CERTAIN DIFFERENTIABLE STRUCTURES ON A MANIFOLD

By

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To

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CERTIFICATE

This is to certify that the thesis entitled *"Certain Differentiable Structures on a Manifold"* submitted by **Mrs. Mayanglambam Saroja Devi** (Registration no. MZU/Ph.D./388 of 07.06.2011), for the degree of Doctor of Philosophy of the Mizoram University, Aizawl, embodies the record of the original investigations carried out by her under my supervision. She has been duly registered and the thesis presented is worthy of being considered for the award of the Ph.D. degree. This work has not been submitted for any degree of any other universities.

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I, Mrs. Mayanglambam Saroja Devi, hereby declare that the subject matter of this thesis entitled "CERTAIN DIFFERENTIABLE STRUCTURES ON A MANIFOLD " is the record of work done by me, that the contents of this thesis do not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in other University/Institute.

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PREFACE

The present thesis entitled "CERTAIN DIFFERENTIABLE STRUCTURES ON A MANIFOLD" is an outcome of the researcher carried out by the author Mrs. Mayanglambam Saroja Devi under the supervision of Dr. Jay Prakash Singh, Department of Mathematics and Computer science, Mizoram University, Aizawl, Mizoram.

This thesis has been divided into six chapters and each chapter is subdivided into a number of articles. The first chapter is general introduction which include basic definitions, Differentiable manifolds, Tangent Vector, Tangent space and Vector field, Tensors, Lie-bracket, Lie derivative, Covariant derivative, Connection, tensors, contracted tensors, Riemannian manifolds, Torsion tensor, Riemannian connection, Exterior derivative, quarter symmetric non-metric connection, semi-symmetric non-metric connection, Ricci tensor, Curvature tensors on Riemannian manifolds, Almost contact metric manifold, P-Sasakian manifold, Almost paracontact metric manifold, Recurrent manifold, Lorentzian paracontact manifold, Kenmotsu manifold, submanifold and hypersurfaces.

The second chapter is related with the characterization of LP-Sasakian manifolds admitting a semi-symmetric non-metric and a quarter symmetric non-metric connection in an LP-Sasakian manifold. We have shown that if an LP-Sasakian manifold admits a semi symmetric non-metric connection $\tilde{D}$, then the necessary and sufficient condition for the conformal curvature tensor of $\tilde{D}$ to coincide with that of the Riemannian connection D is that the conharmonic curvature tensor of $\tilde{D}$ is equal to that of D provided $\psi = -1$. Next, we have proved that if an LP-Sasakian manifold admits a semi-symmetric non-metric connection $\tilde{D}$, then the necessary and sufficient condition for the concircular curvature tensor of $\tilde{D}$ to coincide with that of D is that the curvature tensor of $\tilde{D}$ coincides with that of D only when $\psi = -1$. Later, we have shown that an n-dimensional LP-Sasakian manifold M^n with respect to semi-symmetric non-metric connection is $\xi - m$-projectively flat if and only if the manifold is also $\xi - m$-projectively flat with respect to the Riemannian connection provided the vector fields X and Y are orthogonal to ξ. Further, we have studied about LP-Sasakian manifold admitting a quarter symmetric non-metric connection. An n-dimensional LP-Sasakian manifold with quarter-symmetric non-metric connection is ξ-quasi conformally flat if and only if the manifold is also ξ-quasi conformally flat with respect to the Riemannian connection provided the vector fields X, Y are orthogonal to ξ. An n-dimensional LP-Sasakian manifold is ξ-pseudo projectively flat with respect to the quarter-symmetric non-metric connection if and only if the manifold is also ξ-pseudo projectively flat with respect to the Riemannian connection provided the vector fields X and Y are orthogonal to ξ. We also prove that an n-dimensional LP-Sasakian manifold is globally $\phi - m$-projectively symmetric with respect to the quarter symmetric non-metric connection if and only if the manifold is also globally $\phi - m$- projectively symmetric with respect to the Riemannian connection provided the vector fields X, Y, Z, U are orthogonal to ξ. We also discussed curvature conditions of submanifolds of LP-Sasakian manifolds with respect to quarter symmetric non-metric connection.

The third chapter deals with study of hypersurfaces of LP-Sasakian manifolds. We have shown that if an LP-Sasakian manifold M^n is recurrent, then the totally geodesic hypersurface M^{n-1} of LP-Sasakian manifold M^n is recurrent. Later, we prove that the LP-Sasakian manifold is η- Einstein manifold, then its hypersurface M^{n-1} is A-Einstein whether it is totally geodesic or totally umbilical. Finally, we have got that totally geodesic (totally umbilical) hypersurface M^{n-1} of a generalized Ricci-recurrent LP-Sasakian manifold is a generalized Ricci-recurrent manifold.

In the fourth chapter we have shown that if a Kenmotsu manifold is globally ϕ-m-projectively symmetric, then the manifold is an Einstein manifold. Next, we have proved that a 3-dimensional Kenmotsu manifold is locally ϕ-m-projectively symmetric if and only if the scalar curvature r is constant. Later we prove that an n-dimensional Kenmotsu manifold is ξ-m-projectively flat if and only if it is an Einstein manifold.

After this, we have got that an n-dimensional ϕ-m-Projectively flat Kenmotsu manifold is an η-Einstein manifold with constant curvature and an example of a locally ϕ-m-Projectively symmetric Kenmotsu manifold in 3-Dimension are also discussed in this chapter.

In the fifth chapter firstly, we show that an m-projectively symmetric LP-Sasakian manifold M^n is Ricci-recurrent. Next, we have proved that a $\phi - m$-projectively symmetric LP-Sasakian manifold M^n is an Einstein. Later, we have proved that if an extended generalized concircularly ϕ -recurrent LP-Sasakian manifold M^n, $n \geq 3$, is an extended generalized ϕ-recurrent LP-Sasakian manifold, then the associated vector field corresponding to the 1-form A is given by $\rho_1 = \frac{1}{r} grad\ \ r$, r being the non-zero and non-constant scalar curvature of the manifold. We also show that an extended generalized concircularly ϕ-recurrent LP-Sasakian manifold $M^n, n \geq 3$, is super generalized Ricci recurrent manifold.

The last chapter is summary and conclusion.

In the end, the references of the papers of the authors have been given with surname of the author and their years of the publication, which are decoded in chronological order in the Bibliography.

Some portions of present thesis has been already published in National/International journals. A brief account of published chapters is given in the list of publications.

Contents

Chapter 1

Introduction

1.1 Differentiable Manifold

Let M^n be a Housdorff space. An open chart on M^n is a pair (U, ϕ), where U is an open subset of M^n and ϕ is homeomorphism of U onto an open subset of $\mathbb{R}^n$, where $\mathbb{R}^n$ is an n-dimensional Euclidean space.

A differentiable structure on M^n of dimension n is a collection of open charts $(U_\alpha, \phi_\alpha)_{\alpha \in I}$ on M^n, where $\phi_\alpha(U_\alpha)$ is an open subset of $\mathbb{R}^n$, such that

(1) the union of all domains of charts coincides with itself.

i.e. $\quad M^n = \underset{\alpha \in I}{\cup} U_\alpha$

and

(2) for each pair $\alpha, \beta \in I$, the mapping $\phi_\alpha o \phi_\beta^{-1}$ is a diffeomorphic mapping of $\phi_\alpha(U_\alpha \cap U_\beta)$ onto $\phi_\beta(U_\alpha \cap U_\beta)$.

A differentiable manifold (or C^∞ manifold) of dimension n is a Housdorff space with differentiable structure of dimension n. If M^n is a manifold, a local chart on M^n is by definition a pair (U_α, ϕ_α). If $p \in U_\alpha$ and $\phi_\alpha(p) = (x^1(p), ..., x^n(p))$ then the set U_α is called a coordinate neighbourhood of p and the numbers $x^\alpha(p)$ are called the local coordinate of p. If M^n has a C^∞-differentiable structure, then M^n is called an analytic manifold.

1.2 Tangent vector

If the real valued function f is C^∞ at every point in M^n, then f is said to be a C^∞ or a smooth function on M^n. The set of all smooth functions on M^n will be denoted by $C^\infty(M^n)$.

Tangent Vector: A tangent vector X at a point $p \in M^n$ is a mapping $X : C^\infty(M^n) \to \mathbb{R}$ such that

(i) $Xf \in \mathbb{R}$ for all $f \in C^\infty(M^n)$,

(ii) $X(\alpha f + \beta g) = \alpha(Xf) + \beta(Xg)$ for all $\alpha, \beta \in \mathbb{R}$, and $f, g \in C^\infty(M^n)$,

(iii) $X(fg) = f(Xg) + (Xf)g$.

Tangent Space: The system consisting of

(i) the set T_p of all tangent vectors at p,

(ii) a binary operation $'+'$:

$X, Y \in T_p \Rightarrow X + Y \in T_p$,

satisfying

$(X + Y)f = Xf + Yf$,

(iii) an operation of scalar multiplication $'\cdot'$:

$f \in C^\infty(p), X \in T_p \Rightarrow fX \in T_p$,

satisfying

$(aX)f = aXf; a \in \mathbb{R}$,

is a vector space called tangent space to M^n at p.

The basis of T_p with respect to coordinate system $(x^1, x^2, ...x^n)$ is $(\frac{\partial}{\partial x^i}), i = 1, 2, ...n$.

Let T_p^* be the dual space of T_p whose basis with respect to the basis $(\frac{\partial}{\partial x^i})$ is $(dx^1, dx^2, ...dx^n)$. We observe that the elements of T_p are the contravariant vectors and elements of T_p^* are the covariant vectors with respect to the basis of T_p.

1.3 Vector Field

Vector Field: A vector field X on a smooth manifold M^n is a smooth assignment of a tangent vector $X_p \in T_p(M^n)$ at each point $p \in M^n$. Smooth assignment means for all $f \in C^\infty(M^n)$, the function $Xf : M^n \to \mathbb{R}$ defined by

$$p \to (Xf)(p) = X_p(f)$$

is a C^∞ function, that is, $Xf \in C^\infty(M)$, where X_p is the real valued function $X_p : C^\infty(p) \to \mathbb{R}$, $C^\infty(p)$ is the set of smooth functions at $p \in M^n$.
A vector field X on M^n gives rise to a linear map $X : C^\infty(M^n) \to C^\infty(M^n)$ such that the map $f \to Xf$ satisfies the following properties:

$$X(f+g) = Xf + Xg \tag{1.3.1}$$

$$X(\alpha f) = \alpha Xf \tag{1.3.2}$$

$$X(fg) = (Xf)g + f(Xg) \tag{1.3.3}$$

for all $f, g \in C^\infty(M^n), \alpha \in \mathbb{R}$. This implies that X is also derivation of the algebraic $C^\infty(M^n)$. Thus a vector field X is defined as a derivation of the ring of functions $C^\infty(M^n)$ satisfying (1.3.1) - (1.3.3). Thus to each point $p \in M^n$ such a derivation assigns a linear map $X_p : C^\infty(M^n) \to \mathbb{R}$ defined by $X_p f = (Xf)(p)$ for each $f \in C^\infty(M^n)$ and hence the map $p \in X_p$ assigns a field of tangent vectors.

1.4 Lie-Bracket, Lie Derivative

Lie-Bracket: Let X and Y be C^∞ vector fields on M^n. Then their Lie bracket is a mapping

$$[\ ,\ ] : M^n \times M^n \longrightarrow M^n$$

such that

$$[X, Y]f = X(Yf) - Y(Xf),$$

where f is a C^∞-function.

The Lie-bracket has the following properties:

$$[X, Y](f + g) = [X, Y]f + [X, Y]g, \tag{1.4.1}$$

$$[X, Y](fg) = f[X, Y]g + g[X, Y]f, \tag{1.4.2}$$

$$[X, Y] + [Y, X] = 0 \tag{1.4.3}$$

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X \tag{1.4.4}$$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \qquad (\textit{Jacobi identity}). \tag{1.4.5}$$

Lie Derivative: A linear map $\pounds_X : \chi(M^n) \to \chi(M^n)$ for all $X \in \chi(M^n)$ defined by

$$\pounds_X Y = [X, Y], \tag{1.4.6}$$

is known as Lie derivative of the vector field Y with respect to the vector field X.

It satisfies the following properties:

(a) $\pounds_X f = Xf, \qquad f \in F$

(b) $(\pounds_X A)(Y) = X(A(Y)) - A[X, Y], \qquad$ A is 1-form

(c) $$\begin{aligned}(\pounds_X P)(A_1, ..., A_r, X_1, ..., X_s) &= X(P(A_1, ..., A_r, X_1, ..., X_s)) \\ &- P(\pounds_X A_1, ..., A_r, X_1, ..., X_s)...... \\ &- P(A_1, ..., A_r, [X, X_1], X_2, ..., X_s)...... \\ &.................. \\ &- P(A_1, ..., X_{s-1}, [X, X_s]), \qquad p \in T^r_s.\end{aligned}$$

1.5 Covariant Derivative

A linear affine connection on M^n is a function

$$D : T_p(M^n) \times T_p(M^n) \to T_p(M^n)$$

such that

$$D_{fX+gY} Z = f(D_X Z) + g(D_Y Z), \tag{1.5.1}$$

$$D_X f = Xf, \tag{1.5.2}$$

$$D_X(fY + gZ) = f(D_X Y) + g(D_X Z) + (Xf)Y + (Xg)Z, \tag{1.5.3}$$

for arbitrary vector fields X, Y, Z and smooth functions f, g on M^n. D_X is called covariant derivative operator and $D_X Y$ is called covariant derivative of Y with respect to X.

The covariant derivative of a 1-form w is given by

$$(D_X w)(Y) = X(w(Y)) - w(D_X Y).$$

1.6 Connection

Connection: A connection D is a type preserving mapping that assigns to each pair of C^∞ fields (X, P), $X \in T_p, P \in T_s^r$, a C^∞ vector fields $D_X P$ such that if $X, Y, Z \in T_p$, $A \in T_p^r$ are C^∞ fields and f is a C^∞ function, then

$$(i)\ D_X f = Xf, \tag{1.6.1}$$

$$\begin{aligned} (ii) \quad (a) \quad & D_X(Y+Z) = D_X Y + D_X Z, \\ (b) \quad & D_X(fY) = (Xf)Y + f D_X Y, \end{aligned} \tag{1.6.2}$$

$$\begin{aligned} (iii) \quad (a) \quad & D_{X+Y} Z = D_X Z + D_Y Z, \\ (b) \quad & D_{fX} Z = f D_X Z, \end{aligned} \tag{1.6.3}$$

$$(iv)\ (D_X A)(Y) = X(A(Y)) - A(D_X Y), \tag{1.6.4}$$

$$\begin{aligned} (v)\ (D_X P)(A_1, ..., A_r, X_1, ..., X_s) &= X(P(A_1, ..., A_r, X_1, ..., X_s)) \\ &- P(D_X A_1, ..., A_r, X_1, ..., X_s)... \\ &- P(A_1..., A_r, X_1, ..., D_X X_s). \end{aligned} \tag{1.6.5}$$

1.7 Tensors

If V is an n-dimensional vector space over the field F with dual space V^*, then the elements of the tensor product

$$V_s^r = \underbrace{V \otimes ..... \otimes V}_{r} \otimes \underbrace{V^* \otimes ..... \otimes V^*}_{s}, \tag{1.7.1}$$

are called tensors of type (r, s), where r is the contravariant and s is the covariant order.

The elements of $V_0^r = V^r$ are called contravariant tensors of order r, and those of $V_s^0 = V^s$ are called covariant tensors of order s. We also have $V_0^0 = F$, $V_0^1 = V$, $V_1^0 = V^*$. The elements of $F = V_0^0$ are called scalars and the elements of $V = V_0^1$ are called contravariant vectors whereas, the elements of $V_1^0 = V^*$ are called covariant vectors.

1.8 Contracted Tensors

The linear mapping

$$C_k^h : T_s^r \to T_{s-1}^{r-1} \qquad (i \leq h \leq r \qquad i \leq k \leq s)$$

such that

$$\begin{aligned} C_k^h(\lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_r \otimes \alpha_1 \otimes \cdots \otimes \alpha_s) &= \alpha_k(\lambda_1 \otimes \cdot \cdot \otimes \lambda_{h-1} \otimes \lambda_{h+1} \cdots \otimes \lambda_r \\ &\quad \otimes \alpha_1 \otimes \alpha_2 \otimes \cdot \cdot \alpha_{k-1} \otimes \lambda_{k+1} \otimes \cdot \cdot \alpha_s), \end{aligned}$$

where $\lambda_1, \lambda_2 ... \lambda_r \in T_p$ and $\alpha_1, \alpha_2 .... \alpha_s \in \bar{T}_p$ and $\otimes$ denote tensor product, is called contraction with respect to h^{th} contravariant and k^{th} covariant places. A tensor obtained after contraction is called a contracted tensor.

1.9 Riemannian Manifold, Torsion Tensor and Riemannian Connection

Let us consider a C^∞ real valued, bilinear symmetric, non-singular positive definite function g on the ordered pair of tangent vectors at each point $p \in M^n$, such that

$$g(X, Y) \text{ is a real number,} \tag{1.9.1}$$

$$g \text{ is symmetric} \Rightarrow g(X, Y) = g(Y, X),$$

$$g \text{ is non-singular i.e. } g(X,Y)=0, \forall Y \neq 0 \Rightarrow X=0.$$

$$g \text{ is positive definite i.e. } g(X,Y)>0, \forall X \neq 0.$$

and

$$g(\alpha X+\beta Y, Z)=\alpha g(X,Z)+\beta g(Y,Z);\ \alpha,\beta \in \mathbb{R},$$

then g is said to be Riemannian metric tensor.

The manifold M^n with a Riemannian metric is called a Riemannian manifold and its geometry is called a Riemannian geometry.

Torsion tensor: A vector valued, skew-symmetry, bilinear function T of the type $(1,2)$ defined by

$$T(X,Y) \overset{def}{=} D_X Y - D_Y X - [X,Y] \tag{1.9.2}$$

is called a torsion tensor of the connection D in a C^∞-manifold M^n.

If the torsion tensor of a connection D vanishes, it is said to be symmetric or torsion free.

Riemannian connection: A connection D is said to be Riemannian, if

$$T(X,Y)=0 \tag{1.9.3}$$

and

$$D_X g = 0. \tag{1.9.4}$$

Hence, we can say that a linear connection is symmetric and metric if and only if it is the Riemannian connection.

1.10 Exterior Derivative:

Let V_p be the set of all C^∞ p-forms on an open set A. Then the mapping $d : V_p \to V_{p+1}$ given by

$$(df)(X) = Xf, \quad X \in T_p, \quad f \in F \tag{1.10.1}$$

and

$$\begin{aligned}(dA)(X_1, ..., X_{p+1}) &= X_1\,(A(X_2, ..., X_{p+1}))\\ &- X_2\,(A(X_1, X_3, ...X_{p+1}))\\ &+ X_3\,(A(X_1, X_2, X_4, ..., X_{p+1}))\,....\\ &- A([X_1, X_2], X_3, ..., X_{p+1})\\ &+ A([X_1, X_3], X_2, X_4, ..., X_{p+1})\\ &- A([X_2, X_3], X_1, X_4, ..., X_{p+1}) + ...\end{aligned} \tag{1.10.2}$$

for arbitrary C^∞ vector fields $X's \in V^1$ and $A \in V_p$, is called the exterior derivative.

1.11 Semi-symmetric non-metric connection

(Friedmann and Scouten, 1924) A linear connection $\tilde{\nabla}$ in an n-dimensional differentiable manifold M^n is said to be semi-symmetric connection if its torsion tensor $\tilde{T}$ satisfies

$$\tilde{T}(X,Y) = \eta(Y)X - \eta(X)Y, \tag{1.11.1}$$

where η is a 1-form. Further, a semi-symmetric connection is called a semi-symmetric metric connection if

$$(\tilde{\nabla}_X g)(Y,Z) = 0. \tag{1.11.2}$$

Otherwise, the linear connection is called semi-symmetric non-metric connection.

1.12 Quarter Symmetric Non-Metric Connection

A linear connection ∇ on an n-dimensional Riemannian manifold (M^n, g) is called a quarter symmetric connection if its torsion tensor T of the connection ∇

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{1.12.1}$$

satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.12.2}$$

where η is 1-form and ϕ is a $(1,1)$ tensor field. In particular, if $\phi(X) = X$, then the quarter symmetric connection reduces to a semi symmetric connection. Thus the notion of quarter symmetric connection generalizes the notion of semi symmetric connection. Moreover, if a quarter symmetric connection ∇ satisfies the condition

$$(\nabla_X g)(Y,Z) = 0, \tag{1.12.3}$$

for all $X, Y, Z \in T_p(M^n)$, where $T_p(M^n)$ is the Lie algebra of vector fields of the manifold M^n, then ∇ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter symmetric non-metric connection.

1.13 Ricci-Tensor

Let M^n is a Riemannian manifold with a Riemannian connection D. Then the Ricci tensor field S is the covariant tensor field of degree 2 defined as $Ric(Y,Z) = S(Y,Z) =$ Trace of the linear map $X \to R(X,Y)Z$ for all $X, Y, Z \in T_p(M^n)$.

If $\{e_1, ..., e_n\}$ is an orthonormal basis of the tangent space $T_p, p \in M^n$ and R is the

Riemannian curvature tensor of the Riemannian manifold (M^n, g), then

$$S(X,Y) = \sum_{i=1}^{n} g(R(e_i,X)Y,e_i) \tag{1.13.1}$$

$$= \sum_{i=1}^{n} {'R}(e_i,X,Y,e_i) \tag{1.13.2}$$

$$= \sum_{i=1}^{n} {'R}(X,e_i,e_i,Y),$$

$$= \sum_{i=1}^{n} g(R(X,e_i)e_i,Y),$$

where $'R$ is the Riemannian curvature tensor of the manifold of type $(0,4)$.

The linear map Q of the type $(1,1)$ defined by

$$g(QX,Y) \overset{def}{=} S(X,Y) \tag{1.13.3}$$

is called a Ricci-map.It is self-adjoint,

$$i.e., \quad g(QX,Y) = g(X,QY). \tag{1.13.4}$$

The scalar r defined by

$$r \overset{def}{=} (C_1^1 R) \tag{1.13.5}$$

is called the scalar curvature of M^n at the point p.

A Riemannian manifold M^n is said to be Einstein manifold, if

$$S(X,Y) = \frac{r}{n} g(X,Y). \tag{1.13.6}$$

A Riemannian manifold M^n is said to be η-Einstein manifold, if

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y). \tag{1.13.7}$$

where α and β are smooth functions.

1.14 Curvature Tensor

The curvature tensor R of type $(1, 3)$ with respect to the Riemannian connection D is defined by the mapping

$$R : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \longrightarrow T_p(M^n)$$

given by

$$R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z, \tag{1.14.1}$$

for all $X, Y, Z \in T_p(M^n)$.

Let $'R$ be the associative curvature tensor of the type $(0, 4)$ of the curvature tensor R. Then

$$'R(X, Y, Z, U) = g(R(X, Y, Z)U), \tag{1.14.2}$$

$'R$ is called the Riemann-Christoffel curvature tensor.

The following identities are satisfied by associative curvature tensor $'R$:

$'R$ is skew-symmetric in first two slot

$$i.e., \quad 'R(X, Y, Z, U) = -\ 'R(Y, X, Z, U) \tag{1.14.3}$$

$'R$ is skew-symmetric in last two slot

$$i.e., \quad 'R(X, Y, Z, U) = -\ 'R(X, Y, U, Z) \tag{1.14.4}$$

$'R$ is symmetric in two pair of slot

$$i.e., \quad 'R(X, Y, Z, U) = \quad 'R(Z, U, X, Y) \tag{1.14.5}$$

$'R$ satisfies Bianchi's first identities

$$i.e., \quad 'R(X,Y,Z,U) + \ 'R(Y,Z,X,U) + \ 'R(Z,X,Y,U) = 0 \tag{1.14.6}$$

and $\ 'R$ satisfies Bianchi's second identities

$$i.e., \ (D_X \ 'R)(Y,Z,U,V) + (D_Y \ 'R)(Z,X,U,V) + (D_Z \ 'R)(X,Y,U,V) = 0. \tag{1.14.7}$$

1.15 Important Curvature Tensors on Riemannian Manifold

The m-projective curvature tensor $'W^*$ of the type $(0,4)$, is defined by (Pokhariyal and Mishra, 1970)

$$\begin{aligned} 'W^*(X,Y,Z,U) &= \ 'R(X,Y,Z,U) - \frac{1}{2(n-1)}\{g(X,U)S(Y,Z) - g(Y,U)S(X,Z) \\ &+ \ S(X,U)g(Y,Z) - S(Y,U)g(X,Z)\}. \end{aligned} \tag{1.15.1}$$

It satisfies the following algebraic properties

(a) $'W^*(X,Y,Z,U) = \ 'W^*(Z,U,X,Y)$,

(b) $'W^*(X,Y,Z,U) = - \ 'W^*(Y,X,U,Z)$,

(c) $'W^*(X,Y,Z,U) = - \ 'W^*(X,Y,U,Z)$,

(d) $'W^*(X,Y,Z,U) + \ 'W^*(Y,Z,X,U) + \ 'W^*(Z,X,Y,U) = 0$,

where

$$'W^*(X,Y,Z,U) = g(W^*(X,Y,Z),U).$$

The concircular curvature tensor $'C$ of type $(0,4)$, is given by

$$\begin{aligned} 'C(X,Y,Z,U) &= {}'R(X,Y,Z,U) - \frac{r}{n(n-1)}\{g(Y,Z)g(X,U) \\ &- g(X,Z)g(Y,U)\}. \end{aligned} \tag{1.15.2}$$

It satisfies the following algebraic properties

$$(a)\ 'C(X,Y,Z,U) = -'C(Y,X,Z,U),$$

$$(b)\ 'C(X,Y,Z,U) = -'C(X,Y,U,Z),$$

$$(c)\ 'C(X,Y,Z,U) = {}'C(Z,U,X,Y),$$

$$(d)\ 'C(X,Y,Z,U) + {}'C(Y,Z,X,U) + {}'C(Z,X,Y,U) = 0,$$

where

$$'C(X,Y,Z,U) = g(C(X,Y,Z),U).$$

The conharmonic curvature tensor $'L$ of the type $(0,4)$, is defined as follows

$$\begin{aligned} 'L(X,Y,Z,U) &= {}'R(X,Y,Z,U) - \frac{1}{n-1}\{S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \\ &+ S(X,U)g(Y,Z) - S(Y,U)g(X,Z)\}. \end{aligned} \tag{1.15.3}$$

It satisfies the following properties

$$(a)'L(X,Y,Z,U) = -'L(Y,X,Z,U),$$

$$(b)'L(X,Y,Z,U) = {}'L(X,Y,U,Z),$$

$$(c)'L(X,Y,Z,U) = {}'L(Z,U,X,Y),$$

$$(d)'L(X,Y,Z,U) + {}'L(Y,Z,X,U) + {}'L(Z,X,Y,U) = 0,$$

where

$$'L(X,Y,Z,U) = g(L(X,Y,Z),U).$$

The projective curvature tensor $'P$ of the type $(0,4)$, is defined by

$$\begin{aligned} 'P(X,Y,Z,U) &= {}'R(X,Y,Z,U) - \frac{1}{n-1}\{S(Y,Z)g(X,U) \\ &- S(X,Z)g(Y,U)\}. \end{aligned} \tag{1.15.4}$$

The projective curvature tensor $'P$ satisfies the following identities

$$(a)'P(X,Y,Z,U) = -'P(Y,X,Z,U),$$

$$(b)C_1^1P = C_2^1P = C_3^1P = 0,$$

$$(c)'P(X,Y,Z,U) + 'P(Y,Z,X,U) + 'P(Z,X,Y,U) = 0,$$

where

$$'P(X,Y,Z,U) = g(P(X,Y,Z),U).$$

The conformal curvature tensor $'V$ of the type $(0,4)$, is defined as

$$\begin{aligned} 'V(X,Y,Z,U) &= R(X,Y,Z,U) - \frac{1}{(n-2)}\big[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \\ &+ g(Y,Z)S(X,U) - g(X,Z)S(Y,U)\big] \\ &+ \frac{r}{(n-1)(n-2)}\big[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\big]. \end{aligned} \tag{1.15.5}$$

It satisfies the following properties

$$(a)'V(X,Y,Z,U) = -'V(Y,X,Z,U),$$

$$(b)'V(X,Y,Z,U) = -'V(X,Y,U,Z),$$

$$(c)'V(X,Y,Z,U) = 'V(Z,U,X,Y),$$

$$(d)'V(X,Y,Z,U) + 'V(Y,Z,X,U) + 'V(Z,X,Y,U) = 0,$$

where

$$'V(X,Y,Z,U) = g(V(X,Y,Z),U).$$

The Weyl projective curvature tensor $'W$ of the type $(0,4)$, defined as

$$\begin{aligned} 'W(X,Y,Z,U) &= R(X,Y,Z,U) - \frac{1}{n-1}\{S(Y,Z)g(X,U) \\ &- S(X,Z)g(Y,U)\}. \end{aligned} \tag{1.15.6}$$

It satisfies the following properties

$$(a)'W(X,Y,Z,U) = -'W(Y,X,Z,U),$$

$$(b)'W(X,Y,Z,U) = -'W(X,Y,U,Z),$$

$$(c)'W(X,Y,Z,U) = 'W(Z,U,X,Y),$$

$$(d)'W(X,Y,Z,U) + 'W(Y,Z,X,U) + 'W(Z,X,Y,U) = 0,$$

where

$$'W(X,Y,Z,U) = g(W(X,Y,Z),U).$$

Finally the W_2 curvature tensor $'W_2$ of the type $(0,4)$, is defined as (Pokhariyal and Mishra, 1970)

$$\begin{aligned} 'W_2(X,Y,Z,U) &= R(X,Y,Z,U) + \frac{1}{n-1}\{g(X,Z)g(QY,U) \\ &- g(Y,Z)g(QX,U)\}. \end{aligned} \tag{1.15.7}$$

It satisfies the following identities

$$(a)'W_2(X,Y,Z,U) = -'W_2(Y,X,Z,U),$$

$$(b)'W_2(X,Y,Z,U) = -'W_2(X,Y,U,Z),$$

$$(c)'W_2(X,Y,Z,U) = 'W_2(Z,U,X,Y),$$

$$(d)'W_2(X,Y,Z,U) + 'W_2(Y,Z,X,U) + 'W_2(Z,X,Y,U) = 0,$$

where

$$'W_2(X,Y,Z,U) = g(W_2(X,Y,Z),U).$$

1.16 Almost Contact Metric Manifold

If M^n be an odd dimensional differentiable manifold on which there are defined a real vector valued linear function ϕ, a 1-form η and a vector field ξ satisfying for arbitrary vectors $X, Y, Z, ...$

$$\phi^2 X = -X + \eta(X)\xi, \tag{1.16.1}$$

$$\eta(\xi) = 1, \tag{1.16.2}$$

$$\phi(\xi) = 0, \tag{1.16.3}$$

$$\eta(\phi X) = 0, \tag{1.16.4}$$

and

$$rank(\phi) = n - 1, \tag{1.16.5}$$

is called an almost contact manifold (Sasaki, 1968) and the structure (ϕ, η, ξ) is called an almost contact structure (Hatakeyama *et al.*, 1963; Sasaki and Hatakeyama, 1961).

An almost contact manifold M^n on which a Riemannian metric tensor g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.16.6}$$

and

$$g(X, \xi) = \eta(X), \tag{1.16.7}$$

is called an almost contact metric manifold and the structure (ϕ, ξ, η, g) is called an almost contact metric structure (Sasaki, 1960).

The fundamental 2-form Φ of an almost contact metric manifold M^n is defined by

$$\Phi(X,Y) = g(\phi X, Y). \tag{1.16.8}$$

From the equations (1.16.6) and (1.16.8), we have

$$\Phi(X,Y) = -\ \Phi(Y,X). \tag{1.16.9}$$

If in an almost contact metric manifold

$$2\ \Phi(X,Y) = (D_X\eta)(Y) - (D_Y\eta)(X), \tag{1.16.10}$$

then M^n is called an almost Sasakian manifold.

1.17 P- Sasakian manifolds

An n-dimensional differentiable manifold M^n is a P-Sasakian manifold if it admits a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Riemannian metric g, which satisfy (Matsumoto, 1977; Miyazawa, 1979)

$$\phi^2(X) = X - \eta(X)\xi, \tag{1.17.1}$$

$$\phi\xi = 0, \tag{1.17.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{1.17.3}$$

$$g(X,\xi) = \eta(X), \tag{1.17.4}$$

$$(D_X\phi)(Y) = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{1.17.5}$$

$$D_X\xi = \phi X, \tag{1.17.6}$$

$$(a)\eta(\xi) = 1, (b)\ \ \eta(\phi X) = 0, \tag{1.17.7}$$

$$rank(\phi) = (n-1), \tag{1.17.8}$$

$$(D_X\eta)(Y) = g(\phi X, Y) = g(\phi Y, X), \tag{1.17.9}$$

for any vector fields X, Y where D denotes covariant differentiation with respect to g.

1.18 Almost Paracontact Metric Manifold

Let M^n be an n-dimensional C^∞-manifold. If there exist in M^n a tensor field ϕ of the type $(1,1)$, consisting of a vector field ξ and 1-form η in M^n satisfying

$$\phi^2 X = X - \eta(X)\xi, \tag{1.18.1}$$

$$\phi(\xi) = 0, \qquad \eta(\xi) = 1, \tag{1.18.2}$$

then M^n is called an almost paracontact manifold.

Let g the Riemannian metric satisfying

$$\eta(X) = g(X, \xi), \qquad \eta(\phi X) = 0, \tag{1.18.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.18.4}$$

then the structure (ϕ, ξ, η, g) satisfying (1.18.1) to (1.18.4) is called an almost paracontact Riemannian structure. The manifold with such structure is called an almost paracontact Riemannian manifold (Sato and Matsumoto, 1976).

If we define $\Phi(X,Y) = g(\phi X, Y)$, then the following relations are satisfied

$$\Phi(X,Y) = \Phi(Y,X), \tag{1.18.5}$$

and

$$\Phi(\phi X, \phi Y) = \Phi(X,Y). \tag{1.18.6}$$

If in M^n the relation

$$(D_X\eta)(Y) - (D_Y\eta)(X) = 0, \tag{1.18.7}$$

$$d\eta(X,Y) = 0, \quad \textit{i.e. } \eta \textit{ is closed.} \tag{1.18.8}$$

$$\begin{aligned} (D_X\Phi)(Y,Z) &= -g(X,Z)\eta(Y) - g(X,Y)\eta(Z) \\ &+ 2\,\eta(X)\eta(Y)\eta(Z), \end{aligned} \tag{1.18.9}$$

$$(D_X\eta)(Y) + (D_X\eta)(X) = 2\,\Phi(X,Y), \tag{1.18.10}$$

and

$$D_X\xi = \phi X, \tag{1.18.11}$$

hold, then (M^n, g) is called Para-Sasakian manifold or briefly P-Sasakian manifold.

1.19 Recurrent Manifold

Let M^n be an n-dimensional smooth Riemannian manifold and $T_p(M^n)$ denotes the set of differentiable vector fields on M^n. Let $X, Y \in T_p(M^n)$; D_XY denotes the covariant derivative of Y with respect to X and $R(X,Y,Z)$ be the Riemannian curvature tensor of type $(1,3)$.

A Riemannian manifold M^n is said to be recurrent (Kobayashi and Nomizu, 1963) if

$$(D_U R)(X, Y, Z) = \alpha(U) R(X, Y, Z), \tag{1.19.1}$$

where $X, Y \in T_p(M^n)$ and α is a non-zero 1-form known as recurrence parameter. If the 1-form α is zero in (1.19.1), then the manifold reduces to symmetric manifold (Singh and Khan, 1999).

A Riemannian manifold (M^n, g) is said to be semi-symmetric if it satisfies the relation (Szabo, 1982)

$$(R(X, Y).R)(U, V)W = 0, \tag{1.19.2}$$

where $R(X, Y)$ is considered as the tensor algebra at each point of the manifold i.e. $R(X, Y)$ is curvature transformation or curvature operator.

A Riemannian manifold (M^n, g) is said to be Ricci-recurrent if it satisfies the relation

$$(D_X S)(Y, Z) = A(X) S(Y, Z) \tag{1.19.3}$$

for all $X, Y, Z \in T_p(M^n)$, where D denotes the Riemannian connection and A is a 1-form on M^n. If the 1-form A vanishes identically on M^n, then a Ricci-recurrent manifold becomes a Ricci-symmetric manifold.

A Riemannian manifold (M^n, g) is called a generalized recurrent manifold (De and Guha, 1991) if its curvature tensor R satisfies the condition:

$$(D_X R)(Y, Z)U = A(X) R(Y, Z)U + B(X)[g(Z, U)Y - g(Y, U)Z]. \tag{1.19.4}$$

where A and B are two 1-forms, B is non-zero and these are defined by

$$A(X) = g(X, P_1), \qquad B(X) = g(X, P_2), \tag{1.19.5}$$

P_1 and P_2 are vector fields associated with 1-forms A and B, respectively.

A Riemannian manifold (M^n, g) is called generalized ϕ-recurrent if its curvature tensor R satisfies the condition

$$\begin{aligned} \phi^2((D_W R)(Y,Z)U) &= A(W)R(Y,Z)U \\ &+ B(W)[g(Z,U)Y - g(Y,U)Z] \end{aligned} \tag{1.19.6}$$

where A and B are two 1-forms, B is non-zero.

1.20 Lorentzian Paracontact Metric Manifold

Let M^n be an n-dimensional differentiable manifold endowed with a tensor field ϕ of the type $(1,1)$, a vector field ξ, a 1-form η and a Lorentzian metric g satisfying

$$\phi^2 X = X + \eta(X)\xi, \tag{1.20.1}$$

$$\eta(\xi) = -1, \tag{1.20.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{1.20.3}$$

$$g(X,\xi) = \eta(X), \tag{1.20.4}$$

for arbitrary vector field X and Y, then M^n is called a Lorentzian paracontact (LP-Contact) manifold and the structure (ϕ, ξ, η, g) is called the Lorentzian paracontact structure (Matsumoto, 1989).

Let M^n be a Lorentzian paracontact manifold with structure (ϕ, ξ, η, g). Then it satisfy

$$(a)\ \phi(\xi) = 0, \quad (b)\ \eta(\phi X) = 0, \quad (c)\ rank(\phi) = n-1. \tag{1.20.5}$$

A Lorentzian paracontact manifold is called a Lorentzian Para-Sasakian manifold if (Matsumoto and Mihai, 1988)

$$\nabla_X \xi = \phi X, \tag{1.20.6}$$

$$(\nabla_X \phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\ \eta(X)\eta(Y)\xi, \tag{1.20.7}$$

where ∇ denotes the covariant differentiation with respect to g.

Let us put $\Phi(X,Y) = g(\phi X, Y)$. Then the tensor field Φ is symmetric.

$$i.e.\ \ \Phi(X,Y) = \ \Phi(Y,X), \tag{1.20.8}$$

and

$$\Phi(X,Y) = (\nabla_X \eta)(Y). \tag{1.20.9}$$

Also, in an LP-Sasakian manifold the following relation holds

$$'R(X,Y,Z,\xi) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{1.20.10}$$

and

$$S(X,\xi) = (n-1)\eta(X). \tag{1.20.11}$$

1.21 Kenmotsu Manifold

Let $(M^n, \phi, \xi, \eta, g)$ be an n-dimensional (where n=2m+1) almost contact metric manifold, where ϕ is a $(1,1)$- tensor field, ξ is the structure vector field , η is a 1-form and g is the Riemannian metric. It is well known that the (ϕ, ξ, η, g) structure satisfies the conditions (Blair,1976)

$$\phi^2(X) = -X + \eta(X)\xi, \tag{1.21.1}$$

$$g(X,\xi) = \eta(X), \tag{1.21.2}$$

$$\phi\xi = 0, \eta\phi = 0, \eta(\xi) = 1, \tag{1.21.3}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{1.21.4}$$

for any vector fields X and Y on M^n.

If moreover

$$(D_X\phi)(Y) = -g(X,\phi Y)\xi - \eta(Y)\phi X, \tag{1.21.5}$$

$$D_X\xi = X - \eta(X)\xi, \tag{1.21.6}$$

where D is the Riemannian connection, then $(M^n, \phi, \xi, \eta, g)$ is called a Kenmotsu manifold.

1.22 Submanifold and Hypersurfaces

let M^n be a C^∞-Riemannian manifold. A C^∞ manifold $M^m(m \leq n)$ is called a submanifold of M^n, if for each point in M^n, there is a coordinate neighbourhood $\overline{U}$ of M^n with coordinate function $\{y_\alpha : \alpha = 1, 2, ..., n\}$ such that for the set

$$U = \{p \in \overline{U} : y_{m+1} = ... = y_n = 0 \quad at \quad p\}$$

is a coordinate neighbourhood of P in M^m with coordinate functions

$$x_i = y_\alpha|U, i = 1, 2, ..., m$$

If $n = m + 1$, the submanifold M^m is called a hypersurface.

Let

$$b : M^m \longrightarrow M^n$$

be the inclusion map such that $p \in M^m \Rightarrow bp \in M^n$.

The map b induces a linear transformation B called the Jaccobian map such that

$$b : T_p^m \longrightarrow T_p^n$$

where T_P^m is the tangent space to M^m at point p and T_p^n is the tangent space to M^n at bp, such that

$X \in M^n \quad at \quad p \Rightarrow BX \in M^n \quad at \quad bp.$

Let g be the metric tensor at M^n and G the induced metric tensor of M^m at bp relative to the metric tensor g of M^n at bp. Let X, Y be arbitrary vector fields to M^n. Then

$$G(X,Y) = g(BX, BY) \circ b. \tag{1.22.1}$$

A C^∞ vector field N of M^n satisfying

$$\begin{aligned} &(a) \quad g(N, BX) \circ b = 0 \\ &(b) \quad g(N, N) \circ b = 1, \end{aligned} \tag{1.22.2}$$

for arbitrary vector field X is called field of normal.

Let $\underset{x}{N}, x = m+1, ..., n.$ be a system of C^∞-orthogonal unit normal vector fields to M^m. Then

$$\begin{aligned} &(a) \quad g(\underset{x}{N}, BX) \circ b = 0 \\ &(b) \quad g(\underset{x}{N}, \underset{y}{N}) = \delta xy. \end{aligned} \tag{1.22.3}$$

If M^m is a hypersurface, the equation (1.22.3)((a),(b)) assumes the form of (1.22.2). Let D be the Riemannian connection in M^n and E be the induced connection in M^m. Then the Gauss and the Weingarten equation can be written as

$$\begin{aligned} &(a) \quad D_{BX}BY = BE_XY + {}'\underset{x}{H}(X,Y)\underset{y}{N} \qquad (Gauss\ Equation) \\ &(b) \quad D_{BX}\underset{x}{N} = -B\underset{x}{H}X + \overset{y}{\underset{x}{I}}(X)\underset{y}{N} \qquad (Weingarten\ Equation) \end{aligned} \tag{1.22.4}$$

where

$$
\begin{aligned}
&(a) \quad G(\underset{x}{H}, Y) = {}'\underset{x}{H}(X, Y) = {}'\underset{x}{H}(Y, X) \\
&(b) \quad \overset{y}{\underset{x}{I}} + \overset{x}{\underset{y}{I}} = 0.
\end{aligned} \tag{1.22.5}
$$

${}'H$ is called second fundamental magnitudes in M^m.

For a hypersurface M^m of M^n

$$
\begin{aligned}
&(a) \quad D_{BX}BY = BE_XY + {}'H(X, Y)N \qquad (Gauss\ Equation) \\
&(b) \quad D_{BX}N = -BHX \qquad (Weingarten\ Equation)
\end{aligned} \tag{1.22.6}
$$

1.23 Review of Literature

The idea of semi-symmetric linear connection on a differentiable manifold was introduced by Friedman and Schouten (1924). Later, Hayden (1932) defined a semi-symmetric metric connection on a Riemannian manifold and this was further developed by Yano (1970). This was also studied by many geometers like as Sasaki and Hatakeyama (1961), Hatakeyama (1963), Hatakeyama *et al.* (1963), Yano (1972), Pravanovic (1975), Sato (1976), Sharfuddin and Hussain (1976) and obtained a number of interesting results. Further, semi-symmetric metric connections on a Riemannian manifold have been studied by Amur and Pujar (1978), Binh (1990), De (1990, 1991), De and Biswas (1997), Pathak and De (2002), Jun *et al.* (2005), Barmen and De (2013), Chaubey and Ojha (2012), Singh and Pandey (2008), Singh *et al.* (2012, 2013) and many other geometers.

Sasaki and Hatakeyama (1961), Hatakeyama(1963), Hatakeyama *et al.*(1963), Sasaki (1960, 1968) defined an almost contact manifold. In the meantime, Sasaki(1960), Hatakeyama *et al.* (1963) defined an almost contact metric manifold or an almost Grayan manifold. Tanno(1971) classified connected almost contact metric manifolds whose automorphism group possesses the maximum dimension. A semi-symmetric

metric connection was defined in an almost contact manifold by Sharfuddin and Hussain (1976). De and Sengupta (2001) investigated the curvature tensor of an almost contact metric manifold admitting a type of semi-symmetric metric connection and studied the curvature properties of conformal curvature tensor and projective curvature tensor. Agashe and Chafle (1992) introduced a semi symmetric non-metric connection on a Riemannian manifold and this was further studied by Prasad (1994), Ojha and Prasad (1994), De and Kamilya(1995), Sengupta *et al.* (2000), Pandey and Ojha (2001), Tripathi and Kakar (2001a, b), Prasad and Kumar (2002), Chaturvedi and Pandey (2008), Murathan and Özgür (2008), Chaubey (2011), Singh (2014a) and others.

On the other hand, the notion of quarter-symmetric connection in a Riemannian manifold with an affine connection which generalized the idea of semi-symmetric connection was introduced and studied by Golab(1975). Further this was developed by Rastogi (1978, 1987), Mishra and Pandey (1980), Yano and Imai(1982), Mukhopadhyay, Roy and Barua(1991), Biswas and De (1997), Sengupta and Biswas (2003), Nivas and Verma (2005), Singh and Pandey (2007) and many other geometers.

Matsumoto (1989) introduced the notion of Lorentzian Para Sasakian manifold. Mihai and Rosca (1992) also introduced the same notion independently and they obtained several results on this manifold. Lorentzian Para-Sasakian manifolds had also been studied by Matsumoto and Mihai (1988), Mihai *et al.* (1999a, b), De *et al.* (1999), Shaikh and De (2000), De and Sengupta (2002), Özgür (2003), Shaikh and Biswas (2004), Venkatesha and Bagewadi (2008), Dhruwa *et al.* (2009), Perktas and Tripathi (2010), Taleshian and Asghari (2010), Venkatesha *et al.* (2011), Prakash *et al.* (2011), Taleshian and Asghari (2011) and Singh (2013, 2015) obtained some results on Lorentzian Para-Sasakian manifolds. Prakash and Narain (2011) defined and studied quarter-symmetric non metric connection on an LP-Sasakian manifolds and proved its existence. They found some properties of the curvature tensor and the Ricci tensor of quarter-symmetric non metric connection. Singh (2013) studied weakly symmetric, weakly Ricci symmetric, generalized recurrent LP-Sasakian manifolds admitting a quarter-symmetric non metric connections.

Kenmotsu (1972) studied a class of contact Riemannian manifolds and called them Kenmotsu manifold. He proved that if a Kenmotsu manifolds satisfies the condition $R(X,Y).R = 0$, then the manifold is of negative curvature -1, where R is the Riemannian curvature of type $(1,3)$ and $R(X,Y)$ denotes the derivation of tensor algebra at each point of the tangent space. A space form is said to be elliptic, hyperbolic or Euclidean according as sectional curvature tensor is positive, negative or zero. The properties of Kenmotsu manifold had been studied by authors such as Sinha and Srivastava (1991), De and Pathak (2004), De(2008), De *et al.*(2008) and others. Wang and Liu (2015a, b) studied almost Kenmotsu manifolds with some nullity distributions. Recently, Mandal and De (2017) studied about the geometric conditions in 3-dimensional almost Kenmotsu manifolds such that ξ belongs to the $(k,\mu)'$ -nullity distribution and $h' \neq 0$.

The idea of recurrent manifolds was introduced by Walker (1950). On the other hand, Dubey(1979) introduced the notion of generalized recurrent manifold and then such a manifold was studied by De and Guha (1995). De *et al.*(1995) defined the generalized recurrent Riemannian manifold and generalized Ricci-recurrent Riemannian manifold. The notion of generalized ϕ - recurrency to Sasakian manifolds and Lorentzian α-Sasakian manifolds are respectively studied in Patil *et al.*(2009) and Prakasha and Yildiz (2010). By extending the notion of generalized ϕ -recurrency, Shaikh and Hui (2011), introduced the notion of extended generalized ϕ -recurrent manifolds. Prakasha (2013) considered the extended generalized ϕ -recurrent in Sasakian manifold. Further Shaikh *et al.*(2013) studied this notion for LP-Sasakian manifolds. Prasad (2000) introduced the idea of semi generalized recurrent manifold. Yildiz and

Murathan (2005) studied Lorentzian α-Sasakian manifolds and proved that conformally flat and quasi conformally flat Lorentzian α-Sasakian manifolds are locally isometric with a sphere. Lorentzian α-Sasakian manifolds have been studied by De and Tripathi (2003), Yildiz and Turan (2009), Yildiz *et al.* (2009), Prakasha and Yildiz (2010), Lokesh, *et al.* (2012), Teleshian and Asghari (2012), Yadav and Suthar (2012), Bhattacharya and Patra (2014), Berman (2014), and many others. Shaikh (2015) introduced the notion of generalized $\phi\phi$-recurrent LP-Sasakian manifold and studied its various geometric properties.

Adati and Matsumoto (1977) defined P-Sasakian and Special Para Sasakian manifold, which are special classes of an almost para-contact manifold introduced by Sato (1976). Para Sasakian manifolds have been studied by Matsumoto (1977), Adati and Miyazawa (1979), Matsumoto *et.al.* (1986), De and Pathak (1994), Özgür and Tripathi (2007), De and Sarkar (2009), Shukla and Shukla (2010), Berman (2013), Singh (2014b) and many others.

Takahashi (1977) introduced the notion of ϕ-symmetric Sasakian manifold and obtained some interesting properties. Many authors like Shaikh and De (2000), De and Pathak (2004) and Venkatesha and Bagewadi (2006) have extended this notion to 3-dimensional LP-Sasakian manifold, 3-dimensional Kenmotsu manifold and 3-dimensional trans-Sasakian manifolds respectively. De and Kamilya (1994) studied the generalized concircular recurrent manifolds and De *et al.* (1995) studied the generalized Ricci-recurrent manifolds. Generalizing the notion of recurrency Khan (2004) introduced the notion of generalized recurrent Sasakian manifold. Jaiswal and Ojha (2009) studied generalized ϕ-recurrent and generalized concircular ϕ-recurrent LP-Sasakian manifolds. Sreenivasa *et al.* (2009) define ϕ-recurrent Lorentzian β-Kenmotsu manifold and prove that a concircular ϕ-recurrent Lorentzian β-Kenmotsu manifold is an Einstein manifold. Debnath and Bhattacharya (2013) studied the generalized ϕ-recurrent trans-Sasakian manifolds. Pokhariyal and Mishra (1971) introduced new curvature tensor called m-projective curvature tensor in a Riemannian manifold and studied its properties. Ojha (1975) studied a note on the m-projective curvature tensor. Later, Pokhariyal (1982) studied some properties of this curvature tensor in a Sasakian manifold. Ojha (1986), Chaubey (2012), Chaubey and Ojha (2010), Singh (2009, 2012, 2016) and many other geometers studied this curvature tensor in different manifolds. Tripathi and Dwivedi (2008) studied projective curvature tensor in K -contact and Sasakian manifolds and they proved that (i) if a K-contact manifold is quasi projectively flat then it is Einstein and (ii) a K-contact manifold is ξ -projectively flat if and only if it is Einstein Sasakian.

Chen and Ogive (1973,1974) introduced geometry of submanifolds and real submanifolds. Eum (1968), Blair and Ludden (1969), Goldsberg and Yano (1970), Ludden (1970) and others studied hypersurfaces of an almost contact manifold. Goldsberg and Yano (1970) defined noninvariant hypersurface of the contact almost contact manifolds. Sato (1976) studied a structure similar to the almost contact structure, almost paracontact structure. Adati (1981) studied hypersurfaces of an almost paracontact manifold. Bucki(1989) considered hypersurfaces of an almost r-paracontact Riemannian manifold. Bucki and Miernowski(1989) investigated some properties of invariant hypersurfaces of an almost r-paracontact Riemannian manifold. Yano and Kon (1977) studied anti invariant submanifold of Sasakian space forms. Al and Nivas (2000) studied on submanifolds of a manifold with quarter-symmetric connection. Ahmad *et al.* (2011) studied the properties of hypersurfaces and submanifold on r-paracontact Riemannian manifold with connection.

Chapter 2

Semi-symmetric non-metric and quarter symmetric non-metric connections

In this chapter, we have studied some properties of certain curvatures on LP-Sasakian manifolds admitting semi-symmetric non-metric connection. We also discussed different geometrical properties of LP-Sasakian manifolds admitting quarter-symmetric non- metric connection and obtained some interesting results.

2.1 Introduction

In an n-dimensional LP-Sasakian manifold with structure (ϕ, ξ, η, g) defined in equations (1.20.1-1.20.8) the following relation holds (De *et al.,* 2005, De and Shaikh, 1999):

$$(D_X\eta)(Y) = \Phi(X, Y) = g(\phi X, Y), \Phi(X, \xi) = 0, \tag{2.1.1}$$

for all vector fields X, Y.

Also in an LP-Sasakian manifold the following relations hold (Shaikh and De, 2000):

$$\eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.1.2}$$

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$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.1.3}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.1.4}$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \tag{2.1.5}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{2.1.6}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.1.7}$$

for all vector fields X, Y, Z, where R and S are the Riemannian curvature tensor and the Ricci tensor of the manifold respectively.

Here we consider a semi-symmetric non-metric connection $\tilde{D}$ on M^n given by (Agashe and Chafle, 1992)

$$\tilde{D}_X Y = D_X Y + \eta(Y)X. \tag{2.1.8}$$

The curvature tensor $\tilde{R}$ with respect to semi-symmetric non-metric connection $\tilde{D}$ is defined as

$$\tilde{R}(X, Y, Z) = \tilde{D}_X \tilde{D}_Y Z - \tilde{D}_Y \tilde{D}_X Z - \tilde{D}_{[X,Y]} Z$$

which satisfies

$$\begin{aligned} \tilde{R}(X, Y, Z) = R(X, Y, Z) &+ g(\phi X, Z)Y - g(\phi Y, Z)X \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y. \end{aligned} \tag{2.1.9}$$

Contracting (2.1.9) with respect to X, we have

$$\tilde{S}(Y, Z) = S(Y, Z) - (n-1)g(\phi Y, Z) + (n-1)\eta(Y)\eta(Z), \tag{2.1.10}$$

where $\tilde{S}$ is the Ricci tensor with respect to semi-symmetric non-metric connection $\tilde{D}$. Again from the above relation it follows that

$$\tilde{Q}Y = QY - (n-1)\phi Y + (n-1)\eta(Y)\xi \tag{2.1.11}$$

and

$$\tilde{r} = r - (n-1)\psi - (n-1), \tag{2.1.12}$$

where $\psi = trace\phi$, $\tilde{S}(Y,Z) = g(\tilde{Q}Y,Z)$, $S(Y,Z) = g(QY,Z)$ and $\tilde{r}, r$ are the Ricci tensors and scalar curvatures of the connections $\tilde{D}$ and D respectively.

2.2 Properties of some curvature tensors on an LP-Sasakian manifold admitting semi-symmetric non-metric connection

Definition 2.2.1 *The W_2-curvature tensor of M^n with respect to Reimannian connection D is defined as(Pokhariyal and Mishra, 1970)*

$$\begin{aligned} W_2(X,Y)Z &= R(X,Y)Z + \frac{1}{(n-1)}\{g(X,Z)QY \\ &- g(Y,Z)QX\}. \end{aligned} \tag{2.2.1}$$

Definition 2.2.2 *The m-projective curvature tensor W^* of an LP-Sasakian manifold with respect to Riemannian connection is given as(Pokhariyal and Mishra, 1971)*

$$\begin{aligned} W^*(X,Y)Z &= R(X,Y)Z - \frac{1}{2(n-1)}\{S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\}. \end{aligned} \tag{2.2.2}$$

Theorem 2.2.1 *In an LP-Sasakian manifold admitting a semi-symmetric non-metric connection $\tilde{D}$, the difference between the W_2 curvature tensors of $\tilde{D}$ and D is equal to twice of the difference between the m-projective curvature tensors of $\tilde{D}$ and D.*

Proof: The $\tilde{W}_2$-curvature tensor of M^n with respect to semi-symmetric non-metric connection $\tilde{D}$ is defined as

$$\begin{aligned}\tilde{W}_2(X,Y)Z &= \tilde{R}(X,Y)Z + \frac{1}{(n-1)}\{g(X,Z)\tilde{Q}Y \\ &- g(Y,Z)\tilde{Q}X\}. \end{aligned} \tag{2.2.3}$$

Making use of (2.1.9) and (2.1.11) in the equation (2.2.3), we obtain

$$\begin{aligned}\tilde{W}_2(X,Y)Z &= R(X,Y)Z + g(\phi X,Z)Y \\ &- g(\phi Y,Z)X + \eta(Y)\eta(Z)X \\ &- \eta(X)\eta(Z)Y + \frac{1}{(n-1)}\Big[g(X,Z)\{QY \\ &- (n-1)\phi Y + (n-1)\eta(Y)\xi\} \\ &- g(Y,Z)\{QX - (n-1)\phi X + (n-1)\eta(X)\xi\}\Big] \end{aligned} \tag{2.2.4}$$

which is equivalent to

$$\begin{aligned}\tilde{W}_2(X,Y)Z &= W_2(X,Y)Z - g(\phi Y,Z)X + g(\phi X,Z)Y \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - g(X,Z)\phi Y \\ &+ \eta(Y)g(X,Z)\xi + g(Y,Z)\phi X - \eta(X)g(Y,Z)\xi. \end{aligned} \tag{2.2.5}$$

Taking inner product of the equation (2.2.5) with respect to U, we get

$$\begin{aligned}{}'\tilde{W}_2(X,Y,Z,U) &= {}'W_2(X,Y,Z,U) - g(\phi Y,Z)g(X,U) \\ &+ g(\phi X,Z)g(Y,U) + \eta(Y)\eta(Z)g(X,U) \\ &- \eta(X)\eta(Z)g(Y,U) - g(X,Z)g(\phi Y,U) \\ &+ \eta(Y)g(X,Z)g(\xi,U) + g(Y,Z)g(\phi X,U) \\ &- \eta(X)g(Y,Z)g(\xi,U) \end{aligned} \tag{2.2.6}$$

where

$$'\tilde{W_2}(X,Y,Z,U) \overset{def}{=} g(\tilde{W_2}(X,Y)Z,U)$$

and

$$'W_2(X,Y,Z,U) \overset{def}{=} g(W_2(X,Y,Z),U).$$

The equation (2.2.6) implies that

$$\begin{aligned} '\tilde{W_2}(X,Y,Z,U) - {}'W_2(X,Y,Z,U) &= -g(\phi Y,Z)g(X,U) \\ &+ g(\phi X,Z)g(Y,U) + \eta(Y)\eta(Z)g(X,U) \\ &- \eta(X)\eta(Z)g(Y,U) - g(X,Z)g(\phi Y,U) \\ &+ \eta(Y)g(X,Z)g(\xi,U) + g(Y,Z)g(\phi X,U) \\ &- \eta(X)g(Y,Z)g(\xi,U). \end{aligned} \tag{2.2.7}$$

Again, the m- projective curvature tensor $\tilde{W^*}$ with respect to semi-symmetric non-metric connection $\tilde{D}$ is given by

$$\begin{aligned} \tilde{W^*}(X,Y,Z) &= \tilde{R}(X,Y,Z) - \frac{1}{2(n-1)}\{\tilde{S}(Y,Z)X \\ &- \tilde{S}(X,Z)Y + g(Y,Z)\tilde{Q}X - g(X,Z)\tilde{Q}Y\}. \end{aligned} \tag{2.2.8}$$

If we define

$$'\tilde{W^*}(X,Y,Z,U) = g(\tilde{W^*}(X,Y,Z),U), \tag{2.2.9}$$

then by virtue of the equations (2.1.9), (2.1.10), (2.1.11) and (2.2.8), we get

$$\begin{aligned}
'\tilde{W}^*(X,Y,Z,U) &= 'W^*(X,Y,Z,U)+\frac{1}{2}\{-g(\phi Y,Z)g(X,U)\\
&+ g(\phi X,Z)g(Y,U)+\eta(Y)\eta(Z)g(X,U)\\
&- \eta(X)\eta(Z)g(Y,U)-g(X,Z)g(\phi Y,U)\\
&+ \eta(Y)\eta(U)g(X,Z)+g(Y,Z)g(\phi X,U)\\
&- \eta(X)\eta(U)g(Y,Z)\},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
'\tilde{W}^*(X,Y,Z,U)-'W^*(X,Y,Z,U) &= \frac{1}{2}\{-g(\phi Y,Z)g(X,U)\\
&+ g(\phi X,Z)g(Y,U)+\eta(Y)\eta(Z)g(X,U)\\
&- \eta(X)\eta(Z)g(Y,U)-g(X,Z)g(\phi Y,U)\\
&+ \eta(Y)\eta(U)g(X,Z)+g(Y,Z)g(\phi X,U)\\
&- \eta(X)\eta(U)g(Y,Z)\}, \qquad (2.2.10)
\end{aligned}$$

where

$$'\tilde{W}^*(X,Y,Z,U)\overset{def}{=}g(\tilde{W}^*(X,Y)Z,U).$$

In consequence of (2.2.7) and (2.2.10), we have

$$'\tilde{W}^*(X,Y,Z,U)-'W^*(X,Y,Z,U)=\frac{1}{2}\Big\{'\tilde{W}_2(X,Y,Z,U)-'W_2(X,Y,Z,U)\Big\}.$$

Hence, we obtain the statement of the theorem.

Definition 2.2.3 *The conformal curvature tensor V of Riemannian curvature tensor is defined (Mihai* et al.*, 1999) as*

$$\begin{aligned}
V(X,Y,Z) &= R(X,Y,Z)-\frac{1}{(n-2)}\{S(Y,Z)X\\
&- S(X,Z)Y-g(X,Z)QY+g(Y,Z)QX\}\\
&+ \frac{r}{(n-1)(n-2)}\{g(Y,Z)X-g(X,Z)Y\}. \qquad (2.2.11)
\end{aligned}$$

Definition 2.2.4 *The conharmonic curvature tensor L of Riemannian connection D in an LP-Sasakian manifold is defined as*

$$\begin{aligned} L(X,Y,Z) &= R(X,Y,Z) - \frac{1}{(n-2)}\{S(Y,Z)X - S(X,Z)Y \\ &- g(X,Z)QY + g(Y,Z)QX\}. \end{aligned} \tag{2.2.12}$$

Theorem 2.2.2 *If an LP-Sasakian manifold admits a semi symmetric non-metric connection $\tilde{D}$, then the necessary and sufficient condition for the conformal curvature tensor of $\tilde{D}$ to coincide with that of the Riemannian connection D is that the conharmonic curvature tensor of $\tilde{D}$ is equal to that of D provided $\psi = -1$.*

Proof: The conformal curvature tensor of $\tilde{D}$ is defined as

$$\begin{aligned} \tilde{V}(X,Y,Z) &= \tilde{R}(X,Y,Z) - \frac{1}{(n-2)}\{\tilde{S}(Y,Z)X \\ &- \tilde{S}(X,Z)Y - g(X,Z)\tilde{Q}Y + g(Y,Z)\tilde{Q}X\} \\ &+ \frac{\tilde{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{2.2.13}$$

If we define

$$'\tilde{V}(X,Y,Z,U) = g(\tilde{V}(X,Y,Z),U), \tag{2.2.14}$$

then by virtue of the equations (2.1.9), (2.1.10), (2.1.11), (2.1.12), (2.2.13) and (2.2.14), we obtain

$$\begin{aligned} '\tilde{V}(X,Y,Z,U) &= {}'V(X,Y,Z,U) \\ &- \frac{n}{(n-2)}\{'\tilde{R}(X,Y,Z,U) - {}'R(X,Y,Z,U)\} \\ &- \left\{\frac{1+\psi}{n-2}\right\}[g(X,U)g(Y,Z) - g(Y,U)g(X,Z)] \end{aligned}$$

which is equivalent to

$$
\begin{aligned}
'\tilde{V}(X,Y,Z,U) -' V(X,Y,Z,U) &= -\frac{n}{(n-2)}\{'\tilde{R}(X,Y,Z,U) \\
&- \ 'R(X,Y,Z,U)\} - \left\{\frac{1+\psi}{n-2}\right\}[g(X,U)g(Y,Z) \\
&- \ g(Y,U)g(X,Z)], \qquad (2.2.15)
\end{aligned}
$$

where

$$'V(X,Y,Z,U) \overset{def}{=} g(V(X,Y,Z),U).$$

Again, we define conharmonic curvature tensor $\tilde{L}$ of $\tilde{D}$ as

$$
\begin{aligned}
\tilde{L}(X,Y,Z) &= \tilde{R}(X,Y,Z) - \frac{1}{(n-2)}\{\tilde{S}(Y,Z)X \\
&- \ \tilde{S}(X,Z)Y - g(X,Z)QY + g(Y,Z)QX\}. \qquad (2.2.16)
\end{aligned}
$$

In view of (2.1.9), (2.1.10) and (2.2.12) the equation (2.2.16) takes the form as

$$
\begin{aligned}
'\tilde{L}(X,Y,Z,U) &= \ 'L(X,Y,Z,U) - \frac{n}{(n-2)}\{'\bar{R}(X,Y,Z,U) \\
&- \ 'R(X,Y,Z,U)\}
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
'\tilde{L}(X,Y,Z,U) - 'L(X,Y,Z,U) &= -\frac{n}{(n-2)}\{'\bar{R}(X,Y,Z,U) \\
&- \ 'R(X,Y,Z,U)\}, \qquad (2.2.17)
\end{aligned}
$$

where

$$'\tilde{L}(X,Y,Z,U) \overset{def}{=} g(\tilde{L}(X,Y,Z),U)$$

and

$$'L(X,Y,Z,U) \overset{def}{=} g(L(X,Y,Z),U).$$

In consequence of (2.2.15) and (2.2.17) and the fact $\psi = -1$, we have

$$'\tilde{V}(X,Y,Z,U) - 'V(X,Y,Z,U) = '\tilde{L}(X,Y,Z,U) - 'L(X,Y,Z,U).$$

And thus we have the result.

Definition 2.2.5 *The concircular curvature tensor C of D is defined as*

$$\begin{aligned} C(X,Y,Z) &= R(X,Y,Z) - \frac{r}{n(n-1)}\{g(Y,Z)X \\ &- g(X,Z)Y\}. \end{aligned} \tag{2.2.18}$$

Theorem 2.2.3 *If an LP-Sasakian manifold admits a semi-symmetric non-metric connection $\tilde{D}$, then the necessary and sufficient condition for the concircular curvature tensor of $\tilde{D}$ to coincide with that of D is that the curvature tensor of $\tilde{D}$ coincides with that of D only when $\psi = -1$.*

Proof: The concircular curvature tensor $\tilde{C}$ of $\tilde{D}$ is defined as

$$\begin{aligned} \tilde{C}(X,Y,Z) &= \tilde{R}(X,Y,Z) - \frac{\tilde{r}}{n(n-1)}\{g(Y,Z)X \\ &- g(X,Z)Y\}. \end{aligned} \tag{2.2.19}$$

If we define

$$'\tilde{C}(X,Y,Z,U) = g(\tilde{C}(X,Y,Z),U), \tag{2.2.20}$$

then by virtue of (2.1.9), (2.1.12), (2.2.19) and (2.2.20), we obtain

$$\begin{aligned} '\tilde{C}(X,Y,Z,U) &= 'C(X,Y,Z,U) - g(\phi Y,Z)g(X,U) \\ &+ g(\phi X,Z)g(Y,U) + \eta(Y)\eta(Z)g(X,U) \\ &- \eta(X)\eta(Z)g(Y,U) + \frac{(1+\psi)}{n}\Big\{g(Y,Z)g(X,U) \\ &- g(X,Z)g(Y,U)\Big\} \end{aligned}$$

which is equivalent to

$$
\begin{aligned}
{}'\tilde{C}(X,Y,Z,U) - {}'C(X,Y,Z,U) &= -g(\phi Y,Z)g(X,U) \\
&- g(\phi X,Z)g(Y,U) + \eta(Y)\eta(Z)g(X,U) \\
&- \eta(X)\eta(Z)g(Y,U) + \frac{(1+\psi)}{n}\{g(Y,Z)g(X,U) \\
&- g(X,Z)g(Y,U)\}, \qquad (2.2.21)
\end{aligned}
$$

where

$$
{}'C(X,Y,Z,U) = g(C(X,Y,Z),U).
$$

Taking inner product of (2.1.9) with respect to U, we obtain

$$
\begin{aligned}
{}'\tilde{R}(X,Y,Z,U) - {}'R(X,Y,Z,U) &= -g(\phi Y,Z)g(X,U) \\
&+ g(\phi X,Z)g(Y,U) + \eta(Y)\eta(Z)g(X,U) \\
&- \eta(X)\eta(Z)g(Y,U) \qquad (2.2.22)
\end{aligned}
$$

where

$$
{}'\tilde{R}(X,Y,Z,U) \stackrel{def}{=} g(\tilde{R}(X,Y,Z),U)
$$

and

$$
{}'R(X,Y,Z,U) \stackrel{def}{=} g(R(X,Y,Z),U).
$$

From the equations (2.2.21) and (2.2.22) and using the fact that $\psi = -1$, we obtain the statement of the theorem.

Definition 2.2.6 *The Weyl projective curvature tensor P of an LP-Sasakian manifold M^n with respect to Riemannian connection D is defined (Mishra, 1984) as*

$$
\begin{aligned}
P(X,Y)Z &= R(X,Y)Z - \frac{1}{(n-1)}\{S(Y,Z)X \\
&- S(X,Z)Y\}. \qquad (2.2.23)
\end{aligned}
$$

Theorem 2.2.4 *The Weyl projective curvature tensor of $\tilde{D}$ coincides with that of D in LP-Sasakian manifold.*

Proof: The Weyl projective curvature tensor $\tilde{P}$ with respect to semi-symmetric non-metric connection $\tilde{D}$ is given by

$$\begin{aligned}\tilde{P}(X,Y)Z &= \tilde{R}(X,Y)Z - \frac{1}{(n-1)}\{\tilde{S}(Y,Z)X \\ &- \tilde{S}(X,Z)Y\}. \end{aligned} \tag{2.2.24}$$

Taking inner product of (2.2.24) with respect to U and using (2.1.9) and (2.1.10), we get

$$\begin{aligned}{}'\tilde{P}(X,Y,Z,U) &= g(R(X,Y,Z),U) + g(\phi X,Z)g(Y,U) - g(\phi Y,Z)g(X,U) \\ &+ \eta(Y)\eta(Z)g(X,U) - \eta(X)\eta(Z)g(Y,U) \\ &- \frac{1}{(n-1)}\big[\{S(Y,Z) - (n-1)g(\phi Y,Z) \\ &+ (n-1)\eta(Y)\eta(Z)\}g(X,U) - \{S(X,Z) \\ &- (n-1)g(\phi X,Z) + (n-1)\eta(X)\eta(Z)\}g(Y,U)\big] \end{aligned} \tag{2.2.25}$$

which after simplification reduces to

$${}'\tilde{P}(X,Y,Z,U) = {}'P(X,Y,Z,U). \tag{2.2.26}$$

This completes the proof of the theorem.

2.3 $\xi - m$-projectively flat LP-Sasakian manifolds admitting semi-symmetric non-metric connection

Definition 2.3.1 *An n-dimensional LP-Sasakian manifold M^n is $\xi - m$-projectively flat if*

$$W^*(X,Y)\xi = 0. \tag{2.3.1}$$

Theorem 2.3.1 *An n-dimensional LP-Sasakian manifold M^n with respect to semi-symmetric non-metric connection is $\xi - m$-projectively flat if and only if the manifold is also $\xi - m$-projectively flat with respect to the Riemannian connection provided the vector fields X and Y are orthogonal to ξ.*

Proof: Using the equations (2.1.9), (2.1.10), (2.1.11) in (2.2.8), we obtain

$$\begin{aligned}
\tilde{W^*}(X,Y)Z &= R(X,Y)Z + g(\phi X, Z)Y - g(\phi Y, Z)X \\
&+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - \frac{1}{2(n-1)}\{S(Y,Z)X \\
&- (n-1)g(\phi Y, Z)X + (n-1)\eta(Y)\eta(Z)X - S(X,Z)Y \\
&+ (n-1)g(\phi X, Z)Y - (n-1)\eta(X)\eta(Z)Y + g(Y,Z)QX \\
&- (n-1)g(Y,Z)\phi X + (n-1)g(Y,Z)\eta(X)\xi - g(X,Z)QY \\
&+ (n-1)g(X,Z)\phi Y - (n-1)\eta(Y)g(X,Z)\xi\}
\end{aligned} \tag{2.3.2}$$

which, after simplification reduces to

$$\begin{aligned}
\tilde{W^*}(X,Y)\xi &= W^*(X,Y)\xi - \frac{1}{2}\{\eta(Y)X - \eta(X)Y \\
&- \eta(Y)\phi X - \eta(X)\phi Y\}.
\end{aligned}$$

Suppose X and Y are orthogonal to ξ, then the above relation becomes

$$\tilde{W^*}(X,Y)\xi = W^*(X,Y)\xi. \tag{2.3.3}$$

Thus we obtain the statement of the theorem.

2.4 Einstein manifold with respect to semi-symmetric non-metric connection

A Riemannian manifold is said to be an Einstein manifold with respect to Riemannian connection if

$$S(X,Y) = \frac{r}{n}g(X,Y). \tag{2.4.1}$$

Theorem 2.4.1 *In an LP-Sasakian manifold M^n admitting a semi-symmetric non-metric connection if the relation (2.4.4) holds, then the manifold is an Einstein manifold for the Riemannian connection D if and only if it is an Einstein manifold for the connection $\tilde{D}$.*

Proof: Analogous to (2.4.1), we define Einstein manifold with respect to semi-symmetric non-metric connection $\tilde{D}$ by

$$\tilde{S}(X,Y) = \frac{\tilde{r}}{n}g(X,Y). \tag{2.4.2}$$

With the help of (2.1.10) and (2.1.12), we get

$$\begin{aligned}
\tilde{S}(X,Y) - \frac{\tilde{r}}{n}g(X,Y) &= S(X,Y) - (n-1)g(\phi X,Y) \\
&+ (n-1)\eta(X)\eta(Y) - \Big[\frac{r-(n-1)\psi-(n-1)}{n}\Big]g(X,Y) \\
&= S(X,Y) - \frac{r}{n}g(X,Y) - (n-1)[g(\phi X,Y) \\
&- \eta(X)\eta(Y) - \Big\{\frac{1+\psi}{n}\Big\}g(X,Y)].
\end{aligned} \tag{2.4.3}$$

Let us suppose that

$$g(\phi X,Y) - \eta(X)\eta(Y) - \Big\{\frac{1+\psi}{n}\Big\}g(X,Y) = 0. \tag{2.4.4}$$

Then using (2.4.1) in (2.4.4), we get

$$\tilde{S}(X,Y) - \frac{\tilde{r}}{n} g(X,Y) = 0. \tag{2.4.5}$$

This completes the theorem.

2.5 Quarter-symmetric non- metric connection

Consider a quarter-symmetric non-metric connection ∇ on LP-Sasakian manifolds

$$\nabla_X Y = D_X Y - \eta(X)\phi Y, \tag{2.5.1}$$

given by Mishra and Pandey (1980) which satisfies

$$(\nabla_X g)(Y,Z) = 2\eta(X) g(\phi Y, Z). \tag{2.5.2}$$

The curvature tensor $\bar{R}$ with respect to a quarter-symmetric non-metric connection ∇ and the curvature tensor R with respect to the Riemannian connection D in LP-Sasakian manifolds are related (Singh, 2013) as

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X. \end{aligned} \tag{2.5.3}$$

Contracting the above equation with respect to X we get

$$\bar{S}(Y,Z) = S(Y,Z) - g(Y,Z) - n\eta(Y)\eta(Z). \tag{2.5.4}$$

Putting $Z = \xi$ in the equations (2.5.3) and (2.5.4) we get the following equations

$$\bar{R}(X,Y)\xi = 2R(X,Y)\xi, \tag{2.5.5}$$

$$\bar{S}(Y,\xi) = 2S(Y,\xi) = 2(n-1)\eta(Y), \tag{2.5.6}$$

where $\bar{S}$ is the Ricci tensor of M^n with respect to quarter-symmetric non-metric connection.

Combining (2.5.5) and (2.1.4), it follows that,

$$\eta(\bar{R}(X,Y)\xi) = 0. \tag{2.5.7}$$

Theorem 2.5.1 *In an LP-Sasakian manifold the curvature tensor with respect to quarter-symmetric non -metric connection* ∇ *satisfies the followings*

$$(i) \quad \bar{R}(X,Y,Z,U) = -\bar{R}(Y,X,Z,U),$$

$$(ii) \quad \bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0,$$

where $\bar{R}(X,Y,Z,U) = g(R(X,Y)Z,U)$ *for all vector fields* $X,Y,Z,U \in \chi(M^n)$.

Theorem 2.5.2 *For an LP-Sasakian manifold* M^n *with respect to the quarter-symmetric non- metric connection* ∇*, the following conditions hold:*

(i) *The scalar curvature* $\bar{r}$ *is given by* (2.5.10),

$$(ii)\nabla_X \xi = \phi X, \tag{2.5.8}$$

$$(iii)(\nabla_X \eta)Y = g(\phi X, Y), \tag{2.5.9}$$

$$(iv)(\nabla_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi.$$

Proof: Let $\{e_1, e_2, ..., e_n\}$ be a local orthonormal basis of vector fields in M^n. Then by putting $Y = Z = e_i$ in (2.5.4) and taking summation over $i, 1 \le i \le n$, we have

$$\bar{r} = r, \tag{2.5.10}$$

where $\bar{r}$ and r and the scalar curvature with respect to the quarter symmetric non-metric connection and the Riemannian connection respectively.

From the equation (2.5.4) we obtain

$$\bar{Q}Y = QY - Y - n\eta(Y)\xi. \tag{2.5.11}$$

Using (2.5.1) and (1.20.6), it follows that

$$\nabla_X \xi = \phi X. \tag{2.5.12}$$

Combining (2.5.1) and (2.1.1), it follows that

$$(\nabla_X \eta)Y = g(\phi X, Y). \tag{2.5.13}$$

Again combining (2.5.1) and (1.20.7) we obtain

$$(\nabla_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi. \tag{2.5.14}$$

Definition 2.5.1 *(Yano and Sawaki, 1968)The quasi conformal curvature tensor C_* for an n-dimensional LP-Sasakian manifold M^n with respect to Riemannian connection D is given by*

$$\begin{aligned} C_*(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY] \\ &- \frac{r}{n}\Big\{\frac{a}{n-1} + 2b\Big\}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{2.5.15}$$

where a and b are constants such that $a, b \neq 0$.

Definition 2.5.2 *An n-dimensional LP-Sasakian manifold M^n is ξ-quasi conformally flat if*

$$C_*(X,Y)\xi = 0. \tag{2.5.16}$$

Theorem 2.5.3 *An n-dimensional LP-Sasakian manifold with quarter-symmetric non-metric connection is ξ-quasi conformally flat if and only if the manifold is also ξ-quasi* *conformally flat with respect to the Riemannian connection provided the vector fields X, Y are orthogonal to ξ.*

Proof: The quasi-conformal curvature tensor of an LP-Sasakian manifold with respect to the quarter-symmetric non- metric connection ∇ is given as

$$\begin{aligned}\bar{C}_*(X,Y)Z &= a\bar{R}(X,Y)Z + b[\bar{S}(Y,Z)X \\ &- \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] \\ &- \frac{\bar{r}}{n}\Big\{\frac{a}{n-1} + 2b\Big\}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{2.5.17}$$

Using (2.5.3), (2.5.4), (2.5.10) and (2.5.11) in (2.5.17), we obtain

$$\begin{aligned}\bar{C}_*(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY] - \frac{r}{n}\Big\{\frac{a}{n-1} + 2b\Big\}[g(Y,Z)X \\ &- g(X,Z)Y] + a\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} - b\{g(Y,Z)X \\ &+ n\eta(Y)\eta(Z)X - g(X,Z)Y - n\eta(X)\eta(Z)Y \\ &+ g(Y,Z)X + n\eta(X)g(Y,Z)\xi - g(X,Z)Y \\ &- n\eta(Y)g(X,Z)\xi\} \end{aligned} \tag{2.5.18}$$

which is equivalent to

$$\begin{aligned}\bar{C}_*(X,Y)Z &= C_*(X,Y)Z + a\{g(Y,Z)\eta(X)\xi \\ &- g(X,Z)\eta(Y)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} \\ &- b[2g(Y,Z)X + n\eta(Y)\eta(Z)X - 2g(X,Z)Y \\ &- n\eta(X)\eta(Z)Y + n\eta(X)g(Y,Z)\xi - n\eta(Y)g(X,Z)\xi\}. \end{aligned} \tag{2.5.19}$$

Putting $Z = \xi$ in the equation (2.5.19) and making use of (1.20.2) and (1.20.4), we obtain

$$\bar{C}_*(X,Y)\xi = C_*(X,Y)\xi + [a + b(n-2)]\{\eta(Y)X - \eta(X)Y\}. \tag{2.5.20}$$

Suppose X and Y are orthogonal to ξ, then from (2.5.20), we obtain

$$\bar{C}_*(X,Y)\xi = C_*(X,Y)\xi. \tag{2.5.21}$$

Hence, we obtain the statement of the theorem.

2.6 ξ-pseudo-projectively flat LP-Sasakian manifolds admitting quarter-symmetric non-metric connection

Definition 2.6.1 *The pseudo-projective curvature tensor P_* on an LP-Sasakian manifold is defined as (Prasad, 2002)*

$$\begin{aligned} P_*(X,Y)Z &= a_0 R(X,Y)Z + a_1\{S(Y,Z)X - S(X,Z)Y\} \\ &- \frac{r}{n}\Big\{\frac{a_0}{n-1} + a_1\Big\}[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{2.6.1}$$

where a_0, a_1 are constants and $a_0, a_1 \neq 0$.

Definition 2.6.2 *An n-dimensional LP-Sasakian manifold M^n is ξ-pseudo projectively flat if*

$$P_*(X,Y)\xi = 0. \tag{2.6.2}$$

Theorem 2.6.1 *An n-dimensional LP-Sasakian manifold is ξ-pseudo projectively flat with respect to the quarter-symmetric non-metric connection if and only if the manifold is also ξ-pseudo projectively flat with respect to the Riemannian connection provided the vector fields X and Y are orthogonal to ξ.*

Proof:The pseudo projective curvature tensor of LP-Sasakian manifold M^n with respect to the quarter-symmetric non-metric connection ∇ is given by

$$\begin{aligned}\bar{P}_*(X,Y)Z &= a_0\bar{R}(X,Y)Z + a_1[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y] \\ &- \frac{\bar{r}}{n}\Big\{\frac{a_0}{n-1} + a_1\Big\}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{2.6.3}$$

Using (2.5.3),(2.5.4) and (2.5.10) in (2.6.3), we get

$$\begin{aligned}\bar{P}_*(X,Y,Z) &= a_0[R(X,Y,Z) + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] + a_1[S(Y,Z)X - g(Y,Z)X \\ &- n\eta(Y)\eta(Z)X - S(X,Z)Y + g(X,Z)Y + n\eta(X)\eta(Z)Y] \\ &- \frac{r}{n}\Big\{\frac{a_0}{n-1} + a_1\Big\}[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{2.6.4}$$

which is equivalent to

$$\begin{aligned}\bar{P}_*(X,Y)Z &= P_*(X,Y)Z + a_0\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} + a_1[g(X,Z)Y \\ &- g(Y,Z)X - n\eta(Y)\eta(Z)X + n\eta(X)\eta(Z)Y\}. \end{aligned} \tag{2.6.5}$$

Putting $Z = \xi$ in (2.6.5), we get

$$\bar{P}_*(X,Y)\xi = P_*(X,Y)\xi + \{a_0 + (n-1)a_1\}[\eta(Y)X - \eta(X)Y\}]. \tag{2.6.6}$$

Suppose X and Y are orthogonal to ξ, then equation (2.6.6) implies

$$\bar{P}_*(X,Y)\xi = P_*(X,Y)\xi.$$

Thus the proof of the theorem is over.

2.7 Globally ϕ-m-projectively symmetric LP-Sasakian manifolds with respect to the quarter symmetric non-metric connection

Definition 2.7.1 *An LP-Sasakian manifold M^n with respect to the Riemannian connection is called to be globally $\phi - m$-projectively symmetric if*

$$\phi^2((\nabla_U W^*)(X,Y)Z) = 0. \tag{2.7.1}$$

Theorem 2.7.1 *An n-dimensional LP-Sasakian manifold is globally $\phi - m$-projectively symmetric with respect to the quarter symmetric non-metric connection if and only if the manifold is also globally $\phi - m$- projectively symmetric with respect to the Riemannian connection provided the vector fields X, Y, Z, U are orthogonal to ξ.*

Proof : Taking inner product of the equation (2.2.2) with respect to ξ and using (2.1.2), we obtain

$$\begin{aligned} \eta(W^*(X,Y)Z) &= g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \\ &- \left[\frac{1}{2(n-1)}\right]\{S(Y,Z)\eta(X) - S(X,Z)\eta(Y) \\ &+ g(Y,Z)\eta(QX) - g(X,Z)\eta(QY)\}. \end{aligned} \tag{2.7.2}$$

From the equation (2.5.6) we get

$$\eta(QX) = (n-1)\eta(X). \tag{2.7.3}$$

We know that

$$\begin{aligned} (\nabla_U W^*)(X,Y)Z &= \nabla_U W^*(X,Y)Z - W^*(\nabla_U X,Y)Z \\ &- W^*(X,\nabla_U Y)Z - W^*(X,Y)\nabla_U Z. \end{aligned} \tag{2.7.4}$$

Moreover, using (2.5.1) in (2.7.4) and taking X, Y, Z, U orthogonal to ξ, it follows that

$$(\nabla_U W^*)(X, Y)Z = (D_U W^*)(X, Y)Z. \tag{2.7.5}$$

We define the m-projective curvature tensor $\bar{W}^*$ with respect to the quarter-symmetric non-metric connection on LP-Sasakian manifolds as

$$\begin{aligned}\bar{W}^*(X, Y)Z &= \bar{R}(X, Y)Z - \frac{1}{2(n-1)}\{\bar{S}(Y, Z)X - \bar{S}(X, Z)Y \\ &+ g(Y, Z)\bar{Q}X - g(X, Z)\bar{Q}Y\},\end{aligned} \tag{2.7.6}$$

where

$$\bar{S}(Y, Z) = g(\bar{Q}Y, Z).$$

Using the equations (2.5.3), (2.5.4) and (2.5.11) in (2.7.6), we obtain

$$\begin{aligned}\bar{W}^*(X, Y)Z &= W^*(X, Y)Z + \frac{1}{(n-1)}\Big\{g(Y, Z)X - g(X, Z)Y \ Big\} \\ &+ \frac{(3n-2)}{2(n-1)}\Big\{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi\Big\} \\ &+ \frac{(n-2)}{2(n-1)}\Big[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\Big].\end{aligned} \tag{2.7.7}$$

Taking covariant differentiation of (2.7.7) with respect to U and also taking X, Y, Z, U are orthogonal to ξ and using (2.5.2),(2.5.13), (2.5.14) and (2.7.4), we have

$$\begin{aligned}(\nabla_U \bar{W}^*)(X, Y)Z &= (\nabla_U W^*)(X, Y)Z + \frac{(3n-2)}{2(n-1)}\Big\{g(Y, Z)g(\phi U, X) \\ &- g(X, Z)g(\phi U, Y)\Big\}\xi\end{aligned}$$

which is equivalent to

$$\begin{aligned}(\nabla_U \bar{W}^*)(X, Y)Z &= (D_U W^*)(X, Y)Z + \frac{(3n-2)}{2(n-1)}\Big\{g(Y, Z)g(\phi U, X) \\ &- g(X, Z)g(\phi U, Y)\Big\}\xi.\end{aligned} \tag{2.7.8}$$

Now, applying ϕ^2 on both sides of (2.7.8) we get

$$\phi^2((\nabla_U \bar{W}^*)(X,Y)Z) = \phi^2((D_U W^*)(X,Y)Z). \tag{2.7.9}$$

This completes the proof.

2.8 Symmetric properties of projective Ricci tensor with respect to quarter-symmetric non-metric connection

Projective Ricci tensor in a Riemannian manifold is defined as follows (Chaki and Saha, 1994)

$$P(X,Y) = \left[\frac{n}{n-1}\right]\{S(X,Y) - \frac{r}{n}g(X,Y)\}. \tag{2.8.1}$$

Theorem 2.8.1 *In an LP-Sasakian manifold the projective Ricci tensor $\bar{P}$ with respect to quarter-symmetric non-metric connection ∇ is symmetric.*

Proof: Analogous to the relation (2.8.1), we define the projective curvature with respect to quarter-symmetric non-metric connection ∇ by

$$\bar{P}(X,Y) = \left[\frac{n}{n-1}\right]\{\bar{S}(X,Y) - \frac{\bar{r}}{n}g(X,Y)\}. \tag{2.8.2}$$

With the help of (2.5.4) and (2.5.10), we have

$$\begin{aligned} \bar{P}(X,Y) &= \left[\frac{n}{n-1}\right]\{S(X,Y) - g(X,Y) \\ &- n\eta(X)\eta(Y) - \frac{r}{n}g(X,Y)\}. \end{aligned} \tag{2.8.3}$$

This completes the proof.

2.9 Induced connection on the submanifold

Let $\bar{M}^n$ be a submanifold of M^n of dimension n. Let $b : \bar{M}^n \to M^n$ be the inclusion map such that $p \in \bar{M}^n$ and $bp \in M^n$. The map b induces a Jacobian map $B : \bar{T} \to T$ where $\bar{T}$ is a tangent space to $\bar{M}^n$ at a point p and T is a tangent space to M^n at bp. The Riemannian metric G induced on $\bar{M}^n$ from that of M^n is given

$$g(BU, BV) \circ p = G(U, V), \tag{2.9.1}$$

where $U, V \in \bar{T}\bar{M}^n$.

Let N_1 and N_2 be two mutually orthogonal unit normals to $\bar{M}^n$ such that

$$\begin{aligned} &(a) \quad g(BU, N_1) = g(BU, N_2) = g(N_1, N_2) = 0, \\ &(b) \quad g(N_1, N_1) = g(N_2, N_2) = 1. \end{aligned} \tag{2.9.2}$$

Let the LP-Sasakian manifold M^n admit a quarter-symmetric non-metric connection given by (2.5.1), then we have

$$\begin{aligned} &(a) \quad \phi(BU) = Bf(U) + \alpha(U)N_1 + \gamma(U)N_2, \\ &(b) \quad \xi = Bt + \sigma N_1 + \delta N_2, \end{aligned} \tag{2.9.3}$$

where $t \in \bar{M}^n$ and σ, δ are functions in $\bar{M}^n$.

Let $\bar{\nabla}$ be the induced connection on the submanifold from ∇ with respect to the unit normals N_1, N_2.

Now the Gauss equation is given by

$$\nabla_{BU} BV = B(\bar{\nabla}_U V) + h_1(U, V)N_1 + h_2(U, V)N_2, \tag{2.9.4}$$

where h_1 and h_2 are second fundamental tensors and $U, V \in \bar{M}^n$.

Denoting $\bar{D}$ the connection induced on the submanifold from D with respect to the unit normals N_1, N_2.

The Gauss equation is given as

$$D_{BU}BV = B(\bar{D}_U V) + m_1(U,V)N_1 + m_2(U,V)N_2, \tag{2.9.5}$$

where m_1, m_2 are tensor fields of type $(0,2)$ of submanifold $\bar{M}^n$.

Theorem 2.9.1 *The induced connection on submanifold of an LP-Sasakian manifold with quarter-symmetric non-metric connection is also a quarter-symmetric non-metric connection.*

Proof: From (2.5.1), we have

$$\nabla_{BU}BV = D_{BU}BV - \eta(BU)\phi BV. \tag{2.9.6}$$

Making use of (2.9.4) and (2.9.5) in the above equation, we obtain

$$\begin{aligned} B(\bar{\nabla}_U V) + h_1(U,V)N_1 + h_2(U,V)N_2 &= B(\bar{D}_U V) + m_1(U,V)N_1 \\ &+ m_2(U,V)N_2 - \eta(BU)\phi(BV). \end{aligned} \tag{2.9.7}$$

Comparing the tangent and normal parts from the above equation we get

$$(\bar{\nabla}_U V) = (\bar{D}_U V) - a(U)f(V), \tag{2.9.8}$$

where

$$a(U) = \eta(BU)$$

and

$$\begin{aligned} &(a) h_1(U,V) = m_1(U,V) - a(U)\alpha(V), \\ &(b) h_2(U,V) = m_2(U,V) - a(U)\gamma(V). \end{aligned} \tag{2.9.9}$$

We know that

$$\begin{aligned} Ug(V,W) &= (\bar{\nabla}_U g)(V,W) + g(\bar{\nabla}_U V, W) + g(V, \bar{\nabla}_U W) \\ &= g(\bar{D}_U V, W) + g(V, \bar{D}_U W). \end{aligned}$$

Therefore from the above relations, we have

$$(\bar{\nabla}_U g)(V,W) = g(\bar{D}_U V - \bar{\nabla}_U V, W) + g(V, \bar{D}_U W - \bar{\nabla}_U W).$$

Using (2.5.1) in the above equation, we get

$$(\bar{\nabla}_U g)(V,W) = a(U)g(f(V),W) + a(U)g(V,f(W)).$$

Again in consequence of (2.9.8) we get

$$\bar{\nabla}_U V - \bar{\nabla}_V U - [U,V] = a(V)f(U) - a(U)f(V).$$

Our theorem is thus proved.

Theorem 2.9.2 *(a)The mean curvature of the submanifold $\bar{M}^n$ with respect to the Riemannian connection $\bar{D}$ coincides with mean curvature of the submanifold $\bar{M}^n$ with respect to the quarter-symmetric non-metric connection $\bar{\nabla}$ provided $\alpha = 0$ and $\gamma = 0$.*
(b)The submanifold $\bar{M}^n$ is totally geodesic with respect to the Riemannian connection $\bar{D}$ if and only if it is totally geodesic with respect to the quarter-symmetric non-metric connection $\bar{\nabla}$ provided $\alpha = 0$ and $\gamma = 0$.
(c) The submanifold $\bar{M}^n$ is totally umbilical with respect to the Riemannian connection $\bar{D}$ if and only if it is totally umbilical with respect to the quarter-symmetric non-metric connection $\bar{\nabla}$ provided $\alpha = 0$ and $\gamma = 0$.

Proof: Define $\bar{D}B$ and $\bar{\nabla}B$ respectively by

$$(\bar{D}B)(\lambda,\mu) = (\bar{D}_\lambda B)\mu = (D_{B\lambda})B\mu - B(\bar{D}_\lambda \mu). \tag{2.9.10}$$

$$(\bar{\nabla}B)(\lambda,\mu) = (\bar{\nabla}_\lambda B)\mu = (\nabla_{B\lambda})B\mu - B(\bar{\nabla}_\lambda \mu). \tag{2.9.11}$$

In view of (2.9.4) and (2.9.5), the equation (2.9.10) and (2.9.11) can be rewritten as

$$(\bar{D}_\lambda B)\mu = m_1(\lambda,\mu)N_1 + m_2(\lambda,\mu)N_2, \tag{2.9.12}$$

$$(\bar{\nabla}_\lambda B)\mu = h_1(\lambda,\mu)N_1 + h_2(\lambda,\mu)N_2, \tag{2.9.13}$$

respectively. Let $e_1, e_2, ..., e_{n-2}$ be $(n-2)$ orthonormal local vector fields in the submanifold $\bar{M}^n$. Then the function $\frac{1}{n-2}\sum_{i=1}^{n-2} m(e_i,e_i)$ is called the mean curvature of the submanifold $\bar{M}^{n-2}$ with respect to the Riemannian connection $\bar{D}$ and $\frac{1}{n-2}\sum_{i=1}^{n-2} h(e_i,e_i)$ is called the mean curvature of the submanifold $\bar{M}^n$ with respect to the quarter-symmetric non-metric connection $\bar{\nabla}$.

If m_1, m_2 vanish, then the submanifold $\bar{M}^{n-2}$ is said to be totally geodesic with respect to the Riemannian connection $\bar{D}$ provided $\alpha = 0$ and $\gamma = 0$ and if m_1, m_2 is proportional to g, then the submanifold $\bar{M}^n$ is called totally umbilical with respect to the Riemannian connection $\bar{D}$ provided $\alpha = 0$ and $\gamma = 0$. Similarly if m_1, m_2 vanish, then the submanifold $\bar{M}^n$ is said to be totally geodesic with respect to the quarter-symmetric non-metric connection $\bar{\nabla}$ provided $\alpha = 0$ and $\gamma = 0$ and if h_1, h_2 is proportional to g, then the submanifold $\bar{M}^n$ is called totally umbilical with respect to the quarter-symmetric non-metric connection $\bar{\nabla}$ provided $\alpha = 0$ and $\gamma = 0$.
The proof follows from the equation (2.9.9).

Chapter 3

Some Hypersurfaces of Lorentzian Para-Sasakian Manifolds

In this chapter totally geodesic and totally umbilical hypersurfaces of LP-Sasakian manifolds and that of recurrent LP-Sasakian manifolds have been studied.

3.1 Introduction

A Lorentzian paracontact manifold of $n = 2m+1$-dimension defined in (1.20.1-1.20.5) is called LP-Sasakian manifold with structure (ϕ, ξ, η, g) if (De *et al.*, 1988)

$$\nabla_X \xi = \phi X, \tag{3.1.1}$$

$$(\nabla_X \phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\ \eta(X)\eta(Y)\xi, \tag{3.1.2}$$

where ∇ denotes the covariant differentiation with respect to g.

Let us put $\Phi(X,Y) = g(\phi X, Y)$. Then the tensor field Φ is symmetric

$$i.e.\ \ \Phi(X,Y) = \ \Phi(Y,X), \tag{3.1.3}$$

and

$$(\nabla_X \eta)(Y) = g(X, \phi Y) = g(\phi X, Y), \tag{3.1.4}$$

$$'R(X, Y, Z, \xi) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{3.1.5}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{3.1.6}$$

$$R(X, Y, \xi) = \eta(Y)X - \eta(X)Y, \tag{3.1.7}$$

$$R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi, \tag{3.1.8}$$

$$R(X, \xi)\xi = -X - \eta(X)\xi, \tag{3.1.9}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{3.1.10}$$

for any vector fields X, Y, Z, where R is the curvature tensor, S is the Ricci tensor.
An LP-Sasakian manifold is called a generalized Ricci-recurrent manifold (Bhattacharya, 2003) if

$$(\nabla_X S)(Y, Z) = \eta(X)S(Y, Z) + p(X)g(Y, Z), \tag{3.1.11}$$

where η and p are 1-forms with associated vector-fields ξ and P, respectively.
Further an LP-Sasakian manifold is said to be η-Einstein manifold if its Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{3.1.12}$$

where a and b are scalars. Also, if

$$(\nabla_X S)(\phi Y, \phi Z) = 0, \tag{3.1.13}$$

then the manifold is said to be η-parallel Ricci-tensor.

3.2 Hypersurfaces of an LP-Sasakian manifold

Let $M^{n-1}(n = 2m+1)$ be a C^∞ - manifold imbedded in M^n with the immersion map $b : M^{n-1} \to M^n$ such that any point $x \in M^{n-1}$ is mapped to a point $bx \in M^n$. Let $B : T(M^{n-1}) \to T(M^n)$ be the Jacobian map which maps a vector field X at the point x in M^{n-1} into a vector BX at the point bx in M^n. Then M^{n-1} is called the hypersurface of M^n.

Now, we put

$$\begin{aligned}
&(a) \quad \phi BX = BfX + p(X)N,\\
&(b) \quad \phi N = -BP, \qquad\qquad (3.2.1)
\end{aligned}$$

where N is the unit normal vector to M^{n-1} and f is a vector valued linear function. Now taking accounts of the equations (1.20.1) and (3.2.1)((a),(b)), we obtain

$$BX + \eta(BX)\xi = Bf^2X + p(fX)N - p(X)BP.$$

Putting $\xi = Bt$ and $\eta(BX) \circ b = A(X)$ in the above equation and separating tangential and normal parts, we have

$$\begin{aligned}
&(a) \quad f^2X = X + A(X)t + p(X)P,\\
&(b) \quad p(fX) = 0. \qquad\qquad (3.2.2)
\end{aligned}$$

Also from (3.2.1)((a),(b)) and using (1.20.1) we obtain

$$N + \eta(N)\xi = -BfP - p(P)N.$$

Since $\eta(N) = 0$, therefore, by separating the tangential and normal parts from the above equation, we have

$$(a) \quad p(P) = -1,$$
$$(b) \quad fP = 0. \tag{3.2.3}$$

Further, $\phi\xi = 0$ implies $\phi Bt = Bft + p(t)N = 0$; hence, we get

$$(a) \quad ft = 0,$$
$$(b) \quad p(t) = 0. \tag{3.2.4}$$

Now, the metric tensor g in M^n induces the metric tensor G in M^{n-1} such that

$$(a) \quad g(BX, BY) \circ b = G(X, Y),$$
$$(b) \quad g(BX, N) \circ b = 0. \tag{3.2.5}$$

In view of (3.2.1)(a), $\eta(\phi BX) = 0$ implies $G(t, fX) = 0$

Or

$$(a) A(fX) = 0$$

also $\eta(\xi) = -1$ gives

$$(b) A(t) = -1. \tag{3.2.6}$$

Again, using the equations (3.2.1)(a) and (3.2.5)((a),(b)) in the following relation

$$g(\phi BX, \phi BY) \circ b = g(BX, BY) \circ b + \eta(BX) \circ b\eta(BY) \circ b,$$

we obtain

$$g(BfX, BfY) \circ b + p(X)p(Y)g(N, N) \circ b$$
$$= G(X, Y) + A(X)A(Y)$$

which implies

$$G(fX, fY) = G(X, Y) + A(X)A(Y) - p(X)p(Y). \tag{3.2.7}$$

Now, Guass and Weingarten's equations are given by

$$\begin{aligned} (a) \quad & \nabla_{BX} BY = BD_X Y +' H(X, Y)N, \\ (b) \quad & \nabla_{BX} N = -BHX, \end{aligned} \tag{3.2.8}$$

where $'H(X, Y) = G(HX, Y)$ is symmetric and it is called the second fundamental tensor in M^{n-1} and D is the induced Riemannian connection. Then, from the equation (3.1.2), we have

$$\begin{aligned} (\nabla_{BX}\phi)(BY) &= g(BX, BY) \circ bBt + 2\eta(BX) \circ b\eta(BY) \circ bBt \\ &+ \eta(BY) \circ bBX \end{aligned}$$

or

$$\nabla_{BX}\phi BY - \phi(\nabla_{BX} BY) = G(X, Y)Bt + 2A(X)A(Y)Bt + A(Y)BX.$$

Taking accounts of the equations (3.2.1)((a),(b)) and (3.2.8)((a),(b)) in the above equation, we obtain

$$\begin{aligned} &\nabla_{BX}\{BfY + p(Y)N\} - \phi\{BD_X Y + 'H(X, Y)N\} \\ &\qquad = G(X, Y)Bt + 2A(X)A(Y)Bt + A(Y)BX, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &B(D_X f)(Y) - p(Y)BHX + 'H(X, Y)BP + 'H(X, fY)N + (D_X p)(Y)N \\ &\qquad\qquad = G(X, Y)Bt + 2A(X)A(Y)Bt + BA(Y)BX. \end{aligned}$$

Now separating the tangential and normal parts from both sides of the above equation, we get

$$(a)(D_X f)(Y) - p(Y)HX + {}'H(X,Y)P = G(X,Y)t + 2A(X)A(Y)t + A(Y)X,$$

$$(b){}'H(X, fY) = -(D_X p)(Y)$$

which is equivalent to

$$(a) \quad (D_X f)(Y) = p(Y)HX - {}'H(X,Y)P + G(X,Y)t + 2A(X)A(Y)t + A(Y)X$$

and

$$(b) \quad (D_X p)(Y) = -{}'H(X, fY). \tag{3.2.9}$$

Again, from (3.1.2), we have

$$\begin{aligned}(\nabla_{BX}\phi)N &= g(BX, N) \circ bBt + 2\eta(BX) \circ b\eta(N)Bt + \eta(N)BX \\ &= 0\end{aligned}$$

which implies

$$(\nabla_{BX}\phi)N = 0$$

or,

$$\nabla_{BX}\phi N - \phi(\nabla_{BX}N) = 0.$$

Using (3.2.1)(b) and (3.2.8)((a),(b)) in the above equation, we have

$$\nabla_{BX}(-BP) - \phi(\nabla_{BX}N) = 0$$

$\Rightarrow$

$$-\nabla_{BX}BP - \phi(-BHX) = 0$$

$\Rightarrow$

$$-\{BD_XP + {}'H(X,P)N\} + \phi BHX = 0$$

$\Rightarrow$

$$-BD_XP -' H(X,P)N + BfHX + p(HX)N = 0.$$

Separating tangential and normal parts from the above equation, we have

$$\begin{aligned} &(a) \quad D_XP = fHX \\ &(b) \quad p(HX) = {}'H(X,P). \end{aligned} \tag{3.2.10}$$

In consequence of (3.1.1), we have

$$\nabla_{BX}\xi = \phi BX$$

which is equivalent to

$$\nabla_{BX}Bt = BfX + p(X)N.$$

Using (3.2.8)(a) in the above relation, we obtain

$$BD_Xt + {}'H(X,t)N = BfX + p(X)N$$

which on separation of tangential and normal parts yeilds

$$\begin{aligned} &(a) \quad D_Xt = fX, \\ &(b) \quad {}'H(X,t) = p(X) = A(HX). \end{aligned} \tag{3.2.11}$$

Again, from (3.1.4), we have

$$(\nabla_{BX}\eta) \circ b = g(BX, \phi BY) \circ b$$

$\Rightarrow$

$$g(\nabla_{BX}\xi, BY)\circ b = g(BX, BfY + p(Y)N)\circ b.$$

Making use of (3.2.1)(a) and (3.2.8)(a) in the above equation we get

$$g(BD_Xt + {}'H(X,t)N, BY)\circ b = g(BX, BfY)\circ b + p(Y)g(BX,N)\circ b.$$

From the equation (3.2.5) and above relation, we have

$$g(BD_Xt, BY)\circ b + {}'H(X,t)g(N,BY)\circ b = G(X, fY)$$

which is equivalent to

$$G(D_Xt, Y) = G(X, fY).$$

The above relation can be expressed as

$$(D_XA)Y = G(X, fY). \tag{3.2.12}$$

Now, taking account of the equations (3.2.8)((a),(b)) in the following relation

$$R(BX, BY, BZ) = \nabla_{BX}\nabla_{BY}BZ - \nabla_{BY}\nabla_{BX}BZ - \nabla_{[BX,BY]}BZ$$

we obtain

$$\begin{aligned} R(BX, BY, BZ) &= B\{K(X,Y,Z) - {}'H(Y,Z)HX + {}'H(X,Z)HY\} \\ &+ \{(D_X{}'H)(Y,Z) - (D_Y{}'H)(X,Z)\}N. \end{aligned} \tag{3.2.13}$$

and

$$
\begin{aligned}
'R(BX, BY, BZ, BU) &\overset{def}{=} g(R(BX, BY, BZ), BU) \circ b \\
&= g(B\{K(X, Y, Z) - \ 'H(Y, Z)HX + \ 'H(X, Z)HY\} \\
&+ \{(D_X{}'H)(Y, Z) - (D_Y{}'H)(X, Z)\}N, BU) \circ b \\
&= 'K(X, Y, Z, U) - 'H(Y, Z)'H(X, U) \\
&+ 'H(X, Z)'H(Y, U), \qquad (3.2.14)
\end{aligned}
$$

where K is the Riemannian curvature tensor in M^{n-1} and

$$'K(X, Y, Z, U) \overset{def}{=} G(K(X, Y, Z), U).$$

Theorem 3.2.1 *In the hypersurface M^{n-1} of an LP-Sasakian manifold, the following results hold:*

$$(a) \quad 'K(X, Y, t, Z) = A(Y)G(X, Z) - A(X)G(Y, Z) - p(X)'H(Y, Z) + p(Y)'H(X, Z).$$

Or,

$$(b) \quad K(X, Y, t) = A(Y)X - A(X)Y - p(X)HY + p(Y)HX. \qquad (3.2.15)$$

$$(a) \quad 'K(X, t, Y, Z) = A(Y)G(X, Z) - A(Z)G(X, Y) - p(Z)'H(X, Y) + p(Y)'H(X, Z).$$

Or,

$$(b) \quad K(X, t, Y) = A(Y)X - G(X, Y)t + p(Y)HX - 'H(X, Y)Ht. \qquad (3.2.16)$$

$$(a) \quad 'K(X, t, t, Y) = -G(X, Z) - A(X)A(Y) - p(X)p(Y).$$

Or,

$$(b) \quad K(X, t, t) = -X - A(X)t - p(X)Ht. \qquad (3.2.17)$$

Proof: Taking account of the equation (3.1.7) and using (3.2.11)(b) in (3.2.14), we obtain (3.2.15)(a), which immediately, implies (3.2.15)(b). Further, from (3.1.7), we have

$$'R(BX, \xi, BY, BZ) = \eta(BY) \circ bg(BX, BZ) \circ b - g(BX, BY) \circ bg(\xi, BZ) \circ b.$$

Now, using (3.2.14), for $\xi = Bt$ in the above equation, we easily obtain, in veiw of (3.2.11)(b), the equation (3.2.16)(a). This, further, implies the equation (3.2.16)(b). Again, from (3.2.16)(a), we obtain (3.2.17)(a) which, immediately, implies (3.2.17)(b). Now, from (3.2.10)(a) and taking accounts of the equations (3.2.9)(a) and (3.2.11)(b), we obtain,

$$\begin{aligned} K(X, Y, P) &= p(HY)HX - p(HX)HY + 2A(X)p(Y)t \\ &- 2A(Y)p(X)t + f\{(D_X H)Y - (D_Y H)X\} \\ &+ A(HY)X - A(HX)Y. \end{aligned} \tag{3.2.18}$$

Theorem 3.2.2 *In a hypersurface M^{n-1} of an LP-Sasakian manifold, we have*

$$K(X, Y, P) = 0, \quad if \quad M^{n-1} \textit{ is totally geodesic} \tag{3.2.19}$$

and

$$K(X, Y, P) = 2\{p(Y)X - p(X)Y\}, \quad if \quad M^{n-1} \textit{ is totally umbilical.} \tag{3.2.20}$$

Proof: If the hypersurface M^{n-1} is totally geodesic, then putting $HX = 0$. Taking covariant derivative of $HX = 0$ with respect to Y, we immediately get

$$(D_Y H)X + H(D_Y X) = 0. \tag{3.2.21}$$

Using (3.2.11)(b) in (3.2.18), we obtain

$$\begin{aligned} K(X, Y, P) &= p(HY)HX - p(HX)HY + 2A(X)A(HY)t - 2A(Y)A(HX)t \\ &+ f\{(D_X H)Y - (D_Y H)X\} + A(HY)X - A(HX)Y \end{aligned}$$

which in consequence of the fact $HX = 0$ reduces to

$$K(X, Y, P) = f\{(D_X H)Y - (D_Y H)X\}.$$

Making use of (3.2.21) in the above relation we have

$$K(X, Y, P) = f\{-H(D_X Y) + H(D_Y X)\}$$

which is equivalent to

$$K(X, Y, P) = -f(H[X, Y]).$$

By virtue of the equation (3.2.21) the above relation reduces to the equation (3.2.19).
If the hypersurface is totally umbilical, then $'H(X, Y) = G(X, Y)$ and $HX = X$ implies $(D_X H)(Y) = 0$.
In consequence of (3.2.11)(b) and (3.2.18), we obtain

$$\begin{aligned} K(X, Y, P) &= p(Y)X - p(X)Y + 2A(X)A(Y)t \\ &- 2A(Y)A(X)t + A(Y)X - A(X)Y \end{aligned}$$

which implies the equation (3.2.20).
This completes the proof.

3.3 Hypersurface of recurrent- LP-Sasakian manifolds

Now, taking covariant derivative of the equation (3.2.13) with respect to BU and using (3.2.8)(a) and (3.2.8)(b), we get

$$\begin{aligned} (\nabla_{BU} R)(BX, BY, BZ) &= \nabla_{BU} B\{K(X, Y, Z) - {}'H(Y, Z)HX + {}'H(X, Z)HY\} \\ &+ \nabla_{BU}\{(D_X{}'H)(Y, Z) - (D_Y{}'H)(X, Z)\}N \end{aligned}$$

$\Rightarrow$

$$
\begin{aligned}
(\nabla_{BU}R)(BX,BY,BZ) &= BD_U\{K(X,Y,Z) - {}'H(Y,Z)HX + {}'H(X,Z)HY\} \\
&+ {}'H(U, K(X,Y,Z) - {}'H(Y,Z)HX + {}'H(X,Z)HY)N \\
&+ \Big[\nabla_{BU}\{(D_X{}'H)(Y,Z) - (D_Y{}'H)(X,Z)\}\Big]N \\
&+ \{(D_X{}'H)(Y,Z) - (D_Y{}'H)(X,Z)\}\nabla_{BU}N
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
(\nabla_{BU}R)(BX,BY,BZ) &= B\{(D_UK)(X,Y,Z) - (D_U{}'H)(Y,Z)HX \\
&- {}'H(Y,Z)(D_UH)X + (D_U{}'H)(X,Z)HY \\
&+ {}'H(X,Z)(D_UH)Y\} + \{{}'K(X,Y,Z,HU) \\
&- {}'H(Y,Z){}'H(U,HX) + {}'H(X,Z){}'H(U,HY)\}N \\
&+ \Big[\nabla_{BU}\{(D_X{}'H)(Y,Z) - (D_Y{}'H)(X,Z)\}\Big]N \\
&- B\{(D_X{}'H)(Y,Z) - (D_Y{}'H)(X,Z)\}HU
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
(\nabla_{BU}R)(BX,BY,BZ) &= B[(D_UK)(X,Y,Z) - (D_U{}'H)(Y,Z)HX \\
&- {}'H(Y,Z)(D_UH)X + (D_U{}'H)(X,Z)HY \\
&+ {}'H(X,Z)(D_UH)Y - \{(D_X{}'H)(Y,Z) \\
&- (D_Y{}'H)(X,Z)\}HU] + [{}'K(X,Y,Z,HU) \\
&- {}'H(Y,Z){}'H(U,HX) + {}'H(X,Z){}'H(U,HY) \\
&+ \nabla_{BU}\{(D_X{}'H)(Y,Z) - (D_Y{}'H)(X,Z)\}]N. \quad (3.3.1)
\end{aligned}
$$

Now, we suppose that the LP-Sasakian manifold M^n is a recurrent, then

$$
(\nabla_{BU}R)(BX,BY,BZ) = \nu'(BU)R(BX,BY,BZ), \qquad (3.3.2)
$$

where $\nu^{'}(BU) = g(V', BU)$ and V' is recurrence vector in M^n. Let us put $V' = BV$ such that $g(BV, BU) \circ b = G(V, U) = \nu(U)$ so that $\nu^{'}(BU) \circ b = \nu(U)$.

Taking accounts of the equations (3.2.13) and (3.3.1), and by equating tangential parts on both sides of the above equation (3.3.2), we obtain

$$\begin{aligned}
&\nu(U)\Big[B\{K(X,Y,Z) - {}'H(Y,Z)HX + {}'H(X,Z)HY\} + \{(D_X{}'H)(Y,Z) \\
&-(D_Y{}'H)(X,Z)\}N\Big] = B\Big[(D_UK)(X,Y,Z) - (D_U{}'H)(Y,Z)HX \\
&\qquad -{}'H(Y,Z)(D_UH)X + (D_U{}'H)(X,Z)HY \\
&\qquad +{}'H(X,Z)(D_UH)Y + \{(D_X{}'H)(Y,Z) \\
&\qquad -(D_Y{}'H)(X,Z)\}HU\Big] + [{}'K(X,Y,Z,HU) \\
&\qquad -{}'H(Y,Z){}'H(U,HX) + {}'H(X,Z){}'H(U,HY) \\
&\qquad +\nabla_{BU}\{(D_X{}'H)(Y,Z) - (D_Y{}'H)(X,Z)\}]N
\end{aligned}$$

which implies

$$\begin{aligned}
(D_UK)(X,Y,Z) &- (D_U{}'H)(Y,Z)HX - {}'H(Y,Z)(D_UH)X \\
&+(D_U{}'H)(X,Z)HY + {}'H(X,Z)(D_UH)Y \\
&-\{(D_X{}'H)(Y,Z) - (D_Y{}'H)(X,Z)\}HU \\
&= \nu(U)\{K(X,Y,Z) - {}'H(Y,Z)HX \\
&+{}'HX,Z)HY\}. \qquad (3.3.3)
\end{aligned}$$

Theorem 3.3.1 *If an LP-Sasakian manifold M^n is recurrent, then the totally geodesic hypersurface M^{n-1} of LP-Sasakian manifold M^n is recurrent.*

Proof: For M^{n-1} be totally geodesic hypersurface of recurrent LP-Sasakian manifold M^n, we put $HX = 0$ in (3.3.3), which gives

$$(D_UK)(X,Y,Z) = \nu(U)K(X,Y,Z), \qquad (3.3.4)$$

i.e., M^{n-1} is recurrent.

This completes the statement of the theorem.

Theorem 3.3.2 *If LP-Sasakian manifold is η- Einstein manifold, then its hypersurface M^{n-1} is A-Einstein whether it is totally geodesic or totally umbilical.*

Proof: From (3.2.14), the expression for the relation of Ricci tensor in M^n and M^{n-1} is given by

$$S(BY,BZ)\circ b = Ric(Y,Z) - {'H}(Y,Z)h + {'H}(HY,Z) \tag{3.3.5}$$

where $h = C_1^1 HX$ is the mean curvature in M^{n-1}.

If the LP-Sasakian manifold is η- Einstein, then we have

$$\begin{aligned} S(BY,BZ)\circ b &= \alpha g(BY,BZ)\circ b + \beta\eta(BY)\circ b\eta(BZ)\circ b \\ &= \alpha G(Y,Z) + \beta A(Y)A(Z). \end{aligned} \tag{3.3.6}$$

In consequence of the equation (3.3.6) the equation (3.3.5) assume the following form

$$\alpha G(Y,Z) + \beta A(Y)A(Z) = Ric(Y,Z) - {'H}(Y,Z)h + {'H}(HY,Z). \tag{3.3.7}$$

If the hypersurface M^{n-1} is totally geodesic, then $HX = 0$ and hence the above equation becomes

$$Ric(Y,Z) = \alpha G(Y,Z) + \beta A(Y)A(Z),$$

showing that M^{n-1} is A-Einstein manifold.

Again, if M^{n-1} is totally umbilical, then for ${'H}(X,Y) = G(X,Y)$, the equation (3.3.7) gives

$$Ric(Y,Z) - n\,{'H}(Y,Z) + G(Y,Z) = \alpha G(Y,Z) + \beta A(Y)A(Z),$$

where

$h = C_1^1 HX = C_1^1 X = n$. It implies

$$Ric(Y, Z) = (\alpha + n - 1)G(Y, Z) + \beta A(Y)A(Z),$$

which again shows that M^{n-1} is A-Einstein.

The proof is complete.

Theorem 3.3.3 *A totally geodesic (totally umbilical) hypersurface M^{n-1} of a generalized Ricci-recurrent LP-Sasakian manifold is also a generalized Ricci-recurrent manifold.*

Proof: From the equation (3.3.5), we get

$$\begin{aligned}(\nabla_{BX} S)(BY, BZ) = (D_X Ric)(Y, Z) - (D_X{}'H)(Y, Z)h - {}'H(Y, Z)D_X h \\ +(D_X{}'H)(HY, Z) + {}'H((D_X H)Y, Z).\end{aligned} \tag{3.3.8}$$

If the LP-Sasakian manifold is generalized Ricci-recurrent, then we have

$$(\nabla_{BX} S)(BY, BZ) = [\eta(BX)S(BY, BZ) + \nu(BX)g(BY, BZ)] \circ b.$$

Now, taking $\eta(BX) \circ b = A(X)$ and $\nu(BX) \circ b = p(X)$ in the above equation, we have

$$(\nabla_{BX} S)(BY, BZ) = A(X)S(BY, BZ) \circ b + p(X)g(BY, BZ) \circ b.$$

Using (3.3.5) and (3.3.8) in the above equation, we have

$$\begin{aligned}&(D_X Ric)(Y, Z) - (D_X{}'H)(Y, Z)h - {}'H(Y, Z)D_X h \\ &+(D_X{}'H)(HY, Z) + {}'H((D_X H)Y, Z) \\ &= A(X)S(BY, BZ) \circ b + p(X)g(BY, BZ) \circ b\end{aligned} \tag{3.3.9}$$

$\Rightarrow$

$$\begin{aligned}
&(D_X Ric)(Y,Z) - (D_X{'H})(Y,Z)h - {'H}(Y,Z)D_X h \\
&+(D_X{'H})(HY,Z) + {'H}((D_X H)Y,Z) \\
&= A(X)Ric(Y,Z) - A(X){'H}(Y,Z)h + A(X){'H}(HY,Z) \\
&+p(X)G(Y,Z). \qquad (3.3.10)
\end{aligned}$$

Now, from (3.3.10), for $HX = 0$, we have

$$(D_X Ric)(Y,Z) = A(X)Ric(Y,Z) + p(X)G(Y,Z),$$

which shows that M^{n-1} is generalized Ricci-recurrent.

Further, for ${'H}(X,Y) = G(X,Y)$, we get, $(D_X{'H})(Y,Z) = 0$. Then, the equation (3.3.10) gives,

$$\begin{aligned}
(D_X Ric)(Y,Z) - G(Y,Z)D_X h &= A(X)Ric(Y,Z) - A(X)G(Y,Z)h \\
&+ A(X)G(Y,Z) + p(X)G(Y,Z)
\end{aligned}$$

$\Rightarrow$

$$(D_X Ric)(Y,Z) = A(X)Ric(Y,Z) + B(X)G(Y,Z),$$

where $B(X) = p(X) - (n-1)A(X)$.

This shows that M^{n-1} is generalized Ricci-recurrent.

Hence the proof is over.

Chapter 4

m-Projective Curvature Tensor on a Kenmotsu Manifold

In this chapter, we have studied some properties of m-projective curvature tensor on Kenmotsu manifolds and it has been shown that globally ϕ- m-projectively symmetric Kenmotsu manifold is an Einstein manifold.

4.1 Introduction

In an n-dimensional manifold M^n defined in the (1.21.1-1.21.6), the following relations hold

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{4.1.1}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{4.1.2}$$

$$(D_X\eta)Y = g(X,Y) - \eta(X)\eta(Y), \tag{4.1.3}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{4.1.4}$$

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where X, Y are vector fields.

(Yildiz *et al.*, 2009) A Kenmotsu manifold M^n is said to be η-Einstein if the Ricci tensor S is given by

$$S(X,Y) = \lambda_1 g(X,Y) + \lambda_2 \eta(X)\eta(Y),$$

where λ_1 and λ_2 are functions on M^n. If $\lambda_2 = 0$, then η-Einstein manifold reduces to Einstein manifold.

We know that for a 3-dimensional Kenmotsu manifold (De and Pathak, 2004)

$$\begin{aligned} R(X,Y)Z &= \frac{(r+4)}{2}\Big[g(Y,Z)X - g(X,Z)Y\Big] \\ &- \frac{(r+6)}{2}\Big[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\Big], \end{aligned} \tag{4.1.5}$$

and

$$S(X,Y) = \frac{1}{2}[(r+2)g(X,Y) - (r+6)\eta(X)\eta(Y)], \tag{4.1.6}$$

where r is the scalar curvature of the manifold.

The m-projective curvature tensor W^* is defined by(Pokhariyal and Mishra, 1971)

$$\begin{aligned} W^*(X,Y,Z) &= R(X,Y,Z) - \frac{1}{2(n-1)}\Big\{S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\Big\}. \end{aligned} \tag{4.1.7}$$

4.2 Globally ϕ-m-projectively symmetric Kenmotsu manifolds

Definition 4.2.1 *A Kenmotsu manifold M^n is said to be globally ϕ-m-projectively symmetric if m-projective curvature tensor W^* satisfies*

$$\phi^2((D_U W^*)(X,Y)Z) = 0, \tag{4.2.1}$$

for all vector fields $X, Y, Z, U \in TM^n$.

Theorem 4.2.1 *If a Kenmotsu manifold is globally ϕ-m-projectively symmetric, then the manifold is an Einstein manifold.*

Proof: Let us suppose that M^n is a globally ϕ-m-projectively symmetric Kenmotsu manifold. Then the equation(4.2.1) is satisfied.
Now using (1.21.1) in the equation (4.2.1), we get

$$-(D_U W^*)(X,Y)Z + \eta((D_U W^*)(X,Y)Z)\xi = 0. \tag{4.2.2}$$

From (4.1.7) it follows that

$$\begin{aligned}
&-g((D_U R)(X,Y)Z,V) + \frac{1}{2(n-1)}\Big\{(D_U S)(Y,Z)g(X,V) \\
&-(D_U S)(X,Z)g(Y,V) + (D_U S)(X,V)g(Y,Z) \\
&-(D_U S)(Y,V)g(X,Z)\Big\} + \eta((D_U R)(X,Y)Z)\eta(V) \\
&-\frac{1}{2(n-1)}\Big\{(D_U S)(Y,Z)\eta(X) - (D_U S)(X,Z)\eta(Y) \\
&+g(Y,Z)(D_U S)(X,\xi) - g(X,Z)(D_U S)(Y,\xi)\Big\}\eta(V) = 0.
\end{aligned} \tag{4.2.3}$$

Putting $X = V = e_i$ in the equation (4.2.3), where $\{e_i\}, (i = 1, 2, ...n)$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i, we get

$$-(D_US)(Y,Z)+\frac{n}{2(n-1)}(D_US)(Y,Z)-\frac{1}{2(n-1)}(D_US)(Y,Z)$$
$$+\frac{1}{2(n-1)}dr(U)g(Y,Z)-\frac{1}{2(n-1)}(D_US)(Y,Z)+\eta((D_UR)(e_i,Y)Z)\eta(e_i)$$
$$-\frac{1}{2(n-1)}\Big\{(D_US)(Y,Z)-(D_US)(Z,\xi)\eta(Y)+g(Y,Z)(D_US)(\xi,\xi)$$
$$-(D_US)(Y,\xi)\eta(Z)\Big\}=0.$$

Putting $Z=\xi$ in the above expression and after some computations we obtain,

$$-\frac{n}{2(n-1)}(D_US)(Y,\xi)+\frac{dr(U)}{2(n-1)}\eta(Y)+\eta((D_UR)(e_i,Y)\xi)\eta(e_i)=0. \qquad (4.2.4)$$

We know that

$$\begin{aligned} g((D_UR)(e_i,Y)\xi,\xi) &= g(D_UR(e_i,Y)\xi,\xi)-g(R(D_Ue_i,Y)\xi,\xi) \\ &- g(R(e_i,D_UY)\xi,\xi)-g(R(e_i,Y)D_U\xi,\xi) \end{aligned} \qquad (4.2.5)$$

at $p\in M^n$. Since $\{e_i\}$ is an orthonormal basis, $D_Xe_i=0$ at p. Using (4.1.1) we find

$$\begin{aligned} g(R(e_i,D_UY)\xi,\xi) &= g(\eta(e_i)D_UY-\eta(D_UY)e_i,\xi) \\ &= \eta(e_i)g(D_UY,\xi)-\eta(D_UY)g(e_i,\xi) \\ &= 0. \end{aligned} \qquad (4.2.6)$$

Using (4.2.6) in the equation (4.2.5) we have

$$g((D_UR)(e_i,Y)\xi,\xi)=g(D_UR(e_i,Y)\xi,\xi)-g(R(e_i,Y)D_U\xi,\xi). \qquad (4.2.7)$$

Since

$$g(R(e_i,Y)\xi,\xi)=-g(R(\xi,\xi)Y,e_i)=0,$$

we get

$$g(D_UR(e_i,Y)\xi,\xi)+g(R(e_i,Y)\xi,D_U\xi)=0. \qquad (4.2.8)$$

In consequence of (4.2.8), the equation (4.2.7) becomes

$$g((D_U R)(e_i, Y)\xi, \xi) = -g(R(e_i, Y)\xi, D_U\xi) - g(R(e_i, Y)D_U\xi, \xi).$$

Using (1.21.6) in the above equation, we find

$$\begin{aligned} g((D_U R)(e_i, Y)\xi, \xi) &= -g(R(e_i, Y)\xi, U) + \eta(U)g(R(e_i, Y)\xi, \xi) \\ &- g(R(e_i, Y)U, \xi) + \eta(U)g(R(e_i, Y)\xi, \xi) \\ &= 0 \end{aligned}$$

i.e.

$$g((D_U R)(e_i, Y)\xi, \xi) = 0. \tag{4.2.9}$$

In consequence of (4.2.9) the equation (4.2.4) yields

$$(D_U S)(Y, \xi) = \frac{1}{n} dr(U)\eta(Y). \tag{4.2.10}$$

Putting $Y = \xi$ in (4.2.10), we get $dr(U) = 0$.

This implies that r is constant.

So from (4.2.10) and $dr(U) = 0$, we obtain

$$(D_U S)(Y, \xi) = 0$$

which implies that

$$S(Y, U) = (1 - n)g(Y, U). \tag{4.2.11}$$

Hence proof of the theorem is completed.

Theorem 4.2.2 *A globally ϕ-m-projectively symmetric Kenmotsu manifold is globally ϕ-symmetric.*

Proof: From (4.1.7) we have

$$(D_U W^*)(X,Y)Z = (D_U R)(X,Y)Z.$$

Applying ϕ^2 on both sides of the above equation we have

$$\phi^2((D_U W^*)(X,Y)Z) = \phi^2((D_U R)(X,Y)Z).$$

This proves the statement of the theorem.

Remark 4.2.1 *Since a globally ϕ -symmetric Kenmotsu manifold is always a globally ϕ-m-projectively symmetric manifold, from Theorem (4.2.2),we conclude that on a Kenmotsu manifold, globally ϕ-symmetry and globally ϕ-m-projectively symmetry are equivalent.*

4.3 3-Dimensional locally ϕ-m-projectively symmetric Kenmotsu manifolds

Definition 4.3.1 *A Kenmotsu manifold M^n is said to be locally $\phi - m$-projectively symmetric if m-projective curvature tensor W^* satisfies*

$$\phi^2((D_U W^*)(X,Y)Z) = 0, \tag{4.3.1}$$

where X, Y, Z and U are horizontal vectors.

Theorem 4.3.1 *A 3-dimensional Kenmotsu manifold is locally ϕ-m-projectively symmetric if and only if the scalar curvature r is constant.*

Proof:Using (4.1.5) and (4.1.6) in (4.1.7), in a 3-dimensional Kenmotsu manifold the

m-projective curvature tensor is

$$\begin{aligned} W^*(X,Y)Z &= \left\{\frac{r+6}{4}\right\}[g(Y,Z)X - g(X,Z)Y] \\ &- \left\{\frac{3r+18}{8}\right\}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{4.3.2}$$

Taking the covariant differentiation to the both sides of the equation (4.3.2), we have

$$\begin{aligned} (D_UW^*)(X,Y)Z &= \frac{dr(U)}{4}\Big[g(Y,Z)X - g(X,Z)Y\Big] \\ &- \frac{3}{8}dr(U)\Big[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\Big] \\ &- \left\{\frac{3r+18}{8}\right\}\Big[g(Y,Z)(D_U\eta)(X)\xi + g(Y,Z)\eta(X)D_U\xi \\ &- g(X,Z)(D_U\eta)(Y)\xi - g(X,Z)\eta(Y)D_U\xi + (D_U\eta)(Y)\eta(Z)X \\ &+ \eta(Y)(D_U\eta)(Z)X - (D_U\eta)(X)\eta(Z)Y \\ &- \eta(X)(D_U\eta)(Z)Y\Big]. \end{aligned} \tag{4.3.3}$$

Now assume that X, Y and Z are horizontal vector fields. So the equation (4.3.3) becomes

$$\begin{aligned} (D_UW^*)(X,Y)Z &= \frac{dr(U)}{4}[g(Y,Z)X - g(X,Z)Y] \\ &- \left\{\frac{3r+18}{8}\right\}[g(Y,Z)(D_U\eta)(X)\xi \\ &- g(X,Z)(D_U\eta)(Y)\xi]. \end{aligned} \tag{4.3.4}$$

Applying ϕ^2 on both sides of (4.3.4) and making use of (1.21.3), we obtain

$$\phi^2((D_UW^*)(X,Y)Z) = -\frac{dr(U)}{4}[g(Y,Z)\phi^2X - g(X,Z)\phi^2Y]. \tag{4.3.5}$$

If $dr(U) = 0$ i.e. the manifold is of constant scalar curvature tensor then it is locally $\phi - m$ projectively symmetric. On the other hand if it is locally $\phi - m$ projectively symmetric then from the equation (4.3.5) it is clear that $dr(U) = 0$.

This completes the proof of the theorem.

4.4 ξ-m-Projectively flat Kenmotsu manifolds

Definition 4.4.1 *An n-dimensional Kenmotsu manifold is said to be ξ-m-projectively flat if $W^*(X,Y)\xi = 0$, where $X, Y \in TM^n$.*

Theorem 4.4.1 *An n-dimensional Kenmotsu manifold is ξ-m-projectively flat if and only if it is an Einstein manifold.*

Proof: Let the manifold be $\xi - m$ projectively flat. Then from the equation (4.1.7) we obtain

$$\begin{aligned} R(X,Y)\xi &= \frac{1}{2(n-1)}\{S(Y,\xi)X - S(X,\xi)Y \\ &+ g(Y,\xi)QX - g(X,\xi)QY\}. \end{aligned} \tag{4.4.1}$$

Using (4.1.1),(4.1.2) in (4.4.1), we get

$$\eta(Y)QX - \eta(X)QY + (n-1)\{\eta(Y)X - \eta(X)Y\} = 0. \tag{4.4.2}$$

Putting $Y = \xi$ in (4.4.2) and using (1.21.3), we get

$$QX = -(n-1)X. \tag{4.4.3}$$

Taking inner product of (4.4.3) with U, we obtain

$$S(X,U) = -(n-1)g(X,U). \tag{4.4.4}$$

From the relation (4.4.4), we conclude that the manifold is an Einstein manifold. Conversely , we assume that an n-dimensional Kenmotsu manifold satisfies (4.4.4). Then we easily obtain from (4.1.7) that

$$W^*(X,Y)\xi = 0.$$

Hence the proof of the theorem is completed.

4.5 ϕ-m-Projectively flat Kenmotsu manifolds

In an n-dimensional Kenmotsu manifold, if $\{e_1, ...e_{n-1}, \xi\}$ is a local orthonormal basis of the tangent space of the manifold, then $\{\phi e_1, ...\phi e_{n-1}, \xi\}$ is also a local orthonormal basis. In Kenmotsu manifold it is easy to verify that

$$\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z), \tag{4.5.1}$$

$$\sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = r - n + 1, \tag{4.5.2}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z)S(\phi Y, \phi e_i) = S(\phi Y, \phi Z), \tag{4.5.3}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n - 1, \tag{4.5.4}$$

and

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = g(\phi Y, \phi Z). \tag{4.5.5}$$

Definition 4.5.1 *An n-dimensional Kenmotsu manifold is said to be ϕ-m-projectively flat if*

$$'W^*(\phi X, \phi Y, \phi Z, \phi U) = 0, \tag{4.5.6}$$

where $X, Y, Z, U \in TM^n$.

Theorem 4.5.1 *An n-dimensional ϕ-m-projectively flat Kenmotsu manifold is an η-Einstein manifold with constant curvature.*

Proof: From (4.1.7), we get

$$\begin{aligned} W^*(\phi X, \phi Y, \phi Z) &= R(\phi X, \phi Y, \phi Z) - \frac{1}{2(n-1)}\Big\{S(\phi Y, \phi Z)\phi X \\ &- S(\phi X, \phi Z)\phi Y + g(\phi Y, \phi Z)Q\phi X \\ &- g(\phi X, \phi Z)Q\phi Y\Big\}. \end{aligned} \tag{4.5.7}$$

Taking inner product of the above equation, we obtain

$$\begin{aligned} g(W^*(\phi X, \phi Y, \phi Z), \phi U) &= g(R(\phi X, \phi Y, \phi Z), \phi U) \\ &- \frac{1}{2(n-1)}\Big\{S(\phi Y, \phi Z)g(\phi X, \phi U) - S(\phi X, \phi Z)g(\phi Y, \phi U) \\ &+ g(\phi Y, \phi Z)S(\phi X, \phi U) - g(\phi X, \phi Z)S(\phi Y, \phi U)\Big\} \end{aligned} \tag{4.5.8}$$

which can be written as

$$\begin{aligned} {}'W^*(\phi X, \phi Y, \phi Z, \phi U) &= {}'R(\phi X, \phi Y, \phi Z, \phi U) \\ &- \frac{1}{2(n-1)}\Big\{S(\phi Y, \phi Z)g(\phi X, \phi U) - S(\phi X, \phi Z)g(\phi Y, \phi U) \\ &+ g(\phi Y, \phi Z)S(\phi X, \phi U) - g(\phi X, \phi Z)S(\phi Y, \phi U)\Big\}. \end{aligned} \tag{4.5.9}$$

By virtue of (4.5.6) and (4.5.9), we get

$$\begin{aligned} {}'R(\phi X, \phi Y, \phi Z, \phi U) &= \frac{1}{2(n-1)}\Big\{S(\phi Y, \phi Z)g(\phi X, \phi U) \\ &- S(\phi X, \phi Z)g(\phi Y, \phi U) + g(\phi Y, \phi Z)S(\phi X, \phi U) \\ &- g(\phi X, \phi Z)S(\phi Y, \phi U)\Big\}. \end{aligned} \tag{4.5.10}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of the tangent space of the manifold. Then $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis of the tangent space.

Putting $X = U = e_i$ in (4.5.10) and summing up from 1 to (n-1) we have,

$$\begin{aligned}\sum_{i=1}^{n-1}{'R}(\phi e_i, \phi Y, \phi Z, \phi e_i) &= \frac{1}{2(n-1)}\sum_{i=1}^{n-1}\Big[S(\phi Y, \phi Z)g(\phi e_i, \phi e_i) \\ &- S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) + g(\phi Y, \phi Z)S(\phi e_i, \phi e_i) \\ &- g(\phi e_i, \phi Z)S(\phi Y, \phi e_i)\Big]. \qquad (4.5.11)\end{aligned}$$

Using (4.5.1),(4.5.2),(4.5.3) and (4.5.4) in (4.5.11), we obtain

$$S(\phi Y, \phi Z) = \Big[\frac{r - 3n + 3}{n+1}\Big]g(\phi Y, \phi Z). \qquad (4.5.12)$$

Replacing Y and Z by ϕY and ϕZ in (4.5.12) and using (1.21.1) we obtain

$$S(Y, Z) = \Big\{\frac{r - 3n + 3}{n+1}\Big\}g(Y, Z) + \Big\{\frac{-r - n^2 + 3n - 2}{n+1}\Big\}\eta(Y)\eta(Z). \qquad (4.5.13)$$

Putting $Y = Z = e_i$ in (4.5.13) and taking summation over $i, 1 \leq i \leq n$ we get by using (4.5.4) that

$$r = -(2n^2 - 3n + 1). \qquad (4.5.14)$$

This completes the proof.

4.6 Harmonic m-projective curvature tensor on Kenmotsu manifolds

Let us assume that ξ is a killing vector, then S and r remain invariant under it, i.e.

$$\pounds_\xi S = 0 \qquad (4.6.1)$$

and

$$\pounds_\xi r = 0, \qquad (4.6.2)$$

where $\pounds$ denotes Lie derivation.

Definition 4.6.1 *The Riemannian curvature tensor R is harmonic if*

$$(divR)(X, Y, Z) = 0. \tag{4.6.3}$$

Definition 4.6.2 *A Riemannian manifold M^n is of harmonic m-projective curvature tensor if*

$$(divW^*)(X, Y, Z) = 0. \tag{4.6.4}$$

In a Kenmotsu manifold it is known (Chaubey, 2012) that

$$\begin{aligned}(divW^*)(X, Y, Z) &= \frac{1}{2(n-1)}\Big[(2n-3)\{(D_XS)(Y, Z) - (D_YS)(X, Z)\} \\ &- \frac{1}{2}\{dr(X)g(Y, Z) - dr(Y)g(X, Z)\}\Big]. \end{aligned} \tag{4.6.5}$$

Theorem 4.6.1 *If a Kenmotsu manifold is of harmonic m-projective curvature tensor and ξ is killing vector, then the manifold is an η -Einstein manifold.*

Proof: Let M^n be a Kenmotsu manifold that satisfies $divW^* = 0$.
Then from the equation (4.6.5) we have

$$(D_XS)(Y, Z) - (D_YS)(X, Z) = \frac{1}{2(2n-3)}[dr(X)g(Y, Z) - dr(Y)g(X, Z)]. \tag{4.6.6}$$

From (4.6.1), it follows that

$$(D_\xi S)(Y, Z) = -S(D_Y\xi, Z) - S(Y, D_Z\xi) \tag{4.6.7}$$

and from (4.6.2), we get $dr(\xi) = 0$. Putting $X = \xi$ in (4.6.6), we obtain

$$\begin{aligned}(D_\xi S)(Y, Z) - (D_YS)(\xi, Z) &= \frac{1}{2(2n-3)}[g(Y, Z)dr(\xi) \\ &- \eta(Z)dr(Y)]. \end{aligned} \tag{4.6.8}$$

Making use of (4.6.7) in (4.6.8), we have

$$
\begin{aligned}
-S(D_Y\xi, Z) - S(Y, D_Z\xi) - (D_Y S)(\xi, Z) &= \frac{1}{2(2n-3)}[g(Y, Z)dr(\xi) \\
&- \eta(Z)dr(Y)]. \qquad (4.6.9)
\end{aligned}
$$

In consequence of $dr(\xi) = 0$, the above equation assume the form

$$
-S(Y, D_Z\xi) - D_Y S(\xi, Z) + S(\xi, D_Y Z) = -\frac{1}{2(2n-3)}\eta(Z)dr(Y). \qquad (4.6.10)
$$

Using (1.21.6) and (4.1.3) in the above, we have

$$
\begin{aligned}
&- S(Y, Z - \eta(Z)\xi) + (n-1)D_Y\eta(Z) - (n-1)\eta(D_Y Z) \\
&= -\frac{1}{2(2n-3)}\eta(Z)dr(Y), \qquad (4.6.11)
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&- S(Y, Z) + (n-1)g(Y, Z) - 2(n-1)\eta(Y)\eta(Z) \\
&= -\frac{1}{2(2n-3)}\eta(Z)dr(Y). \qquad (4.6.12)
\end{aligned}
$$

Replacing Z by ϕZ in the above equation, we get

$$
S(Y, \phi Z) = (n-1)g(Y, \phi Z). \qquad (4.6.13)
$$

Again replacing Y by ϕY and using (1.21.4) and (4.1.4) the above equation gives

$$
S(Y, Z) = (n-1)g(Y, Z) - 2(n-1)\eta(Y)\eta(Z),
$$

i.e. the manifold is an η-Einstein manifold.

4.7 Example of a locally ϕ-m-Projectively symmetric Kenmotsu manifold in 3-Dimension

Example 4.7.1 *We consider the 3-dimensional manifold* $M^3 = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$*, where* (x,y,z) *are standard co-ordinate of* $\mathbb{R}^3$*.*

The vector fields

$e_1 = z\frac{\partial}{\partial x}, e_2 = z\frac{\partial}{\partial y}, e_3 = -z\frac{\partial}{\partial z}$

are linearly independent at each point of M^3*.*

Let g *be the Riemannian meric defined by*

$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$

$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$

Let η *be the 1-form defined by* $\eta(Z) = g(Z, e_3)$ *for any* $Z \in TM^n$*.*

Let ϕ *be the* $(1,1)$ *tensor field defined by*

$\phi(e_1) = -e_2, \phi(e_2) = e_1, \phi(e_3) = 0.$

Then using the linearity of ϕ *and* g*, we have*

$$\eta(e_3) = 1,$$

$$\phi^2 Z = -Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in TM^n$*. Then for* $e_3 = \xi$ *, the structure* (ϕ, ξ, η, g) *defines an almost contact metric structure on* M^3*.*

Let D be the Levi-Civita connection with respect to metric g. Then we have

$$
\begin{aligned}
[e_1, e_3] &= e_1e_3 - e_3e_1 \\
&= z\frac{\partial}{\partial x}(-z\frac{\partial}{\partial z}) - (-z\frac{\partial}{\partial z})(z\frac{\partial}{\partial x}) \\
&= -z^2\frac{\partial^2}{\partial x\partial z} + z^2\frac{\partial^2}{\partial z\partial x} + z\frac{\partial}{\partial x} \\
&= e_1. \qquad (4.7.1)
\end{aligned}
$$

Similarly, $[e_1, e_2] = 0$ *and* $[e_2, e_3] = e_2$.

The Riemannian connection D of the metric g is given by

$$
\begin{aligned}
2g(D_XY, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\
&- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \qquad (4.7.2)
\end{aligned}
$$

which is known as Koszul's formula.

Using (4.7.2) we have

$$
\begin{aligned}
2g(D_{e_1}e_3, e_1) &= -2g(e_1, -e_1) \\
&= 2g(e_1, e_1). \qquad (4.7.3)
\end{aligned}
$$

Again by (4.7.3), we have

$$
\begin{aligned}
2g(D_{e_1}e_3, e_2) &= 0 \\
&= 2g(e_1, e_2) \qquad (4.7.4)
\end{aligned}
$$

and

$$
\begin{aligned}
2g(D_{e_1}e_3, e_3) &= 0 \\
&= 2g(e_1, e_3). \qquad (4.7.5)
\end{aligned}
$$

From (4.7.3),(4.7.4) and (4.7.5), we obtain

$$2g(D_{e_1}e_3, X) = 2g(e_1, X), \qquad (4.7.6)$$

for all $X \in TM^n$. Thus $D_{e_1}e_3 = e_1$. Therefore, (4.7.2) further yields

$$\begin{aligned} D_{e_1}e_3 &= e_1, D_{e_1}e_2 = 0, D_{e_1}e_1 = 0, \\ D_{e_2}e_3 &= e_2, D_{e_2}e_2 = -e_3, D_{e_2}e_1 = 0, \\ D_{e_3}e_3 &= 0, D_{e_3}e_2 = 0, D_{e_3}e_1 = 0. \end{aligned} \tag{4.7.7}$$

From the above it follows that the manifold satisfies

$D_X\xi = X - \eta(X)\xi$, for $\xi = e_3$.

Hence the manifold is a Kenmotsu manifold.

Remark 4.7.1 *(De and De, 2012) the authors have shown that the above example establish that a 3-dimensional Kenmotsu manifold is locally ϕ-concircularly symmetric iff the scalar curvature r is constant. Similarly we can show that the above example supports Theorem (4.3.1).*

Chapter 5

Characterization of Lorentzian Para-Sasakian Manifolds

This chapter deals with different geometrical properties of m-projective curvature tensor and the extended generalized concircularly ϕ -recurrent LP-Sasakian manifolds.

5.1 Introduction

In an LP-Sasakian manifold defined in (1.20.1)-(1.20.11), the following relations hold

$$\begin{aligned} g(R(X,Y)Z,\xi) &= \eta(R(X,Y)Z) \\ &= g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \end{aligned} \tag{5.1.1}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{5.1.2}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{5.1.3}$$

$$R(\xi,X)\xi = X + \eta(X)\xi, \tag{5.1.4}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{5.1.5}$$

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for any vector fields X, Y, Z where R and S are the Riemannian curvature and the Ricci tensor of the manifold respectively.

Also the following relation hold good in an LP-Sasakian manifold

$$(\nabla_W R)(X, \xi)Z = g(Z, \phi W)X - g(X, Z)\phi W - R(X, \phi W)Z, \tag{5.1.6}$$

for all vector fields $X, Y, Z, W \in \chi(M^n)$.

An LP-Sasakian manifold M^n is said to be η-Einstein manifold (Blair, 1976) if

$$S(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y), \tag{5.1.7}$$

where α and β are smooth functions.

The m-projective curvature tensor W^* is defined by (Pokhariyal and Mishra, 1971)

$$\begin{aligned} W^*(X, Y, Z) \;\; &= \;\; R(X, Y, Z) - \frac{1}{2(n-1)}\Big\{S(Y, Z)X - S(X, Z)Y \\ &+ \;\; g(Y, Z)QX - g(X, Z)QY\Big\}. \end{aligned} \tag{5.1.8}$$

5.2 m-projectively symmetric LP-Sasakian manifold

Definition 5.2.1 *An LP-Sasakian manifold M^n is said to be m-projectively symmetric if the m-projective curvature tensor W^* satisfies the relation*

$$(\nabla_U W^*)(X, Y), Z = 0, \tag{5.2.1}$$

for all X, Y, Z and U.

Theorem 5.2.1 *An m-projectively symmetric LP-Sasakian manifold M^n is Ricci-recurrent.*

Proof : Let M^n be an m-projectively symmetric LP-Sasakian manifold.

Firstly, taking covariant differentiation of the equation (5.1.8) with respect to U, then making use of the equations (5.2.1) and (5.1.8), we find

$$
\begin{aligned}
(\nabla_U W^*)(X,Y)Z &= (\nabla_U R)(X,Y)Z - \frac{1}{2(n-1)}\Big\{(\nabla_U S)(Y,Z)X \\
&- (\nabla_U S)(X,Z)Y + g(Y,Z)\nabla_U(QX) \\
&- g(X,Z)\nabla_U(QY)\Big\}.
\end{aligned} \tag{5.2.2}
$$

By virtue of (5.2.1), the equation (5.2.2) becomes

$$
\begin{aligned}
(\nabla_U R)(X,Y)Z &= \frac{1}{2(n-1)}\Big\{(\nabla_U S)(Y,Z)X - (\nabla_U S)(X,Z)Y \\
&+ g(Y,Z)\nabla_U(QX) - g(X,Z)\nabla_U(QY)\Big\}.
\end{aligned} \tag{5.2.3}
$$

Taking inner product of the above equation with respect to V, we have

$$
\begin{aligned}
g((\nabla_U R)(X,Y)Z,V) &= \frac{1}{2(n-1)}\Big\{(\nabla_U S)(Y,Z)g(X,V) \\
&- (\nabla_U S)(X,Z)g(Y,V) + g(Y,Z)(\nabla_U S)(X,V) \\
&- g(X,Z)(\nabla_U S)(Y,V)\Big\}.
\end{aligned} \tag{5.2.4}
$$

Taking contraction over X and V, we secure

$$
\begin{aligned}
(\nabla_U S)(Y,Z) &= \frac{1}{2(n-1)}\{n(\nabla_U S)(Y,Z) - (\nabla_U S)(Y,Z) \\
&+ dr(U)g(Y,Z) - (\nabla_U S)(Y,Z)\}
\end{aligned} \tag{5.2.5}
$$

which implies

$$
(\nabla_U S)(Y,Z) = \Big\{\frac{dr(U)}{n}\Big\}g(Y,Z). \tag{5.2.6}
$$

Hence the manifold is Ricci-recurrent.

Suppose the scalar curvature r is constant then we mention the following corollary:

Corollary 5.2.1 *An m-projectively symmetric LP-Sasakian manifold M^n with constant scalar curvature is Einstein.*

5.3 $\phi-m$-projectively symmetric LP-Sasakian manifold

Definition 5.3.1 *An LP-Sasakian manifold M^n is said to be $\phi-m$-projectively symmetric, if the m-projective curvature W^* satisfies the relation*

$$\phi^2(\nabla_U W^*)(X,Y,Z)=0, \tag{5.3.1}$$

for all vector feilds X,Y,Z and U.

Theorem 5.3.1 *A $\phi-m$-projectively symmetric LP-Sasakian manifold M^n is an Einstein.*

Proof: Let us consider M^n is a $\phi-m$ -projectively symmetric LP-Sasakian manifold. Then by the equations (5.3.1) and (1.20.1), we get

$$g((\nabla_U W^*)(X,Y,Z),V)=-\eta((\nabla_U W^*)(X,Y,Z))g(\xi,V). \tag{5.3.2}$$

The existence of the relation (5.1.8), the above equation becomes

$$\begin{aligned}
&g((\nabla_U R)(X,Y,Z),V)-\frac{1}{2(n-1)}\Big\{(\nabla_U S)(Y,Z)g(X,V)\\
&-(\nabla_U S)(X,Z)g(Y,V)+g(Y,Z)(\nabla_U S)(X,V)-g(X,Z)(\nabla_U S)(Y,V)\Big\}\\
&=-g((\nabla_U R)(X,Y,Z),\xi)g(\xi,V)+\frac{1}{2(n-1)}\Big\{(\nabla_U S)(Y,Z)g(X,\xi)\\
&-(\nabla_U S)(X,Z)g(Y,\xi)+(\nabla_U S)(X,\xi)g(Y,Z)\\
&-(\nabla_U S)(Y,\xi)g(X,Z)\Big\}g(\xi,V).
\end{aligned} \tag{5.3.3}$$

After contraction over X and Z, we secure

$$\begin{aligned}
(\nabla_U S)(Y,V)+(\nabla_U S)(Y,\xi)\eta(V) &= \Big[\frac{dr(U)}{n}\Big]\{g(Y,V)\\
&+\ \eta(Y)\eta(V)\}.
\end{aligned} \tag{5.3.4}$$

Putting $Y = \xi$ in the equation (5.3.4) we get

$$(\nabla_U S)(\xi, V) = 0. \tag{5.3.5}$$

By virtue of the relation (5.3.5), we have

$$S(\phi U, V) = (n-1)g(\phi U, V). \tag{5.3.6}$$

We put $U = \phi U$ in the above relation and then using the equation (1.20.1), we find

$$S(U, V) = (n-1)g(U, V). \tag{5.3.7}$$

This shows that the manifold is Einstein and thus the proof of the theorem is over.

5.4 $\phi - m$-projectively flat LP-Sasakian manifold

Definition 5.4.1 *An LP-Sasakian manifold M^n is said to be $\phi - m$-projectively flat if the m-projective curvature tensor W^* satisfies the relation*

$$\phi^2(W^*(\phi X, \phi Y, \phi Z)) = 0, \tag{5.4.1}$$

for all vector fields X, Y and Z.

Theorem 5.4.1 *A $\phi - m$-projectively flat LP-Sasakian manifold M^n is an η-Einstein manifold.*

Proof: Let us assume that M^n be a $\phi - m$-projectvely flat LP-Sasakian manifold. Then by virtue of the relations (5.4.1) and (1.20.1), we have

$$W^*(\phi X, \phi Y, \phi Z) = -\eta(W^*(\phi X, \phi Y, \phi Z))\xi, \tag{5.4.2}$$

which implies

$$g(W^*(\phi X, \phi Y, \phi Z), \phi U) = -\eta(W^*(\phi X, \phi Y, \phi Z))g(\xi, \phi U). \tag{5.4.3}$$

In consequence of (1.20.5) the equation (5.4.3) yields

$$g(W^*(\phi X, \phi Y, \phi Z), \phi U) = 0. \tag{5.4.4}$$

Making use of (5.1.8) in the equation (5.4.4) we obtain

$$\begin{aligned}
&g(R(\phi X, \phi Y, \phi Z), \phi U) - \frac{1}{2(n-1)}\Big\{S(\phi Y, \phi Z)g(\phi X, \phi U) \\
&-S(\phi X, \phi Z)g(\phi Y, \phi U) + g(\phi Y, \phi Z)S(\phi X, \phi U) \\
&-g(\phi X, \phi Z)S(\phi Y, \phi U)\Big\} = 0
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
g(R(\phi X, \phi Y, \phi Z), \phi U) &= \frac{1}{2(n-1)}\{S(\phi Y, \phi Z)g(\phi X, \phi U) \\
&- S(\phi X, \phi Z)g(\phi Y, \phi U) + g(\phi Y, \phi U)S(\phi X, \phi U) \\
&- g(\phi X, \phi Z)S(\phi Y, \phi U)\}. \qquad (5.4.5)
\end{aligned}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M^n. By using the fact that $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is an orthonormal basis, if we put $X = U = e_i$ in the above relation and taking summation over i, $1 \leq i \leq n-1$, then we have

$$\begin{aligned}
\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y, \phi Z), \phi e_i) &= \frac{1}{2(n-1)}\Big[\sum_{i=1}^{n-1} S(\phi Y, \phi Z)g(\phi e_i, \phi e_i) \\
&- \sum_{i=1}^{n-1} S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) \\
&+ \sum_{i=1}^{n-1} g(\phi Y, \phi Z)S(\phi e_i, \phi e_i) \\
&- \sum_{i=1}^{n-1} g(\phi e_i, \phi Z)S(\phi Y, \phi e_i)\Big]. \qquad (5.4.6)
\end{aligned}$$

Now we find that(Özgür, 2003)

$$\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y, \phi Z), \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z), \tag{5.4.7}$$

$$\sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = r + n - 1, \tag{5.4.8}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z) S(\phi Y, \phi e_i) = S(\phi Y, \phi Z), \tag{5.4.9}$$

and

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n + 1. \tag{5.4.10}$$

By virtue of the relations (5.4.7)-(5.4.10), the equation (5.4.6) reduces to

$$S(\phi Y, \phi Z) = \Big[\frac{r}{n-1} - 1\Big] g(\phi Y, \phi Z). \tag{5.4.11}$$

Then by making use of (1.20.3) and (5.1.5), the equation (5.4.11) takes the form

$$S(Y, Z) = \Big\{\frac{r}{n-1} - 1\Big\} g(Y, Z) + \Big\{\frac{r}{n-1} - n\Big\} \eta(Y)\eta(Z), \tag{5.4.12}$$

which implies from (5.1.7) that M^n is an η- Einstein manifold.

This completes the proof of the theorem.

5.5 An extended generalized ϕ- recurrent LP-Sasakian manifold

Definition 5.5.1 *An LP-Sasakian manifold M^n is said to be extended generalized ϕ -recurrent if its curvature tensor R satisfies the relation*

$$\phi^2((\nabla_W R)(X, Y)Z) = A(W)\phi^2(R(X, Y)Z) + B(W)\phi^2(G(X, Y)Z), \tag{5.5.1}$$

where A and B are two 1-forms , B is non zero and these are defined by $g(W, \rho_1) = A(W)$, $g(W, \rho_2) = B(W)$, and

$$G(X, Y, Z) = g(Y, Z)X - g(X, Z)Y,$$

for all $X, Y, Z, W \in \chi(M^n)$ and ρ_1, ρ_2 being vector fields associated to the 1-forms A and B respectively.

Lemma 5.5.1 *In an extended generalized ϕ -recurrent LP-Sasakian manifold*

$$(\nabla_W S)(Y,\xi) = (n-1)g(Y,\phi W) - S(Y,\phi W). \tag{5.5.2}$$

Proof: We know that

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi). \tag{5.5.3}$$

Using (1.20.6), (1.20.9) and (1.20.11) in the above relation, we get

$$\begin{aligned}(\nabla_W S)(Y,\xi) &= (n-1)(\nabla_W \eta)(Y) + (n-1)\eta(\nabla_W Y) \\ &- (n-1)\eta(\nabla_W Y) - S(Y,\phi W)\end{aligned}$$

which gives the expression (5.5.2).

Theorem 5.5.1 *In an extended generalized ϕ-recurrent LP-Sasakian manifold M^n, the 1-forms A and B are in opposite direction.*

Proof: Let us consider that M^n be an extended generalized ϕ-recurrent LP-Sasakian manifold.

Then by virtue of relations (1.20.1), (1.20.2) and (1.20.5), the equation (5.5.1) becomes

$$\begin{aligned}(\nabla_W R)(X,Y,Z) + \eta((\nabla_W R)(X,Y,Z))\xi &= A(W)\{R(X,Y,Z) + \eta(R(X,Y,Z))\xi\} \\ &+ B(W)\{G(X,Y,Z) + \eta(G(X,Y,Z))\xi\}\end{aligned} \tag{5.5.4}$$

Taking inner product of the above relation with U we get

$$\begin{aligned}&g((\nabla_W R)(X,Y,Z),U) + g((\nabla_W R)(X,Y,Z),\xi)g(U,\xi) \\ &= A(W)[g(R(X,Y,Z),U) + g(R(X,Y,Z),\xi)g(U,\xi)] \\ &+ B(W)[g(G(X,Y,Z),U) + g(G(X,Y,Z),\xi)g(U,\xi)].\end{aligned} \tag{5.5.5}$$

Let us suppose that $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of tangent space at any point of the manifold. Setting $X = U = e_i$ in the relation (5.5.5) and taking summation over $i, 1 \leq i \leq n$, we obtain

$$\begin{aligned}(\nabla_W S)(Y,Z) + \eta((\nabla_W R)(\xi, Y, Z)) &= A(W)[S(Y,Z) + \eta(R(\xi, Y, Z))] \\ &+ B(W)[(n-2)g(Y,Z) \\ &- \eta(Y)\eta(Z)]. \end{aligned} \tag{5.5.6}$$

Putting $Z = \xi$ in the above equation, we find

$$(\nabla_W S)(Y,\xi) = \{A(W) + B(W)\}(n-1)\eta(Y) + g(Y, \phi W). \tag{5.5.7}$$

By virtue of the Lemma (5.5.1) in the above relation, we have

$$\begin{aligned}(n-1)g(Y,\phi W) - S(Y,\phi W) = (n-1)\eta(Y)[A(W) \\ +B(W)] + g(Y,\phi W). \end{aligned} \tag{5.5.8}$$

Replacing Y by ξ in equation (5.5.8) and after using the relations (1.20.5) and (1.20.11), we get

$$A(W) + B(W) = 0. \tag{5.5.9}$$

Hence the theorem is proved.

Theorem 5.5.2 *An extended generalized ϕ-recurrent LP-Sasakian manifold M^n is an η- Einstein manifold.*

Proof: Let us consider M^n be an extended generalized ϕ-recurrent LP-Sasakian manifold. In the theorem (5.5.1), we have proved that in an extended generalized ϕ-recurrent LP-Sasakian manifold M^n, the 1-forms A and B are in opposite direction and so the relation (5.5.9) holds.

Now making use of (5.5.9) in (5.5.8) we have

$$(n-2)g(Y,\phi W) - S(Y,\phi W) = 0. \tag{5.5.10}$$

Again replacing W by ϕW in the above relation and then using (1.20.1) and (1.20.11),we obtain

$$S(Y,W) = (n-2)g(Y,W) - \eta(Y)\eta(W). \tag{5.5.11}$$

Our theorem is thus proved.

5.6 Extended generalized concircularly ϕ-recurrent LP-Sasakian manifolds

Definition 5.6.1 *An extended genaralized ϕ-recurrent LP-Sasakian manifold M^n is said to be an extended generalized concircularly ϕ-recurrent LP-Sasakian manifold, if the concircular curvature C satisfies the relation*

$$\phi^2((\nabla_W C)(X,Y,Z)) = A(W)\phi^2(C(X,Y,Z)) + B(W)\phi^2(G(X,Y,Z)), \tag{5.6.1}$$

where A and B are two 1-forms, B is non-zero and

$$C(X,Y,Z) = R(X,Y,Z) - \frac{r}{n(n-1)}G(X,Y,Z) \tag{5.6.2}$$

for all $X,Y,Z \in \chi(M^n)$ and r is the scalar curvature.

Theorem 5.6.1 *An extended genaralized concircularly ϕ-recurrent LP-Sasakian manifold M^n, $n \geq 3$, is extended generalized ϕ-recurrent if and only if*

$$\begin{aligned}&\Big\{\frac{dr(W) - rA(W)}{n(n-1)}\Big\}\big\{g(Y,Z)X + g(Y,Z)\eta(X)\xi - g(X,Z)Y \\ &-g(X,Z)\eta(Y)\xi\big\} = 0.\end{aligned} \tag{5.6.3}$$

Proof: Let us consider an extended generalized concircularly ϕ-recurrent LP-Sasakian manifold M^n, $n \geq 3$. Hence the defining condition of an extended generalized concircularly ϕ-recurrent LP-Sasakian manifold yields by virtue of (5.6.2) that

$$\phi^2((\nabla_W R)(X,Y)Z) - A(W)\phi^2(R(X,Y)Z) - B(W)\phi^2(G(X,Y)Z)$$
$$= \Big[\frac{dr(W) - rA(W)}{n(n-1)}\Big]\{g(Y,Z)X + g(Y,Z)\eta(X)\xi - g(X,Z)Y$$
$$-g(X,Z)\eta(Y)\xi\}. \tag{5.6.4}$$

Using (5.5.1) in the above relation, we get

$$\Big[\frac{dr(W) - rA(W)}{n(n-1)}\Big]\{g(Y,Z)X + g(Y,Z)\eta(X)\xi - g(X,Z)Y$$
$$-g(X,Z)\eta(Y)\xi\} = 0. \tag{5.6.5}$$

This completes the proof of the theorem.

Theorem 5.6.2 *If an extended generalized concircularly ϕ -recurrent LP-Sasakian manifold M^n, $n \geq 3$, is an extended generalized ϕ-recurrent LP-Sasakian manifold, then the associated vector field corresponding to the 1-form A is given by $\rho_1 = \frac{1}{r} grad\ \ r$, r being the non-zero and non-constant scalar curvature of the manifold.*

Proof: Taking inner product of the equation (5.6.5) with U, we obtain

$$\Big[\frac{dr(W) - rA(W)}{n(n-1)}\Big]\{g(Y,Z)g(X,U) + g(Y,Z)\eta(X)g(\xi,U) - g(X,Z)g(Y,U)$$
$$-g(X,Z)\eta(Y)g(\xi,U)\} = 0. \tag{5.6.6}$$

Taking contraction over X and U, we get

$$[dr(W) - rA(W)]\{(n-2)g(Y,Z) - \eta(Y)\eta(Z)\} = 0. \tag{5.6.7}$$

Again contracting the equation (5.6.7) with respect to Y and Z, we obtain

$$[dr(W) - rA(W)]\{(n-2)n + 1\} = 0 \tag{5.6.8}$$

which implies that

$$A(W) = \frac{dr(W)}{r} \tag{5.6.9}$$

for all vector field W and $r \neq 0$

i.e.,

$$\rho_1 = \frac{1}{r} grad \;\; r,$$

where $A(W) = g(W, \rho_1)$.

Our theorem is thus proved.

Theorem 5.6.3 *In an extended generalized concircularly ϕ -recurrent LP-Sasakian manifold M^n, $n \geq 3$, the associated 1-forms A and B are related by the relation*

$$dr(W) = A(W)[r - n(n-1)] + B(W)n(n-1)^2. \tag{5.6.10}$$

Proof: By virtue of (1.20.1), it follows from (5.6.4) that

$$\begin{aligned}(\nabla_W R)(X,Y)Z &= -\eta((\nabla_W R)(X,Y)Z)\xi + A(W)[R(X,Y)Z \\ &+ \eta(R(X,Y)Z)\xi] + B(W)[G(X,Y)Z + \eta(G(X,Y)Z)\xi] \\ &+ \left\{\frac{dr(W) - rA(W)}{n(n-1)}\right\}\Big[g(Y,Z)X + g(Y,Z)\eta(X)\xi \\ &- g(X,Z)Y - g(X,Z)\eta(Y)\xi\Big].\end{aligned} \tag{5.6.11}$$

Taking inner product of the above relation with U and then contracting over X and U, and then using (5.1.6), we get

$$\begin{aligned}(\nabla_W S)(Y,Z) &= A(W)S(Y,Z) + [(n-2)B(W) - A(W)]g(Y,Z) \\ &+ \left[\frac{dr(W)}{n(n-1)}\right]\{(n-2)g(Y,Z) - \eta(Y)\eta(Z)\} \\ &- A(W)\left[\left\{1 - \frac{r}{n(n-1)}\right\}\eta(Y)\eta(Z)\right. \\ &+ \left.\left\{\frac{(n-2)r}{n(n-1)}\right\}g(Y,Z)\right] - B(W)\eta(Y)\eta(Z).\end{aligned} \tag{5.6.12}$$

Again contraction over Y and Z in (5.6.12) yields

$$dr(W) = [r - n(n-1)]A(W) + n(n-1)^2 B(W). \tag{5.6.13}$$

Corollary 5.6.1 *In an extended generalized concircularly ϕ-recurrent LP-Sasakian manifold M^n, $n \geq 3$, with constant scalar curvature, the associated 1-forms A and B are related by*

$$\{r - n(n-1)\}A + n(n-1)^2 B = 0. \tag{5.6.14}$$

Definition 5.6.2 *(Shaikh and Helaluddin, 2011) An n-dimensional Riemannian manifold M^n, $n > 2$, is called a super generalized Ricci-recurrent if its Ricci tensor S of type $(0,2)$ satisfies the relation*

$$DS = \alpha \otimes S + \beta \otimes g + \gamma \otimes \pi, \tag{5.6.15}$$

where α, β, γ are nowhere vanishing unique 1-forms and $\pi = \eta \otimes \eta$.

Theorem 5.6.4 *An extended generalized concircularly ϕ-recurrent LP-Sasakian manifold $M^n, n \geq 3$, is super generalized Ricci recurrent manifold.*

Proof: Using the equation (5.6.13) in (5.6.12), we get

$$\begin{aligned}
(\nabla_W S)(Y,Z) &= A(W)S(Y,Z) + [(n-2)B(W) - A(W)]g(Y,Z) \\
&+ \left\{\frac{[r - n(n-1)]A(W) + n(n-1)^2 B(W)}{n(n-1)}\right\}\{(n-2)g(Y,Z) \\
&- \eta(Y)\eta(Z)\} - A(W)\left[\left\{1 - \frac{r}{n(n-1)}\right\}\eta(Y)\eta(Z)\right. \\
&+ \left.\left\{\frac{(n-2)r}{n(n-1)}\right\}g(Y,Z)\right] - B(W)\eta(Y)\eta(Z)
\end{aligned}$$

which after simplification reduces to

$$\begin{aligned}
(\nabla_W S)(Y,Z) &= A(W)S(Y,Z) + n(n-2)B(W)g(Y,Z) \\
&- (n-1)A(W)g(Y,Z) - nB(W)\eta(Y)\eta(Z).
\end{aligned} \tag{5.6.16}$$

From (5.6.16), it follows that the Ricci tensor S satisfies the condition

$$DS = \alpha \otimes S + \beta \otimes g + \gamma \otimes \pi, \tag{5.6.17}$$

where $\alpha(W) = A(W)$, $\beta(W) = n(n-2)B(W) - (n-1)A(W)$, $\gamma(W) = -nB(W)$ and $\pi = \eta \otimes \eta$.

This completes the proof of the theorem.

Theorem 5.6.5 *In an extended generalized concircularly ϕ-recurrent LP-Sasakian manifold M^n, $n \geq 3$, the Ricci tensor in the direction of ρ_1 is given by*

$$\begin{aligned} S(Y, \rho_1) &= \Big[\frac{r-(n-1)(n-2)}{2}\Big]A(Y) \\ &+ \Big[\frac{n(n^2-4n+5)}{2}\Big]B(Y) - n\eta(Y)B(\xi). \end{aligned} \tag{5.6.18}$$

Proof: Taking contraction of (5.6.16) over W and Z, we get

$$\frac{1}{2}dr(Y) = S(Y, \rho_1) + n(n-2)B(Y) - (n-1)A(Y) + n\eta(Y)B(\xi). \tag{5.6.19}$$

By virtue of (5.6.13), the above relation takes the form

$$\begin{aligned} S(Y, \rho_1) &= \Big[\frac{r-(n-1)(n-2)}{2}\Big]A(Y) \\ &+ \Big[\frac{n(n^2-4n+5)}{2}\Big]B(Y) - n\eta(Y)\beta(\xi). \end{aligned} \tag{5.6.20}$$

This completes the desired result.

Theorem 5.6.6 *In an extended generalized concircularly ϕ-recurrent LP-Sasakian manifold M^n, $n \geq 3$, the vector field ρ_2 associated with the 1-form B and the characteristic vector field ξ are in opposite direction.*

Proof: By setting $Z = \xi$ in (5.6.16) and then using (5.5.2) and (5.1.5) we obtain

$$S(Y, \phi W) = (n-1)g(Y, \phi W) - n(n-1)B(W)\eta(Y). \tag{5.6.21}$$

Making replace of Y by ϕY in the equation (5.6.21) and using (1.20.3) and (5.1.5), we have

$$S(Y,W) = (n-1)g(Y,W). \tag{5.6.22}$$

Again, replacing W by ϕW in the above relation (5.6.21) and then using (1.20.1), we get

$$S(Y,W) = (n-1)g(Y,W) - n(n-1)B(\phi W)\eta(Y). \tag{5.6.23}$$

From (5.6.22) and (5.6.23) we have

$$B(\phi W) = 0,$$

which implies that

$$B(W) = -\eta(W)B(\xi).$$

This shows that the vector field ρ_2 associated with the 1-form B and the characteristic vector field ξ are in opposite direction.

Chapter 6

Summary and Conclusion

Chapter 1 is all about the definitions which we use later. The chapter 2 is about the study of some properties of semi-symmetric non-metric connections as well as quarter symmetric non-metric connections on an LP-Sasakian manifold. We have obtained a number of interesting results. We have established the relationship between different curvature tensors with respect to semi-symmetric non-metric connection $\bar{D}$ to the same with respect to Riemannian connection D. One of the relation is that the necessary and sufficient condition for the conformal curvature tensor of $\tilde{D}$ to coincide with that of the Riemannian connection D is that the conharmonic curvature tensor of $\tilde{D}$ is equal to that of D provided $\psi = -1$. In an LP-Sasakian manifold admitting quarter-symmetric non-metric connection, we have shown that the manifold is ξ-quasi conformally flat with respect to the quarter symmetric non-metric connection if and only if the manifold is also ξ-quasi conformally flat with respect to the Riemannian connection provided the vector fields X, Y are orthogonal to ξ. Again we found the result that ξ-pseudo projectively flat LP-Sasakian manifold with respect to the quarter-symmetric non-metric connection is also ξ-pseudo projectively flat with respect to the Riemannian connection provided the vector fields X and Y are orthogonal to ξ and vice versa. The same condition happens to the property of globally $\phi - m$-projectively symmetric. Finally, we have shown that in a submanifold of an LP-Sasakian manifold, the mean curvature with respect to the Riemannian connection coincides with mean curvature with respect to the quarter-symmetric non-metric connection provided $\alpha = 0$ and $\gamma = 0$. In addition to this, we found that the submanifold is totally geodesic (umbilical) with respect to the Riemannian connection if and only if it is totally geodesic (umbilical) with respect to the quarter-symmetric non-metric connection provided $\alpha = 0$ and $\gamma = 0$.

The chapter 3 is devoted to the study of hypersurfaces of LP-Sasakian manifolds. We have shown that the totally geodesic hypersurface M^{n-1} of the LP-Sasakian recurrent manifold M^n is also recurrent. Next, it is also proved that a hypersurface M^{n-1} of an LP-Sasakian η- Einstein manifold is A-Einstein whether it is totally geodesic or totally umbilical. Finally we obtain that a totally umbilical (totally geodesic) hypersurface M^{n-1} of a generalized Ricci-recurrent LP-Sasakian manifold is also a generalized Ricci-recurrent.

Chapter 4 deals with m-projective curvature tensor on Kenmotsu manifolds. It is shown that a globally ϕ-m-projectively symmetric Kenmotsu manifold is an Einstein manifold as well as globally ϕ-m-symmetric. Further we have shown that in 3-dimensional locally ϕ-m-projectively symmetric Kenmotsu manifold, the scalar curvature r is constant. It is also shown that n-dimensional ξ-m-projectively flat Kenmotsu manifold is an Einstein manifold and vice versa. An n-dimensional ϕ-m-projectively flat Kenmotsu manifold is an η- Einstein manifold with constant curvature is obtained later. Lastly, we have shown that a Kenmotsu manifold of harmonic m-projective curvature tensor with killing vector ξ is an η -Einstein manifold. One example of a locally ϕ-m-Projectively symmetric Kenmotsu manifold in 3-Dimension is also shown.

Chapter 5 is about the characterization of LP-Sasakian manifolds. Firstly we obtained that if an LP-Sasakian manifold M^n is

(i) an m-projectively symmetric, then the manifold is Ricci-recurrent.

(ii) a $\phi - m$-projectively symmetric, then the manifold is an Einstein.

(iii) a $\phi - m$-projectively flat, then the manifold is an η-Einstein.

Next, we have proved that an extended generalized ϕ-recurrent LP-Sasakian manifold M^n

(i) is of which the 1-forms A and B are in opposite direction.

(ii) is an η- Einstein manifold.

In continuation of these, we have shown that an extended generalized concircularly ϕ -recurrent LP-Sasakian manifold M^n, $n \geq 3$, is an extended generalized ϕ-recurrent LP-Sasakian manifold, then the associated vector field corresponding to the 1-form A is given by $\rho_1 = \frac{1}{r} grad\ \ r$, r being the non-zero and non-constant scalar curvature of the manifold. Further, we have proved that an extended generalized concircularly ϕ-recurrent LP-Sasakian manifold $M^n, n \geq 3$, is super generalized Ricci recurrent manifold and in which the vector field ρ_2 associated with the 1-form B and the characteristic vector field ξ are in opposite direction.

Finally we conclude that whole work of this thesis gives the properties and geometrical structure of the LP -Sasakian manifolds equipped with semi-symmetric non-metric connection and quarter symmetric non-metric connection respectively and geometrical results of certain curvature tensors in LP-Sasakian manifolds, hypersurface of an LP-Sasakian manifolds, extended generalized ϕ-recurrent LP-Sasakian manifolds and Kenmotsu manifolds.

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Appendix

(A) LIST OF RESEARCH PUBLICATIONS

(1) M. Saroja Devi & Jay Prakash Singh (2015): On a type of m-projective curvature tensor on Kenmotsu manifolds, International Journal of Mathematical Sciences and Engineering Applications, (9)III, 37-49. ISSN: 0973-9424.

(2) J.P. Singh & M. S. Devi (2016): On a type of quarter-symmetric non- metric connection in LP-Sasakian manifolds, Science and Technology Journal, (4)I, 66-68. ISSN: 2321-3388.

(3) Jay Prakash Singh & Saroja Devi Mayanglambam (2017): On extended generalized ϕ-recurrent LP-Sasakian manifolds, Global Journal of Pure and Applied Mathematics, 13(9), 5551-5563. ISSN: 0973-1768.

(B) CONFERENCES/ SEMINAR/ WORKSHOPS

(1) Participated on " National conference on Mathematical Sciences " sponsored by UGC (NERO), Department of Mathematics, Pachhunga University College Mizoram University in collaboration with "Mizoram Mathematics Society" on November 24-25, 2011.

(2) Participated on "Workshop on Modelling Biological System II " jointly organized by "PAMU, Indian Statistical Institute,Kolkata" and Department of Physics, Mizoram University during August 21-25, 2012.

(3) Parcipated on "ISI-MZU School on Soft Computing and Applications" organized jointly by Machine Intelligence Unit, ISI, Kolkata and Department of Mathematics and Computer Science, Mizoram University during November 5-9, 2012.

(4) Participated on "National workshop on Mathematical analysis" organized by Department of Mathematics and Computer Science, Mizoram University during March 7-8, 2013.

(5) Participated on "National workshop on dynamical systems" organized by department of Mathematics and Computer Science during November 26-27, 2013.

(6) Participated on "Innovations in Science and Technology for Inclusive Development"organized by ISCA Imphal Chapter and Manipur University with Indian Science Congress Association during Decenber 30-31, 2013.

(7) Participated on "North-East ISI-MZU winter school on Algorithms with special focus on graphs" organized by Advanced Computing and Microelectronics unit IndianStatistical Institute and Department of Mathematics and Computer Science, Mizoram University during March 6-11, 2017.

(8) Presented a paper "On a quarter symmetric non-metric connection in an LP-Sasakian manifolds" Second Mizoram Mathematics Congress organized by Mizoram Mathematical Society (MMS) in Collaboration with Department of Mathematics (UG & PG), Mizoram University, August $13-14$, 2015.