

**CERTAIN STRUCTURES AND CONNECTIONS ON
DIFFERENTIABLE MANIFOLDS**

**A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
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**CERTAIN STRUCTURES AND CONNECTIONS
ON DIFFERENTIABLE MANIFOLDS**

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Submitted

**In partial fulfillment of the requirement of the Degree of Doctor of
Philosophy in Mathematics of Mizoram University, Aizawl**

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE



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CERTIFICATE

This is to certify that the thesis entitled “Certain Structures and Connections on Differentiable Manifolds” submitted by Ms. Bidyabati Thangjam (Registration No: MZU/Ph.D./1709 of 07.11.2020) for the degree of Doctor of Philosophy (Ph.D.) of the Mizoram University, embodies the record of original investigation carried out by her under my supervision. She has been duly registered and the thesis presented is worthy of being considered for the award of the Ph.D. degree. This work has not been submitted for any degree from any other university.

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DECLARATION
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I, Bidyabati Thangjam, hereby declare that the subject matter of this thesis is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to do the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in any other University/Institute.

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PREFACE

The present thesis entitled “**Certain Structures and Connections on Differentiable Manifolds** ” is an outcome of the research carried out by me under supervisor, Dr. M. Saroja Devi, Assistant Professor, Department of Mathematics and Computer Science, Mizoram University, Aizawl-796004, Mizoram.

This thesis has been divided into six chapters. The first chapter is the introduction which includes the basic definitions and formulas of differential geometry such as differentiable manifolds, Riemannian manifolds, almost contact metric manifolds, Kenmotsu manifolds, Lorentzian almost contact manifolds, Lorentzian Para-Contact metric manifold, Ricci solitons, Lorentzian manifolds, Relativistic Thermodynamical Fluid spacetime, hypersurface and review of literature.

The second chapter is the study of generalized Tanaka-Webster connection. We examine generalized Tanaka Webster connection with Kenmotsu manifolds and study the conditions of projectively flat, ξ -projectively flat and locally projectively ϕ -symmetric Kenmotsu manifolds with respect to generalized Tanaka Webster connection.

In the third chapter, we deal with the transversal hypersurfaces of Lorentzian Kenmotsu manifolds. We show that all transversal hypersurfaces of Lorentzian Kenmotsu manifolds admit an almost Hermitian structure and (f, g, u, v, δ) -structure. Furthermore, we give some characterizations of totally geodesic transversal hypersurfaces of such manifolds. Finally, we introduce a transversal hypersurface example that admits (f, g, u, v, δ) structure.

The fourth chapter is divided into two section. The first section is devoted to examined Ricci solitons in Lorentzian Para-Kenmotsu manifolds. The study reveals that if $\mathcal{L}_V g + 2S$ is parallel to the Levi-Civita connection associated with g , where V is a given vector field, then (g, V, λ) is a Ricci soliton and also a Ricci soliton in a W_2 -semi-symmetric Lorentzian Para-Kenmotsu manifold is ex-

panding. Furthermore, certain curvature properties of Lorentzian Para-Kenmotsu manifolds admitting Ricci solitons are studied. In the second section, we have discussed how Ricci soliton interact with Relativistic Thermodynamical fluid space-time and its influence on the dynamic behavior and long term evolution of space-time. Moreover, the Ricci soliton metric is used to determine its behavior under different vector fields.

The fifth chapter is divided into two sections. The first section deals with geometrical properties of hyper generalized ϕ -recurrent manifold. The study demonstrates the Ricci semi-symmetry of a hyper generalized ϕ -recurrent LP-Sasakian manifold with non vanishing scalar curvature. Further, we examine quasi generalized ϕ -recurrent LP-Sasakian manifold and their different characteristics. We have also demonstrated that a quasi generalized ϕ -recurrent LP-sasakian manifold is a quasi-generalized Ricci-recurrent manifold. In the second section, we investigate the concept of hyper generalized weakly symmetric Para-Sasakian manifolds. Additionally, we study a hyper generalized weakly symmetric Para-Sasakian manifolds admitting quarter symmetric metric connection.

In Chapter 6, we present the summary and conclusion followed by a list of references.

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Chapter 1

Introduction

1.1 Differentiable Manifold

A topological space M^n is said to be a differentiable manifold of dimension n if it satisfies the following:

- (1) M^n is a Hausdorff space,
- (2) each point $x \in M^n$ has a neighborhood U which is homeomorphic to an open subset V of $\mathbb{R}^n$, i.e., M^n is locally Euclidean,
- (3) M^n is a second countable space i.e., M^n has a countable basis of open sets,
- (4) M^n is endowed with a collection $\{(U_\alpha, \varphi_\alpha) : \alpha \in \Lambda\}$ of coordinate charts such that
 - (i) $\cup_{\alpha \in \Lambda} U_\alpha = M^n$ i.e., $\{U_\alpha : \alpha \in \Lambda\}$ covers M^n ,
 - (ii) the mapping $\varphi_\alpha \circ \varphi_\beta^{-1}$ or $\varphi_\beta \circ \varphi_\alpha^{-1}$ is C^∞ for all $\alpha, \beta \in \Lambda$ and
 - (iii) if (U, φ) is any other coordinate charts for which $\varphi \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi^{-1}$ are C^∞ , then $(U, \varphi) \in \{(U_\alpha, \varphi_\alpha) : \alpha \in \Lambda\}$.

The properties (1)-(3) are the conditions of a topological manifold while property (4) (with (i)-(iii)) is the condition of a differentiable structure. If M^n has a C^∞ -differentiable structure, then M^n is called an analytic manifold.

1.2 Tangent Vector

Let M^n be an n -dimensional differentiable manifold and $p \in M^n$ and $C^\infty(p)$ be the set of all real valued C^∞ function on some neighborhood $\cup$ of p . Let us consider a vector X at p such that

- (i) $X \in M^n$, $f \in C^\infty(p)$ implies that $Xf \in C^\infty(p)$,
- (ii) $X(f + g) = Xf + Xg$, $f, g \in C^\infty(p)$,
- (iii) $X(fg) = f(Xg) + (Xf)g$, and
- (iv) $X(af) = a(Xf)$, $a \in \mathbb{R}$,

then X is called a tangent vector of M^n at p .

1.3 Tangent Space

For tangent vectors X, Y of M^n at p and for $\lambda \in \mathbb{R}$, we define the sum $X + Y$ and the scalar multiple λX by

- (i) $(X + Y)(f) = X(f) + Y(f)$,
- (ii) $(\lambda X)(f) = \lambda(X(f))$, $f \in C^\infty(p)$.

Then $(X + Y)$ and λX are also tangent vectors of M^n at p . Hence, defining the sum and the scalar multiple of tangent vectors at p in this manner, the set of all tangent vectors at p becomes a vector space over $\mathbb{R}$. This vector space is denoted

as T_p and call it the tangent vector space or tangent space of M^n at p .

Let $(x^1, \dots, x^n)$ be a local coordinate system of X and at a point p of X . Let

$$\left(\frac{\partial}{\partial x^i}\right)_p f = \frac{\partial f}{\partial x^i}(p), \quad (i = 1, \dots, n).$$

Then $\frac{\partial}{\partial x^i}$ is a tangent vector at p . If M^n is a manifold of dimension n , then the tangent space T_p is also of dimension n . The basis of T_p with respect to coordinate system $(x^1, x^2, \dots, x^n)$ is $\left(\frac{\partial}{\partial x^i}\right)$, $i = 1, 2, \dots, n$.

Let T'_p be the dual space of T_p whose basis with respect to the basis $\left(\frac{\partial}{\partial x^i}\right)$ is $(dx^1, dx^2, \dots, dx^n)$. We observe that the elements of T_p are the contravariant vectors and elements of T'_p are covariant vectors with respect to the basis of T_p .

1.4 Vector Field

A vector field X on a smooth manifold M^n is a smooth assignment of a tangent vector $X_p \in T_p(M^n)$ at each point $p \in M^n$. Smooth assignment means for all $f \in C^\infty(M^n)$, the function $Xf : M^n \rightarrow \mathbb{R}$ defined by

$$p \rightarrow (Xf)(p) = X_p(f)$$

is a C^∞ function, that is, $Xf \in C^\infty(M)$, where X_p is the real valued function $X_p : C^\infty(p) \rightarrow \mathbb{R}$, $C^\infty(p)$ is the set of smooth functions at $p \in M^n$.

A vector field X on M^n gives rise to a linear map $X : C^\infty(M^n) \rightarrow C^\infty(M^n)$ such that the map $f \rightarrow Xf$ satisfies the following properties:

- i) $X(f + g) = Xf + Xg$,
- ii) $X(\alpha f) = \alpha Xf$,
- iii) $X(fg) = (Xf)g + f(Xg)$

for all $f, g \in C^\infty(M^n)$, $\alpha \in \mathbb{R}$. This implies that X is also derivation of the algebraic $C^\infty(M^n)$. Thus a vector field X is defined as a derivation of the ring

of functions $C^\infty(M^n)$ satisfying (i) - (iii). Thus to each point $p \in M^n$ such a derivation assigns a linear map $X_p : C^\infty(M^n) \rightarrow \mathbb{R}$ defined by $X_p f = (Xf)(p)$ for each $f \in C^\infty(M^n)$ and hence the map $p \in X_p$ assigns a field of tangent vectors.

1.5 Riemannian Manifold

The Riemannian metric enables the definition of geometric concepts such as lengths, angles and distances on smooth manifolds. Much like the inner product in a vector space, the Riemannian metric on a manifold is a smoothly varying inner product on each tangent space.

Definition 1.1. Consider an n -dimensional C^∞ with the tangent space T_p at $p \in M^n$. A real valued, bilinear symmetric, non-singular positive definite function g on the ordered pair X, Y of the tangent vectors T_p at each point p , such that

(i) $g(X, Y)$ is a real number,

(ii) g is symmetric $\implies g(X, Y) = g(Y, X)$,

(iii) g is non-singular i.e. $g(X, Y) = 0$, for all $Y \neq 0 \implies X = 0$,

(iv) g is positive definite i.e. $g(X, X) \geq 0 \quad \forall X \neq 0$, and

(v) $g(aX + bY, Z) = ag(X, Z) + bg(Y, Z)$; $a, b \in \mathbb{R}$,

then g is said to be Riemannian metric tensor or fundamental tensor of type $(0, 2)$.

Then, the manifold M^n with a Riemannian metric g is called a Riemannian manifold and its geometry is called a Riemannian geometry denoted by (M^n, g) or (M, g) or simply by M .

1.6 Connection on Riemannian Manifold

An affine or linear connection on a smooth manifold M is an $\mathbb{R}$ -bilinear mapping

$$\nabla : \chi(M) \times \chi(M) \rightarrow \chi(M),$$

which satisfies the following conditions:

1. $\nabla_{fX}Y = f\nabla_XY$,
2. $\nabla_X(fY) = f\nabla_XY + (Xf)Y$,

for any vector fields $X, Y \in \chi(M)$ and smooth function f . On a Riemannian manifold M of dimension n , the affine connection ∇ is said to be a Levi-Civita connection (Myers, 1935) or a Riemannian connection if it meets the following criteria:

1. ∇ is symmetric or torsion-free, i.e., $\nabla_XY - \nabla_YX = [X, Y]$, and
2. ∇ is metric-compatible, i.e., $(\nabla_Xg)(Y, Z) = 0$, for all $X, Y, Z \in \chi(M)$.

Let M^n be an n -dimensional Riemannian manifold with Riemannian metric g . A linear connection $\bar{\nabla}$ on an n -dimensional Riemannian manifold (M, g) is defined as a quarter-symmetric connection (Golab, 1975), if its torsion tensor T of the connection $\bar{\nabla}$, given by

$$T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y],$$

satisfies the condition

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

where η is a 1-form and ϕ is a (1,1)-tensor field.

When a quarter-symmetric connection $\bar{\nabla}$ satisfies

$$(\bar{\nabla}_X g)(Y, Z) = 0,$$

it is known as a quarter symmetric metric connection. Otherwise, it is referred to as a quarter-symmetric non metric connection. Mandal and De (2015) introduced a special type of quarter-symmetric metric connection defined as

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \quad (1.1)$$

The Tanaka-Webster connection has been introduced by Tanno (1989) as a generalization of well-known connection defined at the end of the 1970's by Tanaka (1976) and independently by Webster (1978). The generalized Tanaka-Webster connection is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(X, Y)\xi - \eta(Y)X - \eta(X)\phi Y, \quad (1.2)$$

where $\tilde{\nabla}$ and ∇ denote the Tanaka-Webster connection and Levi-Civita connection respectively.

1.7 Covariant Derivative

A linear affine connection on M^n is a function

$$\nabla : \chi(M) \times \chi(M) \rightarrow \chi(M)$$

such that

$$\nabla_{fX+gY} Z = f(\nabla_X Z) + g(\nabla_Y Z),$$

$$\nabla_X f = Xf,$$

$$\nabla_X (fY + gZ) = f(\nabla_X Y) + g(\nabla_X Z) + (Xf)Y + (Xg)Z,$$

for an arbitrary vector fields $X, Y, Z \in \chi(M^n)$ and smooth functions f, g . ∇_X is called covariant derivative operator and $\nabla_X Y$ is called covariant derivative of Y with respect to X .

The covariant derivative of a 1-form η is given by

$$(\nabla_X \eta)(Y) = \nabla_X(\eta(Y)) - \eta(\nabla_X Y).$$

1.8 Symmetric Manifolds

Riemannian symmetric spaces have been a primary topic of research in differential geometry theories. It was Cartan, who first initiated the study of Riemannian symmetric spaces and gave its classification (Cartan, 1926). According to him, an n -dimensional Riemannian manifold M is said to be locally symmetric if its curvature tensor R satisfies $R_{hijk,l} = 0$, where “,” represent the covariant differentiation with respect to the metric tensor and R_{hijk} are the components of the curvature tensor of the manifold M . The notion of locally symmetric spaces has been extended by many geometer throughout the year, some of which are Recurrent (Walker, 1950), locally ϕ -symmetric (Takahashi, 1977), pseudo symmetric (Chaki, 1987) and weakly symmetric (Tamásy and Binh, 1989). Sen and Chaki (1967) obtained an expression for the covariant derivative of the curvature tensor while studying conformally flat space of class one with certain curvature restrictions on the curvature tensor, which is as follows:

$$R_{ijk,l}^h = 2\lambda_l R_{ijk}^h + \lambda_i R_{ilk}^h + \lambda_j R_{il k}^h + \lambda_k R_{ij l}^h + \lambda^h R_{ijk}^l, \quad (1.3)$$

whose λ_i is a non-zero covariant vector. Later, Chaki (1987) introduced a manifold whose curvature tensor satisfies (1.3) and called it a pseudo symmetric manifold $(PS)_n$. Extending $(PS)_n$, De and Gazi (2008) introduced almost pseudo symmetric manifold $(APS)_n$. A Riemannian manifold (M^n, g) , $(n > 2)$ is said to

be an almost pseudo symmetric (De and Gazi, 2008) if its curvature tensor R of type $(0, 4)$ satisfies the following relation

$$\begin{aligned} (\nabla_E R)(X, Y, W, V) &= [A(E) + B(E)]R(X, Y, W, V) + A(X)R(E, Y, W, V) \\ &+ A(Y)R(X, E, W, V) + A(W)R(X, Y, E, V) \\ &+ A(V)R(X, Y, W, E), \end{aligned} \quad (1.4)$$

for all $X, Y, W, V \in \chi(M)$, where A, B are non-zero 1-forms given by

$$g(E, \rho) = A(E), \quad g(E, \sigma) = B(E), \quad (1.5)$$

for all vector field $E \in \chi(M)$, where $\chi(M)$ being the Lie algebra of vector fields on M . Further extending the notion of $(APS)_n$, Tamaáßsy and Binh (1989) introduced weakly symmetric manifolds.

1.9 Recurrent Manifolds

Walker (1950) introduced the notion of a locally recurrent Riemannian manifold as an extension to locally symmetric spaces. A non flat Riemannian manifold is said to be locally recurrent (Walker, 1950) if there exists a non-zero 1-form A such that

$$(\nabla_X R)(Y, Z)U = A(X)R(Y, Z)U, \quad (1.6)$$

for all $X, Y, Z, U \in \chi(M)$. Then, De et al. (2003) studied ϕ -recurrent Sasakian manifolds as an extension to locally ϕ -symmetric manifolds (Takashi, 1977). As a weaker version of the locally recurrent Riemannian manifold, Dubey (1979) introduced the notion of the generalized recurrent manifold if its curvature tensor R satisfies

$$\nabla R = A \otimes R + B \otimes G, \quad (1.7)$$

where A and B are non-vanishing 1-forms defined as (1.5) and the tensor G is defined by

$$G(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \quad (1.8)$$

for any $X, Y, Z \in \chi(M)$. Shaikh and Ahmad (2011) extended this concept to generalized ϕ -recurrent Sasakian manifold. Hui (2017) studied generalized ϕ -recurrent generalized (κ, μ) -contact metric manifold and obtained interesting results. A non-flat Riemannian manifold is said to be a generalized ϕ -recurrent manifold if the curvature tensor R satisfies the condition

$$\phi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z + B(W)G(X, Y)Z, \quad (1.9)$$

for any $X, Y, Z, W \in \chi(M)$.

1.10 Almost Contact Metric Manifolds

A $(2n+1)$ -dimensional smooth manifold M is called an almost contact metric manifold if it admits a $(1, 1)$ -tensor field ϕ , a unit vector field ξ (called the Reeb vector field) and a 1-form η such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \cdot \phi = 0, \quad \phi\xi = 0, \quad (1.10)$$

which is equivalent to a reduction of the structural group of the tangent bundle to $U(n) \times 1$ (Sasaki, 1960; Sasaki and Hatakeyama, 1961). A Riemannian metric g is said to be an associated (or compatible) metric if it satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (1.11)$$

$$g(X, \xi) = \eta(X), \quad (1.12)$$

for all $X, Y \in \chi(M)$. An almost contact manifold $M(\phi, \xi, \eta)$ together with a compatible metric g is known as an almost contact metric manifold (Blair, 1976).

Chinea and Gonzalez (1990) obtained a complete classification for almost contact metric manifolds through the study of the covariant derivative of the fundamental 2-form. The fundamental 2-form Φ of an almost contact metric manifold (M, ϕ, ξ, η, g) is defined by

$$\Phi(X, Y) = g(X, \phi Y),$$

for all $X, Y \in \chi(M)$, and this form satisfies $\eta \wedge \Phi^n \neq 0$. This means that every almost contact metric manifold is orientable. Moreover, an almost contact metric manifold is said to be a contact metric manifold (Perrone, 2004) if $d\eta = \Phi$. The following formula holds on a contact metric manifold (Blair, 2002):

$$\nabla_X \xi = -\phi X - \phi hX. \quad (1.13)$$

Further, we define two self-adjoint operators h and ℓ by $h = \frac{1}{2}(\mathcal{L}_\xi \phi)$, where $\mathcal{L}_\xi \phi$ denotes the Lie-derivative of ϕ along ξ , and $\ell = R(\cdot, \xi)\xi$. These operators satisfy

$$h\xi = \ell\xi = 0, \quad h\phi + \phi h = 0, \quad \text{Tr } h = \text{Tr}(h\phi) = 0, \quad (1.14)$$

$$\text{Tr } \ell = \text{Ric}(\xi, \xi) = 2n - \|h\|^2. \quad (1.15)$$

Here, “Tr” denotes trace. When a unit vector ξ is Killing, i.e., $h = 0$ or $\text{Tr } \ell = 2n$, then the contact metric manifold is called K -contact. On the K -contact manifold, the following condition holds:

$$R(X, \xi)\xi = X - \eta(X)\xi. \quad (1.16)$$

An almost contact structure (ϕ, η, ξ) and almost contact manifold M are said to be normal if the almost complex structure on $M \times \mathbb{R}$ defined by $J(X, f\partial_t) = (\phi X - f\xi, \eta(X)\partial_t)$, where f is a real function on $M \times \mathbb{R}$ and t a coordinate on $\mathbb{R}$, is integrable (Blair, 1976, 2002). The necessary and sufficient conditions for the

almost contact structure (ϕ, η, ξ) to be normal is

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where the pair $[\phi, \phi]$ is the Nijenhuis tensor of ϕ defined by

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y],$$

for all $X, Y \in \chi(M)$. A normal almost contact metric manifold is a Sasakian manifold. It is shown that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \quad (1.17)$$

for any $X, Y \in \chi(M)$. A Sasakian manifold is K -contact, but the converse is true only in dimension 3. Olszak (1986) showed that a 3-dimensional almost contact metric manifold M is normal if and only if $\nabla \xi \cdot \phi = \phi \cdot \nabla \xi$ or equivalently

$$\nabla_X \xi = -\alpha \phi X + \beta(X - \eta(X)\xi), \quad (1.18)$$

where $2\alpha = \operatorname{div} \xi$, $2\beta = \operatorname{Tr}(\phi \nabla \xi)$, $\operatorname{div} \xi$ is the divergence of ξ defined by $\operatorname{div} \xi = \operatorname{Tr}\{X \rightarrow \nabla_X \xi\}$ and $\operatorname{Tr}(\phi \nabla \xi) = \operatorname{Tr}\{X \rightarrow \phi \nabla_X \xi\}$. On a 3-dimensional normal almost contact metric manifold, the following relations hold (Olszak, 1986):

$$S(X, \xi) = -X\alpha - (\phi X)\beta - \{\xi\alpha + 2(\alpha^2 - \beta^2)\}\eta(X), \quad (1.19)$$

$$\xi\alpha + 2\alpha\beta = 0, \quad (1.20)$$

for any $X \in \chi(M)$.

1.11 Curvatures on Riemannian Manifold

A conformal transformation is a map that preserves angles. Suppose g and $\bar{g}$ are two metrics on an n -dimensional Riemannian manifold M related by

$$\bar{g}(X, Y) = e^{2\sigma} g(X, Y), \quad (1.21)$$

for all vector fields X, Y on M and some scalar function σ . The angle between any two tangent vectors at a point $p \in M$ remains unchanged under this transformation. In this case, M and $\bar{M}$ (where $\bar{M}$ is the same manifold M , but considered with the new conformally metric $\bar{g}$) are said to be conformally related, and the mapping between them is called a conformal transformation (Obata, 1970). One significant curvature tensor for investigating the intrinsic properties of a Riemannian manifold is the Weyl conformal curvature tensor, introduced by Yano and Kon (1984). This curvature tensor is invariant under conformal transformations. The conformal curvature tensor W of type (1,3) on a $(2n + 1)$ -dimensional Riemannian manifold (M, g) , where $n > 1$, is defined by

$$\begin{aligned} W(X, Y)Z &= R(X, Y)Z - \frac{1}{2n-1} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ &\quad - g(X, Z)QY] + \frac{r}{2n(2n-1)} [g(Y, Z)X - g(X, Z)Y], \end{aligned} \quad (1.22)$$

where r is the scalar curvature of M , Q is the Ricci operator, and S is the Ricci tensor.

An interesting invariant of a concircular transformation is the concircular curvature tensor C . It is defined by (Yano, 1940; Yano and Bochner 1953)

$$\begin{aligned} C(X, Y)Z &= R(X, Y)Z \\ &\quad - \frac{r}{2n(2n+1)} [g(Y, Z)X - g(X, Z)Y]. \end{aligned} \quad (1.23)$$

A harmonic function is defined as a function whose Laplacian is zero. Generally,

harmonic functions are not invariant under conformal transformations. To address this, Ishi (1957) identified the condition under which a harmonic function remains invariant by introducing the conharmonic transformation as a subgroup of conformal transformations (1.21) satisfying

$$\sigma_{,i}^i + \sigma_{,i}\sigma^{,i} = 0.$$

The tensor H , which remains invariant under conharmonic transformation, is known as the conharmonic curvature tensor. For a Riemannian manifold M of dimension $(2n + 1)$, the conharmonic curvature tensor is given by

$$\begin{aligned} H(X, Y)Z &= R(X, Y)Z - \frac{1}{(2n - 1)} \left[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \right. \\ &\quad \left. - g(X, Z)QY \right], \end{aligned} \quad (1.24)$$

for all vector fields X, Y, Z on M . Also, Yano and Kon (1984) defined the projective curvature tensor as

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{(n - 1)} [g(QY, Z)X - g(QX, Z)Y]. \quad (1.25)$$

for any $X, Y, Z \in \chi(M)$.

1.12 Kenmotsu Manifold

To investigate manifolds with negative curvature, Bishop and O'Neill (1969) introduced the concept of a warped product as a generalization of the Riemannian product. Tanno (1969) classified connected $(2n + 1)$ -dimensional almost contact metric manifolds M based on their automorphism groups having the maximum dimension of $(n + 1)^2$. For such manifolds, if the sectional curvature of plane sections containing ξ is a constant k , then there are three categories:

1. If $k > 0$, M is a homogeneous Sasakian manifold with constant holomorphic

sectional curvature.

2. If $k = 0$, M is the global Riemannian product of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature.
3. If $k < 0$, M is a warped product space $\mathbb{R} \times_f \mathbb{C}^n$.

Kenmotsu (1972) studied the third category and derived its geometrical properties which lead to what is now known as the Kenmotsu structure, and the corresponding manifolds are called Kenmotsu manifolds (Janssens and Vanhecke, 1981). Generally, a Kenmotsu manifold is not Sasakian. A Kenmotsu manifold can be characterized as a normal almost contact metric manifold satisfying $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. Kenmotsu demonstrated that such a manifold is locally a warped product $I \times_f N^{2n}$, where I is an open interval with coordinate t , $f = ce^t$ is the warping function for some positive constant c , and N^{2n} is a Kählerian manifold (Kenmotsu, 1972). A Kenmotsu manifold can be further characterized by its Levi-Civita connection ∇ satisfying:

$$\nabla_X \xi = X - \eta(X)\xi, \quad (1.26)$$

$$(\nabla_X \phi)Y = -\eta(Y)\phi X + g(X, \phi Y)\xi, \quad (1.27)$$

for any $X, Y \in \chi(M)$. In a Kenmotsu manifold M , the following properties hold (Kenmotsu, 1972):

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (1.28)$$

$$S(X, \xi) = -2n\eta(X), \quad Q\xi = -2n\xi, \quad (1.29)$$

for any vector fields X, Y on M .

1.13 Lorentzian Almost Contact Manifold

A $(2n+1)$ -differentiable manifold M with a $(1, 1)$ tensor field ϕ , contravariant vector field ξ , an 1-form η , and a Lorentzian metric g which satisfies the following conditions:

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad (1.30)$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y), \quad (1.31)$$

$$\eta(X) = \epsilon g(X, \xi), \quad g(\xi, \xi) = \epsilon = -1, \quad (1.32)$$

$$\text{and } \phi(\xi) = 0, \quad \eta(\phi X) = 0. \quad (1.33)$$

Then (ϕ, η, ξ, g) is referred to as a Lorentzian almost contact structure and M is said to be Lorentzian almost contact manifold.

We consider a 2-form Φ , defined by $\Phi(X, Y) = g(X, \phi Y)$, for any vector fields X and Y , called the fundamental 2-form.

Moreover, a Lorentzian almost contact manifold is classified as normal if

$$N = [\phi, \phi] + 2d\eta \otimes \xi = 0, \quad (1.34)$$

where $[\phi, \phi]$ denotes the Nijenhuis tensor field associated with ϕ .

A Lorentzian almost contact manifold of dimension $(2n+1)$, with (ϕ, ξ, η, g) is identified as Lorentzian almost Kenmotsu manifold if 1-form η is closed, and $dF = -2\eta \wedge \phi$. When such a Lorentzian almost Kenmotsu manifold M is normal; it is called a Lorentzian Kenmotsu manifold (Rosca, 1991).

Let (M, ϕ, ξ, η, g) be a Lorentzian almost contact manifold then M is a Lorentzian Kenmotsu manifold if and only if

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi + \eta(Y)\phi X, \quad (1.35)$$

$$\text{and } (\nabla_X \xi) = -\phi^2 X, \quad (1.36)$$

for any vector fields X and Y on M and ∇ is the Levi-civita connection (Rosca, 1991).

1.14 Lorentzian Para-Contact Manifold

Let M be an $(2n + 1)$ -dimensional differentiable manifold endowed with a tensor field ϕ of the type $(1, 1)$, a vector field ξ , a 1-form η and a Lorentzian metric g satisfying

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad (1.37)$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (1.38)$$

$$g(X, \xi) = \eta(X), \quad (1.39)$$

for any arbitrary vector fields X and Y , then M is called a Lorentzian Para (or LP)-contact manifold and the structure (ϕ, ξ, η, g) is called the Lorentzian Para-contact structure (Matsumoto, 1989).

Let M be a Lorentzian Para-contact manifold with structure (ϕ, ξ, η, g) . Then it satisfies

$$(a) \phi(\xi) = 0, \quad (b) \eta(\phi X) = 0, \quad (c) \text{rank } \phi = 2n, \quad (1.40)$$

A Lorentzian Para-contact manifold is called a Lorentzian Para-Sasakian manifold if (Matsumoto and Mihai, 1988)

$$\nabla_X \xi = \phi X, \quad (\nabla_X \eta)(Y) = g(X, \phi Y) = g(\phi X, Y), \quad (1.41)$$

$$(\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \quad (1.42)$$

where ∇ denotes the covariant differentiation with respect to g .

Also, in an LP -Sasakian manifold the following relation holds

$$\eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \quad (1.43)$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad (1.44)$$

$$S(X, \xi) = 2n\eta(X), \quad (1.45)$$

$$S(\phi X, \phi Y) = S(X, Y) + 2n\eta(X)\eta(Y), \quad (1.46)$$

$$\text{where } S(X, Y) = g(QX, Y). \quad (1.47)$$

1.15 Lorentzian Para-Kenmotsu Manifold

A $(2n+1)$ -differentiable manifold M with a $(1, 1)$ tensor field ϕ , contravariant vector field ξ , an 1-form η , and a Lorentzian metric g is referred as a Lorentzian almost Para-contact manifold (Matsumoto, 1989) if g satisfies the following conditions:

$$\phi^2 X = X + \eta(X)\xi, \quad \phi\xi = 0, \quad (1.48)$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (1.49)$$

$$\eta(\xi) = -1, \quad g(X, \xi) = \eta(X), \quad (1.50)$$

$$\text{and } \eta(\phi X) = 0. \quad (1.51)$$

A Lorentzian almost para-contact manifold M is a Lorentzian Para-Kenmotsu manifold if it satisfies (Haseeb and Prasad, 2020)

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \quad (1.52)$$

for any vector fields X and Y on M and ∇ is the operator of covariant differentiation with respect to the Lorentzian metric g .

In Lorentzian Para-Kenmotsu manifolds, the following relations hold (Pandey et

al., 2021):

$$\nabla_X \xi = -X - \eta(X)\xi, \quad (1.53)$$

and

$$(\nabla_X \eta)Y = -g(X, Y) - \eta(X)\eta(Y). \quad (1.54)$$

In addition to these, the following relations also hold (Haseeb and Prasad, 2021):

$$\eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \quad (1.55)$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \quad (1.56)$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad (1.57)$$

$$S(X, \xi) = 2n\eta(X), \quad Q\xi = 2n, \quad (1.58)$$

$$S(\phi X, \phi Y) = S(X, Y) + 2n\eta(X)\eta(Y), \quad (1.59)$$

for all vector fields X and Y on M .

1.16 Para-Sasakian Manifold

Consider M as $(2n + 1)$ -dimensional almost para-contact metric manifold admitting an almost para-contact metric structure (ϕ, ξ, η, g) , where ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric. Then

$$\phi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad (1.60)$$

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \quad (1.61)$$

for all vector fields X, Y on TM .

An almost Para-contact manifold is called a Para-Sasakian manifold if

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X, \quad (1.62)$$

$$d\eta = 0 \quad \text{and} \quad \nabla_X \xi = -\phi X. \quad (1.63)$$

In a Para-Sasakian manifold equipped with the structure (ϕ, ξ, η, g) , the following relations also hold (Singh and Kishor, 2022)

$$(\nabla_X \eta)Y = g(X, \phi Y), \quad (1.64)$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \quad (1.65)$$

$$R(\xi, X)Y = -g(X, Y)\xi + \eta(Y)X, \quad (1.66)$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (1.67)$$

$$S(X, \xi) = -2n\eta(X), \quad (1.68)$$

$$Q\xi = -2n\xi, \quad (1.69)$$

for all $X, Y, Z \in TM$, where S is the Ricci tensor, R is a Riemannian curvature tensor.

1.17 Hypersurface

Let M' and M be smooth manifolds such that $\dim(M') \leq \dim(M)$. Let $F : M' \rightarrow M$ be a smooth map and let p be a point in M' . We say that F is an immersion at p if the differential map $d_p(F) : T_p(M') \rightarrow T_{F(p)}(M)$ is injective and F is an immersion if it is an immersion at every p in M' .

Definition 1.2. Suppose (M, g) is a Riemannian manifold of dimension m , M' is a manifold of dimension n and $\iota : M' \rightarrow M$ is an immersion. If M' is given the induced Riemannian metric $g' := \iota^*g$, then ι is said to be an isometric immersion.

If, in addition, ι is injective, so that M' is an immersed submanifold of M , then M' is said to be a Riemannian submanifold of M . If $\dim(M) - \dim(M') = 1$, then the submanifold M' is said to be a hypersurface.

In recent decades, the geometry of submanifolds has garnered substantial interest due to its significant applications in both applied mathematics and theoretical physics. Submanifolds constitute a fundamental concept in the realm of differential geometry, serving as essential building blocks for understanding the geometric properties of higher-dimensional spaces. Essentially, a submanifold is a subset of a manifold that retains its own intrinsic manifold structure, inheriting certain geometric properties from the ambient space. Notably, the study of invariant submanifolds offers insights into the properties of non-linear autonomous systems and such submanifolds inherit “almost all geometric properties of the ambient manifold”. Another crucial type of submanifold is the totally geodesic submanifold which is distinguished by the fact that geodesics of the ambient manifold remain geodesics within these submanifolds. The concept of geodesics plays a pivotal role in the theory of relativity.

Consider that M' is an orientable hypersurface, ∇ and D denote the Levi-Cevita connection and induced Levi-Cevita connection on M and M' , with N being a vector field normal to M' . Then, the Gauss and Weingarten formulas are given by

$$\nabla_X Y = D_X Y + h(X, Y) \tag{1.70}$$

and

$$\nabla_X N = -A_N X, \tag{1.71}$$

for all $X, Y \in TM$, where h denotes the second fundamental form and A_N represents the shape operator associated with the normal vector N . These are related

by:

$$g'(A_N X, Y) = h(X, Y). \quad (1.72)$$

where g' is the induced metric on the transversal hypersurface M' . The hypersurface M' is totally geodesic if the second fundamental form vanishes identically. A point p of M' is called umbilical if, at that point, $h(X, Y)|_p = \rho g(X, Y)|_p$, $\forall X, Y \in TM$ where $\rho \in \mathbb{R}$ and depends on M' . The hypersurface M' is said to be totally umbilical if every point of M' is umbilical, that is, $h = \zeta g$, where ζ is a smooth function (Srivastava et al., 2020).

1.18 Lorentzian Manifold

An n -dimensional pseudo-Riemannian (or sometimes called semi-Riemannian) manifold is a pair (M, g) , where M is an n -dimensional smooth manifold and g is a symmetric, nondegenerate 2-tensor field on M (called the metric). A pseudo-Riemannian manifold is said to be Riemannian if g has a signature $(+, \dots, +)$, and is said to be Lorentzian if g has signature $(-, +, \dots, +)$ or $(+, -, \dots, -)$.

1.19 Relativistic Thermodynamical Fluid Space-time

The exploration of spacetime geometry and its connections to physical phenomena has been a captivating subject of investigation spanning the domains of both physics and mathematics. Over time, the intricate interplay between geometry and gravity has evolved into a vibrant field of research and give rises to various mathematical frameworks which are aimed to understand the complex interrelationships. With its profound implications in mathematical physics and cosmology, the study of spacetime has experienced significant growth in recent

years. Albert Einstein’s iconic theory of general relativity (Einstein, 1915) serves as a pivotal bridge between the physical attributes of spacetime and its underlying geometrical structure which is expressed by the field equation

$$S - \frac{r}{2}g + \Lambda g = \frac{8\pi G}{c^4}T, \quad (1.73)$$

where Λ represents the cosmological constant, while c represents the speed of light in a vacuum, G , S , r and T are the gravitational constant, the Ricci tensor, the scalar curvature and the energy-momentum tensor respectively. The study of spacetime and thermodynamics are pillars of modern physics. Previously, these two fields were studied separately: spacetime in the study of general relativity and thermodynamics in the study of the behavior of macroscopic systems in statistical mechanics. However, recently some advancements have been made, showing fascinating links between these two areas. One fascinating concept that emerged is that of “thermodynamical fluid spacetime”, in which spacetime is modeled as thermodynamical fluid.

Originating from Albert Einstein’s theory of relativity, spacetime is a four-dimensional continuum with three spatial dimensions (length, width, and height) and one temporal dimension (time). In general relativity and cosmology, spacetime is presented as a connected four-dimensional pseudo-Riemannian manifold equipped with a Lorentzian metric g that has the signature $(-, +, +, +)$ (Novello and Reboucas, 1978). Therefore, the connected four-dimensional time-oriented Lorentzian manifold is modeled using the thermodynamical spacetime manifold (Hawking and Ellis, 1973).

Thermodynamics branches from physics and it deals with heat, work, temperature, and the statistical behavior of systems with many particles. In thermodynamic spacetime, spacetime itself is treated as a thermodynamic system. So, for the thermodynamic aspect of the spacetime manifold from which Einstein’s equation is derived, the fundamental relationship between the horizon area and

entropy, as well as their proportionality (Bekestein, 1974) is as follows:

$$\delta Q = \mathcal{T} d\mathbb{S}, \quad (1.74)$$

where $\mathcal{T}$ is the temperature, $\mathbb{S}$ is the entropy and Q is the heat. The energy flux is denoted by δQ .

Jacobson (1995) proposed that the Einstein equation can be derived as a constitutive equation for the equilibrium (1.74) of the thermodynamical spacetime point of view with a cosmological constant, which can be given as:

$$S - \frac{1}{2}rg + \Lambda g = \frac{2\pi}{\hbar\alpha}T. \quad (1.75)$$

Here, the length $\alpha^{-\frac{1}{2}}$ is twice the Planck length $(\hbar G)^{\frac{1}{2}}$, where Newton's constant, $G = (4\hbar\alpha)^{-1}$ is determined by the constant of proportionality α between the entropy and the area (Bekestein, 1974).

Definition 1.3. *A four-dimensional Lorentzian spacetime manifold that includes thermal energy density, thermal flux and thermal stress is called thermodynamical fluid spacetime (TFS) (Havas and Swenson, 1979). Heat is described by the general theory of relativity energy tensor T_H , often known as the thermal energy tensor (Eckart, 1940).*

So, in a TFS, the thermodynamic matter tensor T is of the form (Eckart, 1940):

$$T = (\rho + \epsilon)\eta \otimes \eta + 2\eta \otimes \gamma + \tau, \quad (1.76)$$

where ρ is an effective density, ϵ is the thermal energy density, τ is the thermal stress tensor and γ is the thermal flux. Moreover, for a vector field X , we have $g(X, \xi) = \eta(X)$ and $g(X, \zeta) = \gamma(X)$, where ξ and ζ are orthogonal vector fields.

1.20 Lie Derivative

For any smooth vector fields X and Y on a manifold M , let θ be the flow of X , we define a vector $(\mathcal{L}_X Y)_p$ at each $p \in M$, called the Lie Derivative of Y with respect to X at p , by

$$(\mathcal{L}_X Y)_p = \left. \frac{d}{dt} \right|_{t=0} (\theta_{-t})_* Y_{\theta_t(p)} = \lim_{t \rightarrow 0} \frac{(\theta_{-t})_* Y_{\theta_t(p)} - Y_p}{t},$$

provided the derivative exists.

1.21 Some Vector Fields and η -Einstein Manifold

Definition 1.4 (Barros and Ribeiro, 2012). *A smooth vector field X on a Riemannian manifold is said to be a conformal vector field if there exists a smooth function ψ on M that satisfies*

$$\mathcal{L}_X g = 2\Omega g.$$

We say that X is non-trivial if X is not Killing, that is, $\psi \neq 0$.

Definition 1.5 (Yano, 1944). *A Lorentzian manifold features a torse-forming vector field (TFV) denoted as ζ . This vector field satisfies*

$$\nabla_X \zeta = \omega X + \gamma(X)\zeta, \tag{1.77}$$

where ω represents a scalar function and γ is a non-zero 1-form, for any $X \in \chi(M)$.

A time-like TFV unit is $\xi = X$ on a n -dimensional Lorentzian manifold M

and has the following form (Yano, 1944):

$$\nabla_X Y = \omega[X + \eta(X)Y], \quad (1.78)$$

where η is a 1-form and $g(X, Y) = \eta(Y)$ for all Y .

Definition 1.6 (Hinterleitner and Kiosak, 2008). *On a Lorentzian manifold M , a vector field ϕ is termed as $\phi(\text{Ric})$ -vector field when it satisfies the following equation*

$$\nabla_X \phi = \mu SX, \quad (1.79)$$

where ∇ represents the Levi-Civita connection and μ is a constant. In case of a non-zero μ , the vector field ϕ is specifically identified as a proper $\phi(\text{Ric})$ -vector field. Conversely, in the case when $\mu = 0$ in (1.79), the vector field ϕ is categorized as covariantly constant.

Definition 1.7 (Haseeb and Prasad, 2021). *A $(2n+1)$ -dimensional manifold M is called an η -Einstein manifold if its Ricci tensor S satisfies the following equation:*

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \quad (1.80)$$

where a and b are scalars.

1.22 Ricci Solitons

Henri Poincaré, in 1904, wondered if there is a way to recognize a three dimensional sphere while the necessary measurements are operated from inside the shape, which later becomes the famous Poincaré conjecture. Since then, to prove or disprove the conjecture has become the core agenda of many researchers. The study of geometric flows amongst other methods is one of the main attraction for geometers due to its application in mathematical physics and it helps us to

understand shapes in three or more dimensional spaces. One such significant flow is the Ricci flow introduced by Hamilton (1988), who used it to prove a three-dimensional sphere theorem (Hamilton, 1982) and which later become the heart of proof of the Poincaré conjecture by Perelman (2002). The Ricci soliton (Ghosh, 2011) on a Riemannian manifold (M, g) are the self-similar solutions to Ricci flow and is defined by

$$\frac{1}{2}\mathcal{L}_V g + S = \lambda g, \quad (1.81)$$

where $\mathcal{L}_V g$ denotes the Lie-derivative of g along potential vector field V , S is the Ricci tensor of M^{2n+1} and λ , a real constant. When the vector field V is the gradient of a smooth function f on M^{2n+1} , that is, $V = \nabla f$, then we say that Ricci soliton is a gradient which is given by

$$S + \nabla^2 f + \lambda g = 0, \quad (1.82)$$

where $\nabla^2 f$ is the Hessian of f , the second derivative of f with respect to the metric g . Gradient Ricci soliton represents self similar solution to the Ricci flow, which mean they evolve under the flow of diffeomorphisms and scaling. This property makes them critical in understanding the long term behavior of the Ricci flow.

1.23 Review of Literature

Tanno (1989) introduced Tanaka-Webster connection. This connection is a generalization of the well-known connection which were defined by Tanaka (1976) and independently by Webster (1978). This connection was found to be coincided with the Tanaka-Webster connection if associated CR-structure is integrable. The generalized Tanaka-Webster connection were studied with Kenmotsu manifolds and obtained various results by geometers like Acet et al. (2013), Ghosh and De (2017). Kazan and Karadag (2018) investigated the curvature tensors of a trans-

Sasakian manifold with respect to this connection and also obtained some special curvature conditions of a trans-Sasakian manifold with respect to generalized Tanaka-Webster connection. Kiran Kumar et al. (2019) investigated the concircular curvature tensor of Kenmotsu manifolds admitting a generalized Tanaka-Webster connection. They demonstrated that if the curvature tensor vanishes, the manifold is locally isometric to hyperbolic space $H^{2n+1}(-1)$. $N(k)$ -contact metric manifolds with a generalized Tanaka-Webster connection was studied by Ünal and Altin (2020). They obtained some curvature properties and conditions under which these manifolds become generalized Sasakian space forms. Erdogan and Alo (2022) defined the Tanaka-Webster connection on Lorentzian Sasakian manifolds, introducing ξ -conformally and ξ -pseudo projectively flat structures, and proving that such manifolds are η -Einstein if the connection is ξ -conformally flat. Hyperbolic Ricci solitons on nearly cosymplectic manifolds endowed with the Tanaka-Webster connection was studied by Ayar (2023). Mburu and Wahome (2024) analysed the W_8 -curvature tensor on Kenmotsu manifolds admitting generalized Tanaka-Webster connection, establishing relationships between various curvature conditions and Einstein manifolds.

Zamkovoy (2009) introduced a connection called Zamkovoy connection which plays the role of (generalized) Tanaka-Webster connection in paracontact geometry. This connection was studied by Mandal and Das (2020, 2021) on LP-Sasakian manifolds and obtained various results on Projective curvature tensor and Conharmonic curvature tensor, respectively. Prakasha et al. (2021) investigated the Zamkovoy connection on a para-Kenmotsu manifold.

Tashiro (1963) initiated the geometry of hypersurfaces in a complex manifold. After this, Blair and Ludden (1969) started the study of hypersurfaces in almost contact manifolds. Later, Yamaguchi (1969) investigated hypersurfaces in Sasakian manifold. Goldberg and Yano (1970) found various result in non-invariant hypersurfaces of almost contact manifolds. Hypersurface of an almost

r -paracontact Riemannian manifolds with quarter symmetric metric connection were explored by Ahmad et al. (2009a, 2009b). Yano and Okumura (1970) defined (f, U, V, u, v, λ) -structure and found that a hypersurface of an almost contact manifold admits (f, U, V, u, v, λ) -structure. Yano et al. (1972) further introduced the concept of transversal hypersurfaces in almost contact manifolds, describing it as a hypersurface that does not contain the vector field ξ defining the almost contact structure. Prasad and Tripathi (2003) investigated Transversal hypersurfaces of Kenmotsu manifolds, while Prasad (2007) explored transversal hypersurfaces with (f, g, u, v, λ) -structure manifolds. Again, Prasad et al. (2008) investigated transversal hypersurfaces of almost hyperbolic contact metric manifold and found that it admits an almost product semi-Riemannian structure. Later, Rahaman (2013) proved that transversal hypersurfaces of almost hyperbolic contact manifold with a quarter symmetric non metric connection admits an almost product structure. He also found some properties of transversal hypersurfaces with a quarter symmetric connection. Prasad and Yadav (2016) studied Transversal hypersurfaces with (f, g, u, v, λ) -structure of a nearly trans-Sasakian manifold. Later, transversal hypersurfaces of $(LCS)_n$ -manifolds were studied by Ahmad and Shaikh (2018). Srivastava et al. (2020) investigated some results on transversal hypersurfaces of a Parasasakian manifold. Recently, Prasad and Gupta (2025) studied Transversal hypersurfaces on Lorentzian para-Kenmotsu manifolds and derived some results allied with relationship between induced almost product Lorentzian metric structure (J, G) and induced Lorentzian $(f, g, \mu, \nu, \lambda)$ -structure.

A Ricci soliton is a self-similar solution to the Ricci flow, which was introduced by Hamilton (1982). Ivey (1993) showed that there are no compact three-dimensional Ricci solitons other than spaces of constant curvature. The Ricci flow has been widely applied in various fields, including Ricci flow gravity (Graf, 2007), nonlinear reaction-diffusion systems in biology, chemistry, and physics (Ivance-

vic and Ivancevic, 2011), brain surface conformal parametrization (Wang et al., 2012), and economic modeling (Sandhu et al., 2016). Wylie (2008) established that complete shrinking Ricci solitons possess a finite fundamental group, while Cao and Zhou (2010) analyzed complete shrinking Ricci solitons in further detail. Munteanu and Wang (2017) demonstrated that shrinking Ricci solitons with positive curvature exhibit compactness. Additionally, Ricci solitons associated with Jacobi-type vector fields were investigated by Deshmukh (2012), and trivial Ricci solitons were characterized by Deshmukh et al. (2020). Ghosh (2013, 2019, 2020) studied Ricci solitons on η -Einstein Kenmotsu metrics and Kenmotsu manifolds, as well as Ricci almost solitons and almost Yamabe solitons on Kenmotsu manifolds. Almost Ricci solitons and its application were investigated by Duggal (2017).

In recent years, many geometers have studied Ricci solitons on contact and almost contact manifolds. Sharma (2008) investigated Ricci solitons in Sasakian manifolds and proved that if the metric is a Ricci soliton, then it must be Einstein. Ingalahalli and Bagewadi (2012) studied Ricci solitons on α -Sasakian manifolds. Pokhariyal et al. (2018) found some results on Trans-Sasakian manifolds. Ayar and Demirhan (2019) provided basic information about Ricci solitons on nearly Kenmotsu manifolds and obtained some structures on this manifold satisfying a semi-symmetric metric connection. Later, Shah (2020) studied Ricci solitons in Lorentzian Para-Sasakian manifolds, while Chen et al. (2022) studied Ricci solitons and certain related metrics on a three-dimensional Trans-Sasakian manifolds. Many other geometers had also studied Ricci solitons on various manifolds. Pandey et al. (2021) investigated the geometric properties of η -Ricci solitons on Lorentzian para-Kenmotsu manifolds.

Recently, many geometers investigated Ricci solitons to various geometric settings. For instance, Ali and Ahsan (2014) studied spacetime manifold in GTR. In addition, the perfect fluid spacetimes were depicted by Blaga (2020), with η -

Ricci soliton and η -Einstein solitons. Venkatesha and Kumara (2019) used Ricci soliton to study perfect fluid spacetime with torse forming vector field. Recently, Chhakchhuak and Singh (2024) studied Ricci solitons and string cloud spacetime in $f(R)$ -gravity.

The study of Ricci solitons in the context of relativistic thermodynamical fluid spacetimes (TFS) provides a rich array of applications, bridging fundamental physics, cosmology, and astrophysics. In cosmological modeling, Ricci solitons enable a deeper understanding of the universe's large-scale structure and evolution, offering explanations for dynamic behaviors such as expansion, contraction, or steady states (Siddiqi, 2019, 2020). They provide insights into inflationary models, shedding light on the rapid early expansion of the universe, and help elucidate the role of dark energy and cosmological constants in driving the current acceleration of cosmic expansion (Aghanim et al., 2018). In black hole physics, Ricci solitons enhance our understanding of event horizon thermodynamics, particularly the relationship between entropy and horizon area (Brizio et al., 2014; Bardeem et al., 1973). They also provide tools for analyzing the thermal stability of black holes, exploring phenomena such as Hawking radiation and singularity formation under extreme conditions (Wald, 2001; Gibbons and Hawking, 1977). In the field of relativistic fluid dynamics, Ricci solitons allow for precise modeling of highenergy astrophysical systems, such as accretion disks around black holes and the interior structures of neutron stars (Abramowics et al., 1978). Their integration into the modified Einstein field equations accommodates the influence of thermal energy density, heat flux, and stress, creating a more comprehensive description of spacetime behavior under high-energy conditions (Thorne, 1974). Furthermore, the inclusion of tools like gradient Ricci solitons, conformal Killing vector fields, and torse-forming vector fields aids in designing stable, self-consistent models of spacetime, accounting for the effects of geometry and thermodynamics. These frameworks also contribute to predicting and analyzing

gravitational wave behavior in regions of extreme curvature and energy, such as during supernova explosions, neutron star mergers, or gamma-ray bursts (Abbott, 2017).

From a theoretical perspective, Ricci solitons serve as a bridge between classical general relativity and quantum gravity. They offer a thermodynamic interpretation of gravity, where spacetime geometry emerges as a result of fundamental thermodynamic principles, and they aid in constructing semi-classical models that incorporate quantum corrections (Eling et al., 2006). The modified Poisson and Liouville equations derived within the TFS framework provide new tools for studying gravitational and thermal potentials, offering greater precision in describing how matter and energy evolve under extreme conditions (Hirano and Fujita, 2022; Horowitz and Hubeny, 2000). These equations also help characterize systems in thermal equilibrium and out-of-equilibrium states, which are essential for understanding phenomena near black holes and in the early universe (Brechet and Girad, 2022). Finally, Ricci solitons provide a unified framework to explore the interplay between geometry, thermodynamics, and gravitational dynamics. They enable the prediction of critical thresholds for spacetime behaviors, such as the conditions under which a spacetime contracts, expands, or remains steady, based on the interplay of thermal energy density and effective density. This versatility not only advances our understanding of the universe's fundamental structure but also opens avenues for practical applications in modeling extreme astrophysical systems and designing stable configurations in highenergy environments. In essence, Ricci solitons and TFS together form a cornerstone for unraveling the universe's most complex and dynamic phenomena.

The idea of local symmetry of a Riemannian manifold was studied under certain views by several authors. Takahashi (1977) introduced the notion of local ϕ -symmetry on a Sasakian manifold as a weaker version of local symmetry. This notion was generalized by De et al. (2003) and they called it ϕ -recurrent mani-

fold. Dubey (1978) introduced the notion of generalized recurrent manifold. After this De et al. (1995) introduced the generalized recurrent manifold, Shaikh and Ahmad (2011) extended this concept to generalized ϕ -recurrent Sasakian manifold. Thereafter, they (Shaikh et al., 2015) introduced the notion of generalized ϕ -recurrent LP-Sasakian manifold and studied its various geometric properties.

In the meantime, Shaikh and Patra (2010) introduced a generalized class of recurrent manifold called hyper generalized recurrent manifold. Venkatesha et al. (2019) extended this notion to the hyper generalized ϕ -recurrent Sasakian manifold and at the same time they worked on the quasi generalized ϕ -recurrent Sasakian manifold. Recently, Khatri and Singh (2020) studied hyper generalized ϕ recurrent and quasi generalized ϕ recurrent on (κ, μ) -contact metric manifold.

The study of symmetric and generalized symmetric manifolds plays an important role in differential geometry due to their rich structures and diverse applications in both mathematics and theoretical physics. Among the various extensions of symmetric spaces, the concept of weakly symmetric manifolds, which was first introduced by Tamassy and Binh (1989), has been a focal point. Prvanovic (1995) initiated the study of weakly symmetric Riemannian manifolds and established several properties, including relationships between the curvature tensor and its covariant derivative. Since then, this concept has been extensively studied by many geometers. Özen and Altay (2001) examined weakly symmetric and pseudo-symmetric Riemannian spaces, providing a comparative analysis of these two generalizations of symmetric spaces. Their research helped clarify the distinctions and interrelations between these notions of symmetry. Malek and Samawaki (2008) further contributed to the understanding of weakly symmetric Riemannian manifolds by exploring their geometric properties in more depth.

The class of weakly quasi-conformally symmetric manifolds was studied by Shaikh and Baishya (2005), who investigated conditions involving the quasi-conformal curvature tensor—a concept closely tied to weak symmetry. Meanwhile, Jaiswal

and Ojha (2010) introduced and studied weakly pseudo-projectively symmetric manifolds, offering another generalization of symmetric spaces where the pseudo-projective curvature tensor satisfies a condition associated with weak symmetry. Later, Baishya et al. (2016) studied almost generalized weakly symmetric Kenmotsu manifolds, integrating the ideas of weak symmetry with the framework of Kenmotsu manifolds, which are a class of almost contact metric manifolds. In recent years, Blaga et al. (2021) studied hyper generalized pseudo Q-symmetric semi- Riemannian manifold. And Bakshi et al. (2021) investigated the existence of hyper-generalized weakly symmetric Lorentzian Para-Sasakian manifold.

Chapter 2

Generalized Tanaka-Webster Connection

This chapter deals with generalized Tanaka-Webster connection in Kenmotsu manifold and some curvature properties.

2.1 Kenmotsu Manifold with Generalized Tanaka-Webster Connection

Kenmotsu (1972) studied a class of contact metric manifolds satisfying certain tensorial relations and called it as Kenmotsu manifold. He showed that a semisymmetric Kenmotsu manifold satisfying the relation $R(X, Y).R = 0$ is a manifold of negative curvature -1 , where R is a Riemannian curvature tensor and $R(X, Y)$ denotes the derivative of the tensor algebra at each point of the tangent space. Since then, many authors like De and Pathak (2004), Jun et al. (2005), Özgür (2010), Basari and Murathan (2008), Chaubey and Ojha (2010) and many others, have studied various results and properties of Kenmotsu manifold.

M.S. Devi and Bidyabati Th (2023). Some Curvature Properties of Kenmotsu Manifolds with Generalized Tanaka Webster Connection, *Science and Technology Journal* **2(1)**, 114–120.

Recently, Ünal and Altin (2020) studied $N(k)$ -contact metric manifold with a generalized Tanaka-Webster connection and obtained various results. Mburu and Wahome (2024) analysed the W_8 -curvature tensor on Kenmotsu manifolds admitting generalized Tanaka-Webster connection, establishing relationships between various curvature conditions and Einstein manifolds. Continuing this, we studied some curvature properties of Kenmotsu manifold with generalized Tanaka-Webster connection.

2.2 Some results of a Kenmotsu manifold

In a $(2n + 1)$ -dimensional Kenmotsu manifold with structure (ϕ, ξ, η, g) , the following relations hold (Acet et al., 2013):

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \quad g(X, \xi) = \eta(X), \quad (2.1)$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (2.2)$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \quad (2.3)$$

$$S(X, \xi) = -2n\eta(X), \quad Q\phi = \phi Q, \quad (2.4)$$

$$S(\phi X, \phi Y) = S(X, Y) + 2n\eta(X)\eta(Y), \quad S(QX, Y) = S^2(X, Y), \quad (2.5)$$

for all $X, Y, Z \in \chi(M^{2n+1})$, where R and S are the curvature tensor and Ricci tensor respectively.

In this chapter, we consider a generalized Tanaka-Webster connection $\tilde{\nabla}$ defined by (1.2). Using this connection in Kenmotsu manifold, Ghosh and De (2017) obtained the following results:

$$\tilde{R}(X, Y)Z = R(X, Y)Z + g(Y, Z)X - g(Z, X)Y, \quad (2.6)$$

$$\tilde{S}(Y, Z) = S(Y, Z) + 2ng(Y, Z), \quad (2.7)$$

$$\tilde{S}(\xi, Z) = \tilde{S}(Z, \xi) = 0, \quad (2.8)$$

$$\tilde{Q}Y = QY + 2nY, \quad \tilde{Q}\xi = 0, \quad (2.9)$$

$$\tilde{r} = r + 2n(2n + 1). \quad (2.10)$$

Proposition 2.1. *A Kenmotsu manifold admitting generalized Tanaka-Webster connection $\tilde{\nabla}$ satisfy*

$$(i) \quad \tilde{R}(X, Y)\xi = 0,$$

$$(ii) \quad \tilde{R}(\xi, Y)Z = 0,$$

$$(iii) \quad \eta(\tilde{R}(X, Y)Z) = 0.$$

Proof. Using equations (2.1), (2.2), (2.3), and (2.6), we get the results (i), (ii) and (iii). $\square$

2.3 Conharmonically flat and ξ -conharmonically flat Kenmotsu manifolds

The conharmonic curvature tensor $\tilde{H}$ of a Kenmotsu manifold M^{2n+1} with generalized Tanaka-Webster connection is given by

$$\begin{aligned} \tilde{H}(X, Y)Z &= \tilde{R}(X, Y)Z - \frac{1}{2n-1} \left[\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y \right. \\ &\quad \left. + g(Y, Z)\tilde{Q}X - g(X, Z)\tilde{Q}Y \right]. \end{aligned} \quad (2.11)$$

Definition 2.1. *A Kenmotsu manifold M^{2n+1} is known as conharmonically flat with respect to the generalized Tanaka-Webster connection if $\tilde{H}(X, Y)Z = 0$, for all $X, Y, Z \in \chi(M^{2n+1})$.*

Definition 2.2. *A Kenmotsu manifold M^{2n+1} is known as ξ -conharmonically flat with respect to the generalized Tanaka-Webster connection if $\tilde{H}(X, Y)\xi = 0$, for all $X, Y \in \chi(M^{2n+1})$.*

Theorem 2.1. *If a Kenmotsu manifold M^{2n+1} is conharmonically flat with respect to the generalized Tanaka-Webster connection, then the scalar curvature is given by $r = -2n(2n + 1)$.*

Proof. Let us assume that a Kenmotsu manifold M^{2n+1} is conharmonically flat equipped with generalized Tanaka-Webster connection. Then from equation (2.11), we get

$$\begin{aligned}\tilde{R}(X, Y)Z &= \frac{1}{2n-1} \left[\tilde{S}(Y, Z)X + \tilde{S}(X, Z)Y + g(Y, Z)\tilde{Q}X \right. \\ &\quad \left. - g(X, Z)\tilde{Q}Y \right].\end{aligned}\tag{2.12}$$

Taking inner product on both sides of equation (2.12) with respect to vector field V , we have

$$\begin{aligned}g(\tilde{R}(X, Y)Z, V) &= \frac{1}{2n-1} \left[\tilde{S}(Y, Z)g(X, V) + \tilde{S}(X, Z)g(Y, V) \right. \\ &\quad \left. + g(Y, Z)\tilde{S}(X, V) - g(X, Z)\tilde{S}(Y, V) \right].\end{aligned}\tag{2.13}$$

Let $\{e_1, e_2, \dots, e_{2n+1}\}$ be an orthonormal basis of the manifold. Setting $X = V = e_i$ in equation (2.13) and taking summation over i , $1 \leq i \leq 2n+1$, and then using equation (2.10), we have

$$r = -2n(2n + 1).\tag{2.14}$$

Hence, the proof is completed. $\square$

Theorem 2.2. *If a Kenmotsu manifold M^{2n+1} is ξ -conharmonically flat with respect to generalized Tanaka-Webster connection if and only if it is ξ -conharmonically flat with respect to Levi-Civita connection.*

Proof. From equation (1.24) and (2.11) and using (2.6), (2.7) and (2.9) we get

$$\tilde{H}(X, Y)Z = H(X, Y)Z - \frac{2n+1}{2n-1} \left[g(Y, Z)X - g(X, Z)Y \right].\tag{2.15}$$

Putting $Z = \xi$ and taking $X, Y \in (\text{span } \xi)^\perp$ in (2.15), we have

$$\tilde{H}(X, Y)\xi = H(X, Y)\xi. \quad (2.16)$$

This completes the proof of theorem. $\square$

Theorem 2.3. *If a Kenmotsu manifold M^{2n+1} is ξ -conharmonically flat with respect to generalized Tanaka-Webster connection, then its scalar curvature equipped with generalized Tanaka-Webster connection disappears.*

Proof. Setting $Z = \xi$ in equation (2.11), we get

$$\begin{aligned} \tilde{H}(X, Y)\xi &= \tilde{R}(X, Y)\xi - \frac{1}{2n-1} \left[\tilde{S}(Y, \xi)X - \tilde{S}(X, \xi)Y \right] - g(Y, \xi)\tilde{Q}X \\ &\quad + g(X, \xi)\tilde{Q}Y. \end{aligned} \quad (2.17)$$

If M^{2n+1} is ξ -conharmonically flat then using (2.8) and proposition 2.1 in (2.17), it follows that

$$\eta(Y)\tilde{Q}X - \eta(X)\tilde{Q}Y = 0. \quad (2.18)$$

Taking inner product of equation (2.18) with vector field V , we have

$$\eta(Y)\tilde{S}(X, V) - \eta(X)\tilde{S}(Y, V) = 0. \quad (2.19)$$

Putting $Y = \xi$ in (2.19), we get

$$\tilde{S}(X, V) = 0. \quad (2.20)$$

Let $\{e_1, e_2, \dots, e_{2n+1}\}$ be an orthonormal basis of the manifold and contracting over X and V in equation (2.20), we obtain

$$\tilde{r} = 0. \quad (2.21)$$

Hence, proof is completed. $\square$

2.4 Quasi conharmonically flat Kenmotsu manifolds

Theorem 2.4. *If a Kenmotsu manifold M^{2n+1} is quasi-conharmonically flat with respect to generalized Tanaka-Webster connection, then its scalar curvature equipped with generalized Tanaka-Webster connection disappears.*

Proof. Let us assume that a Kenmotsu manifold M^{2n+1} which is quasi-conharmonically flat with respect to generalized Tanaka-Webster connection, i.e.,

$$g(\tilde{H}(\phi X, Y)Z, \phi V) = 0, \quad (2.22)$$

for all $X, Y, Z, V \in \chi(M^{2n+1})$.

Then, from equations (2.11) and (2.22), we get

$$\begin{aligned} g(\tilde{R}(\phi X, Y)Z, \phi V) &= \frac{1}{2n-1} \left[\tilde{S}(Y, Z)g(\phi X, \phi V) - \tilde{S}(\phi X, Z)g(Y, \phi V) \right. \\ &\quad \left. + g(Y, Z)\tilde{S}(\phi X, \phi V) - g(\phi X, Z)\tilde{S}(Y, \phi V) \right]. \end{aligned} \quad (2.23)$$

Let $\{e_1, e_2, \dots, e_{2n+1}\}$ be an orthonormal basis of the manifold and contracting equation (2.23) over Y and Z we obtain

$$\begin{aligned} &\sum_{i=1}^{2n+1} g(\tilde{R}(\phi X, e_i)e_i, \phi V) \\ &= \frac{1}{2n-1} \left[\sum_{i=1}^{2n+1} \tilde{S}(e_i, e_i)g(\phi X, \phi V) - \sum_{i=1}^{2n+1} \tilde{S}(\phi X, e_i)g(e_i, \phi V) \right. \\ &\quad \left. + \sum_{i=1}^{2n+1} g(e_i, e_i)\tilde{S}(\phi X, \phi V) - \sum_{i=1}^{2n+1} g(\phi X, e_i)\tilde{S}(e_i, \phi V) \right]. \end{aligned} \quad (2.24)$$

We know that (Mandal and Das, 2020)

$$\sum_{i=1}^{2n+1} g(e_i, e_i) = 2n+1, \quad (2.25)$$

$$\sum_{i=1}^{2n+1} \tilde{S}(\phi X, e_i)g(e_i, \phi V) = \tilde{S}(\phi X, \phi V), \quad (2.26)$$

$$\sum_{i=1}^{2n+1} \tilde{S}(e_i, e_i) = \tilde{r}. \quad (2.27)$$

Using (2.25), (2.26) and (2.27) in (2.24), we get

$$\tilde{r} = 0.$$

Hence the theorem is proved. $\square$

2.5 Generalized conharmonic ϕ -recurrent Kenmotsu manifolds

Definition 2.3. (Mandal and Das, 2020) A Kenmotsu manifold M^{2n+1} is said to be generalized conharmonic ϕ -recurrent with respect to generalized Tanaka-Webster connection if

$$\phi^2(\tilde{\nabla}_W \tilde{H})(X, Y)Z = A(W)\tilde{H}(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y], \quad (2.28)$$

for all $X, Y, Z, W \in \chi(M^{2n+1})$, where $A(W) = g(W, \alpha_1)$ and $B(W) = g(W, \alpha_2)$ are 1-forms and B is non-vanishing and α_1, α_2 are vector fields associated with 1-forms A and B , respectively.

Theorem 2.5. If a Kenmotsu manifold M^{2n+1} is generalized conharmonic ϕ -recurrent with respect to generalized Tanaka-Webster connection, then the relation of 1-forms A and B are given by $B(W) = \left[\frac{r+4n^2+4n}{2n(2n-1)} \right] A(W)$, where W is an arbitrary vector field on M^{2n+1} .

Proof. Let us consider a generalized conharmonic ϕ -recurrent Kenmotsu manifold M^{2n+1} equipped with the generalized Tanaka-Webster connection.

Using (1.10), in (2.28), we have

$$(\tilde{\nabla}_W \tilde{H})(X, Y)Z = \eta((\tilde{\nabla}_W \tilde{H})(X, Y)Z)\xi - A(W)\tilde{H}(X, Y)Z$$

$$- B(W)[g(Y, Z)X - g(X, Z)Y]. \quad (2.29)$$

Taking inner product of (2.29) with vector field V , we get

$$\begin{aligned} & g((\tilde{\nabla}_W \tilde{H})(X, Y)Z, V) \\ &= \eta((\tilde{\nabla}_W \tilde{H})(X, Y)Z)g(\xi, V) - A(W)g(\tilde{H}(X, Y)Z, V) \\ & - B(W)[g(Y, Z)g(X, V) - g(X, Z)g(Y, V)] \end{aligned} \quad (2.30)$$

In view of equation (2.11), we have

$$\begin{aligned} & g((\tilde{\nabla}_W \tilde{H})(X, Y)Z, V) \\ &= g((\tilde{\nabla}_W \tilde{R})(X, Y)Z, V) - \frac{1}{2n-1}[(\tilde{\nabla}_W \tilde{S})(Y, Z)g(X, V) \\ & - (\tilde{\nabla}_W \tilde{S})(X, Z)g(Y, V) - g(Y, Z)(\tilde{\nabla}_W \tilde{S})(X, V) + g(X, Z)(\tilde{\nabla}_W \tilde{S})(Y, V)], \end{aligned} \quad (2.31)$$

$$\begin{aligned} \text{and } g(\tilde{H}(X, Y)Z, V) &= g(\tilde{R}(X, Y)Z, V) - \frac{1}{2n-1}[\tilde{S}(Y, Z)g(X, V) \\ & - \tilde{S}(X, Z)g(Y, V)] - g(Y, Z)\tilde{S}(X, V) + g(X, Z)\tilde{S}(Y, V)]. \end{aligned} \quad (2.32)$$

Using equations (1.12), (2.8) and putting $V = \xi$ in (2.31), we have

$$\begin{aligned} \eta((\tilde{\nabla}_W \tilde{H})(X, Y)Z) &= \eta((\tilde{\nabla}_W \tilde{R})(X, Y)Z) \\ & - \frac{1}{2n-1}[(\tilde{\nabla}_W \tilde{S})(Y, Z)\eta(X) - (\tilde{\nabla}_W \tilde{S})(X, Z)\eta(Y)]. \end{aligned} \quad (2.33)$$

Using equations (2.31), (2.32) and (2.33) in (2.30), we get

$$\begin{aligned} & g((\tilde{\nabla}_W \tilde{R})(X, Y)Z, V) \\ &= \frac{1}{2n-1} \left[(\tilde{\nabla}_W \tilde{S})(Y, Z)g(X, V) - (\tilde{\nabla}_W \tilde{S})(X, Z)g(Y, V) \right. \\ & + g(Y, Z)(\tilde{\nabla}_W \tilde{S})(X, V) - g(X, Z)(\tilde{\nabla}_W \tilde{S})(Y, V) - \{(\tilde{\nabla}_W \tilde{S})(Y, Z)\eta(X) \\ & - (\tilde{\nabla}_W \tilde{S})(X, Z)\eta(Y)\}\eta(V) + \{\tilde{S}(Y, Z)g(X, V) - \tilde{S}(X, Z)g(Y, V) \\ & \left. + g(Y, Z)\tilde{S}(X, V) - g(X, Z)\tilde{S}(Y, V)\}A(W) \right] + \eta((\tilde{\nabla}_W \tilde{R})(X, Y)Z)\eta(V) \end{aligned}$$

$$- A(W)g(\tilde{R}(X, Y)Z, V) - B(W)[g(Y, Z)g(X, V) - g(X, Z)g(Y, V)]. \quad (2.34)$$

Let $\{e_1, e_2, \dots, e_{2n+1}\}$ be an orthonormal basis of the manifold and contracting over Y and Z in (2.34), we obtain

$$\begin{aligned} (\tilde{\nabla}_W \tilde{S})(X, V) &= \frac{1}{2n-1} \left[(\tilde{\nabla}_W \tilde{r})g(X, V) - 2(\tilde{\nabla}_W \tilde{S})(X, V) + \{\tilde{r}g(X, V) \right. \\ &\quad - \tilde{S}(X, V) + (2n+1)g(X, V) - g(X, V)\}A(W) - \{(\tilde{\nabla}_W \tilde{r})\eta(X) \\ &\quad - (\tilde{\nabla}_W \tilde{S})(X, \xi)\}\eta(V) + (2n+1)(\tilde{\nabla}_W \tilde{S})(X, V) \Big] \\ &\quad + g((\tilde{\nabla}_W \tilde{S})(X, \xi)\eta(V) - A(W)\tilde{S}(X, V) \\ &\quad - B(W)[(2n+1)g(X, V) - g(X, V)]]. \end{aligned} \quad (2.35)$$

Putting $V = \xi$ in equation (2.35), we have

$$B(W) = \left[\frac{r + 4n^2 + 4n}{2n(2n-1)} \right] A(W). \quad (2.36)$$

Hence, we prove the theorem. $\square$

2.6 Kenmotsu manifolds satisfying $\tilde{H}(\xi, U).\tilde{R} = 0$.

Theorem 2.6. *If in a $(2n+1)$ -dimensional Kenmotsu manifold M^{2n+1} , the condition $\tilde{H}(\xi, U).\tilde{R} = 0$ holds, then the equation $S^2(Y, U) + 4nS(Y, U) + 4n^2g(Y, U) = 0$, is satisfied on M^{2n+1} .*

Proof. Let us consider a Kenmotsu manifold M^{2n+1} , satisfying the condition

$$(\tilde{H}(\xi, U).\tilde{R})(X, Y)Z = 0. \quad (2.37)$$

Then, we have

$$0 = \tilde{H}(\xi, U)\tilde{R}(X, Y)Z - \tilde{R}(\tilde{H}(\xi, U)X, Y)Z - \tilde{R}(X, \tilde{H}(\xi, U)Y)Z$$

$$- \tilde{R}(X, Y)\tilde{H}(\xi, U)Z. \quad (2.38)$$

Putting $Z = \xi$ in (2.38), we get

$$\begin{aligned} 0 &= \tilde{H}(\xi, U)\tilde{R}(X, Y)\xi - \tilde{R}(\tilde{H}(\xi, U)X, Y)\xi - \tilde{R}(X, \tilde{H}(\xi, U)Y)\xi \\ &- \tilde{R}(X, Y)\tilde{H}(\xi, U)\xi. \end{aligned} \quad (2.39)$$

Using (2.6), (2.9) and (2.11) in (2.39), we have

$$\begin{aligned} 0 &= \tilde{R}(X, Y)\tilde{H}(\xi, U)\xi \\ &= \tilde{R}(X, Y)\tilde{Q}U \\ &= \tilde{R}(X, Y)QU + 2n\tilde{R}(X, Y)U. \end{aligned} \quad (2.40)$$

Taking inner product of the equation (2.40) with vector field V , we obtain

$$g(\tilde{R}(X, Y)QU, V) + 2ng(\tilde{R}(X, Y)U, V) = 0. \quad (2.41)$$

Let $\{e_1, e_2, \dots, e_{2n+1}\}$ be an orthonormal basis of the manifold and contracting over X and V in (2.41) and using (2.5) and (2.6), we get

$$S^2(Y, U) + 4nS(Y, U) + 4n^2g(Y, U) = 0. \quad (2.42)$$

Hence, we get the result. $\square$

2.7 Projectively flat Kenmotsu manifolds

The Projective curvature tensor $\tilde{P}$ of a Kenmotsu manifold $M^{(2n+1)}$ equipped with the generalized Tanaka-Webster connection is given by

$$\tilde{P}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{2n}[\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y]. \quad (2.43)$$

Definition 2.4. A Kenmotsu manifold M^{2n+1} is known as Projectively flat if

$P(X, Y)Z = 0$ for all $X, Y, Z \in \chi(M^{2n+1})$.

Definition 2.5. A Kenmotsu manifold M^{2n+1} is known as ξ -Projectively flat if $P(X, Y)\xi = 0$ for all $X, Y \in \chi(M^{2n+1})$.

Theorem 2.7. If a $(2n + 1)$ - dimensional Kenmotsu manifold M^{2n+1} is Projectively flat with respect to the generalized Tanaka-Webster connection, then it is an Einstein manifold.

Proof. Using equation (2.6) and (2.7) in equation (2.43), we have

$$\begin{aligned} \tilde{P}(X, Y)Z &= R(X, Y)Z + g(Y, Z)X - g(Z, X)Y - \frac{1}{2n} \left[S(Y, Z)X \right. \\ &\quad \left. + 2ng(Y, Z)X - S(X, Z)Y - 2ng(X, Z)Y \right]. \end{aligned} \quad (2.44)$$

Let M^{2n+1} be a projectively flat with respect to generalized Tanaka-Webster connection, then from equation (2.44), we get

$$\begin{aligned} R(X, Y)Z &= -g(Y, Z)X + g(Z, X)Y + \frac{1}{2n} \left[S(Y, Z)X \right. \\ &\quad \left. + 2ng(Y, Z)X - S(X, Z)Y - 2ng(X, Z)Y \right]. \end{aligned} \quad (2.45)$$

Taking inner product of (2.45) with vector field V , we obtain

$$\begin{aligned} g(R(X, Y)Z, V) &= -g(Y, Z)g(X, V) + g(Z, X)g(Y, V) \\ &\quad + \frac{1}{2n} \left[S(Y, Z)g(X, V) + 2ng(Y, Z)g(X, V) \right. \\ &\quad \left. - S(X, Z)g(Y, V) - 2ng(X, Z)g(Y, V) \right]. \end{aligned} \quad (2.46)$$

Putting $X = V = \xi$ and using (1.10) and (2.4) in (2.46), we have

$$S(Y, Z) = -2ng(Y, Z), \quad (2.47)$$

this shows that M^{2n+1} is an Einstein manifold. □

Corollary 2.1. A $(2n+1)$ -dimensional Kenmotsu manifold M^{2n+1} is ξ -Projectively flat with respect to the generalized Tanaka-Webster connection if and only if it is

ξ -Projectively flat with respect to Levi-Civita connection.

Proof. Using (1.25) in (2.44), we have

$$\begin{aligned}\tilde{P}(X, Y)Z &= P(X, Y)Z + g(Y, Z)X - g(Z, X)Y - \frac{1}{2n} \left[2ng(Y, Z)X \right. \\ &\quad \left. - 2ng(X, Z)Y \right].\end{aligned}\tag{2.48}$$

Putting $Z = \xi$ in (2.48), we get

$$\tilde{P}(X, Y)\xi = P(X, Y)\xi.\tag{2.49}$$

Hence, the statement of the corollary is proved. $\square$

2.8 Locally projectively ϕ -symmetric Kenmotsu manifolds

Definition 2.6. A $(2n + 1)$ -dimensional Kenmotsu manifold M^{2n+1} is known as locally projectively ϕ -symmetric equipped with generalized Tanaka-Webster connection $\tilde{\nabla}$ if and only if the projective curvature tensor $\tilde{P}$ with respect to the connection satisfies

$$\phi^2(\tilde{\nabla}_W \tilde{P})(X, Y)Z = 0,\tag{2.50}$$

where X, Y, Z and W are orthornormal to ξ on the Manifold M^{2n+1} .

Theorem 2.8. A $(2n + 1)$ - dimensional Kenmotsu manifold M^{2n+1} is locally projectively ϕ -symmetric equipped with generalized Tanaka-Webster connection if and only if it is locally projectively ϕ -symmetric with respect to the Levi-Civita connection.

Proof. From equation (1.2), we get

$$(\tilde{\nabla}_W \tilde{P})(X, Y)Z = (\nabla_W \tilde{P})(X, Y)Z + g(W, \tilde{P}(X, Y)Z)\xi$$

$$-\eta(\tilde{P}(X, Y)Z)W - \eta(W)\phi\tilde{P}(X, Y)Z. \quad (2.51)$$

Taking covariant derivative in the direction of W in (2.48), we have

$$(\nabla_W \tilde{P})(X, Y)Z = (\nabla_W P)(X, Y)Z. \quad (2.52)$$

From equation (1.25), (2.3) and (2.48), we get

$$\begin{aligned} \eta(\tilde{P}(X, Y)Z) &= -\frac{1}{2n} \left[S(Y, Z)\eta(X) + 2ng(Y, Z)\eta(X) - S(X, Z)\eta(Y) \right. \\ &\quad \left. - 2ng(X, Z)\eta(Y) \right]. \end{aligned} \quad (2.53)$$

Putting the value of (2.52) and (2.53) in (2.51), we get

$$\begin{aligned} (\tilde{\nabla}_W \tilde{P})(X, Y)Z &= (\nabla_W P)(X, Y)Z + g(W, P(X, Y)Z)\xi + \frac{1}{2n} \left[\{S(Y, Z) \right. \\ &\quad \left. + 2ng(Y, Z)\}\eta(X) - \{S(X, Z) + 2ng(X, Z)\}\eta(Y) \right] W \\ &\quad - \eta(W)\phi P(X, Y)Z. \end{aligned} \quad (2.54)$$

Applying ϕ^2 on both sides of (2.54), we have

$$\phi^2(\tilde{\nabla}_W \tilde{P})(X, Y)Z = \phi^2(\nabla_W P)(X, Y)Z, \quad (2.55)$$

where X, Y, Z, W are orthonormal to ξ on the manifold M^{2n+1} .

Hence, the proof is completed. $\square$

2.9 Conclusion

In this chapter, we have studied some curvature properties of Kenmotsu manifolds with respect to generalized Tanaka-Webster connection. We observed that the scalar curvature of conharmonically flat Kenmotsu manifold equipped with generalized Tanaka-Webster connection is of the form $r = -2n(2n + 1)$. Also the scalar curvature of ξ -conharmonically flat Kenmotsu manifold and quasi-conharmonically flat Kenmotsu manifold disappear in generalized Tanaka-Webster

connection. We have deduced the relation between 1-form C and D in a generalized conharmonic ϕ -recurrent Kenmotsu manifold with respect to generalized Tanaka-Webster connection. Hence, we conclude that the projectively flat Kenmotsu manifold with respect to generalized Tanaka-Webster connection is an Einstein manifold. In the last theorem, we have found that a Kenmotsu manifold M^{2n+1} is locally projectively ϕ -symmetric equipped with generalized Tanaka-Webster connection if and only if it is locally projectively ϕ -symmetric with respect to the Levi-Civita connection.

Chapter 3

Transversal hypersurfaces of a manifold

This chapter deals with transversal hypersurface of a Lorentzian Kenmotsu manifold and its geometrical properties.

3.1 Transversal Hypersurfaces of a Lorentzian Kenmotsu manifold

Kenmotsu (1972) introduced Kenmotsu manifolds as a class of contact Riemannian manifolds that satisfy certain conditions. According to Rosca (1991), a Kenmotsu manifold together with a Lorentzian metric is called Lorentzian Kenmotsu manifold. Sari and Vanli (2019) delved into the study of slant submanifolds of a Lorentz Kenmotsu manifold. Later, Sari (2020) extended the study of some curvature properties of Lorentzian Kenmotsu manifolds.

B. Thangjam and M.S. Devi (2025). Transversal Hypersurfaces of Lorentzian Kenmotsu Manifolds (communicated).

3.2 Some properties of Transversal Hypersurfaces of a Lorentzian Kenmotsu manifold

Let us consider M' be a transversal hypersurface of a Lorentzian Kenmotsu manifold M such that the vector field ξ never belongs to a tangent hyperplane of the hypersurface M' . Under these conditions, the structure vector field ξ can be taken as affine normal to the hypersurface (Prasad, 2007). For any $X \in TM'$ where TM' is the tangent bundle on M' , since X and ξ are linearly independent, then we can write

$$\phi X = JX + \alpha(X)\xi, \quad (3.1)$$

where J is a $(1, 1)$ tensor field and α is a 1-form on M' .

Theorem 3.1. *In a Lorentzian Kenmotsu manifold, each transversal hypersurface admits an almost Hermitian structure.*

Proof. Operating ϕ to (3.1) and using (1.33), we have

$$\phi^2 X = \phi JX. \quad (3.2)$$

Employing eq. (1.30) and eq. (3.1), we obtain

$$-X + \eta(X)\xi = J^2 X + (\alpha \circ J)(X)\xi.$$

Considering the normal and tangential parts, we have

$$J^2 = -I, \quad \alpha \circ J = \eta. \quad (3.3)$$

From eq. (3.3), we can conclude that

$$-\alpha = \eta \circ J. \quad (3.4)$$

We also denote by g' the induced metric on the transversal hypersurface M . Then, from eqs. (1.31) and (3.1), we have

$$g(JX, JY) = g'(X, Y) - \alpha(X)\alpha(Y) + \eta(X)\eta(Y), \quad (3.5)$$

for all $X, Y \in TM'$.

Let us define

$$H(X, Y) = g'(\phi X, \phi Y). \quad (3.6)$$

We assert that H is a Hermitian metric. Using eqs. (1.31), (3.4), (3.5) in (3.6), we get

$$H(JX, JY) = g'(X, Y) + \eta(X)\eta(Y). \quad (3.7)$$

In view of eq. (3.6) and eq. (3.7), we find

$$H(JX, JY) = H(X, Y). \quad (3.8)$$

Thus, the proof is completed. $\square$

Let us consider that M' is an orientable hypersurface, ∇ and D denote the Levi-Civita connection and induced Levi-Civita connection on M and M' respectively, with N being a unit vector field normal to M' .

Let us define

$$\phi X = fX + \mu(X)N, \quad (3.9)$$

$$\phi N = -U, \quad \xi = V + \delta N, \quad \eta(X) = \nu(X), \quad \eta(N) = \delta, \quad (3.10)$$

for $X \in TM'$, where U and V are vector fields in TM' , ν is a 1-form, and δ is a smooth function on M' . Clearly, $\delta \neq 0$, because if $\delta = 0$, $g(\xi, N) = 0$, this implies that $\xi \in \Gamma(TM')$, which contradicts the fact that M' is a transversal

hypersurface. Substituting U in place of X in eq. (3.9), we get

$$\phi U = fU + \mu(U)N. \quad (3.11)$$

In view of eq. (3.10), we obtain

$$\phi^2 N = -f(U) - \mu(U)N. \quad (3.12)$$

Using eq. (1.30) in the above equation and applying eq. (3.10), we get

$$-N + \delta V + \delta^2 N = -f(U) - \mu(U)N. \quad (3.13)$$

Considering the normal and tangential parts of the above expression, we obtain

$$f(U) = -\delta V, \quad \mu(U) = 1 - \delta^2. \quad (3.14)$$

On the other hand, substituting $X = V$ in eq. (3.9) and solving in the same way as we solve in U , we deduce

$$fV = \delta U, \quad \mu(V) = 0. \quad (3.15)$$

By the consequences of eqs. (3.9) and (3.10), we get $\mu(X) = g'(U, X)$ and

$$\mu(f(X)) = g'(fX, U) = -g'(X, N) + \eta(X)\eta(N).$$

Using eq. (3.10) in the above equation, we find

$$\mu \circ f = -\delta \nu. \quad (3.16)$$

Similarly, we can solve

$$\nu \circ f = -\delta \mu, \quad \nu(U) = 0, \quad \nu(V) = 1 - \delta^2. \quad (3.17)$$

Replacing X by fX in eq. (3.9), we have

$$\phi(fX) = f(fX) + \mu(fX)N.$$

Using eqs. (3.9) and (3.10) in the above equation, we can deduce

$$f^2 = -I + \mu \otimes U + V \otimes \nu. \quad (3.18)$$

With the help of eq. (1.31) and (3.9), we find that g' satisfies

$$g'(fX, fY) = g'(X, Y) + \eta(X)\eta(Y) + \mu(X)\mu(Y), \quad (3.19)$$

and

$$g'(X, fY) + g'(fX, Y) = 0, \quad (3.20)$$

$\forall X, Y \in \Gamma(TM')$. The above computations lead to the following result:

Proposition 3.1. *Every transversal hypersurface of a Lorentzian Kenmotsu manifold also admits an (f, g, μ, ν, δ) -structure.*

Theorem 3.2. *If M' is a transversal hypersurface of a Lorentzian Kenmotsu manifold then*

$$\delta\alpha = \mu, \quad (3.21)$$

$$J = f - \frac{1}{\delta}\mu \circ V, \quad (3.22)$$

$$JU = -\frac{1}{\delta}V, \quad (3.23)$$

$$\mu \circ J = \mu \circ f = -\delta\nu, \quad (3.24)$$

$$\nu \circ J = -\frac{1}{\delta}\mu, \quad (3.25)$$

$$JV = fV = \delta U. \quad (3.26)$$

Proof. From eqs. (3.1) and (3.10), we have

$$\phi X = fX + \alpha(X)V + \delta\alpha(X)N. \quad (3.27)$$

Comparing eqs. (3.9) and (3.27), we obtain

$$\delta\alpha = \mu,$$

which is (3.21), and

$$JX = fX - \alpha(X)V.$$

Using eq. (3.21) in the above relation, we have

$$J = f - \frac{1}{\delta}\mu \circ V,$$

which is (3.22).

Using eqs. (3.14) and (3.22), we get

$$JU = -\frac{1}{\delta}V,$$

which is (3.23).

Also, using eq. (3.22), we have

$$\mu \circ (J(X)) = \mu(f(X)) - \frac{1}{\delta}\mu(X)\mu(V),$$

Using eqs. (3.15) and (3.16) in the above equation, we get

$$\mu \circ J = \mu \circ f = -\delta\nu,$$

which is (3.24).

From eqs. (3.17) and (3.22), we obtain

$$\nu \circ J = -\frac{1}{\delta}\mu,$$

which is (3.25).

Again, from eqs. (3.15) and (3.22), we get

$$JV = fV = \delta V,$$

which is (3.26). $\square$

Lemma 3.1. *If M' is a transversal hypersurface with an (f, g, μ, ν, δ) -structure of a Lorentzian Kenmotsu manifold. Then*

$$\begin{aligned} (\nabla_X \phi)Y &= (D_X f)(Y) - \mu(Y)A_N X + h(X, Y)U \\ &\quad + \{(D_X \mu)Y + h(X, fY)\}N, \end{aligned} \quad (3.28)$$

$$\nabla_X \xi = D_X V + h(X, V)N - \delta A_N X + X.\delta N, \quad (3.29)$$

$$(\nabla_X \phi)N = -D_X U + fA_N X + (\mu(A_N X) - h(X, U))N, \quad (3.30)$$

$$(\nabla_X \eta)(Y) = (D_X \nu)(Y) - \delta h(X, Y), \quad (3.31)$$

for all $X, Y, U \in TM'$.

Proof. Using eqs. (1.70), (1.71), (1.72), (3.9), and (3.10), we can proof the above lemma (Srivastava et al., 2020). $\square$

Theorem 3.3. *If M' is a transversal hypersurface with an (f, g, μ, ν, δ) -structure of a Lorentzian Kenmotsu manifold M , then*

$$\begin{aligned} (D_X f)Y &= \mu(Y)A_N X - h(X, Y)U - g(fX, Y)V \\ &\quad + \nu(Y)fX, \end{aligned} \quad (3.32)$$

$$(D_X \mu)Y = -\delta g(fX, Y) - h(X, fY) + \nu(Y)\mu(X), \quad (3.33)$$

$$D_X V = X + \delta A_N X - \nu(X)V, \quad (3.34)$$

$$h(X, V) + X.\delta = -\delta\nu(X), \quad (3.35)$$

$$D_X U = fA_N X + \mu(X)V - \delta fX, \quad (3.36)$$

for all $X, Y \in TM'$.

Proof. Using eqs. (1.35), (3.9), and (3.10) in (3.28), we get

$$\begin{aligned} & (D_X f)Y - \mu(Y)A_N X + h(X, Y)U \\ & + (D_X \mu)Y + h(X, fY)N = -g(fX, Y)V \\ & - \delta g(fX, Y)N + \nu(Y)fX + \nu(Y)\mu(X)N. \end{aligned}$$

Comparing the tangential and normal components of the above equation, we have (3.32) and (3.33).

Using eqs. (1.30) and (1.36) in (3.29), we obtain

$$\begin{aligned} & X - \nu(X)V - \delta\nu(X)N \\ & = D_X V - \delta(A_N X) + (h(X, V) + X.\delta)N. \end{aligned}$$

Considering the normal and tangential parts of the above equation, we get (3.34) and (3.35).

Using eqs. (3.9) and (3.10) in (1.35), we obtain

$$(\nabla_X \phi)N = -\mu(X)V + \delta fX,$$

Using eq. (3.30) in the above equation, we have

$$-D_X U + fA_N X + (\mu(A_N X) - h(X, U))N = -\mu(X)V + \delta fX.$$

Equating the normal and tangential parts, we get (3.36). □

3.3 A characterization of totally geodesic transversal hypersurfaces

Theorem 3.4. *A totally umbilical transversal hypersurface M' of a Lorentzian Kenmotsu manifold M is totally geodesic if and only if*

$$\delta\nu(X) + X\delta = 0. \quad (3.37)$$

for all $X \in TM$.

Proof. If M' is totally umbilical, then $A_N = \zeta I$, where ζ is a smooth function.

From eq. (1.72), we have

$$h(X, Y) = \zeta g'(X, Y), \quad (3.38)$$

Thus,

$$h(X, V) = \zeta g'(X, V) = \zeta \nu(X),$$

Using eq. (3.35) in the above equation, we obtain

$$-X\delta - \delta\nu(X) = \zeta\nu(X). \quad (3.39)$$

If M' is totally geodesic, then $\zeta = 0$. Thus, (3.39) reduces to

$$\delta\nu(X) + X\delta = 0.$$

□

Theorem 3.5. *If M' is a totally geodesic transversal hypersurface of a Lorentzian Kenmotsu manifold with parallel tensor field f of type $(1,1)$, then*

$$X\delta = 0.$$

Proof. If f is a parallel tensor field, then $D_X f = 0$.

From eq. (3.32), we have

$$0 = \mu(Y)A_N X - h(X, Y)U - g(fX, Y)V + \nu(Y)fX. \quad (3.40)$$

Applying μ on both sides of (3.40) and using (3.15), we get

$$h(X, Y) = \frac{1}{1 - \delta^2} [\mu(Y)\mu(A_N X) - \delta\nu(X)\nu(Y)]. \quad (3.41)$$

Putting $Y = V$ in the above equation and using eq. (3.35), we have

$$-\delta\nu(X) - X\delta = \frac{1}{1 - \delta^2} [\mu(V)\mu(A_N X) - \delta\nu(X)\nu(V)]. \quad (3.42)$$

Using eqs. (3.15) and (3.17) in (3.42), we have

$$X\delta = 0. \quad (3.43)$$

Hence, we get the result. $\square$

Theorem 3.6. *If M' is a transversal hypersurface of a Lorentzian Kenmotsu manifold with parallel vector field U , then M' is totally geodesic if*

$$\mu(X)V - \delta fX = 0.$$

Proof. If U is parallel, then $D_X U = 0$.

From eq. (3.36), we get

$$0 = fA_N X + \mu(X)V - \delta fX. \quad (3.44)$$

If M' is totally geodesic, then $A_N X = 0$.

Thus,

$$\mu(X)V - \delta fX = 0.$$

$\square$

Theorem 3.7. *If M' is a transversal hypersurface of a Lorentzian Kenmotsu*

manifold with a parallel vector field V , then M' is totally geodesic if

$$X - \nu(X)V = 0.$$

Proof. If V is parallel, then $D_X V = 0$.

From eq. (3.34), we get

$$0 = X + \delta A_N X - \nu(X)V. \quad (3.45)$$

If M' is totally geodesic, then $A_N X = 0$.

Thus,

$$X - \nu(X)V = 0.$$

□

3.4 An example of a transversal hypersurface of a Lorentzian Kenmotsu manifold

Example 3.1. Let $M = (R - 0, 1) \times R_2^4 \subset R_2^5$ with standard cartesian coordinates $(x_1, x_2, x_3, x_4, x_5)$. Define ϕ , ξ , η , and g on M by

$$\phi \partial x_1 = -\partial x_1, \quad \phi \partial x_2 = \partial x_1, \quad \phi \partial x_3 = -\partial x_4, \quad \phi \partial x_4 = \partial x_3, \quad \phi \partial x_5 = 0,$$

$$\xi = \partial x_5, \quad \eta = dx_5, \quad \text{and } g = [(dx_2^2 - dx_1^2) + (dx_4^2 - dx_3^2)] + \eta \otimes \eta,$$

We can find that (M, ϕ, ξ, η, g) is an almost contact pseudo-metric manifold.

Consider (M', g') as a pseudo-Riemannian hypersurface of M which is given by

$$\mathcal{J}(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3, x_4, x_2).$$

Then the local basis of the tangent hyperplane of M is given by

$$E_1 = \partial x_1, \quad E_2 = \partial x_2 + \partial x_5, \quad E_3 = \partial x_3, \quad E_4 = \partial x_4,$$

and the normal vector field N of the hypersurface is given by $N = \partial x_1 - x_2^2 \partial x_5$. Here, it is clear that ξ_p , $p \in M'$, is not tangent to the hypersurface. Therefore, M' is a transversal hypersurface of M . Here, we find

$$\eta(N) = x_2^2 = \delta, \quad V = -x_2^2 \delta x_2 + (1 + x_2^4) \partial x_5 \text{ and } U = -\partial x_1.$$

Further, any tangent vector field E of the hypersurface M can be expressed as $E = \sum_{i=1}^4 a_i E_i$, where a_1, a_2, a_3 , and a_4 are smooth functions. Operating ϕ on both sides, we have

$$\begin{aligned} \phi E &= a_2(1 + x_2^2) \partial x_1 - a_1 \partial x_2 + a_4 \partial x_3 - a_3 \partial x_4 - a_2 x_2^4 \partial x_5 - x_2^2 a_2 N. \\ &= fE + \mu(E)N. \end{aligned}$$

where $\mu = -x_2^2 a_2$ and f is given by

$$f = \begin{pmatrix} 0 & (1 + x_2^2) & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -x_2^4 & 0 & 0 \end{pmatrix}.$$

Hence M' is a transversal hypersurface of M which admits an (f, g, μ, ν, δ) -structure.

3.5 Conclusion

We investigate the properties and characteristics of transversal hypersurface in Lorentzian Kenmotsu manifolds in this chapter. In section 3.2, we demonstrate that each transversal hypersurface of a Lorentzian Kenmotsu manifold admits an almost Hermitian structure. Additionally, every transversal hypersurface of a Lorentzian Kenmotsu manifold also admits a structure described by the pa-

rameters (f, g, μ, ν, δ) . We define criteria for totally umbilical hypersurface and parallel tensor fields, as well as totally geodesic hypersurface through conditions using δ and ν . Finally, we provide one example of transversal hypersurfaces with (f, g, μ, ν, δ) -structure.

Chapter 4

Some Results On Ricci solitons

This chapter is divided into two sections. First section is devoted to the study of some properties of Ricci solitons in LP-Kenmotsu manifolds and in the next section, we study Ricci solitons on a Thermodynamical fluid spaceetime.

4.1 Some properties of Ricci solitons in LP-Kenmotsu manifolds

Sato (1976) introduced the concept of an almost Para-contact Riemannian manifold. Subsequently, Sinha and Prasad (1995) defined a specific class of almost Para-contact metric manifolds, namely Para Kenmotsu manifold and special Para Kenmotsu manifold. Recently, Haseeb and Prasad (2020),(2021) focused on studying the properties of Lorentzian Para-Kenmotsu manifolds, particularly in terms of Ricci-pseudosymmmetricity and Ricci-generalized pseudosymmetry conditions. These studies motivate us to investigate Ricci solitons on Lorentzian Para-Kenmotsu manifolds.

B. Thangjam and M.S. Devi (2024). Some properties of Ricci solitons in LP-Kenmotsu manifolds, *Communication on Applied Nonlinear Analysis* **32(3s)**, 699–710.

4.1.1 Some Results of Ricci solitons in Lorentzian Para-Kenmotsu manifolds(LP-Kenmotsu manifolds)

Let (g, V, λ) be a Ricci soliton in a $(2n + 1)$ -dimensional Lorentzian Para-Kenmotsu manifold M . Then, we have

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_\xi X, Y) + g(X, \nabla_\xi Y). \quad (4.1)$$

Using (1.53) in (4.1), we have

$$(\mathcal{L}_\xi g)(X, Y) = -2g(X, Y) - 2\eta(X)\eta(Y). \quad (4.2)$$

From (1.81) and (4.2), we get

$$S(X, Y) = (1 - \lambda)g(X, Y) + \eta(X)\eta(Y), \quad (4.3)$$

$$QX = (1 - \lambda)X + \eta(X)\xi, \quad r = 2n - \lambda(2n + 1). \quad (4.4)$$

In view of (1.50) and (4.3), we have

$$S(X, \xi) = -\lambda\eta(X), \quad (4.5)$$

$$Q\xi = -\lambda\xi. \quad (4.6)$$

4.1.2 Parallel Symmetric Second Order Tensor and Ricci Solitons in Lorentzian Para-Kenmotsu manifolds

In this subsection, we study parallel symmetric second order tensor and Ricci solitons in Lorentzian Para-Kenmotsu manifolds.

Theorem 4.1. *A symmetric parallel second order covariant tensor in a Lorentzian Para-Kenmotsu manifold is a constant multiple of the metric tensor.*

Proof. Let h be a symmetric tensor field of $(0, 2)$ -type which is parallel with respect to ∇ that is $\nabla h = 0$. Then by applying the Ricci identity (Ingalahalli and Bagewadi, 2012), we obtain

$$\nabla^2 h(X, Y; Z, U) - \nabla^2 h(X, Y; U, Z) = 0, \quad (4.7)$$

which implies that

$$h(R(X, Y)Z, U) + h(Z, R(X, Y)U) = 0. \quad (4.8)$$

Replacing $Z = U = \xi$ in (4.8) and using (1.57) and the symmetry of h , we get

$$2 [\eta(Y)h(X, \xi) - \eta(X)h(Y, \xi)] = 0. \quad (4.9)$$

Putting $X = \xi$ in (4.9), we have

$$h(Y, \xi) = -\eta(Y)h(\xi, \xi). \quad (4.10)$$

Now, differentiating (4.10) covariantly with respect to X , we obtain

$$\begin{aligned} & (\nabla_X h)(Y, \xi) + h(\nabla_X Y, \xi) + h(Y, \nabla_X \xi) \\ &= - [(\nabla_X \eta)(Y) + \eta(\nabla_X Y)] h(\xi, \xi) \\ &+ \eta(Y) [(\nabla_X h)(\xi, \xi) + 2h(\nabla_X \xi, \xi)]. \end{aligned} \quad (4.11)$$

Using (1.53), (4.10) and the parallel condition $\nabla h = 0$ in (4.11), we get

$$h(Y, \nabla_X \xi) = -(\nabla_X \eta)(Y)h(\xi, \xi). \quad (4.12)$$

In consequences of (1.53), (1.54), (4.10) and (4.12), it yields

$$h(X, Y) = -g(X, Y)h(\xi, \xi). \quad (4.13)$$

From the above (4.13) and (4.10), we can conclude that $h(\xi, \xi)$ is a constant.

This completes the proof of the theorem. $\square$

Theorem 4.2. *Let M be an Lorentzian Para-Kenmotsu manifold. Assume that a symmetric metric tensor field $h = \mathcal{L}_V g + 2S$ is parallel with respect to the Levi-Civita connection associated with g . Then (g, V, λ) yields a Ricci-soliton on M .*

Proof. Let us assume that a symmetric tensor field $h = \mathcal{L}_V g + 2S$ is parallel with respect to the Levi-Civita connection associated with g .

Then, $h(\xi, \xi) = 2\lambda$, this shows that $\lambda = \frac{1}{2}h(\xi, \xi)$.

Now, as h is parallel with respect to g , then from (4.13) we get

$$h(X, Y) = -2\lambda g(X, Y), \quad (4.14)$$

for all vector fields X and Y on M , which leads to

$$\mathcal{L}_V g(X, Y) = -2\lambda g(X, Y) - 2S(X, Y). \quad (4.15)$$

Hence, we have proved the statement of the theorem. $\square$

Theorem 4.3. *A Ricci semi-symmetric Lorentzian Para-Kenmotsu manifold is an Einstein manifold.*

Proof. We consider a Ricci semi-symmetric Lorentzian Para-Kenmotsu manifold, i.e, $R \circ S = 0$.

We have (Ingalahalli and Bagewadi, 2012),

$$\begin{aligned} (R(X, Y) \circ S)(Z, U) &= -S(R(X, Y)Z, U) \\ &\quad - S(Z, R(X, Y)U). \end{aligned} \quad (4.16)$$

Putting $X = \xi$ and using $R \circ S = 0$ in (4.16), we have

$$S(R(\xi, Y)Z, U) + S(Z, R(\xi, Y)U) = 0. \quad (4.17)$$

Using (1.56) in (4.17), we obtain

$$\begin{aligned} & -g(Y, Z)S(\xi, U) - \eta(Z)S(Y, U) \\ & + S(Z, \xi)g(Y, U) + \eta(U)S(Z, X) = 0. \end{aligned} \quad (4.18)$$

Setting $Z = \xi$ in (4.18) and by making used of (1.50) and (1.58), we have

$$S(Y, U) = 2ng(Y, U). \quad (4.19)$$

Thus, we complete the proof. $\square$

Proposition 4.1. *If a $(2n+1)$ -dimensional Lorentzian Para-Kenmotsu manifold is an η -Einstein manifold, then the Ricci soliton with constant scalar curvature is shrinking.*

Proof. Suppose that the Lorentzian Para-Kenmotsu manifold is an η -Einstein manifold. Then, we will find the values of α and β .

Let $\{e_1, e_2, \dots, e_{2n+1}\}$ be an orthonormal basis of the tangent at any point of the manifold. Putting $X = Y = e_i$ in (1.80) and taking summation over i , we get

$$r = (2n+1)\alpha - \beta. \quad (4.20)$$

Again, setting $X = Y = \xi$ in (1.80), and using (1.58), we have

$$-2n = -\alpha + \beta. \quad (4.21)$$

Then from (4.20) and (4.21), we get

$$\alpha = \left[\frac{r}{2n} - 1 \right], \quad \beta = \left[-2n - 1 + \frac{r}{2n} \right]. \quad (4.22)$$

Substituting the value of α and β in (1.80), we have

$$\begin{aligned} S(X, Y) &= \left[\frac{r}{2n} - 1 \right] g(X, Y) \\ &+ \left[-2n - 1 + \frac{r}{2n} \right] \eta(X)\eta(Y). \end{aligned} \quad (4.23)$$

For a $(2n + 1)$ -dimensional Lorentzian Para-Kenmotsu manifold the symmetric parallel covariant tensor $h(X, Y)$ of type $(0, 2)$ is given by

$$h(X, Y) = (\mathcal{L}_\xi g)(X, Y) + 2S(X, Y). \quad (4.24)$$

Using (4.1) and (4.23) in (4.24), we get

$$\begin{aligned} h(X, Y) &= \left[\frac{2r}{2n} - 4 \right] g(X, Y) \\ &\quad + \left[-4n + \frac{2r}{2n} - 4 \right] \eta(X)\eta(Y). \end{aligned} \quad (4.25)$$

Taking covariant derivative of (4.25) with respect to Z , we have

$$\begin{aligned} (\nabla_Z h)(X, Y) &= \left[\frac{2(\nabla_Z r)}{2n} \right] [g(X, Y) + \eta(X)\eta(Y)] \\ &\quad + \left[\frac{2r}{2n} - 4n - 4 \right] [g(X, \nabla_Z \xi)\eta(Y) + g(Y, \nabla_Z \xi)\eta(X)]. \end{aligned} \quad (4.26)$$

By putting $Z = \xi$ and $X = Y \in (\text{span}\xi)^\perp$ in (4.26) and by using $\nabla h = 0$, we obtain

$$2\nabla_\xi r = 0. \quad (4.27)$$

On integrating (4.27), we get

$$r = c, \quad (4.28)$$

where c is some integral constant.

Thus, from (4.28) we have r is constant scalar curvature.

Finally, we will check the nature of the Ricci-soliton.

From (4.24), we have $h(X, Y) = -2\lambda g(X, Y)$, then putting $X = Y = \xi$, we have

$$h(\xi, \xi) = 2\lambda. \quad (4.29)$$

If we put $X = Y = \xi$ in (4.25), that is

$$h(\xi, \xi) = - \left[\frac{2r}{2n} - 4 \right] + \left[-4n + \frac{2r}{2n} - 4 \right]. \quad (4.30)$$

The above equation is reduced as

$$h(\xi, \xi) = -4n. \quad (4.31)$$

Equating (4.29) and (4.31), we obtain

$$\lambda = -2n < 0, \quad (4.32)$$

that is the Ricci soliton in a Lorentzian Para-Kenmotsu manifold is shrinking.

Hence, the theorem is proved. $\square$

4.1.3 Ricci solitons in a W_2 -semisymmetric Lorentzian Para-Kenmotsu manifold

Here, we study the conditions of Ricci solitons in a W_2 - semisymmetric Lorentzian Para-Kenmotsu manifold.

Definition 4.1. *(Pokhariyal and Mishra, 1970) In a $(2n+1)$ -dimensional Lorentzian Para-Kenmotsu manifold M , the W_2 -curvature tensor is defined as*

$$\begin{aligned} W_2(X, Y)Z &= R(X, Y)Z \\ &+ \frac{1}{2n} [g(X, Z)QY - g(Y, Z)QX], \end{aligned} \quad (4.33)$$

for all X, Y and Z in M .

Theorem 4.4. *A Ricci soliton in a W_2 -semi symmetric Lorentzian Para-Kenmotsu manifold M of dimension $(2n + 1)$ is shrinking.*

Proof. Putting $X = \xi$ in (4.33) and using (1.50) and (1.56), we have

$$\begin{aligned} W_2(\xi, Y)Z &= g(Y, Z)\xi - \eta(Z)Y \\ &\quad + \frac{1}{2n} [\eta(Z)QY - g(Y, Z)Q\xi]. \end{aligned} \quad (4.34)$$

Taking inner product on both sides of (4.33) with respect to ξ , we get

$$\begin{aligned} \eta(W_2(X, Y)Z) &= \eta(R(X, Y)Z) \\ &\quad + \frac{1}{2n} [g(X, Z)g(QY, \xi) - g(Y, Z)g(QX, \xi)]. \end{aligned} \quad (4.35)$$

Using (1.55) and (4.5) in (4.35), we obtain

$$\eta(W_2(X, Y)Z) = (1 + \frac{\lambda}{2n}) [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \quad (4.36)$$

Suppose that the condition, $R(\xi, X) \circ W_2(Y, Z)U = 0$ holds in M . Then by definition, we have

$$\begin{aligned} R(\xi, X)W_2(Y, Z)U - W_2(R(\xi, X)Y, Z)U \\ - W_2(Y, R(\xi, X)Z)U - W_2(Y, Z)R(\xi, X)U = 0, \end{aligned} \quad (4.37)$$

for all vector fields X, Y, Z and U on M .

In view of (1.56) and (4.37), we get

$$\begin{aligned} g(X, W_2(Y, Z)U)\xi - \eta(W_2(Y, Z)U)X \\ - g(X, Y)W_2(\xi, Z)U + \eta(Y)W_2(X, Z)U \\ - g(X, Z)W_2(Y, \xi)U + \eta(Z)W_2(Y, X)U \\ - g(X, U)W_2(Y, Z)\xi + \eta(U)W_2(Y, Z)X = 0. \end{aligned} \quad (4.38)$$

Again, taking inner product on both sides of (4.38) with ξ and using (1.50), we have

$$\begin{aligned} -g(X, W_2(Y, Z)U) - \eta(W_2(Y, Z)U)\eta(X) \\ - g(X, Y)\eta(W_2(\xi, Z)U) + \eta(Y)\eta(W_2(X, Z)U) \end{aligned}$$

$$\begin{aligned}
& -g(X, Z)\eta(W_2(Y, \xi)U) + \eta(Z)\eta(W_2(Y, X)U) \\
& -g(X, U)\eta(W_2(Y, Z)\xi) + \eta(U)\eta(W_2(Y, Z)X) = 0.
\end{aligned} \tag{4.39}$$

In consequence of (4.34), (4.36) and (4.39), it yields

$$\begin{aligned}
& g(R(Y, Z)U, X) \\
& + \frac{1}{2n} [g(Y, U)S(X, Z) - g(Z, U)S(X, Y)] \\
& - (1 + \frac{\lambda}{2n}) [\eta(X)\{g(U, Z)\eta(Y) - g(Y, U)\eta(Z)\}] \\
& + (1 + \frac{\lambda}{2n}) [g(X, Y)\{\eta(U)\eta(Z) + g(Z, U)\}] \\
& + (1 + \frac{\lambda}{2n}) [\eta(Y)\{g(Z, U)\eta(X) - g(X, U)\eta(Z)\}] \\
& - (1 + \frac{\lambda}{2n}) [g(X, Z)\{\eta(U)\eta(Y) + g(Y, U)\}] \\
& + (1 + \frac{\lambda}{2n}) [\eta(Z)\{g(X, U)\eta(Y) - g(Y, U)\eta(X)\}] \\
& + (1 + \frac{\lambda}{2n}) [\eta(U)\{g(Z, X)\eta(Y) - g(Y, X)\eta(Z)\}] = 0.
\end{aligned} \tag{4.40}$$

Let $\{e_1, e_2, \dots, e_{2n+1}\}$ be an orthonormal basis. Putting $X = Y = e_i$ in (4.40) and taking summation over i , where $1 \leq i \leq (2n+1)$, we have

$$\begin{aligned}
\left[\frac{2n+1}{2n} \right] S(Z, U) &= \frac{r}{2n} g(Z, U) - (1 + \frac{\lambda}{2n}) 2n g(Z, U) \\
&+ 2(1 + \frac{\lambda}{2n})(2n+1)\eta(Z)\eta(U).
\end{aligned} \tag{4.41}$$

Again, taking orthonormal frame field over Z and U , we get

$$\lambda = -2n < 0,$$

which implies that the soliton is shrinking.

Hence, the proof is completed. $\square$

Theorem 4.5. *Let M be a $(2n+1)$ -dimensional Lorentzian Para-Kenmotsu manifold and (g, V, λ) be a Ricci soliton satisfying the condition $W_2(\xi, X) \circ S = 0$ in M , then the Ricci soliton is steady.*

Proof. Let M be a $(2n+1)$ -dimensional Lorentzian Para-Kenmotsu manifold and (g, V, λ) be a Ricci soliton in M .

We assume that the condition $W_2(\xi, X) \circ S = 0$ holds in M , then we have

$$S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z) = 0. \quad (4.42)$$

Using (4.5), (4.34) and (4.42), we obtain

$$\begin{aligned} & -\lambda g(X, Y)\eta(Z) - \lambda g(X, Z)\eta(Y) \\ & + \frac{1}{2n} [S(QX, Z)\eta(Y) + S(QX, Y)\eta(Z)] \\ & - S(X, Z)\eta(Y) - S(X, Y)\eta(Z) \\ & - \frac{1}{2n} [g(X, Y)S(Q\xi, Z) + g(X, Z)S(Q\xi, Y)] = 0. \end{aligned} \quad (4.43)$$

Setting $Z = \xi$ in (4.43) and using (1.50), (4.5) and (4.6), we get

$$\begin{aligned} & (\lambda + \frac{1}{2n}\lambda^2)g(X, Y) + S(X, Y) \\ & + \frac{1}{2n} [\lambda^2\eta(X)\eta(Y) - S(QX, Y)] \\ & - \frac{1}{2n}\eta(X)S(Q\xi, Y) = 0. \end{aligned} \quad (4.44)$$

Again, putting $Y = \xi$ in (4.44) and using (4.5) and (4.6), we have

$$\lambda = 0.$$

The above equation implies that the Ricci soliton is steady.

Hence, we complete the proof. $\square$

4.1.4 Ricci tensor of a Lorentzian Para-Kenmotsu manifold admitting a Ricci soliton

In this subsection, we study Ricci tensor of a Lorentzian Para-Kenmotsu manifold admitting a Ricci soliton.

Theorem 4.6. *Let M be a Lorentzian Para-Kenmotsu manifold admitting a Ricci soliton (g, V, λ) . If the Ricci tensor S of the manifold is η -recurrent, then the Ricci soliton is steady.*

Proof. Suppose that the Ricci tensor of the Lorentzian Para-Kenmotsu manifold is η -recurrent, i.e,

$$(\nabla_X S)(Y, Z) = \eta(X)S(Y, Z), \quad (4.45)$$

for all vector fields X, Y, Z on M .

Then, by using (4.3), we have

$$\begin{aligned} (\nabla_X S)(Y, Z) &= -2\eta(X)\eta(Y)\eta(Z) - g(X, Y)\eta(Z) \\ &\quad - g(X, Z)\eta(Y). \end{aligned} \quad (4.46)$$

Using (4.3) in (4.45) and comparing with (4.46), we get

$$\begin{aligned} &-g(X, Y)\eta(Z) - g(X, Z)\eta(Y) - (1 - \lambda)g(Y, Z)\eta(X) \\ &= 3\eta(X)\eta(Y)\eta(Z). \end{aligned} \quad (4.47)$$

Setting $Y = Z = \xi$ in (4.47), we obtain

$$-\lambda\eta(X) = 0. \quad (4.48)$$

Since $\eta(X) \neq 0$, we have $\lambda = 0$.

Therefore, the Ricci soliton is steady. $\square$

Theorem 4.7. *Let M be an Lorentzian Para-Kenmotsu manifold, admitting a Ricci soliton (g, V, λ) . Then Q and S are parallel along ξ , where Q is the Ricci operator, defined by $S(X, Y) = g(QX, Y)$ and S is the Ricci tensor of M .*

Proof. We can express the equations for the Ricci operator and Ricci tensor along ξ as follows,

$$(\nabla_\xi Q)(X) = \nabla_\xi Q(X) - Q(\nabla_\xi X), \quad (4.49)$$

and

$$(\nabla_\xi S)(X, Y) = \nabla_\xi S(X, Y) - S(\nabla_\xi X, Y) - S(X, \nabla_\xi Y). \quad (4.50)$$

Using (4.4) in (4.49), we can simplify to obtain

$$(\nabla_\xi Q)(X) = 0. \quad (4.51)$$

Similarly, applying (4.3) to (4.50), we get

$$(\nabla_\xi S)(X, Y) = 0. \quad (4.52)$$

Therefore, from (4.51) and (4.52), we can conclude that Q and S are parallel along ξ , which completes the proof. $\square$

4.1.5 Ricci soliton in a Lorentzian Para-Kenmotsu manifold and its curvature properties

In this subsection, we explore some curvature properties of an Lorentzian Para-Kenmotsu manifold admitting a Ricci soliton.

Proposition 4.2. *A Lorentzian Para-Kenmotsu manifold M , admitting a Ricci soliton (g, V, λ) is ξ -projectively flat iff the soliton is shrinking.*

Proof. Putting $Z = \xi$ in (1.25) and by using (1.57) and (1.58), we get

$$P(X, Y)\xi = \left[\frac{2n + \lambda}{2n} \right] [\eta(Y)X - \eta(X)Y]. \quad (4.53)$$

This implies that $P(X, Y)\xi = 0$ if and only if $\lambda = -2n$, which proves the proposition. $\square$

Proposition 4.3. *A Lorentzian Para-Kenmotsu manifold M , admitting a Ricci soliton (g, V, λ) is ξ -concentrically flat if the soliton is shrinking.*

Proof. Setting $Z = \xi$ in (1.23) and using the equations (1.50), (1.57) and (4.4), we get

$$C(X, Y)\xi = \left[\frac{4n^2 + \lambda(2n + 1)}{2n(2n + 1)} \right] [\eta(Y)X - \eta(X)Y]. \quad (4.54)$$

This shows that $C(X, Y)\xi = 0$ if and only if $\lambda = -\frac{4n^2}{2n+1}$.

Hence, we prove the proposition. $\square$

Proposition 4.4. *A Lorentzian Para-Kenmotsu manifold M , admitting a Ricci soliton (g, V, λ) is ξ -conharmonically flat if the soliton is shrinking.*

Proof. Putting $Z = \xi$ in (1.24), we obtain

$$\begin{aligned} H(X, Y)\xi &= R(X, Y)\xi \\ &\quad - \frac{1}{2n-1} [g(Y, \xi)QX - g(X, \xi)QY] \\ &\quad - \frac{1}{2n-1} [S(Y, \xi)X - S(X, \xi)Y]. \end{aligned} \quad (4.55)$$

Using (1.50), (4.4) and (4.5) in (4.55), we have

$$H(X, Y)\xi = \frac{2n-2+2\lambda}{2n-1} [\eta(Y)X - \eta(X)Y]. \quad (4.56)$$

Thus, $H(X, Y)\xi = 0$ if and only if $\lambda = -(n-1)$.

Hence, the proof is completed. $\square$

Proposition 4.5. *A Lorentzian Para-Kenmotsu manifold M , admitting a Ricci soliton (g, V, λ) is ξ -conformally flat if the soliton is shrinking.*

Proof. Putting $Z = \xi$ in (1.22), we obtain

$$\begin{aligned} W(X, Y)\xi &= R(X, Y)\xi \\ &\quad - \frac{1}{2n-1} [g(Y, \xi)QX - g(X, \xi)QY + S(Y, \xi)X \\ &\quad - S(X, \xi)Y] + \frac{r}{2n(2n-1)} [g(Y, \xi)X - g(X, \xi)Y]. \end{aligned} \quad (4.57)$$

Using (1.50), (4.4) and (4.5) in (4.57), we have

$$W(X, Y)\xi = \frac{2n + \lambda}{2n} [\eta(Y)X - \eta(X)Y]. \quad (4.58)$$

This implies that $W(X, Y)\xi = 0$ if and only if $\lambda = -2n$.

This completes the proof of the proposition. $\square$

4.1.6 Example of a Lorentzian Para-Kenmotsu manifold

In this subsection we establish an example of a Lorentzian Para-Kenmotsu manifold. We consider the 3-dimensional manifold $M = \{(x, y, z) \in R^3 : z \neq 0\}$, where (x, y, z) are the standard coordinates in R^3 . Let E_1, E_2 and E_3 be a linearly independent vector fields in M which satisfy

$$[E_1, E_2] = E_2, [E_2, E_3] = 0, [E_1, E_3] = E_3.$$

Let g be the Lorentzian metric defined by

$$\begin{aligned} g(E_1, E_1) &= -1, \quad g(E_2, E_2) = g(E_3, E_3) = 1, \\ g(E_1, E_2) &= g(E_2, E_3) = g(E_1, E_3) = 0. \end{aligned}$$

Let η be the 1-form defined by

$$\eta(X) = g(X, E_3),$$

for any vector field X . Let ϕ be $(1, 1)$ -tensor field defined by

$$\phi E_1 = 0, \quad \phi E_2 = E_3, \quad \phi E_3 = E_2.$$

Then we have

$$\eta(E_1) = -1, \quad \phi^2(X) = X + \eta(X)E_1,$$

and

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

Thus for $\xi = E_1$, (ϕ, ξ, η, g) defines a Lorentzian almost Para-contact metric structure on M . Let ∇ be the Levi-Civita connection of the Lorentzian metric g . Then using Koszul's formula, we obtain

$$\begin{aligned}\nabla_{E_1} E_1 &= 0, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = 0, \\ \nabla_{E_2} E_1 &= -E_2, \quad \nabla_{E_2} E_2 = -E_1, \quad \nabla_{E_2} E_3 = 0, \\ \nabla_{E_3} E_1 &= -E_3, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = -E_1.\end{aligned}$$

From the above calculation, one can easily verify that

$$\nabla_X \xi = -\phi^2 X,$$

and

$$(\nabla_X \phi)Y = -g(\phi X, Y) - \eta(Y)\phi X.$$

Therefore, the manifold (M, g, ξ, ϕ, η) is a Lorentzian Para-Kenmotsu manifold.

On this manifold (M, g, ξ, ϕ, η) , we can easily verify our results.

4.2 Ricci Soliton and Relativistic Thermodynamical Fluid Spacetime

Relativistic thermodynamical fluid spacetime (TFS) merges thermodynamic principles with the geometric framework of general relativity to explore how thermal properties influence the dynamics of spacetime. This area of study examines

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the interplay between heat flux, energy density, and stress within a relativistic context. TFS is characterized by its evolution under thermal influences, leading to a generalized Ricci recurrent structure that is crucial for understanding cosmological effects. Research in TFS reveals solitonic properties through the Ricci metric, enhancing our comprehension of spacetime geometry in thermodynamic scenarios (Siddiqi et al., 2023).

The relativistic continuity equations derived from Minkowski space provide a robust foundation for the first and second laws of thermodynamics, emphasizing the relationship between heat and work in perfect fluids. This framework also facilitates the derivation of transformation laws for thermodynamic variables, ensuring consistency across different reference frames (Brechet and Girard, 2022). Studies on the equilibrium of quantum relativistic fluids in curved spacetime have resulted in corrections to flat spacetime models and clarified the significance of curvature in thermodynamic behavior (Becattini and Grossi, 2015). The Israel-Stewart formalism is employed to generalize key thermodynamic relations, making them applicable even in the presence of dissipative processes (Majhi, 2023). Although the integration of thermodynamics and relativity offers significant insights into fluid dynamics within spacetime, challenges remain in fully understanding the impact of dissipative effects and non-ideal fluid behaviors within these frameworks. In view of (1.75), the Einstein's equation of state without cosmological constant for thermodynamical fluid spacetime is given as:

$$S - \frac{1}{2}rg = \frac{2\pi}{\hbar\alpha}T, \quad (4.59)$$

where S and r denote the Ricci tensor and scalar curvature, and T is the energy-momentum tensor of thermodynamical fluid spacetime, respectively.

Using (1.76) in the above equation. we have

$$S(U, V) = \frac{r}{2}g(U, V) + \frac{2\pi}{\hbar\alpha}(\rho + \epsilon)\eta(U)\eta(V) + \frac{4\pi}{\hbar\alpha}\eta(U)\gamma(V) + \frac{2\pi}{\hbar\alpha}\tau(U, V), \quad (4.60)$$

for any vector fields $U, V \in \chi(M)$.

With the vector ξ serves as a unit timelike vector field, which accomodate the condition $g(\xi, \xi) = -1$, while ζ is the spacelike vector field, ensuring $g(\zeta, \zeta) = 1$ and ξ, ζ are orthogonal unit vector fields.

So, from (4.60), we obtain the following results:

$$S(U, \xi) = \left(\frac{r}{2} + \frac{2\pi}{\hbar\alpha} \right), \quad (4.61)$$

$$S(U, \zeta) = \frac{r}{2}\gamma(U) + \frac{4\pi}{\hbar\alpha}\eta(U) + \frac{2\pi}{\hbar\alpha}\tau(U, \zeta), \quad (4.62)$$

$$S(\xi, \xi) = -\frac{r}{2} + \frac{2\pi}{\hbar\alpha}(\rho + \epsilon), \quad (4.63)$$

$$S(\zeta, \zeta) = \frac{r}{2} + \frac{2\pi}{\hbar\alpha}I, \quad (4.64)$$

$$S(\xi, \zeta) = -\frac{4\pi}{\hbar\alpha}, \quad (4.65)$$

where $\tau(\zeta, \zeta) = I$. Also, we shall assume that the spacetime is of n -dimensions where $n > 2$.

In this section, we investigate the effects and impacts of a Ricci soliton (1.81) within a TFS. So, for any vector fields U and V , the Ricci soliton equation becomes

$$\begin{aligned} S(U, V) &= -\lambda g(U, V) - \frac{1}{2}\mathcal{L}_X g(U, V) \\ &= -\lambda g(U, V) - \frac{1}{2}[g(\nabla_U X, V) + g(\nabla_V X, U)]. \end{aligned}$$

Putting $X = \xi$ in the above equation and then contracting, we have

$$r = -n\lambda - \text{div}\xi. \quad (4.66)$$

Contracting the eq. (4.60), we obtain

$$r = \frac{2}{n-2} \left[\frac{2\pi}{\hbar\alpha} (\rho + \epsilon - J) \right], \quad (4.67)$$

where $tr(\tau) = J$.

Comparing eqs. (4.66) and (4.67), we get

$$\frac{2}{n-2} \left[\frac{2\pi}{\hbar\alpha} (\rho + \epsilon - J) \right] = -n\lambda - div\xi. \quad (4.68)$$

And also putting $U = V = \xi$ in eqs. (1.81) and (4.60) and considering the spacetime is in thermal equilibrium condition, we get

$$\begin{aligned} S(\xi, \xi) &= -\frac{r}{2} + \frac{2\pi}{\hbar\alpha} (\rho + \epsilon + J), \\ S(\xi, \xi) &= \lambda. \end{aligned}$$

Then, we have

$$\lambda = -\frac{r}{2} + \frac{2\pi}{\hbar\alpha} (\rho + \epsilon + J) \quad (4.69)$$

Solving eqs. (4.68) and (4.69), we obtain

$$\lambda = \frac{2\pi}{\hbar\alpha} \left[\frac{1}{(n-1)(n-2)} J - \frac{\rho + \epsilon}{n-2} \right] - \frac{1}{n-1} div\xi \quad (4.70)$$

Assuming that ξ is a Killing vector, then

$$\lambda = \frac{2\pi}{\hbar\alpha} \left[\frac{1}{(n-1)(n-2)} J - \frac{\rho + \epsilon}{n-2} \right]$$

This results in the following theorem:

Theorem 4.8. *Let M be a relativistic TFS admitting a Ricci soliton (M, g, X, λ) with a killing vector field ξ , then the soliton is shrinking, steady or expanding according to $J < (n-1)(\rho + \epsilon)$, $J = (n-1)(\rho + \epsilon)$ and $J > (n-1)(\rho + \epsilon)$ respectively.*

Remark 4.1. *Assume that the soliton is shrinking, the timelike convergence con-*

dition implies null convergence condition by Hawking and Ellis, (1973). Take some black holes contained in TFS and cauchy surface S trapped within no naked singularities. Then S is completely surrounded by the event horizon (Vilenkin and Wall, 2014).

A vector field X is said to be a conformal Killing vector field if and only if the following relation holds (Venkatesha and Kumara, 2019):

$$(\mathcal{L}_X g)(U, V) = 2\Omega g(U, V), \quad (4.71)$$

where Ω is some function of the coordinates (conformal scalar). If $\Omega \neq \text{constant}$, then X is termed a proper vector field. When $\Omega = \text{constant}$, X is classified as a homothetic vector field. If the constant $\Omega \neq 0$, X is specifically a proper homothetic Killing vector field. In the case where $\Omega = 0$, X is a Killing vector field.

Assume that X is a conformal Killing vector field. Eqs. (1.81) and (4.71) results in

$$S(U, V) = -(\Omega + \lambda)g(U, V), \quad (4.72)$$

this shows that the spacetime is an Einstein. Conversely, suppose that the spacetime with conformal Killing vector field is an Einstein spacetime, that is, $S(U, V) = ag(U, V)$. Then equation (1.74) becomes

$$(\mathcal{L}_X g)(U, V) = 2\psi g(U, V), \quad (4.73)$$

where $\psi = -(\lambda + a)$.

Now contracting eq. (4.72), we obtain

$$r = -(\Omega + \lambda)n. \quad (4.74)$$

Now, comparing eqs. (4.67) and (4.74), we have

$$\frac{2}{n-2} \left[\frac{2\pi}{\hbar\alpha} (\rho + \epsilon - J) \right] = -(\Omega + \lambda)n. \quad (4.75)$$

Solving eq. (4.75), we get

$$\Omega = -\frac{2}{n(n-2)} \left[\frac{2\pi}{\hbar\alpha} (\rho + \epsilon - J) \right] - \lambda, \quad (4.76)$$

this implies that $\Omega \neq 0$, so X is a proper homothetic vector field. Hence, we state the following:

Theorem 4.9. *In a relativistic thermodynamical fluid spacetime admitting a Ricci soliton, if the potential vector field X is conformal Killing vector field, then X is a proper homothetic vector field.*

Now, we recall the following definition:

Definition 4.2. (Yano, 1944) *A torse forming vector field is a special type of vector field on a Lorentzian manifold. If ζ is a torse forming vector field, then the following condition holds:*

$$\nabla_U \zeta = \omega U + \gamma(U)\zeta, \quad (4.77)$$

where γ denotes a non-zero 1-form, for any $U \in \chi(M)$.

It is observed that a unit timelike torse-forming vector field unit is $\xi = V$ on a n -dimensional Lorentzian manifold has the following form (Yano, 1944)

$$\nabla_U V = \omega [U + \eta(U)V], \quad (4.78)$$

where η is a 1-form and $g(U, V) = \eta(U)$ for all U .

Now, from the foregoing expressions, and taking the Lie derivative of g for relativistic Thermodynamical fluid spacetime along with a unit timelike torse

forming vector ξ yields

$$\mathcal{L}_\xi g(U, V) = 2\omega [g(U, V) + \eta(U)\eta(V)]. \quad (4.79)$$

In view of eq. (1.81), we have

$$(\lambda + \omega)g(U, V) + \eta(U)\eta(V) + S(U, V) = 0. \quad (4.80)$$

Using eq. (4.60) in eq. (4.80), we obtain

$$\begin{aligned} & (\lambda + \omega)g(U, V) + \omega\eta(U)\eta(V) + \frac{r}{2}g(U, V) + \frac{2\pi}{\hbar\alpha}(\rho + \epsilon)\eta(U)\eta(V) \\ & + \frac{4\pi}{\hbar\alpha}\eta(U)\gamma(V) + \frac{2\pi}{\hbar\alpha}\tau(U, V) = 0. \end{aligned} \quad (4.81)$$

Substituting $U = V = \xi$ in eq. (4.81) and considering thermal equilibrium condition, we get

$$\lambda = \frac{2\pi}{\hbar\alpha} - \frac{r}{2}.$$

Thus, we can state the following theorem:

Theorem 4.10. *If a relativistic thermodynamical fluid spacetime with torse-forming vector field ξ admits a Ricci soliton, then the soliton is expanding, steady and shrinking according as $\frac{2\pi}{\hbar\alpha} > \frac{r}{2}$, $\frac{2\pi}{\hbar\alpha} = \frac{r}{2}$ and $\frac{2\pi}{\hbar\alpha} < \frac{r}{2}$ respectively.*

In view of eqs. (1.81), (4.60) and (4.71), we have

$$\begin{aligned} & \Omega g(U, V) + \frac{r}{2}g(U, V) + \frac{2\pi}{\hbar\alpha}(\rho + \epsilon)\eta(U)\eta(V) \\ & + \frac{4\pi}{\hbar\alpha}\eta(U)\gamma(V) + \frac{2\pi}{\hbar\alpha}\tau(U, V) + \lambda g(U, V) = 0. \end{aligned} \quad (4.82)$$

Putting $U = \xi$ in the above equation, we obtain

$$\gamma(V) = \frac{\hbar\alpha}{4\pi} \left[\Omega + \frac{r}{2} - \frac{2\pi}{\hbar\alpha}(\rho + \epsilon) + \lambda \right] \eta(V) \quad (4.83)$$

As a result, we get the following

Theorem 4.11. *A relativistic thermodynamical fluid spacetime with conformal*

Killing vector field, admits thermal flux, provided

$$\Omega + \frac{r}{2} - \frac{2\pi}{\hbar\alpha}(\rho + \epsilon) + \lambda \neq 0.$$

Corollary 4.1. *A relativistic thermodynamical fluid spacetime with Killing vector field, admits thermal flux, provided*

$$\frac{r}{2} - \frac{2\pi}{\hbar\alpha}(\rho + \epsilon) + \lambda \neq 0.$$

4.2.1 Modified Equations in Relativistic Thermodynamical Fluid Spacetime

In this part, we will derive the modified Poisson and Liouville equations that describe a Ricci soliton within the context of a Relativistic Thermodynamical fluid spacetime, taking into account its harmonic nature. The research is critical for precisely modeling and understanding the behavior of matter and energy in severe conditions. By analyzing these modified equations, we can get profound insights into the fundamental nature of the cosmos.

So, taking $\xi = \nabla h$, where ∇h is a gradient of a smooth function ‘ h ’, then eq. (4.70) yields:

Theorem 4.12. *Consider a relativistic thermodynamical fluid spacetime with a Ricci soliton as its metric. Suppose that ξ is of the gradient type, then the modified Poisson equation is*

$$\nabla^2 h = \frac{2\pi}{\hbar\alpha} \left[\frac{1}{n-2} J - \frac{n-1}{n-2} (\rho + \epsilon) \right] - (n-1)\lambda.$$

Consider $h \in C^\infty(M)$ and a vector field ξ , then we have

$$\text{div}(h\xi) = \xi(dh) + h\text{div}\xi.$$

In Riemannian geometry, when considering a smooth function h in the form

$h = \ln|\xi|$ and a vector field ξ , the Liouville equation can be written as

$$\xi(d \ln h) = -\operatorname{div} \xi.$$

Considering these equation in eq. (4.70), the following result arises:

Theorem 4.13. *Consider a thermodynamical spacetime, admitting a Ricci soliton. Assume that ξ is of the gradient type, then the modified Liouville equation is given by*

$$\xi(d \ln h) = (n-1)\lambda - \frac{2\pi}{\hbar\alpha} \left[\frac{1}{n-2}J - \frac{n-1}{n-2}(\rho + \epsilon) \right].$$

Next, if a function h is said to be harmonic, i.e, $\nabla^2 h = 0$. Then, an immediate result from Theorem 4.12 gives:

Corollary 4.2. *Consider a relativistic thermodynamical fluid spacetime admitting a Ricci soliton. Also, assume that the velocity vector field ξ is gradient and h is harmonic, then the spacetime is either decreasing, steady, or expanding behaviour, depending on*

1. $J < (n-1)(\rho + \epsilon)$,
2. $J = (n-1)(\rho + \epsilon)$, or
3. $J > (n-1)(\rho + \epsilon)$,

respectively.

4.2.2 Thermodynamical Fluid Spacetime and Gradient Ricci Soliton

Studying gradient Ricci solitons is crucial for understanding spacetime geometry, stability, and evolution under the Ricci flow. They provide insights into long-term behavior, contribute to general relativity and cosmology, and offer

valuable mathematical and physical interpretations, particularly in relation to thermal effects and geometric analysis.

Consider $X = Dh$ where D is the gradient operator. Then, (1.81) can be expressed as

$$\nabla_U Dh + QU + \lambda U = 0, \quad (4.84)$$

for any vector field U . Utilizing the relationship

$$R(U, V)Dh = \nabla_U \nabla_V Dh - \nabla_V \nabla_U Dh - \nabla_{[U, V]} Dh, \quad (4.85)$$

for any vector fields U and V . Equation (4.84) transforms into

$$R(U, V)Dh = (\nabla_U Q)V - (\nabla_V Q)U. \quad (4.86)$$

Differentiating (4.84) covariantly along V results in

$$\nabla_V \nabla_U Dh = -[(\nabla_V Q)U - Q(\nabla_V U)] - \lambda \nabla_V U. \quad (4.87)$$

Swapping U and V in the above equation results in

$$\nabla_U \nabla_V Dh = -[(\nabla_U Q)V - Q(\nabla_U V)] - \lambda \nabla_U V. \quad (4.88)$$

Now, the Ricci tensor can be expressed as

$$QU = \frac{\pi}{\hbar\alpha}(J - \rho - \epsilon)\eta(U)\xi. \quad (4.89)$$

Again, differentiating (4.89) covariantly along U , we obtain

$$(\nabla_U Q)(V) = \frac{\pi}{\hbar\alpha}(J - \rho - \epsilon)[(\nabla_U \eta)(V)\xi + \eta(V)\nabla_U \xi]. \quad (4.90)$$

By virtue of (4.86) and (4.90), we have

$$\begin{aligned} R(U, V)Dh &= \frac{\pi}{\hbar\alpha}(J - \rho - \epsilon)[(\nabla_U \eta)(V)\xi + \eta(V)\nabla_U \xi - (\nabla_V \eta)(U)\xi \\ &\quad - \eta(U)\nabla_V \xi]. \end{aligned} \quad (4.91)$$

Now, contracting the above expression results in

$$S(V, Dh) = \frac{\pi}{\hbar\alpha}(J - \rho - \epsilon)[(\nabla_\xi\eta)(V) - (\nabla_V\eta)(\xi) + \eta(V)\text{div}\xi]. \quad (4.92)$$

Also, from the expression of the Ricci tensor, we have

$$S(V, Dh) = \frac{\pi}{\hbar\alpha}(J + \rho + \epsilon)V(h) + \frac{2\pi}{\hbar\alpha}(\rho + \epsilon)\eta(V)\xi(h). \quad (4.93)$$

Putting $V = \xi$ in (4.92) and (4.93) and then comparing their results gives

$$\frac{\pi}{\hbar\alpha}(J - \rho - \epsilon)\xi(h) = \frac{\pi}{\hbar\alpha}(\rho + \epsilon - J) \text{div}\xi. \quad (4.94)$$

Let us assume now that ξ is Killing, i.e., $\mathcal{L}_\xi g = 0$. Therefore, this implies that $\text{div}\xi = 0$. Hence, we have from (4.94) that $\xi(h) = 0$. Then, differentiating $g(\xi, Dh) = 0$ covariantly along U with $S(U, \xi) = \frac{\pi}{\hbar\alpha}(J - \rho - \epsilon)\eta(U)$ and (4.84) gives

$$g(\nabla_U \xi, Dh) = -[\lambda + \frac{\pi}{\hbar\alpha}(J - \rho - \epsilon)]\eta(U). \quad (4.95)$$

Since ξ is Killing, hence, we have $g(\nabla_U \xi, V) + g(U, \nabla_V \xi) = 0$. Again, by setting $V = \xi$, we get $g(U, \nabla_\xi \xi) = 0$ since $g(\nabla_U \xi, \xi) = 0$. Therefore, substituting $U = \xi$ in (4.95) gives

$$\lambda = \frac{\pi}{\hbar\alpha}(\rho + \epsilon - J). \quad (4.96)$$

Hence, we state:

Theorem 4.14. *Consider that a thermodynamical fluid spacetime admits a gradient Ricci soliton. Assuming the velocity vector field ξ is Killing, then the expression for the λ is determined by (4.96), or alternatively, the soliton behaves as shrinking, steady, or expanding based on:*

1. $J < \rho + \epsilon$,
2. $J = \rho + \epsilon$, or
3. $J > \rho + \epsilon$,

respectively.

Remark 4.2. *From the above theorem, we see that the trace of the thermal stress tensor τ solely depends on the thermal energy density ϵ and effective density ρ . Also, when the spacetime is steady, the trace of the thermal stress tensor is equal to the sum of the thermal energy density and effective density. In other words, this result shows that the total pressure or stress in a system, influenced by heat, can be directly determined by the amount of thermal energy and mass present. In a stable, unchanging situation, the pressure is simply the sum of these two factors, making it easier to predict how the system will behave. This understanding is crucial for designing stable systems in various fields, from engineering to astrophysics, by linking energy and mass to the forces within the system.*

4.2.3 Example of thermodynamical fluid spacetime

In this subsection we illustrate an example of Thermodynamical fluid spacetime satisfying one of the theorem.

Example 4.1. *Consider $\mathcal{M} = \{(u, v, w, t) \in \mathbb{R}^4 : t \neq 0\}$, where (u, v, w, t) are the local coordinates of $\mathbb{R}^4$.*

Assume that (e_1, e_2, e_3, e_4) are the set of linearly independent vector fields of $\mathcal{M}$, and is defined as:

$$e_1 = t \frac{\partial}{\partial u}, \quad e_2 = \left(u \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right), \quad e_3 = t \frac{\partial}{\partial w}, \quad e_4 = -t \frac{\partial}{\partial t}$$

The Riemannian metric g of $\mathcal{M}$ is defined as the following:

$$\begin{aligned} g(e_1, e_1) &= g(e_2, e_2) = g(e_3, e_3) = 1, & g(e_4, e_4) &= -1 \\ g(e_i, e_j) &= 0 \quad \text{if } i \neq j, \quad i, j = 1, 2, 3, 4. \end{aligned}$$

Taking the 1-form η defined by $\eta(W) = g(W, e_4)$ for any $W \in \chi(\mathcal{M})$. Further-

more, the $(1,1)$ -tensor ϕ is defined by

$$\phi(e_1) = e_1, \quad \phi(e_2) = e_2, \quad \phi(e_3) = e_3, \quad \phi(e_4) = 0,$$

Thus, by the linearity of ϕ and g , we obtain:

$$[e_1, e_2] = e_1, \quad [e_1, e_4] = e_1, \quad [e_2, e_4] = 0, \quad [e_3, e_4] = e_3.$$

Then, for $e_4 = \xi$ and using Koszul's formula for Lorentzian metric g , we obtain

$$\nabla_{e_1} e_1 = -e_4 - e_2, \quad \nabla_{e_1} e_2 = e_1, \quad \nabla_{e_1} e_4 = e_1,$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_4 = 0,$$

$$\nabla_{e_3} e_3 = e_4, \quad \nabla_{e_3} e_4 = e_3,$$

where ∇ denotes the Levi-Civita connection with respect to the Lorentzian metric g . The non-zero components of Riemannian curvature and the Ricci tensors are shown below:

$$R(e_1, e_4)e_1 = e_4 - e_2, \quad R(e_3, e_4)e_3 = -e_4,$$

$$R(e_1, e_3)e_3 = e_1, \quad R(e_1, e_3)e_1 = -e_3,$$

$$R(e_1, e_4)e_4 = -e_1, \quad R(e_1, e_2)e_2 = e_1,$$

$$R(e_3, e_4)e_4 = -e_3, \quad R(e_1, e_2)e_1 = e_2 - e_4.$$

Now, from these expressions, it can be easily verified that the non-vanishing Ricci tensor S are

$$S(e_1, e_1) = -1, \quad S(e_2, e_2) = 1.$$

Similarly, we get

$$S(e_3, e_3) = 0, \quad S(e_4, e_4) = -2.$$

Therefore,

$$R = \sum_{i,j=1}^4 S(e_i, e_j) = -2.$$

In view of eqs. (4.60) and (4.72), we have

$$\left(\frac{r}{2} + \Omega + \lambda\right) g(e_i + e_i) + \frac{2\pi}{\hbar\alpha}(\rho + \epsilon)\eta(e_i)\eta(e_i) + \frac{4\pi}{\hbar\alpha}\eta(e_i)q(e_i) + \frac{2\pi}{\hbar\alpha}\tau(e_i, e_i) = 0,$$

for all $i \in \{1, 2, 3, 4\}$, and we have:

$$\left(\frac{r}{2} + \Omega + \lambda\right) g(e_i + e_i) + \frac{2\pi}{\hbar\alpha}(\rho + \epsilon)\delta_{i4} = 0,$$

for all $i \in \{1, 2, 3, 4\}$, we get $\Omega = \frac{4\pi}{\hbar\alpha}(\rho + \epsilon) - \frac{r}{2} - \lambda$. This means that $\Omega = \frac{4\pi}{\hbar\alpha}(\rho + \epsilon) - \lambda + 1$. It is obvious that $\Omega \neq 0$ and is a constant, therefore the homothetic vector field X must be proper. This satisfies Theorem 4.9.

4.3 Conclusion

In Section 4.1, the study suggests that in a Lorentzian Para-Kenmotsu manifold, a symmetric parallel second-order covariant tensor is proportional to the metric tensor. Additionally, if $\mathcal{L}_V g + 2S$ is parallel, where V is a vector field, then (g, V, λ) is a Ricci soliton. The study further states that a Ricci soliton in a W_2 -semi-symmetric Lorentzian Para-Kenmotsu manifold is expanding, whereas a Ricci soliton satisfying the condition $W_2(\xi, X) \circ S = 0$ in an Lorentzian Para-Kenmotsu manifold is steady. The study concludes by analyzing certain curvature properties of Lorentzian Para-Kenmotsu manifolds admitting Ricci soliton and provides an example of a 3-dimensional Lorentzian Para-Kenmotsu manifold.

Section 4.2 deals with Ricci solitons on relativistic thermodynamical fluid spacetime. The study of Ricci solitons within the framework of relativistic thermodynamical fluid spacetime (TFS) uncovers a range of important insights into how matter and energy behave in extreme conditions, such as those found in cos-

mological models and black hole physics. By integrating thermodynamic principles with spacetime geometry, this research highlights the crucial role that thermal properties play in influencing the dynamics of spacetime.

One of the key findings of this study is the ability of Ricci solitons to drive the expansion, steadiness, or contraction of spacetime, depending on the interplay between thermal energy density and effective density. This behavior is particularly significant because it demonstrates how thermodynamic factors, such as heat flux and stress, directly influence the overall structure and evolution of the universe. The generalized Ricci recurrent structure observed in TFS under thermal effects is essential for understanding cosmological phenomena, suggesting that the inclusion of thermodynamical properties is vital for a more accurate and comprehensive model of spacetime. Moreover, the study reveals interesting properties when different vector fields, such as conformal Killing and torse-forming vector fields, are introduced into the analysis. These fields provide deeper insight into the conditions necessary for spacetime to exhibit Ricci soliton behavior. For instance, when the potential vector field X is conformal Killing, the spacetime not only admits a Ricci soliton but also behaves as a proper homothetic vector field. This result has far-reaching implications, suggesting that the presence of specific vector fields can dictate the solitonic nature of spacetime, which is critical for understanding its geometric structure and evolution.

We also derive modified Poisson and Liouville equations within the TFS framework, highlighting their significance in modeling the behavior of matter and energy in environments such as black holes. These equations account for the harmonic nature of the fluid spacetime and provide a more precise description of how spacetime evolves under the influence of thermal properties. Notably, when the velocity vector field associated with the Ricci soliton is of the gradient type, the spacetime can exhibit decreasing, steady, or expanding behavior, depending on the relative values of thermal energy density and effective density. This

demonstrates the importance of understanding the thermodynamical aspects of spacetime to accurately predict its behavior in extreme conditions.

Moreover, we have constructed an example to illustrate how Ricci solitons behave within a four-dimensional thermodynamical fluid spacetime (TFS). By defining a specific spacetime model with standard coordinates and a Riemannian metric, we have simplified the calculations and explored how vector fields interact with the geometric structure. We have applied Theorem 4.9 to show how thermal properties affect the Ricci soliton structure, validating the theoretical framework and demonstrating the complex relationship between geometry and thermodynamics. This example bridges abstract theory with practical application, providing educational value and insights into the impact of thermal energy on spacetime dynamics.

This study underscores the necessity of studying spacetime using a thermodynamical approach, as it provides a more holistic understanding of how thermal properties influence spacetime geometry and dynamics. By exploring Ricci solitons in the context of relativistic thermodynamical fluid spacetime, the study offers critical insights into the role of thermodynamics in shaping the universe's structure and evolution. The findings demonstrate that thermal properties, such as energy density and stress, are not merely passive components but actively contribute to the dynamics of spacetime. This integration of thermodynamics with general relativity enriches the theoretical framework for understanding cosmological phenomena and the behavior of black holes, providing a more comprehensive model that accounts for the complex interplay of forces at work in the universe.

Chapter 5

Properties of Hyper Generalized manifold

This chapter is divided into two sections. First section deals with the geometrical properties of hyper generalised ϕ -recurrent LP-Sasakian manifold and also quasi generalized ϕ -recurrent LP-Sasakian manifold with their various properties. And second section investigated hyper generalised weakly symmetric Para-Sasakian manifold.

5.1 Certain geometrical properties of generalized ϕ -recurrent LP-Sasakian manifold

After Cartan's (1926) introduction of locally symmetric spaces, many authors introduced weaker version of symmetric spaces, one of which is a hyper generalised recurrent manifold which is an extension of a generalised recurrent manifold (Dubey, 1979). A Riemannian manifold is said to be hyper generalised

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recurrent manifold if its curvature tensor R satisfies the condition

$$\nabla R = A \otimes R + B \otimes (g\Lambda S), \quad (5.1)$$

where A and B are 1-forms defined in (1.5).

Recently, Venkatesha et al. (2019) extended the notion of hyper generalised recurrent manifolds (resp. quasi generalized recurrent manifolds) to hyper generalised ϕ -recurrent Sasakian manifolds (resp. quasi generalized ϕ -recurrent manifolds). After this, Khatri and Singh (2020) investigated hyper generalised ϕ -recurrent (k, μ) -contact metric manifolds as well as quasi generalised ϕ -recurrent (k, μ) -contact metric manifolds. Continuing this, we studied hyper generalised ϕ -recurrent LP-Sasakian manifolds and prove its existence by giving a proper example. Similarly, quasi generalised ϕ -recurrent LP-Sasakian manifolds are investigated.

5.1.1 Hyper generalised ϕ -recurrent LP-Sasakian manifold

Definition 5.1. A $(2n + 1)$ -dimensional LP-Sasakian manifold M^{2n+1} is called a hyper generalized ϕ -recurrent if its curvature tensor R satisfies

$$\begin{aligned} \phi^2(\nabla_U R)(X, Y)Z &= A(U)R(X, Y)Z \\ &+ B(U)L(X, Y)Z \end{aligned} \quad (5.2)$$

for all $X, Y, Z, U \in \chi(M^{2n+1})$, where $A(U) = g(U, \rho_1)$ and $B(U) = g(U, \rho_2)$ are 1-forms and the tensor L is defined by

$$\begin{aligned} L(X, Y)Z &= S(Y, Z)X - S(X, Z)Y \\ &+ g(Y, Z)QX - g(X, Z)QY. \end{aligned} \quad (5.3)$$

If the 1-form B vanishes, then equation (5.2) reduces to the notion of ϕ -recurrent manifolds.

Theorem 5.1. *Let M^{2n+1} be a hyper generalized ϕ -recurrent LP-Sasakian manifold of dimension $(2n + 1)$ then*

- i). M^{2n+1} is Ricci recurrent if the scalar curvature is zero everywhere on M^{2n+1} ,
- ii). M^{2n+1} is generalized Ricci recurrent if the scalar curvature is non zero everywhere on M^{2n+1} .

Proof. Let us take a hyper generalized ϕ -recurrent LP-Sasakian manifold M^{2n+1} of dimension $(2n + 1)$.

By making use of equation (1.37) in equation (5.2) we obtain

$$\begin{aligned}
 (\nabla_U R)(X, Y)Z + \eta((\nabla_U R)(X, Y)Z)\xi \\
 = A(U)R(X, Y)Z \\
 + B(U)L(X, Y)Z.
 \end{aligned} \tag{5.4}$$

Taking inner product on both sides of (5.4) with respect to V , we get

$$\begin{aligned}
 g((\nabla_U R)(X, Y)Z, V) + \eta((\nabla_U R)(X, Y)Z)g(\xi, V) \\
 = A(U)g(R(X, Y)Z, V) \\
 + B(U)g(L(X, Y)Z, V).
 \end{aligned} \tag{5.5}$$

Contracting over X and V and then using equations (5.3), (1.47), we get

$$\begin{aligned}
 (\nabla_U S)(Y, Z) + \eta((\nabla_U R)(\xi, Y)Z) \\
 = \{A(U) + (2n - 1)B(U)\}S(Y, Z) \\
 + rB(U)g(Y, Z).
 \end{aligned} \tag{5.6}$$

We know that,

$$\eta((\nabla_U R)(\xi, Y)Z) = g((\nabla_U R)(\xi, Y)Z, \xi). \quad (5.7)$$

By using equations (1.41), (1.44) and the relation

$$g((\nabla_U R)(X, Y)Z, V) = -g((\nabla_U R)(X, Y)V, Z), \text{ we obtain}$$

$$g((\nabla_U R)(\xi, Y)Z, \xi) = 0. \quad (5.8)$$

By making use of (5.7) and (5.8) in (5.6), we have

$$(\nabla_U S)(Y, Z) = V(U)S(Y, Z) + \psi(U)g(Y, Z) \quad (5.9)$$

where $V(U) = (A(U) + (2n - 1)B(U))$ and $\psi(U) = rB(U)$. From equation (5.9), we can say that whether a hyper generalized ϕ -recurrent LP-Sasakian manifold is Ricci recurrent or generalized Ricci recurrent depending on if the scalar curvature is zero or non-zero respectively everywhere on M^{2n+1} . Hence, this completes the proof. $\square$

Theorem 5.2. *A hyper generalized ϕ -recurrent LP-Sasakian manifold M^{2n+1} with non vanishing scalar curvature is an Einstein manifold and moreover the associated vector fields ρ_1 and ρ_2 of the 1-forms A and B respectively are co-directional.*

Proof. Putting $Z = \xi$ in equation (5.9) and then using equation (1.45), we obtain

$$\begin{aligned} (\nabla_U S)(Y, \xi) &= 2n(A(U) + (2n - 1)B(U))\eta(Y) \\ &\quad + rB(U)\eta(Y). \end{aligned} \quad (5.10)$$

We know that

$$(\nabla_U S)(Y, \xi) = \nabla_U S(Y, \xi) - S(\nabla_U Y, \xi) - S(Y, \nabla_U \xi). \quad (5.11)$$

In view of (1.41) and (1.45) in (5.11), it follows that

$$(\nabla_U S)(Y, \xi) = 2ng(Y, \phi U) - S(Y, \phi U). \quad (5.12)$$

Comparing (5.10) and (5.12), we have

$$\begin{aligned} 2ng(Y, \phi U) - S(Y, \phi U) &= 2n[A(U) + (2n - 1)B(U)]\eta(Y) \\ &\quad + rB(U)\eta(Y). \end{aligned} \quad (5.13)$$

Putting ϕY instead of Y in (5.13) and using (1.38), (1.40) and (1.46), we have

$$S(Y, U) = 2ng(Y, U). \quad (5.14)$$

Contracting Y and U in the above equation, we obtain

$$r = 2n(2n + 1). \quad (5.15)$$

Substituting Y by ξ in (5.13), we have

$$A(U) = - \left[\frac{2n(2n - 1) + r}{2n} \right] B(U). \quad (5.16)$$

Using (5.15) in (5.16) we get

$$A(U) = -4nB(U). \quad (5.17)$$

From (5.14) and (5.17), we obtain the statement of the theorem. $\square$

We know that an LP-Sasakian manifold is Ricci semisymmetric if and only if it is Einstein manifold (Shaikh and Baishya, 2006). So from the above theorem, we get the following corollary:

Corollary 5.1. *A hyper generalized ϕ -recurrent LP-Sasakian manifold with non vanishing scalar curvature is Ricci Semisymmetric.*

Theorem 5.3. *A hyper generalized ϕ -recurrent LP-Sasakian manifold of non vanishing scalar curvature is a space of constant curvature +1.*

Proof. In LP-Sasakian manifold it is easy to verify the following equation

$$\begin{aligned} (\nabla_U R)(X, Y)\xi &= g(U, \phi Y)X - g(U, \phi X)Y \\ &\quad - R(X, Y)\phi U. \end{aligned} \quad (5.18)$$

Using (1.43) in equation (5.18), we have

$$\eta(\nabla_U R)(X, Y)\xi = 0. \quad (5.19)$$

From equations (5.4) and (5.19), we obtain

$$(\nabla_U R)(X, Y)\xi = 0. \quad (5.20)$$

Comparing equations (5.18) and (5.20), we get

$$g(U, \phi Y)X - g(U, \phi X)Y = R(X, Y)\phi U. \quad (5.21)$$

In LP-Sasakian manifold the following relation holds(Shaikh and Baishya, 2006):

$$\begin{aligned} -R(X, Y)\phi U &= g(Y, U)\phi X + g(X, \phi U)Y \\ &\quad + 2g(X, \phi U)\eta(Y)\xi + 2\eta(Y)\eta(U)\phi X \\ &\quad - g(X, U)\phi Y - g(Y, \phi U)X \\ &\quad - 2g(Y, \phi U)\eta(X)\xi - 2\eta(X)\eta(U)\phi Y \\ &\quad - \phi R(X, Y)U. \end{aligned} \quad (5.22)$$

In view of equations (5.21) and (5.22), we have

$$\begin{aligned} &g(Y, U)\phi X + 2g(X, \phi U)\eta(Y)\xi + 2\eta(Y)\eta(U)\phi X \\ &\quad - g(X, U)\phi Y - 2g(Y, \phi U)\eta(X)\xi - 2\eta(X)\eta(U)\phi Y \\ &= \phi R(X, Y)U. \end{aligned} \quad (5.23)$$

Let X and Y are orthogonal to ξ , then equation (5.23) gives

$$g(Y, U)\phi X - g(X, U)\phi Y = \phi R(X, Y)U. \quad (5.24)$$

Operating ϕ on both sides of the above equation and using (1.37), we get

$$g(Y, U)X - g(X, U)Y = R(X, Y)U. \quad (5.25)$$

Hence, the theorem is proved. $\square$

Theorem 5.4. *In a hyper generalized ϕ -recurrent LP-Sasakian manifold, the associated vector fields ρ_1 and ρ_2 corresponding to 1-forms A and B satisfy the relation*

$$(r - 4n)\eta(\rho_1) + 2(2n - 1)(r - 2n)\eta(\rho_2) = 0.$$

Proof. Changing X, Y, Z cyclically in (5.5) and using Bianchi's identity we get

$$\begin{aligned} & A(U)g(R(X, Y)Z, V) + A(X)g(R(Y, U)Z, V) \\ & + A(Y)g(R(U, X)Z, V) + B(U)g(L(X, Y)Z, V) \\ & + B(X)g(L(Y, U)Z, V) + B(Y)g(L(U, X)Z, V) = 0 \end{aligned} \quad (5.26)$$

Contracting over Y and Z in (5.26) and using (5.3) we have

$$\begin{aligned} & A(U)S(X, V) - A(X)S(U, V) + g(R(V, X)\rho_1, V) \\ & + B(W)\{rg(X, V) + (2n - 1)S(X, V)\} \\ & - B(X)\{rg(U, V) + (2n - 1)S(U, V)\} \\ & + S(U, \rho_2)g(U, V) - S(U, \rho_2)g(X, V) \\ & + B(X)S(U, V) - B(U)S(X, V) = 0. \end{aligned} \quad (5.27)$$

Again contracting over X and V in (5.27), we obtain

$$\begin{aligned} & rA(U) - 2S(U, \rho_1) + 2r(2n - 1)B(U) \\ & - 2(2n - 1)S(U, \rho_2) = 0. \end{aligned} \quad (5.28)$$

Putting $U = \xi$ in equation (5.28) and using (1.45) we have

$$(r - 4n)\eta(\rho_1) + 2(2n - 1)(r - 2n)\eta(\rho_2) = 0. \quad (5.29)$$

This gives the proof. $\square$

Using (5.17) in (5.28) we obtain the following corollary:

Corollary 5.2. *In a hyper generalized ϕ -recurrent LP-Sasakian manifold, $\frac{8n^2-r}{(2n-1)}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ_2 .*

5.1.2 An Example of generalized ϕ -recurrent LP-Sasakian manifold

In this section, we establish an example of hyper generalized ϕ -recurrent LP-Sasakian manifold. Let us consider a 3-dimensional manifold $M^3 = \{(a, b, c) \in \mathbb{R}^3 : a \neq 0\}$ where (a, b, c) are the standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be linearly independent vector fields in M^3 which satisfy

$$[E_1, E_2] = e^a E_1, [E_2, E_3] = 0, [E_1, E_3] = 0.$$

Let us define a Lorentzian metric g as follows

$$g(E_1, E_1) = 1, g(E_2, E_2) = 1, g(E_3, E_3) = -1,$$

$$g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0.$$

Let η be the 1-form defined by

$$\eta(X) = g(X, E_3)$$

for any vector field X . Let ϕ be (1,1)-tensor field defined by

$$\phi E_1 = -E_1, \phi E_2 = -E_2, \phi E_3 = 0.$$

Then we get

$$\eta(E_3) = -1, \phi(X) = X + \eta(X)E_3$$

and

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y).$$

for any $X, Z \in \chi(M^3)$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines a Lorentzian para-contact stucture on M^3 .

Let ∇ be the Levi-Civita connection with respect to Lorentzian metric g . Then using Koszul formula we obtain

$$\nabla_{E_1} E_1 = -e^a E_2, \nabla_{E_1} E_2 = e^a E_1, \nabla_{E_1} E_3 = 0,$$

$$\nabla_{E_2} E_1 = 0, \nabla_{E_2} E_2 = 0, \nabla_{E_2} E_3 = 0,$$

$$\nabla_{E_3} E_1 = 0, \nabla_{E_3} E_2 = 0, \nabla_{E_3} E_3 = 0.$$

It is clear from the preceding equations that for $E_3 = \xi$, (ϕ, ξ, η, g) is an LP-Sasakian structure on M^3 . As a result, $M^3(\phi, \xi, \eta, g)$ is a LP-Sasakian manifold. Now, we will show that it is a 3-dimensional hyper generalized ϕ -recurrent LP-Sasakian manifold. The non-vanishing components of the curvature tensor, Ricci tensor and Ricci operator are

$$R(E_1, E_2)E_1 = e^a E_2, \quad R(E_1, E_2)E_2 = -e^a E_1,$$

$$S(E_1, E_1) = S(E_2, E_2) = -e^{2a},$$

$$QE_1 = -e^{2a} E_1, \quad QE_2 = -e^{2a} E_2.$$

Since $\{E_1, E_2, E_3\}$ forms the orthonormal basis of a 3-dimensional LP-Sasakian manifold, any vector field can be expressed as

$$X = x_1 E_1 + y_1 E_2 + z_1 E_3,$$

$$Y = x_2 E_1 + y_2 E_2 + z_2 E_3,$$

$$Z = x_3 E_1 + y_3 E_2 + z_3 E_3.$$

Then,

$$R(X, Y)Z = u_1 E_1 + u_2 E_2, \tag{5.30}$$

where $u_1 = e^{2a}y_3(x_2y_1 - x_1y_2)$ and $u_2 = -e^{2a}Z(x_2y_1 - x_1y_2)$.

And

$$L(X, Y)Z = v_1E_1 + v_2E_2 + v_3E_3, \quad (5.31)$$

where

$$v_1 = e^{2a} [2y_3(x_2y_1 - x_1y_2) + z_3(x_1z_2 - x_2z_1)],$$

$$v_2 = e^{2a} [2x_3(x_1y_2 - x_2y_1) + z_3(y_1z_2 - y_2z_1)],$$

$$v_3 = e^{2a} [y_3(z_2y_1 - z_1y_2) + x_3(x_1z_2 - x_2z_1)].$$

In view of (5.30), we have the following

$$\begin{aligned} (\nabla_{E_1}R)(X, Y)Z &= e^{3a}(x_1y_2 - x_2y_1)(x_3E_1 + y_3E_2), \\ (\nabla_{E_2}R)(X, Y)Z &= 0, \\ (\nabla_{E_3}R)(X, Y)Z &= 0. \end{aligned} \quad (5.32)$$

From (5.30) we can easily obtain the following

$$\phi^2(\nabla_{E_i}R)(X, Y)Z = p_iE_1 + q_iE_2, \quad i = 1, 2, 3, \quad (5.33)$$

where $p_1 = e^{3a}x_3(x_1y_2 - y_2x_1)$, $q_1 = e^{3a}y_3(x_1y_2 - y_2x_1)$, $p_2 = 0$, $q_2 = 0$, $p_3 = 0$, $q_3 = 0$.

Let the 1-form be defined as

$$\begin{aligned} A(E_1) &= \frac{v_1q_1 - p_1v_2}{v_1u_2 - u_1v_2}, \quad B(E_1) = \frac{p_1u_2 - u_1q_1}{v_1u_2 - u_1v_2}, \\ A(E_2) &= 0, \quad B(E_2) = 0, \\ A(E_3) &= 0, \quad B(E_3) = 0 \end{aligned} \quad (5.34)$$

satisfying $v_1q_1 - p_1v_2 \neq 0$, $v_1u_2 - u_1v_2 \neq 0$, $p_1u_2 - u_1q_1 \neq 0$ and $v_3 \neq 0$. By virtue of equations (5.30), (5.31) and (5.33) we can easily show the following

relation:

$$\begin{aligned}\phi^2(\nabla_{E_i}R)(X, Y)Z &= A(E_i)R(X, Y)Z + B(E_i)L(X, Y)Z, \\ i &= 1, 2, 3.\end{aligned}\tag{5.35}$$

Hence the manifold under consideration is a 3-dimensional hyper generalized ϕ -recurrent LP-Sasakian manifold which is neither ϕ -symmetric nor ϕ -recurrent. This leads to the following:

Theorem 5.5. *There exists a 3-dimensional hyper generalized ϕ -recurrent LP-Sasakian manifold which is neither ϕ -symmetric nor ϕ -recurrent.*

5.1.3 Quasi generalized ϕ -recurrent LP-Sasakian manifold

Definition 5.2. *An LP-Sasakian manifold M^{2n+1} is called quasi generalized ϕ -recurrent manifold if its curvature tensor R satisfies the condition*

$$\begin{aligned}\phi^2((\nabla_U R)(X, Y)Z) &= C(U)R(X, Y)Z \\ &+ D(U)F(X, Y)Z\end{aligned}\tag{5.36}$$

for all $X, Y, Z, U \in \chi(M^{2n+1})$, where λ and γ are vector fields associated with two 1-forms C and D respectively and the tensor F is defined by

$$\begin{aligned}F(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + \eta(Y)\eta(Z)X \\ &- \eta(X)\eta(Z)Y + g(Y, Z)\eta(X)\xi \\ &- g(X, Z)\eta(Y)\xi.\end{aligned}\tag{5.37}$$

If the 1-form D vanishes, then (5.36) becomes the notion of ϕ -recurrent manifold.

Theorem 5.6. *A $(2n+1)$ -dimensional quasi generalized ϕ -recurrent LP-Sasakian manifold M^{2n+1} is an Einstein manifold with Einstein factor $2n$.*

Proof. We consider quasi generalized ϕ -recurrent LP-Sasakian manifold M^{2n+1}

of dimension $(2n + 1)$. Using (1.37) in (5.36) we get

$$\begin{aligned} (\nabla_U R)(X, Y)Z + \eta((\nabla_U R)(X, Y)Z)\xi \\ = C(U)R(X, Y)Z + D(U)F(X, Y)Z. \end{aligned} \quad (5.38)$$

Taking inner product on both sides of (5.38) with respect to V , we get

$$\begin{aligned} g((\nabla_U R)(X, Y)Z, V) + \eta((\nabla_U R)(X, Y)Z)g(\xi, V) \\ = C(U)g(R(X, Y)Z, V) + D(U)g(F(X, Y)Z, V). \end{aligned} \quad (5.39)$$

Contracting over X and V and then using (5.37), we get

$$\begin{aligned} (\nabla_U S)(Y, Z) = C(U)S(Y, Z) + (2n - 1)D(U)g(Y, Z) \\ + (2n - 1)D(U)\eta(Y)\eta(Z). \end{aligned} \quad (5.40)$$

Setting $Z = \xi$ in (5.40) and then using (1.37), (1.39) and (1.45), we obtain

$$(\nabla_U S)(Y, \xi) = 2nC(U)\eta(Y). \quad (5.41)$$

Comparing (5.12) and (5.41), we have

$$2ng(Y, \phi U) - S(Y, \phi U) = 2nC(U)\eta(Y). \quad (5.42)$$

Putting ϕY instead of Y in (5.42) and using (1.38), (1.41) and (1.46), we have

$$S(Y, U) = 2ng(Y, U). \quad (5.43)$$

This completes the proof of the theorem. □

Equation (5.40) yields the following corollary:

Corollary 5.3. *A quasi generalized ϕ -recurrent LP-Sasakian manifold M^{2n+1} is a super generalized Ricci-recurrent manifold.*

Also from (5.40), it leads to the following:

Corollary 5.4. *In a quasi-generalized ϕ -recurrent LP-Sasakian manifold M^{2n+1} with non-zero constant scalar curvature, the relation of the associated 1-forms C and D is given by*

$$rC(U) + 2n(2n - 1)D(U) = 0.$$

Theorem 5.7. *A quasi generalized ϕ -recurrent LP-Sasakian manifold is a quasi-generalized Ricci-recurrent manifold.*

Proof. Suppose that quasi generalized ϕ -recurrent LP-Sasakian manifold M^{2n+1} is quasi generalized Ricci-recurrent (Shaikh and Roy, 2010). Then in (5.40) we see that $b = 2n - 1 = c$, which is true. Hence proof of the theorem is completed. $\square$

We recall some definition given in Venkatesha et al., (2019).

Definition 5.3. *In a $(2n + 1)$ dimensional LP-Sasakian manifold M^{2n+1} , the $*$ -Ricci tensor and $*$ -scalar curvature of M^{2n+1} are defined respectively by Blair and Yildirim, (2016).*

$$S^*(X, Y) = \sum_{i=1}^{2n+1} g(R(X, e_i)\phi e_i, \phi Y) \text{ and } r^* = \sum_{i=1}^{2n+1} S^*(e_i, e_i). \quad (5.44)$$

Definition 5.4. *An LP-Sasakian manifold M^{2n+1} is known as weakly ϕ -Einstein if $S^\phi(X, Y) = \nu g^\phi(X, Y)$, for some function ν . Here S^ϕ is the symmetric part of S^* , i.e.,*

$$S^\phi(X, Y) = \frac{1}{2}S^*(X, Y) + S^*(Y, X),$$

where S^ϕ is the ϕ -Ricci tensor on M^{2n+1} and g^ϕ is defined by $g^\phi(X, Y) = g(\phi X, \phi Y)$. When ν is constant, M^{2n+1} is called ϕ -Einstein (Cho and Inoguchi, 2010).

Theorem 5.8. *A quasi generalized ϕ -recurrent LP-Sasakian manifold M^{2n+1} is a ϕ -Einstein manifold.*

Proof. In LP-Sasakian manifold we can easily verify that

$$(\nabla_U R)(X, Y)\xi = g(U, \phi Y)X - g(U, \phi X)Y$$

$$- R(X, Y)\phi U. \quad (5.45)$$

Using (5.45) and the relation

$$g((\nabla_U R)(X, Y)Z, \xi) = -g((\nabla_U R)(X, Y)\xi, Z)$$

in (5.38), we get

$$\begin{aligned} & (\nabla_U R)(X, Y)Z \\ &= g(U, \phi Y)g(X, Z)\xi - g(U, \phi X)g(Y, Z)\xi \\ & \quad - g(R(X, Y)\phi U, Z)\xi + C(U)R(X, Y)Z \\ & \quad + D(U)F(X, Y)Z. \end{aligned} \quad (5.46)$$

From the above equation, it follows that

$$\begin{aligned} & g((\nabla_U R)(X, Y)Z, V) \\ &= g(U, \phi Y)g(X, Z)\eta(V) - g(U, \phi X)g(Y, Z)\eta(V) \\ & \quad - g(R(X, Y)Z, \phi U)\eta(V) + C(U)g(R(X, Y)Z, V) \\ & \quad + D(U)g(F(X, Y)Z, V). \end{aligned} \quad (5.47)$$

Putting Z by ϕZ in (5.47), we obtain

$$\begin{aligned} & g((\nabla_U R)(X, Y)\phi Z, V) \\ &= g(U, \phi Y)g(X, \phi Z)\eta(V) - g(U, \phi X)g(Y, \phi Z)\eta(V) \\ & \quad - g(R(X, Y)\phi Z, \phi U)\eta(V) + C(U)g(R(X, Y)\phi Z, V) \\ & \quad + D(U)g(F(X, Y)\phi Z, V). \end{aligned} \quad (5.48)$$

Since $g(R(X, Y)\phi Z, V) = g(R(X, Y)Z, \phi V)$ and

$$g((\nabla_U R)(X, Y)\phi Z, V) = g((\nabla_U R)(X, Y)Z, \phi V),$$

using this equation in (5.48), we get

$$\begin{aligned}
& g((\nabla_U R)(X, Y)Z, \phi V) \\
&= g(U, \phi Y)g(X, \phi Z)\eta(V) - g(U, \phi X)g(Y, \phi Z)\eta(V) \\
&\quad - g(R(X, Y)\phi Z, \phi U)\eta(V) + C(U)g(R(X, Y)Z, \phi V) \\
&\quad + D(U)g(F(X, Y)\phi Z, V).
\end{aligned} \tag{5.49}$$

By contracting equation (5.49) over Y and Z and using equations (5.37) and (5.44), we obtain

$$(\nabla_U S)(X, \phi V) = g(\phi X, \phi U)\eta(V) - S^*(X, U)\eta(V) + C(U)S(X, \phi V) - D(U)g(X, \phi V). \tag{5.50}$$

Substituting $V = \xi$ in the above equation

$$S^*(X, U) = g(\phi X, \phi U). \tag{5.51}$$

which completes the proof. $\square$

In consequence of equations (1.38) and (5.51), we get the following:

Corollary 5.5. *A quasi generalized ϕ -recurrent LP-Sasakian manifold M^{2n+1} is an $*$ - η -Einstein manifold.*

Remark 5.1. *We can prove the existence of quasi generalized ϕ -recurrent LP-Sasakian manifold with an example by following the similar step as we did in the example of hyper generalized ϕ -recurrent LP-Sasakian manifold.*

5.2 Hyper generalized weakly symmetric Para-Sasakian manifold and their geometric properties

This section deals with hyper generalized weakly symmetric Para-Sasakian manifold. Recently, there has been a growing interest in almost para-contact geometry, in particular, para-Sasakian geometry, due to its connections with the theory of para-Kähler manifolds. The study of almost para-contact and para-Sasakian structures is also growing traction because of their applications in pseudo-Riemannian geometry and mathematical physics. Almost para-contact structures facilitate the exploration of new geometric invariants and curvature conditions that differ significantly from their contact counterparts. In recent year, Buluta and İnseilöz (2024) studied Para-Sasaki-like Riemannian manifolds with generalized symmetric metric connection. Motivated by this, we study Para-Sasakian manifold whose curvature tensor satisfies the hyper-generalized weakly symmetric condition.

A non-flat, $(2n + 1)$ -dimensional Riemannian manifold (M^n, g) $n > 2$ is called a generalized weakly symmetric manifold (Baishya et al., 2018) if its curvature tensor R of type $(0, 4)$ is non zero and satisfies the condition:

$$\begin{aligned}
 (\nabla_U R)(V, X, Y, Z) = & A(U)R(V, X, Y, Z) + B(V)R(U, X, Y, Z) + B(X)R(V, U, Y, Z) \\
 & + D(Y)R(V, X, U, Z) + D(Z)R(V, X, Y, U) + \alpha(U)G(V, X, Y, Z) \\
 & + \beta(V)G(U, X, Y, Z) + \beta(X)G(V, U, Y, Z) + \gamma(Y)G(V, X, U, Z) \\
 & + \gamma(Z)G(V, X, Y, U),
 \end{aligned}
 \tag{5.52}$$

B. Thangjam and M.S. Devi (2025). Hyper generalized weakly symmetric Para-Sasakian manifold and their geometric properties, *Bulletin of the Karaganta University. Mathematics series* 118(2), 241–251.

where $G(X, Y)Z = g(Y, Z)X - g(X, Z)Y$, and $A, B, D, \alpha, \beta, \gamma$ are non zero 1-forms defined as follows: $A(X) = g(X, \theta_1)$, $B(X) = g(X, \phi_1)$, $D(X) = g(X, \pi_1)$, $\alpha(X) = g(X, \theta_2)$, $\beta(X) = g(X, \phi_2)$ and $\gamma(X) = g(X, \pi_2)$, in which $\theta_1, \phi_1, \pi_1, \theta_2, \phi_2$ and π_2 are associated vector fields of A, B, D, α, β and γ respectively.

Keeping in tune with Shaikh and Patra (2010), we shall call a Riemannian manifold of dimension n , hyper generalized weakly symmetric if it admits the equation:

$$\begin{aligned}
 (\nabla_U R)(V, X, Y, Z) &= A(U)R(V, X, Y, Z) + B(V)R(U, X, Y, Z) \\
 &+ D(Y)R(V, X, U, Z) + D(Z)R(V, X, Y, U) \\
 &+ \alpha(U)(g \wedge S)(V, X, Y, Z) + \beta(V)(g \wedge S)(U, X, Y, Z) \quad (5.53) \\
 &+ \beta(X)(g \wedge S)(V, U, Y, Z) + \gamma(Y)(g \wedge S)(V, X, U, Z) \\
 &+ \gamma(Z)(g \wedge S)(V, X, Y, U) + B(X)R(V, U, Y, Z),
 \end{aligned}$$

where

$$\begin{aligned}
 (g \wedge S)(V, X, Y, Z) &= g(V, Z)S(X, Y) + g(X, Y)S(V, Z) - g(V, Y)S(X, Z) \\
 &- g(X, Z)S(V, Y),
 \end{aligned} \quad (5.54)$$

and $A, B, D, \alpha, \beta, \gamma$ are non zero 1-forms defined as follows: $A(X) = g(X, \theta_1)$, $B(X) = g(X, \phi_1)$, $D(X) = g(X, \pi_1)$, $\alpha(X) = g(X, \theta_2)$, $\beta(X) = g(X, \phi_2)$ and $\gamma(X) = g(X, \pi_2)$.

5.2.1 Hyper Generalized Weakly Symmetric Para-Sasakian Manifolds

A Para-sasakian manifold is considered to be hyper-generalized weakly symmetric if it satisfies the following curvature condition (Shaikh and Patra, 2010)

$$\begin{aligned}
 (\nabla_U R)(V, X, Y, W) &= A(U)R(V, X, Y, W) + B(V)R(U, X, Y, W) \\
 &+ B(X)R(V, U, Y, W) + D(Y)R(V, X, U, W) \\
 &+ D(W)R(V, X, Y, U) + \alpha(U)(g \wedge S)(V, X, Y, W) \\
 &+ \beta(V)(g \wedge S)(U, X, Y, W) + \beta(X)(g \wedge S)(V, U, Y, W) \\
 &+ \gamma(Y)(g \wedge S)(V, X, U, W) + \gamma(W)(g \wedge S)(V, X, Y, U),
 \end{aligned} \tag{5.55}$$

where $g \wedge S$ is defined in (5.54).

By applying (5.54) and then contracting it with U and V in (5.55), we obtain

$$\begin{aligned}
 (\nabla_U S)(V, W) &= A(U)S(V, W) + B(V)S(U, W) + D(W)S(U, V) \\
 &+ B(R(U, V)W) + D(R(U, W)V) + \alpha(U)[(2n-1)S(V, W) \\
 &+ rg(V, W)] + \beta(V)[(2n-1)S(U, W) + rg(U, W)] \\
 &+ \gamma(W)[(2n-1)S(U, V) + rg(U, V)] + \tilde{\beta}(U)g(V, W) \\
 &+ \beta(U)S(V, W) - \beta(V)S(U, W) - \tilde{\beta}(V)g(U, W) \\
 &+ \tilde{\gamma}(U)g(V, W) + \gamma(U)S(V, W) - \tilde{\gamma}(W)g(V, U) - \gamma(W)S(V, U).
 \end{aligned} \tag{5.56}$$

By further contracting the above equation, we get

$$\begin{aligned}
 dr(U) &= rA(U) + 2\tilde{B}(U) + 2\tilde{D}(U) + 2(2n)r\alpha(U) + 2(2n-1)[\beta(\tilde{U}) \\
 &+ \tilde{\gamma}(U)] + 2r[\beta(U) + \gamma(U)].
 \end{aligned} \tag{5.57}$$

where $\tilde{B}(U) = S(U, \phi_1)$, $\tilde{D}(U) = S(U, \pi_1)$, $\tilde{\beta}(U) = S(U, \phi_2)$ and $\tilde{\gamma}(U) = S(U, \pi_2)$.

Assuming that the scalar curvature of a hyper-generalized weakly symmetric para-Sasakian manifold is a non-zero constant, equation (5.57) reduces to

$$\begin{aligned} & r[A(U) + 2(2n)\alpha(U) + 2\beta(U) + 2\gamma(U)] \\ & = -2\tilde{B}(U) - 2\tilde{D}(U) - 2(2n-1)[\beta(\tilde{U}) + \tilde{\gamma}(U)]. \end{aligned} \quad (5.58)$$

This leads to the following result

Theorem 5.9. *Let M be a hyper generalized weakly symmetric para-Sasakian manifold with non-zero constant scalar curvature. The 1-form are then related by the equation (5.58).*

Putting $V = \xi$ in (5.56) and using eqs. (1.61) and (1.68), we get

$$\begin{aligned} (\nabla_U S)(\xi, W) &= -2nA(U)\eta(W) + B(\xi)S(U, W) - 2nD(W)\eta(U) + B(\xi)g(U, W) \\ &\quad - \eta(W)B(U) + D(W)\eta(U) - \eta(W)D(U) - 2n(2n-1)\alpha(U)\eta(W) \\ &\quad + r\eta(W)\alpha(U) + (2n-1)\beta(\xi)S(U, W) + r\beta(\xi)g(U, W) \\ &\quad + r\gamma(W)\eta(U) + \tilde{\beta}(U)\eta(W) - 2n\beta(U)\eta(W) - \beta(\xi)S(U, W) \\ &\quad + \tilde{\gamma}(U)\eta(W) - 2n\gamma(U)\eta(W) - \tilde{\gamma}(W)\eta(U) - \tilde{\beta}(\xi)g(U, W) \\ &\quad + 2n\gamma(W)\eta(U) - 2n(2n-1)\gamma(W)\eta(U). \end{aligned} \quad (5.59)$$

Setting $W = \xi$ in (5.56) and applying eqs. (1.60), (1.61) and (1.68), we have

$$\begin{aligned} (\nabla_U S)(V, \xi) &= -2nA(U)\eta(V) - 2nB(V)\eta(U) + D(\xi)S(U, V) + B(V)\eta(U) \\ &\quad - B(U)\eta(V) + D(\xi)g(U, V) - D(U)\eta(V) - 2n(2n-1)\alpha(U)\eta(V) \\ &\quad + r\alpha(U)\eta(V) - 2n(2n-1)\beta(V)\eta(U) + r\beta(V)\eta(U) + r\gamma(\xi)g(U, V) \\ &\quad + (2n-1)\gamma(\xi)S(U, V) + \tilde{\beta}(U)\eta(V) - 2n\beta(U)\eta(V) - \tilde{\beta}(V)\eta(U) \\ &\quad + 2n\beta(V)\eta(U) + \tilde{\gamma}(U)\eta(V) - 2n\gamma(U)\eta(V) - \tilde{\gamma}(\xi)\eta(V) \\ &\quad - \gamma(\xi)S(U, V). \end{aligned} \quad (5.60)$$

Replacing $V = W$ in the above equation, we obtain

$$\begin{aligned}
(\nabla_U S)(W, \xi) &= -2nA(U)\eta(W) - 2nB(W)\eta(U) + D(\xi)S(U, W) + B(W)\eta(U) \\
&\quad - B(U)\eta(W) + D(\xi)g(U, W) - D(U)\eta(W) + r\gamma(\xi)g(U, W) \\
&\quad + r\alpha(U)\eta(W) - 2n(2n-1)\beta(W)\eta(U) + r\beta(W)\eta(U) \\
&\quad + 2n\gamma(\xi)S(U, W) + \tilde{\beta}(U)\eta(W) - 2n\beta(U)\eta(W) - \tilde{\beta}(W)\eta(U) \\
&\quad + 2n\beta(W)\eta(U) + \tilde{\gamma}(U)\eta(W) - 2n\gamma(U)\eta(W) - \tilde{\gamma}(\xi)\eta(W) \\
&\quad - \gamma(\xi)S(U, W) - 2n(2n-1)\alpha(U)\eta(W). \tag{5.61}
\end{aligned}$$

Comparing (5.59) and (5.61), we have

$$\begin{aligned}
&[B(\xi) + 2(n-1)\beta(\xi) - D(\xi) - 2n\gamma(\xi)]S(U, W) \\
&= [(2n-1)D(Z) + (2n-1)^2\gamma(W) - r\gamma(W) - (2n-1)B(W) + 4n(n-1)\beta(W) \\
&\quad + \beta(W) - \tilde{\beta}(W)]\eta(U) + [\tilde{\beta}(\xi) - (r+1)\beta(\xi) + r\gamma(\xi) - \tilde{\gamma}(\xi)]g(U, W). \tag{5.62}
\end{aligned}$$

Putting $U = \xi$ in (5.62) and using eqs. (1.61) and (1.68), we get

$$\begin{aligned}
&(2n-1)D(W) + (2n-1)^2\gamma(W) - r\gamma(W) - (2n-1)B(W) + 4n(n-1)\beta(W) \\
&\quad + r\beta(W) - \tilde{\beta}(W) = [2nB(\xi) + 4n(n-1)\beta(\xi) - 2nD(\xi) \\
&\quad - (2n)^2\gamma(\xi)\tilde{\beta}(\xi) - (r+1)\beta(\xi) + r\gamma(\xi) - \tilde{\gamma}(\xi)]\eta(W). \tag{5.63}
\end{aligned}$$

Substituting (5.63) in (5.62), we deduce

$$\begin{aligned}
&[B(\xi) + 2(n-1)\beta(\xi) - D(\xi) - 2n\gamma(\xi)]S(U, W) \\
&= [2nB(\xi) + 4n(n-1)\beta(\xi) - 2nD(\xi) \\
&\quad - (2n)^2\gamma(\xi)\tilde{\beta}(\xi) - (r+1)\beta(\xi) + r\gamma(\xi) - \tilde{\gamma}(\xi)]\eta(U)\eta(W) \\
&\quad + [\tilde{\beta}(\xi) - (r+1)\beta(\xi) + r\gamma(\xi) - \tilde{\gamma}(\xi)]g(U, W). \tag{5.64}
\end{aligned}$$

Thus, we can state the following theorem:

Theorem 5.10. *A hyper generalized Para-Sasakian manifold is an η -Einstein*

manifold if

$$B(\xi) + 2(n-1)\beta(\xi) \neq D(\xi) + 2n\gamma(\xi).$$

We can also observe that,

$$(\nabla_X S)(Y, \xi) = -2ng(X, \phi Y) + S(Y, \phi X). \quad (5.65)$$

From (5.60) and (5.65), we derive

$$\begin{aligned} & -2ng(U, \phi V) + S(V, \phi U) \\ &= -2nA(U)\eta(V) - 2nB(V)\eta(U) + D(\xi)S(U, V) + B(V)\eta(U) \\ & - B(U)\eta(V) + D(\xi)g(U, V) - D(U)\eta(V) - 2n(2n-1)\alpha(U)\eta(V) \\ & + r\alpha(U)\eta(V) - 2n(2n-1)\beta(V)\eta(U) + r\beta(V)\eta(U) + r\gamma(\xi)g(U, V) \quad (5.66) \\ & + (2n-1)\gamma(\xi)S(U, V) + \tilde{\beta}(U)\eta(V) - (n-1)\beta(U)\eta(V) - \tilde{\beta}(V)\eta(U) \\ & + (n-1)\beta(V)\eta(U) + \tilde{\gamma}(U)\eta(V) - (n-1)\gamma(U)\eta(V) - \tilde{\gamma}(\xi)\eta(V) \\ & - \gamma(\xi)S(U, V). \end{aligned}$$

Setting $U = V = \xi$ in (5.66) and applying eqs. (1.60), (1.61) and (1.68) yields

$$\begin{aligned} & 2n[A(\xi) + B(\xi) + D(\xi) + (2n-1)\alpha(\xi) + (2n-1)\beta(\xi) + (2n-1)\gamma(\xi)] \\ & - r[\alpha(\xi) + \beta(\xi) + \gamma(\xi)] = 0. \end{aligned} \quad (5.67)$$

Hence, the following theorem can be stated:

Theorem 5.11. *In Para-Sasakian manifold with hyper generalized weakly symmetric curvature condition, the relation (5.67) holds.*

5.2.2 Codazzi Type Of Ricci Tensor

A Ricci tensor is of Codazzi type when its covariant derivative is symmetric with respect to its indices, i.e, for any vector fields U, V, W it satisfies

$$(\nabla_U S)(V, W) = (\nabla_W S)(V, U), \quad (5.68)$$

where ∇ is the Levi-Civita connection of the manifold's metric.

In view of (5.56) and (5.68), we have

$$\begin{aligned}
0 &= [A(U) - D(U) + (2n-1)\alpha(U) - (2n-3)\gamma(U) + \beta(U)]S(V, W) \\
&\quad + [D(W) + (2n-3)\gamma(W) - A(W) - (2n-1)\alpha(W) - \beta(W)]S(V, U) \\
&\quad + [r\alpha(U) + \tilde{\beta}(U) + 2\tilde{\gamma}(U) - r\gamma(U)]g(V, W) + [r\gamma(W) - \tilde{\gamma}(W) \\
&\quad - r\alpha(W) - \tilde{\beta}(W) - \tilde{\gamma}(W)]g(V, U) + B(R(U, V)W) \\
&\quad + 2D(R(U, W)V) - B(R(W, V)U).
\end{aligned} \tag{5.69}$$

Now by substituting $W = \xi$ in (5.69) and from eqs. (1.61) and (1.68), we obtain

$$\begin{aligned}
0 &= -2n[A(U) - D(U) + (2n-1)\alpha(U) - (2n-3)\gamma(U) + \beta(U)]\eta(V) \\
&\quad + [D(\xi) + (2n-3)\gamma(\xi) - A(\xi) - (2n-1)\alpha(\xi) - \beta(\xi)]S(V, U) \\
&\quad + [r\alpha(U) + \tilde{\beta}(U) + 2\tilde{\gamma}(U) - r\gamma(U)]\eta(V) + [r\gamma(\xi) - \tilde{\gamma}(\xi) - r\alpha(\xi) \\
&\quad - \tilde{\beta}(\xi) - \tilde{\gamma}(\xi)]g(V, U) - B(V)\eta(U) + B(U)\eta(V) - 2D(U)\eta(V) \\
&\quad + 2D(\xi)g(U, V) + B(\xi)g(V, U) - B(V)\eta(U).
\end{aligned} \tag{5.70}$$

Putting $U = \xi$ in (5.70) and applying (1.60) and (1.61), we get

$$B(V) = B(\xi)\eta(V). \tag{5.71}$$

Replacing $V = \xi$ in (5.70), we deduce

$$\begin{aligned}
&2n[A(U) - D(U) + (2n-1)\alpha(U) - (2n-3)\gamma(U) + \beta(U)] \\
&\quad - [r\alpha(U) + \tilde{\beta}(U) + 2\tilde{\gamma}(U) - r\gamma(U)] - B(U) + 2D(U) \\
&= \{-2n[D(\xi) + (2n-3)\gamma(\xi) - A(\xi) - (2n-1)\alpha(\xi) - \beta(\xi)] \\
&\quad + [r\gamma(\xi) - 2\tilde{\gamma}(\xi) - r\alpha(\xi) - \tilde{\beta}(\xi) - B(\xi) + D(\xi)]\}\eta(U).
\end{aligned} \tag{5.72}$$

Using (5.71) and (5.72) in (5.70), we obtain

$$\begin{aligned}
& [D(\xi) + (2n - 3)\gamma(\xi) - A(\xi) - (2n - 1)\alpha(\xi) - \beta(\xi)]S(U, V) \\
& = \{-2n[D(\xi) + (2n - 3)\gamma(\xi) - A(\xi) - (2n - 1)\alpha(\xi) - \beta(\xi)] \\
& \quad + [r\gamma(\xi) - 2\tilde{\gamma}(\xi) - r\alpha(\xi) - \tilde{\beta}(\xi) - B(\xi) + D(\xi)]\}\eta(U)\eta(V) \\
& \quad - [r\gamma(\xi) - 2\tilde{\gamma}(\xi) - r\alpha(\xi) - \tilde{\beta}(\xi) - B(\xi) + D(\xi)]g(U, V).
\end{aligned} \tag{5.73}$$

From this, we can present the following theorem:

Theorem 5.12. *A hyper generalized weakly symmetric Para-Sasakian manifold is an η -Einstein if it admits Codazzi type of Ricci tensor, provided that*

$$D(\xi) + (2n - 3)\gamma(\xi) = A(\xi) + (2n - 1)\alpha(\xi) + \beta(\xi).$$

5.2.3 Recurrent Ricci Tensor

If the hyper generalized weakly symmetric Para-Sasakian manifold has a recurrent Ricci tensor then

$$(\nabla_U S)(V, W) = \lambda(U)S(V, W). \tag{5.74}$$

In view of (5.56) and (5.74), we have

$$\begin{aligned}
\lambda(U)S(V, W) &= A(U)S(V, W) + B(V)S(U, W) + D(W)S(U, V) + B(R(U, V)W) \\
&\quad + D(R(U, W)V) + \alpha(U)[(2n - 1)S(V, W) + rg(V, W)] \\
&\quad + \beta(V)[(2n - 1)S(U, W) + rg(U, W)] + \gamma(W)[(2n - 1)S(U, V) \\
&\quad + rg(U, V)] + \tilde{\beta}(U)g(V, W) + \beta(U)S(V, W) - \beta(V)S(U, W) \\
&\quad - \tilde{\beta}(V)g(U, W) + \tilde{\gamma}(U)g(V, W) + \gamma(U)S(V, W) \\
&\quad - \tilde{\gamma}(W)g(V, U) - \gamma(W)S(V, U).
\end{aligned} \tag{5.75}$$

Placing $W = \xi$ in (5.75), we obtain

$$-2n\lambda(U)\eta(V) = -2nA(U)\eta(V) - 2nB(V)\eta(U) + D(\xi)S(U, V)$$

$$\begin{aligned}
& + B(V)\eta(U) - B(U)\eta(V) + D(\xi)g(U, V) - D(U)\eta(V) \\
& - 2n(2n-1)\alpha(U)\eta(V) + r\alpha(U)\eta(V) + \tilde{\beta}(U)\eta(V) \\
& + r\beta(V)\eta(U) + (2n-1)\gamma(\xi)S(U, V) + r\gamma(\xi)g(U, V) \\
& - 2n\beta(U)\eta(V) + 2n\beta(V)\eta(U) - \tilde{\beta}(V)\eta(U) \\
& + \tilde{\gamma}(U)\eta(V) - 2n\gamma(U)\eta(V) - \tilde{\gamma}(\xi)g(V, U) \\
& - \gamma(\xi)S(V, U) - 2n(2n-1)\beta(V)\eta(U). \tag{5.76}
\end{aligned}$$

Putting $U = \xi$ in (5.76), we have

$$\begin{aligned}
& - (2n-1)B(V) - 4n(n-1)\beta(V) + r\beta(V) - \tilde{\beta}(V) \\
& = [-2n\lambda(\xi) + (n-1)A(\xi) + 2nD(\xi) + B(\xi) + 2n(2n-1)\alpha(\xi) \\
& - r\alpha(\xi) - 2n(2n-1)\gamma(\xi) - r\gamma(\xi) - \tilde{\beta}(\xi) + 2n\beta(\xi)]\eta(V). \tag{5.77}
\end{aligned}$$

Considering $V = \xi$ in (5.76), we get

$$\begin{aligned}
& 2nA(U) - 2n\lambda(U) + B(U) + D(U) + 2n(2n-1)\alpha(U) \\
& - r\alpha(U) + 2n\beta(U) - \tilde{\beta}(U) - \tilde{\gamma}(U) + 2n\gamma(U) \\
& = [-(2n-1)\beta(\xi) - (2n-1)D(\xi) - 2n(2n-1)\beta(\xi) + r\beta(\xi) \\
& - (2n)^2\gamma(\xi) + r\gamma(\xi) - \tilde{\beta}(\xi) - \tilde{\gamma}(\xi)]\eta(U). \tag{5.78}
\end{aligned}$$

Now, setting $U = V = \xi$ in (5.76), we deduce

$$\begin{aligned}
& - 2n[A(\xi) + B(\xi) + D(\xi) + (2n-1)\alpha(\xi) + (2n-1)\beta(\xi) \\
& + (2n-1)\gamma(\xi) - \lambda(\xi)] + r[\alpha(\xi) + \beta(\xi) + \gamma(\xi)] = 0. \tag{5.79}
\end{aligned}$$

Putting the value of (5.77), (5.78), (5.79) in (5.76), we obtain

$$\begin{aligned}
& [D(\xi) + 2(n-1)\gamma(\xi)]S(U, V) \\
& = [B(\xi) + 2(n-1)D(\xi) - 4n(n-1)\gamma(\xi) + 2n\beta(\xi) \\
& + r\gamma(\xi)]\eta(U)\eta(V) - [D(\xi) + (r-1)\gamma(\xi)]g(U, V). \tag{5.80}
\end{aligned}$$

Theorem 5.13. *In a hyper-generalized weakly symmetric Para-Sasakian manifold, if the Ricci tensor is recurrent then the manifold becomes an η -Einstein, provided that*

$$D(\xi) \neq 2(n-1)\gamma(\xi).$$

5.2.4 Hyper-generalized Weakly Symmetric Para-Sasakian Manifold admitting a Quarter Symmetric Metric Connection

In this subsection, we are using quarter symmetric metric connection $\bar{\nabla}$ on a Para-Sasakian manifold given by (1.1).

The curvature tensor $\bar{R}$ associated with the quarter-symmetric connection relates to the curvature tensor R of the Levi-Civita connection by (Vishnuvardhana and Venkatesha, 2020):

$$\begin{aligned} \bar{R}(U, V)W &= R(U, V)W + 3g(\phi U, W)\phi V - 3g(\phi V, W)\phi U + [\eta(U)V \\ &\quad - \eta(V)U]\eta(W) - [g(V, W)\eta(U) - g(U, W)\eta(V)]\xi, \end{aligned} \quad (5.81)$$

which yields

$$\bar{S}(V, W) = S(V, W) + 2g(V, W) - (n+1)\eta(V)\eta(W) - 3tr\phi g(\phi V, W), \quad (5.82)$$

where $\bar{S}$ and S are Ricci tensor of $\bar{\nabla}$ and ∇ , respectively.

For a $[(H(GWS))_n, \bar{\nabla}]$ (Shaikh and Patra, 2010), we have

$$\begin{aligned} &(\nabla_U \bar{R})(V, X, Y, W) \\ &= A(U)\bar{R}(V, X, Y, W) + B(V)\bar{R}(U, X, Y, W) \\ &+ B(X)R(V, U, Y, W) + D(Y)\bar{R}(V, X, U, W) \\ &+ D(W)\bar{R}(V, X, Y, U) + \alpha(U)(g \wedge \bar{S})(V, X, Y, W) \\ &+ \beta(V)(g \wedge \bar{S})(U, X, Y, W) + \beta(X)(g \wedge \bar{S})(V, U, Y, W) \end{aligned}$$

$$+ \gamma(Y)(g \wedge \bar{S})(V, X, U, W) + \gamma(W)(g \wedge \bar{S})(V, X, Y, U), \quad (5.83)$$

for all $X, Y, U, V, W \in TM$

Making use of (1.1), we can find

$$\begin{aligned} & (\bar{\nabla}_U \bar{R})(V, X, Y, W) \\ &= (\nabla_U \bar{R})(V, X, Y, W) + \eta(\bar{R}(V, X)Y)g(\phi U, W) - \eta(V)\bar{R}(\phi U, X, Y, W) \\ & \quad - \eta(X)\bar{R}(V, \phi U, Y, W) - \eta(Y)\bar{R}(V, X, \phi U, W) - \eta(Y)\bar{R}(V, X, Y, \phi U) \\ & \quad + g(\phi U, V)\bar{R}(\xi, X, Y, W) + g(\phi U, X)\bar{R}(V, \xi, Y, W) \\ & \quad + g(\phi U, Y)\bar{R}(V, X, \xi, W) + g(\phi U, W)\bar{R}(V, X, Y, \xi). \end{aligned} \quad (5.84)$$

Now, using (5.81) in the foregoing equation, we have

$$\begin{aligned} & (\bar{\nabla}_U \bar{R})(V, X, Y, W) \\ &= (\nabla_U R)(V, X, Y, W) + 3(\nabla_U g)(\phi V, Y)g(\phi X, W) + 3g(\phi V, Y)(\nabla_U g)(\phi X, W) \\ & \quad - 3(\nabla_U g)(\phi X, Y)g(\phi V, W) - 3g(\phi X, Y)\nabla_U g(\phi V, W) + (\nabla_U \eta)(V)g(X, W)\eta(Y) \\ & \quad + \eta(V)g(X, W)(\nabla_U \eta)(Y) - (\nabla_U \eta)(X)g(V, W)\eta(Y) - \eta(X)g(V, W)(\nabla_U \eta)(W) \\ & \quad - g(X, Y)(\nabla_U \eta)(V)\eta(W) - g(X, Y)\eta(V)(\nabla_U \eta)(W) + g(V, Y)(\nabla_U \eta)(X)\eta(W) \\ & \quad + g(V, Y)\eta(X)(\nabla_U \eta)(W) - [\eta(R(V, X, Y)) - \eta(V)g(X, Y) \\ & \quad + \eta(X)g(V, Y)]g(\phi U, W) - \eta(V)[R(\phi U, X, Y, W) + 3g(\phi U, \phi Y)g(\phi X, W) \\ & \quad - 3g(\phi V, Y)g(\phi V, \phi W) - \eta(X)g(\phi U, W)\eta(Y) - \eta(X)g(\phi U, Y)\eta(W)] \\ & \quad - \eta(X)[R(V, \phi U, Y, W) + 3g(\phi V, Y)g(\phi U, \phi W) - 3g(\phi U, \phi Y)g(\phi V, W) \\ & \quad + \eta(V)g(\phi U, W)\eta(Y) - \eta(V)g(\phi U, Y)\eta(W)] - \eta(Y)[R(V, X, \phi U, W) \\ & \quad + 3g(\phi V, \phi U)g(\phi X, W) - 3g(\phi X, \phi U)g(\phi V, W) - \eta(V)g(X, \phi U)\eta(W) \\ & \quad - \eta(X)g(V, \phi U)\eta(W)] - \eta(W)[R(V, X, \phi U) + 3g(\phi V, Y)g(\phi X, \phi U) \\ & \quad - 3g(\phi X, Y)g(\phi V, \phi U) - \eta(V)g(X, \phi U)\eta(Y) - \eta(X)g(V, \phi U)\eta(Y)] \\ & \quad + g(\phi U, V)[R(\xi, X, Y, W) + g(X, W)\eta(Y) - g(X, Y)\eta(W)] \end{aligned}$$

$$\begin{aligned}
& + g(\phi U, X)[R(V, \xi, Y, W) + g(X, W)\eta(V) - g(V, W)\eta(X)] \\
& + g(\phi U, Y)[R(V, X, \xi, W) + g(X, W)\eta(V) - g(V, W)\eta(Y)] \\
& + g(\phi U, W)[R(V, X, Y, \xi) - g(X, Y)\eta(V) + g(V, Y)\eta(X)].
\end{aligned} \tag{5.85}$$

A hyper-generalized weakly symmetric para-Sasakian manifold with quarter symmetric metric connection $\bar{\nabla}$ simplifies to a hyper-generalized weakly symmetric para-Sasakian manifold with a Riemannian metric connection ∇ , provided the following condition is satisfied:

$$(\bar{\nabla}_U \bar{R})(V, X, Y, W) = (\nabla_U \bar{R})(V, X, Y, W). \tag{5.86}$$

Using (5.81), (5.83) and (5.85) in (5.86), it yields

$$A(\xi) + B(\xi) + D(\xi) = 0,$$

for $U = X = Y = \xi$. From the above, we can state the following:

Theorem 5.14. *A $[(H(GWS))_n, \bar{\nabla}]$ is a $[(H(GWS))_n, \nabla]$, if the following relations hold*

$$A(\xi) + B(\xi) + D(\xi) = 0.$$

5.3 Conclusion

In Section 5.1, we study hyper generalized ϕ -recurrent LP-Sasakian manifold and quasi generalized ϕ -recurrent LP-Sasakian manifold. In hyper generalized ϕ -recurrent LP-Sasakian manifold we get some results viz., if the scalar curvature is zero everywhere on it, then the manifold is Ricci recurrent, otherwise the manifold is generalized Ricci recurrent and Ricci semisymmetric manifold. The existence of hyper generalized ϕ -recurrent LP-Sasakian manifold is proved with an example. In quasi generalized ϕ -recurrent LP-Sasakian manifold, we find that quasi generalized ϕ -recurrent LP-Sasakian manifold is a quasi-generalized

Ricci-recurrent manifold and this manifold is a ϕ -Einstein manifold as well as an $*\text{-}\eta$ -Einstein manifold.

And in the next Section 5.2, we have explored the geometric properties of hyper-generalized weakly symmetric (H(GWS)) para-Sasakian manifolds. We have shown that, under certain conditions, (H(GWS)) para-Sasakian manifolds can reduce to simpler forms, such as η -Einstein manifolds, when specific relations between their 1-forms are satisfied. The study also highlights the role of quarter symmetric metric connections, showing how these connections influence the manifold's curvature and overall geometric behavior.

Chapter 6

Summary and Conclusion

In the present thesis, we give classification of almost contact metric manifolds admitting some geometrical structures and also studied their submanifold. The following objectives are taken up in the study:

1. To determine some results on a certain manifold with generalized Tanaka-Webster connection.
2. To characterize the geometrical properties of some hypersurfaces of a manifold.
3. To investigate curvature conditions of Ricci soliton on a manifold.
4. To study hypergeneralized ϕ -recurrent manifold.

Chapter 1 provides a general introduction, including basic definitions and fundamental formulas of differential geometry. Topics covered include differentiable manifolds, Riemannian manifolds, almost contact metric manifolds, Kenmotsu manifolds, Lorentzian almost contact manifolds, Lorentzian Para-contact manifolds, Lorentzian Para-Kenmotsu manifolds, submanifolds, thermodynamical fluid spacetimes, Ricci solitons, and a review of the relevant literature.

Chapter 2 explores Kenmotsu manifolds equipped with a generalized Tanaka-

Webster connection and investigates their curvature properties. We observe that the scalar curvature of a conharmonically flat Kenmotsu manifold under this connection is given by $r = -2n(2n + 1)$. Additionally, the scalar curvature of ξ -conharmonically flat and quasi-conharmonically flat Kenmotsu manifolds vanishes in this context. We also derive the relationship between the 1-forms C and D in a generalized conharmonic ϕ -recurrent Kenmotsu manifold with respect to the generalized Tanaka-Webster connection. Furthermore, it is shown that a projectively flat Kenmotsu manifold under this connection is an Einstein manifold. In the concluding theorem, we demonstrate that a Kenmotsu manifold M^{2n+1} is locally projectively ϕ -symmetric with respect to the generalized Tanaka-Webster connection if and only if it is locally projectively ϕ -symmetric with respect to the Levi-Civita connection.

Chapter 3 investigates the properties and characteristics of transversal hypersurfaces of Lorentzian Kenmotsu manifolds. In Section 3.2, we show that each transversal hypersurface admits an almost Hermitian structure. Additionally, such hypersurfaces also admit a structure characterized by the parameters (f, g, μ, ν, δ) . We define criteria for totally umbilical hypersurfaces and parallel tensor fields, as well as totally geodesic hypersurfaces, through conditions involving δ and ν . An example of a transversal hypersurface with a (f, g, μ, ν, δ) -structure is also provided.

Chapter 4 is divided into two sections. In the first section, we show that in a Lorentzian Para-Kenmotsu manifold, a symmetric parallel second-order covariant tensor is proportional to the metric tensor. Furthermore, if $\mathcal{L}_V g + 2S$ is parallel, where V is a vector field, then (g, V, λ) forms a Ricci soliton. The study also establishes that a Ricci soliton in a W_2 -semi-symmetric Lorentzian Para-Kenmotsu manifold is expanding, while a Ricci soliton satisfying $W_2(\xi, X) \circ S = 0$ is steady. We analyze various curvature properties of such manifolds admitting Ricci solitons and present an example of a 3-dimensional Lorentzian Para-Kenmotsu man-

ifold.

The second section investigates Ricci solitons in thermodynamical fluid spacetime (TFS). This study provides insights into the behavior of matter and energy under extreme conditions, such as those in cosmological models and black hole physics. By integrating thermodynamic principles with spacetime geometry, it highlights the essential role thermal properties play in influencing spacetime dynamics.

One key finding is that Ricci solitons can drive the expansion, steadiness, or contraction of spacetime, depending on the interplay between thermal energy and effective density. This highlights how thermodynamic factors—such as heat flux and stress—directly influence the structure and evolution of the universe. The generalized Ricci recurrent structure observed in TFS under thermal effects underscores the importance of incorporating thermodynamical properties into spacetime models.

We also examine the role of various vector fields, such as conformal Killing and torse-forming fields, in shaping Ricci soliton behavior. Notably, when the potential vector field X is conformal Killing, the spacetime not only supports a Ricci soliton but also acts as a proper homothetic vector field—revealing how specific vector fields impact the geometric evolution of spacetime.

Modified Poisson and Liouville equations are derived within the TFS framework, capturing how thermal influences affect spacetime evolution, especially in extreme environments like black holes. For gradient-type velocity vector fields, the behavior of spacetime—contracting, steady, or expanding—depends on thermal and effective energy densities.

To illustrate these concepts, we construct an example of a four-dimensional thermodynamical fluid spacetime, using standard coordinates and a Riemannian metric. Applying Theorem 4.9, we validate the theoretical framework and explore the interaction between vector fields and geometry, linking abstract theory with

practical implications. This section highlights the value of a thermodynamical perspective in spacetime studies, showing that thermal properties such as energy density and stress are active components in spacetime dynamics. Integrating thermodynamics with general relativity deepens our understanding of cosmological phenomena and black hole behavior, leading to a more complete model of the universe.

Chapter 5 investigates hyper-generalized ϕ -recurrent and hyper-generalized weakly symmetric manifolds. This chapter is divided into two main sections. In the first section, we explore hyper-generalized ϕ -recurrent and quasi-generalized ϕ -recurrent LP-Sasakian manifolds. In the former, we show that if the scalar curvature vanishes everywhere, the manifold is Ricci recurrent; otherwise, it is both generalized Ricci recurrent and Ricci semi-symmetric. The existence of such manifolds is confirmed with a concrete example. For the quasi-generalized ϕ -recurrent LP-Sasakian manifold, we establish that it is a quasi-generalized Ricci recurrent manifold and simultaneously a ϕ -Einstein and $\ast\eta$ -Einstein manifold.

In the second section, we study the geometric properties of hyper-generalized weakly symmetric (H(GWS)) Para-Sasakian manifolds. We demonstrate that under specific conditions, these manifolds can simplify to structures such as η -Einstein manifolds when certain relationships between their 1-forms are met. The role of quarter-symmetric metric connections is also examined, showing their influence on the curvature and overall geometric behavior of the manifold. This work contributes significantly to the understanding of Para-Sasakian geometry and provides a solid foundation for further exploration and applications of weakly symmetric manifolds in differential geometry and theoretical physics.

Chapter 6 is a summary and conclusion.

Finally, a list of references is given at the end.

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LIST OF RESEARCH PUBLICATIONS/ ACCEPTED/ COMMUNICATIONS

- (1) M.S. Devi and **Bidyabati Th** (2023). Some Curvature Properties of Kenmotsu Manifolds with Generalised Tanaka Webster Connection, *Science and Technology Journal* **2(1)**, 114–120.
- (2) **B. Thangjam** and M.S. Devi (2024). Some properties of Ricci solitons in LP-Kenmotsu manifolds, *Communication on Applied Nonlinear Analysis* **32(3s)**, 699–710.
- (3) **B. Thangjam**, Z. Chhakchhuak and M.S. Devi (2025). Ricci Soliton and Relativistic Thermodynamical Fluid Spacetime, *Chaos, Solitons & Fractals* **194**, 116202.
- (4) M.S. Devi, **B. Thangjam** and J.P.Singh (2025). Certain geometrical properties of generalized ϕ -recurrent LP-Sasakian manifolds, *Panamerican Mathematical Journal* **35(4s)**, 322–332.
- (5) **B. Thangjam** and M.S. Devi (2025). Hyper Generalized Weakly Symmetric Para-Sasakian Manifolds and their Geometric Properties, *Bulletin of the Karaganda University. Mathematics series* 118(2), 241–251.
- (6) **B. Thangjam** and M.S. Devi (2025). Transversal Hypersurfaces of Lorentzian Kenmotsu Manifolds (communicated).

CONFERENCES/ SEMINARS/ WORKSHOPS

- (1) Attended “One Week Faculty Development Programme on Latest Research Software & ICT Tools for Communicating Research,” organized by Department of Basic Sciences & Humanities, GMR Institute of Technology. Rajam, Amdhra Pradesh, India during 20th to 26th June, 2021.
- (2) Attended “NASI TMC Summer School on Differential Geometry” organised by the Department of Mathematics and Statistics, Central University of Punjab, Bathinda during July 05-24, 2021.
- (3) Attended “International Workshop-cum-Conference on Mathematical Education” organized by Mizoram University, India in association with University of Guadalajara, Mexico during 15–20 November, 2021.
- (4) Trainee in the “One Week Training Program on Mathematical Modelling and Computing” organised by Department of Mathematics and Computer Science, Mizoram University, Aizawl-796004, Mizoram (India) held during 26th April, 2022 to 2nd May, 2022.
- (5) Attended “International Faculty Development Programme on Mathematical Modelling of Biosystems with Special Focus on Epidemiology,” organized by Department of Mathematics and Computer Science, Mizoram University during 22 - 27 August, 2022.
- (6) Presented a paper entitled “Study on a generalized ϕ -recurrent Lorentzian para Sasakian manifold ” at 4th Mizoram Science Congress organized by Mizoram Science Congress 2022 Organising Committee held during 24-25 November, 2022.
- (7) Attended “Teacher’s Enrichment Workshop on Groups, Rings and Number Theory (TEW 2022)” organised by Department of Mathematics and Com-

puter Science, Mizoram University, Aizawl-796004, Mizoram (India) held during 12th December, 2022 to 17th December, 2022.

- (8) Presented a paper entitled “A study on some curvature properties of Kenmotsu manifolds with generalised Tanaka-Webster connection” at the 2nd International Conference on Recent Trends in Applied Sciences and Computing Engineering (RTASCE -2023) organized by VIT Bhopal University in association with National Institute of Technology, Warangal during July 7-9, 2023.
- (9) Presented a paper entitled “Some Properties of Ricci Solitons in Lorentzian Para Kenmotsu Manifolds” at the International Conference on Recent Advances in Mathematical, Physical and Chemical Sciences (ICRAMPC-2024), organized by School of Physical Sciences, Mizoram University, Aizawl-796004, during February 21–23, 2024.
- (10) Presented a paper entitled “Transversal Hypersurface of a Lorentzian Kenmotsu Manifolds” at International Conference on Mathematical Challenges and Novel Applications-2024 organized by Department of Mathematics, Ayya Nadar Janaki Ammal College, Ramco Institute of Technology and The Standard Fireworks Rajaratnam College for Women, during March 21–22, 2024.
- (11) Mentor at “Initiation into Mathematics (Initmath) 2024” held at Mizoram University, Aizawl, organized by MTTS Trust during 8th to 13th July, 2024.
- (12) Presented a paper entitled “Ricci soliton and thermodynamical fluid space-time” at 39th Annual Conference of the Ramanujan Mathematical Society (RMS 2024) organized by CHRIST(Deemed to be University), Bengaluru during 27th December to 29th December, 2024.

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DEPARTMENT : MATHEMATICS AND COMPUTER
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APPROVAL OF RESEARCH PROPOSAL :

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**CERTAIN STRUCTURES AND CONNECTIONS ON
DIFFERENTIABLE MANIFOLDS**

**AN ABSTRACT SUBMITTED IN PARTIAL
FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY**

BIDYABATI THANGJAM

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**DEPARTMENT OF MATHEMATICS AND
COMPUTER SCIENCE**

SCHOOL OF PHYSICAL SCIENCES

JUNE, 2025

**CERTAIN STRUCTURES AND CONNECTIONS
ON DIFFERENTIABLE MANIFOLDS**

By

Bidyabati Thangjam

Department of Mathematics and Computer Science

Supervisor : Dr. M. Saroja Devi

Submitted

**In partial fulfillment of the requirement of the Degree of Doctor of
Philosophy in Mathematics of Mizoram University, Aizawl**

ABSTRACT

In the present thesis, we have investigated some geometrical structures and connection such as Ricci solitons, hyper generalized manifold, and generalized Tanaka-Webster connection in differentiable manifold. We also introduced geometrical structures of certain spacetime in the framework of Ricci solitons, which might be beneficial in theoretical physics, particularly, in the field of general relativity and spacetime. Furthermore, we constructed certain examples to validate the findings.

This thesis aims to deepen the understanding of Riemannian manifolds, almost contact manifolds, submanifold and spacetime by addressing the following objectives:

1. To determine some results on a certain manifold with generalized Tanaka-Webster connection.
2. To characterize the geometrical properties of some hypersurfaces of a manifold.
3. To investigate curvature conditions of Ricci solitons on a manifold.
4. To study hyper generalized ϕ -recurrent manifold.

In Chapter 1, we provide a general introduction which include basic definitions and formulas of differential geometry such as topological manifold, smooth manifold, Riemannian manifold, almost contact metric manifold, Kenmotsu manifold, Lorentzian almost contact manifold, Lorentzian Para-Contact metric manifold, Ricci solitons, Lorentzian manifold, Relativistic Thermodynamical Fluid spacetime, and submanifolds, as well as a review of the literature.

Chapter 2 delves into the study of some results on conharmonically flat, ξ conharmonically flat and Quasi conharmonically flat Kenmotsu Manifold with respect to generalized Tanaka Webster Connection. We study generalized conharmonic ϕ -recurrent Kenmotsu manifolds equipped with generalized Tanaka Webster connection. Moreover, we also study Kenmotsu manifolds satisfying $\tilde{R}(\xi, X).\tilde{K} = 0$ where $\tilde{R}$ denotes conharmonic curvature tensor and $\tilde{K}$ denotes Riemannian curvature tensor

equipped with generalized Tanaka Webster connection respectively. And this paper also includes the study of projectively flat, ξ -projectively flat and locally projectively ϕ -symmetric Kenmotsu manifolds with respect to generalized Tanaka Webster connection.

In Chapter 3, we deal with the transversal hypersurfaces of Lorentzian Kenmotsu manifold. We show that all transversal hypersurfaces of Lorentzian Kenmotsu manifolds admit an almost Hermitian structure and (f, g, u, v, δ) -structure. Furthermore, we give some characterizations of totally geodesic transversal hypersurfaces of such manifolds. Finally, we introduce a transversal hypersurface example that admits (f, g, u, v, δ) structure.

Chapter 4 is divided into two sections. In the first section, we examine Ricci solitons in Lorentzian para-Kenmotsu manifolds. We establish that a symmetric parallel second-order covariant tensor in a Lorentzian para-Kenmotsu manifolds is a constant multiple of the metric tensor. We have shown that if $\mathcal{L}_V g + 2S$ is parallel to the Levi-Civita connection associated with g , where V is a given vector field, then (g, V, λ) is a Ricci solitons. We observe that a Ricci solitons in a W_2 -semi-symmetric Lorentzian para-Kenmotsu manifolds is expanding. Furthermore, certain curvature properties of Lorentzian para-Kenmotsu manifolds admitting Ricci solitons are studied. Finally, we have provided an example of a 3-dimensional Lorentzian para-Kenmotsu manifolds. In the second section, we discussed how Ricci soliton interact with Relativistic Thermodynamical fluid spacetime and its influence on the dynamic behavior and long term evolution of spacetime. Moreover, the Ricci soliton metric is used to determine its behavior under different vector fields.

In Chapter 5, we present two sections. The first section examines the hyper generalized ϕ -recurrent LP-Sasakian manifold's geometrical features. Among other things, we have demonstrated the Ricci semi-symmetry of a hyper generalized ϕ -recurrent LP-Sasakian manifold with non vanishing scalar curvature. An example is shown to demonstrate the existence of a hyper generalized ϕ -recurrent LP-Sasakian mani-

fold. We then examine quasi generalized ϕ -recurrent LP-Sasakian manifold and their different characteristics. We also demonstrate that a quasi generalized ϕ -recurrent LP-sasakian manifold is a quasi-generalized Ricci-recurrent manifold. And in the second section, we investigate the concept of hyper-generalized weakly symmetric Para-Sasakian manifolds. In particular, we establish specific criteria under which a hyper-generalized weakly symmetric para-Sasakian manifold becomes an η -Einstein manifold. Moreover, we explore the behavior of these manifolds when they are equipped with a quarter-symmetric metric connection.

Chapter 6 is devoted to the summary and conclusion.

A list of references is given at the end.