMC) in the tangent bundle. In the first Section, we introduce a QSNMC defined by Prakash and Narain. The complete lifts of LP-Sasakian manifolds and QSNMC in an LP-Sasakian manifold to its tangent bundle are investigated in Sections 2 and 3. In sections 4 and 5, the complete lifts of the curvature tensor and symmetric and skew-symmetric condition of the Ricci tensor in an LP-Sasakian manifold endowed with QSNMC to its tangent bundle are investigated. The skew-symmetric properties of the projective Ricci tensor and Einstein manifold endowed with QSNMC in an LP-Sasakian manifold to its tangent bundle are studied in Sections 6 and 7. Lastly, an example of the lift of four-dimensional LP-Sasakian manifolds to its tangent bundle is shown in Section 8.

In chapter 4, we study the properties of statistical manifold and Sasakian statistical manifold in the tangent bundle. This chapter is divided into two sections. The first section is devoted to the study of the lifts of statistical manifold and its submanifold in the tangent bundle with semi symmetric non-metric connection (briefly SSNMC) introduced by Yildirim. The second section deals with the study of the lifts of Sasakian statistical manifold in the tangent bundle with semi symmetric metric connection (briefly SSMC) introduced by Yano.

In Chapter 5, we obtain some properties of the Kenmotsu manifold in the tangent bundle, This chapter is divided into two sections, the first section is concerned with the study of the lifts of the Kenmotsu manifold in the tangent bundle with non-symmetric non-metric connection (briefly NSNMC) introduced by Singh et al.. The second section deals with the study of the lifts of the Kenmotsu manifold in the tangent bundle with generalized Tanaka Webster connection (briefly GTWC) introduced by Ghosh and De.

In chapter 6, we study the CL - transformation on LP-Kenmotsu manifolds and 3-dimensional Quasi-Sasakian manifolds. This chapter is again divided into two sections. The first section addresses the CL -transformation on LP-Kenmotsu manifolds, whereas the second section is dedicated to the analysis of the CL -transformation on 3-dimensional quasi-Sasakian manifolds, with particular attention to their Ricci soliton.

Chapter 7 is devoted for summary and conclusion.

A list of reference is given at the end.