A STUDY ON ALMOST GRAYAN MANIFOLDS

By

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Submitted in partial fulfillment of the requirement of the Degree of Doctor of Philosophy in Mathematics of Mizoram University, Aizawl.

CERTIFICATE

This is to certify that the thesis entitled "A Study on Almost Grayan Manifolds" submitted by Ms. R. Zosangliani is in partial fulfillment of the requirements for the degree of Doctor of Philosophy (Ph. D.) in Mathematics. She has been duly registered and her thesis is worthy of being considered for the award of the Ph. D. degree.

I, the undersigned, declare that this research work has been done under my supervision and has not been submitted for any degree to any other University.

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Month: November **Year: 2018**

DECLARATION

I, R. Zosangliani, hereby declare that the subject matter of this thesis entitled, "A Study on Almost Grayan Manifolds" is the record of work done by me, that the contents of this thesis do not form basis of the award of any previous degree to me or to the best of my knowledge, to anybody else, and that the thesis has not been submitted by me for any research degree to any other University/Institute.

This thesis is being submitted to Mizoram University for fulfillment of the Degree of Doctor of Philosophy (Ph. D.) in Mathematics.

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ACKNOWLEDGEMENT

As I sat down to write the acknowledgement for my Ph.D. thesis, I realised how fortunate and blessed I am to be able to come to the final stage of my study. I have longed for this moment all through my research work. It gives me immense pleasure to express my thoughts and feelings in writing. During my research period, I have experienced a lot of things, mentally and physically. I also learnt a lot of lessons from my colleagues and friends from both inside and outside my academic circle. I feel a deep sense of gratitude towards those people, for their help and support throughout my learning process.

Firstly, I thank the Almighty God for his guidance all through the research period. He has blessed me with good health and an environment suitable for my academic pursuits.

I would like to thank my respected supervisor Dr. Jay Prakash Singh, for giving me the wonderful opportunity to complete my Ph.D. under his supervision. It is an honour for me to be taught by a man of such stature. I thank him for his enthusiasm and kindness through all these years. Without his supervision, I would not have been able to complete this challenging research work.

I am also highly obliged to Prof. Jamal Hussain, Head of Department, Mathematics and Computer Science, Mizoram University, for providing all the necessary facilities and for his reliable advice during the course of my research work.

I would like to thank Prof. Zaithanzauva Pachuau, Dean, School of Physical Sciences, Mizoram University, for his help and co-operation during my research work.

I am also very grateful to Dr. S. Sarat Singh, Mr. Laltanpuia, Dr. M. Saroja Devi and Mr. Sajal Kanti Das for their valuable suggestions and co-operation.

I am also thankful to Mr. F.P. Laikhama, Miss C. Lalthangpuii and Mr. Vanlalhmuaka for their support, encouragement and co-operation.

I extend my sincere gratitude to all my colleagues and friends, Dr. R. Lianngenga, *Mr. Lalawmpuia Tochhawng, Miss Laldinsangi, Mr. C. Lalmalsawma, Miss K. Lalnunsiami, Miss C. Lalnunmawii, Mr. A. Lalchhuangliana, Miss Lalsangliani Ralte , Mr. Lalthlengliana, Mr. Lalthasanga and many more for their help, support and assistance in my research work.*

I am extremely thankful to all my family members for their support, help and encouragement without which the completion of my work would not have been possible. I especially thank my dearest mother Lalropuii who has always been there for me. Her prayers, unconditional love and support, sense of humour, patience and advice are more valuable than she could ever imagine. She has helped me come this far. I thank her for everything.

Last but not the least, I would like to express my gratitude to everybody who has shown their constant support and has remembered me in their prayers.

Dated: 22.11.2018

Place: Aizawl *R. Zosangliani*

PREFACE

The present thesis entitled "A Study on Almost Grayan Manifolds" is an outcome of the research work carried out by the author under the supervision of Dr. Jay Prakash Singh, Assistant Professor, Department of Mathematics and Computer Science, Mizoram University.

This thesis consists of six chapters and each chapter is subdivided into a number of sections. The first chapter is devoted to the general introduction which includes the basic definitions, literature review, some mathematical tools used for solving the problems.

In the second chapter, we studied an Einstein manifold admitting a Ricci quarter symmetric metric connection in Sasakian manifolds and we also obtained some interesting results. We have discussed and obtained an equivalent relation between the locally symmetric, conharmonically symmetric and m-projectively symmetric manifolds. We also examined and obtained equivalency relation between the locally bi-symmetric, conharmonically bi-symmetric and m-projectively bi-symmetric manifolds. Here, we showed that a generalized conharmonically 2-recurrent Einstein manifold admitting a Ricci quarter symmetric metric connection is conharmonically flat and a generalized conharmonically 2-recurrent Einstein manifold admitting a Ricci quarter symmetric metric connection is m-projectively flat.

The third chapter deals with the study of curvature tensor W_1. Here, we concentrated on weakly W_1 symmetric manifolds and W_1 flat weakly Ricci-symmetric manifolds. We also examined and investigated the nature of the scalar curvature of a $(WW_1S)_n$. Here, we have proved that the Ricci tensor S in a Riemannian manifold (M^n, g) $(n > 2)$ is codazzi type and the Ricci tensor S in $(WW_1S)_n$ has an eigenvalue $-r$ corresponding to the eigenvector $\widehat{\rho}$. We also proved that in a W_1 flat $(WRS)_n$, $(n > 2)$ with $\mu(X) \neq 0$, the vector field ρ defined by $g(X, \rho) = H(X)$ is not a proper concircular vector field and W_1 flat $(WRS)_n$ $(n > 2)$ is a quasi Einstein manifold.

The fourth chapter deals with ϕ-symmetric and ϕ-recurrent curvature tensor in LP-Sasakian manifolds. We proved that ϕ-symmetric LP-Sasakian manifold is an Einstein manifold. It is shown that the scalar curvature r is constant if and only if a 3-dimensional

LP-Sasakian manifold is locally ϕ-symmetric. We constructed an example of 3-dimensional ϕ-Ricci symmetric LP-Sasakian manifolds. In this chapter, we also studied and discussed m-projective ϕ-recurrent LP-Sasakian manifolds. We showed that m-projective ϕ-recurrent LP-Sasakian manifold is an η-Einstein manifold.

We explored some properties of K-contact quasi Einstein manifolds in the fifth chapter. We have obtained some conditions on K-contact manifold which satisfy semi-symmetric, Ricci symmetric and Ricci-recurrent. We also studied Ricci Solitons in K-contact quasi Einstein manifolds. Here, we have proved that in K-contact manifold admitting Ricci solitons $g(X, \phi Y) = 0$. We also showed that a Ricci soliton in K-contact quasi Einstein manifold could not be steady.

The last chapter is summary and conclusion.

Throughout the preparation of the manuscript, we have gone through several text books and research papers which are cited in the bibliography. Towards the end of the thesis, the references of the mentioned papers are listed with the surnames of the authors and the year of publication of their works, which are decoded in chronological order in the Bibliography.

Contents

Chapter 1

Introduction

1.1 Differentiable Manifold

A locally Euclidean space is a topological space M^n such that each point has a neighborhood homeomorphic to an open subset of the Euclidean space $\mathbb{R}^n$. If ϕ is a homeomorphism of a connected open set $U \subset M^n$ onto an open subset of $\mathbb{R}^n$, then U is called a coordinate neighborhood; ϕ is called a coordinate map; the functions $x^i = t^i \circ \phi$, where t^i denotes the i^{th} canonical coordinate function on $\mathbb{R}^n$ are called the coordinate functions and the pair (U, ϕ) is called a coordinate system or a (local) chart. An atlas $\mathscr{A}$ of class C^∞ on a locally Euclidean space M^n is a collection of coordinate systems $(U_\alpha, \phi_\alpha, \alpha \in A)$ satisfying the following two properties

(1) $\cup_{\alpha \in A} U_\alpha = M^n$.

(2) $\phi_\alpha \circ \phi_\beta^{-1}$ is C^∞ for all $\alpha, \beta \in A$.

A differentiable structure (or maximal atlas) $\mathscr{C}$ on a locally Euclidean space M^n is an atlas $\mathscr{A} = (U_\alpha, \phi_\beta) : \alpha \in A$ of class C^∞, satisfying the above two properties (1) and (2) and moreover the condition.

(3) The collection $\mathscr{C}$ is maximal with respect to (2); that is, if (U, ϕ) is a coordinate system such that $\phi \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ \phi^{-1}$ are C^∞, then $(U, \phi) \in \mathscr{C}$.

A topological manifold of dimension n is a Hausdorff, second countable, locally Euclidean space of dimension n. A differentiable manifold of class C^∞ of dimension n (or simply differentiable manifold of dimension n or C^∞ manifold or n manifold) is a pair of $(M^n, \mathscr{C})$ consisting of a topological manifold M^n of dimension n, together with a differentiable structure $\mathscr{C}$ of class C^∞ on M^n.

Let M^n and N^m be differentiable manifolds of respective dimensions n and m. A map $\Phi : M^n \to N^m$ is said to be C^∞ provided for every coordinate system (U, ϕ) on M^n and (V, ψ) on N^n, the composite map $\psi \circ \Phi \circ \phi^{-1}$ is C^∞.

1.2 Tangent Vector

A vector at a point has a direction and a magnitude. If a point moves along a regular curve then its velocity can be interpreted as a certain vector which is tangent to the curve at that point. Again, if a point moves across a 2-dimensional surface then its velocity is interpreted as a certain vector tangent to the given surface.

A tangent vector at a point p in a manifold M^n is a derivation at p. Let M^n be a differentiable manifold and p a point on M^n. Consider the set of all real-valued C^∞ functions, each defined on some neighborhood of p is denoted by $\mathscr{C}(p)$. If $f, g \in \mathscr{C}(p)$, then $f + g$ and $f.g$ are defined on the intersection of the neighborhood where f is defined and the neighborhood where g is defined; λf is defined on the neighborhood where f is defined. If for each $f \in \mathscr{C}(p)$, there corresponds a real number $v(f)$ satisfying

(1) $v(\lambda f + \mu g) = \lambda v(f) + \mu v(g)$,

(2) $v(fg) = v(f)g(p) + f(p)v(g)$,

where $\lambda, \mu \in \mathbb{R}; f, g \in \mathscr{C}(p)$. Then the map $v : \mathscr{C}(p) \to \mathbb{R}$ is called a tangent vector of M^n at p.

1.3 Tangent Space

For tangent vectors v, v' of M^n at p and for $\lambda \in \mathbb{R}$, we define the sum $v + v'$ and the scalar multiple λv by

(i) $(v + v')(f) = v(f) + v'(f)$,

(ii) $(\lambda v)(f) = \lambda(v(f)), \quad f \in \mathscr{C}(p)$.

Then $(v + v')$ and λv are also tangent vectors of M^n at p. Hence, defining the sum and the scalar multiple of tangent vectors at p in this manner, the set of all tangent vectors at p becomes a vector space over $\mathbb{R}$. This vector space is denoted as $T_p(M^n)$ and call it the tangent vector space or tangent space of M^n at p.

Let $(x^1, x^2, ..., x^n)$ be a local coordinate system on U and at a point p of U. Let

$$(\frac{\partial}{\partial x^i})_p f = \frac{\partial f}{\partial x^i}(p), \quad (i = 1, ..., n).$$

Then $\frac{\partial}{\partial x^i}$ is a tangent vector at p. If M^n is a manifold of dimension n, then the tangent space $T_p(M^n)$ is also of dimension n. The basis of T_p with respect to coordinate system $(x^1, x^2, ...x^n)$ is $(\frac{\partial}{\partial x^i}), i = 1, 2, ...n$.
Let $T_p^{'}$ be the dual space of T_p whose basis with respect to the basis $(\frac{\partial}{\partial x^i})$ is $(dx^1, dx^2, ...dx^n)$. We observe that the elements of T_p are the contravariant vectors and elements of $T_p^{'}$ are the covariant vectors with respect to the basis of T_p.

1.4 Vector Field

A vector field X on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U a tangent vector X_p in $T_p(\mathbb{R}^n)$. Since $T_p(\mathbb{R}^n)$ has basis $\frac{\partial}{\partial x^i}|_p$, the vector X_p is a linear combination

$$X_p = \sum a_i(p) \frac{\partial}{\partial x^i}|_p, \quad p \in U, \quad a^i(p) \in \mathbb{R}.$$

Omitting p, we may write $X = \sum a^i \partial/\partial x^i$, where a^i are functions on U. We can say that the vector field X in C^∞ on U if the coefficient functions a^i are all C^∞ on U.

A vector field X on M^n is a linear mapping $X : C^\infty(M^n) \to C^\infty(M^n)$ such that the map $f \to Xf$ satisfies

$$X(f+g) = Xf + Xg, \tag{1.4.1}$$

$$X(af) = aXf, \tag{1.4.2}$$

$$X(fg) = (Xf)g + f(Xg), \tag{1.4.3}$$

for all $f, g \in C^\infty(M^n), a \in \mathbb{R}^n$ which implies that X is also derivation of the algebraic $C^\infty(M^n)$. Thus a vector field X is defined as a derivation of the functions $C^\infty(M^n)$ satisfying (1.4.1) - (1.4.3). Thus to each point $p \in M^n$ such a derivation assigns a linear map $X_p : C^\infty(M^n) \to \mathbb{R}$ defined by $X_{(p)}f = (Xf)(p)$ for each $f \in C^\infty(M^n)$ and hence the map $p \in X_p$ assigns a field of tangent vectors.

1.5 Lie Bracket

If X, Y are C^∞ vector fields, then we define a C^∞ vector field called the Lie bracket (or Poisson Bracket) of X and Y on the intersection of their domain by

$$[X, Y] = XY - YX.$$

The Lie bracket satisfying the following properties:

$$[X, Y](f+g) = [X, Y]f + [X, Y]g, \tag{1.5.1}$$

$$[X, Y](fg) = f[X, Y]g + g[X, Y]f, \tag{1.5.2}$$

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X, \tag{1.5.3}$$

$$[X, Y] = -[Y, X], \qquad (anticommutativity/skew\ \ symmetric) \tag{1.5.4}$$

$$[X, aY + bZ] = a[X, Y] + b[X, Z], \qquad (bilinear) \tag{1.5.5}$$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \qquad (Jacobi \quad identity) \tag{1.5.6}$$

where $f, g \in \mathscr{C}$ and $X, Y, Z \in C^\infty$ and a, b are scalars.

1.6 Lie Derivative

Let X be a C^∞ vector field on an open set A. Lie derivative via. X is a type preserving linear mapping

$$\mathcal{L}_X : T^r_s \to T^r_s,$$

such that

$$\mathcal{L}_X f = Xf, \qquad f \in F \tag{1.6.1}$$

$$\mathcal{L}_X Y = [X, Y], \tag{1.6.2}$$

$$(\mathcal{L}_X u)(Y) = X(u(Y)) - u([X, Y]), \qquad u\ \ is\ 1 - form \tag{1.6.3}$$

$$\mathcal{L}_X a = 0, \qquad a \in \mathbb{R} \tag{1.6.4}$$

$$
\begin{aligned}
(\mathcal{L}_X P)(u_1, ..., u_r, X_1, ..., X_s) = X(P(u_1, ..., u_r, X_1, ..., X_s)) \\
-P(\mathcal{L}_X u_1, ..., u_r, X_1, ..., X_s)... \\
-P(u_1, ..., u_r, [X, X_1], X_2, ..., X_s)... \\
-P(u_1, ..., X_{s-1}, [X, X_s]), \quad P \in T_s^r.
\end{aligned} \tag{1.6.5}
$$

$$
\begin{aligned}
(\mathcal{L}_X P)(u_1, ..., u_r, X_1, ..., X_s) = X(P(u_1, ..., u_r, X_1, ..., X_s)) \\
-P(\mathcal{L}_X u_1, ..., u_r, X_1, ..., X_s)... \\
-P(u_1, ..., u_r, [X, X_1], X_2, ..., X_s)... \\
-P(u_1, ..., u_{s-1}, [X, X_s]), P \in T_s^r.
\end{aligned} \tag{1.6.6}
$$

1.7 Covariant Derivative

A linear affine connection on M^n is a function

$$\nabla : T_p(M^n) \times T_p(M^n) \to T_p(M^n)$$

such that

$$\nabla_{fX+gY} Z = f(\nabla_X Z) + g(\nabla_Y Z), \tag{1.7.1}$$

$$\nabla_X f = Xf, \tag{1.7.2}$$

$$\nabla_X (fY + gZ) = f(\nabla_X Y) + g(\nabla_X Z) + (Xf)Y + (Xg)Z, \tag{1.7.3}$$

for an arbitrary vector fields X, Y, Z and smooth functions $f, g \in M^n$. ∇_X is called covariant derivative operator and $\nabla_X Y$ is called covariant derivative of Y with respect to X.

The covariant derivative of a 1-form u is given by

$$(\nabla_X u)(Y) = X(u(Y)) - u(\nabla_X Y). \tag{1.7.4}$$

1.8 Exterior Derivative

Let V_p be the set of all $C^\infty \quad p$-forms on an open set A. Then the mapping

$$d : V_p \to V_{p+1}$$

such that

$$(df)(\lambda) = \lambda f, \quad \lambda \in V^1, \quad f \in F \tag{1.8.1}$$

$$\begin{aligned}(dA)(\lambda_1, ..., \lambda_{p+1}) &= \sum_{1\leq j}(-)^{j+1}\lambda_j \left(A(\lambda_1, ..., \overline{\lambda}_j, ..., \lambda_{p+1})\right) \\ &+ \sum_{i\leq j}(-)^{j+i}A\left([(\lambda_i, \lambda_j], \lambda 1, ..., \overline{\lambda}_i, ..., \overline{\lambda}_j, ..., \lambda_{p+1})\right)\end{aligned} \tag{1.8.2}$$

for arbitrary C^∞ fields $\lambda^s \in V^1$, where $A \in V_p$.

and

$$\begin{aligned}(dA)(\lambda_1, ..., \lambda_{p+1}) &= \lambda_1 \left(A(\lambda_2, ..., \lambda_{p+1})\right) \\ &- \lambda_2 \left(A(\lambda_1, X_3, ...\lambda_{p+1})\right) \\ &+ X_3 \left(A(\lambda_1, \lambda_2, \lambda_4, ..., \lambda_{p+1})\right) ... \\ &- A([\lambda_1, \lambda_2], \lambda_3, ..., \lambda_{p+1}) \\ &+ A([\lambda_1, \lambda_3], \lambda_2, \lambda_4, ..., \lambda_{p+1}) \\ &- A([\lambda_2, \lambda_3], \lambda_1, \lambda_4, ..., \lambda_{p+1}) + ...\end{aligned} \tag{1.8.3}$$

for arbitrary C^∞ vector fields $X's \in V^1$ and $A \in V_p$, is called the exterior derivative.

1.9 Connection

A connection ∇ is a type preserving mapping $\nabla : T_p * T^r_s \to T^r_s$ that assigns to each pair of C^∞-vector fields (X, P), $X \in T_p, P \in T^r_s$, a C^∞ vector field $\nabla_X P$ such that if

$X, Y, Z \in T_p,\ A \in T_p^r$ are C^∞ field and f is a C^∞ function, then

$$\nabla_X f = Xf, \tag{1.9.1}$$

$$\nabla_X a = 0,\ a \in \mathbb{R} \tag{1.9.2}$$

$$\begin{aligned} (a) \quad & \nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z, \\ (b) \quad & \nabla_X(fY) = (Xf)Y + f\nabla_X Y, \end{aligned} \tag{1.9.3}$$

$$\begin{aligned} (a) \quad & \nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z, \\ (b) \quad & \nabla_{fX} Z = f\nabla_X Z, \end{aligned} \tag{1.9.4}$$

$$(\nabla_X \lambda)(Y) = X(\lambda(Y)) - \lambda(\nabla_X Y), \tag{1.9.5}$$

and

$$\begin{aligned} (\nabla_X P)(\lambda_1, ..., \lambda_r, X_1, ..., X_s) &= X(P(\lambda_1, ..., \lambda_r, X_1, ..., X_s)) \\ &- P(\nabla_X \lambda_1, ..., \lambda_r, X_1, ..., X_s)... \\ &- P(\lambda_1 ..., \lambda_r, X_1, ..., \nabla_X X_s). \end{aligned} \tag{1.9.6}$$

1.10 Riemannian Manifold

Let us consider an n-dimensional C^∞ with the tangent space T_p at $p \in M^n$. A real valued, bilinear symmetric, non-singular positive definite function g on the ordered pair X, Y of tangent vectors $T_{(p)}$ at each point p such that

(1) $g(X, Y)$ is a real number,

(2) g is symmetric $\Rightarrow g(X, Y) = g(Y, X)$,

(3) g is non-singular i.e., $g(X, Y) = 0, \quad$ for all $\quad Y \neq 0 \Rightarrow X = 0$,

(4) g is positive definite i.e., $g(X,X) \geqslant 0$, for all tangent vector $X \in C^\infty$ and $g(X,X) = 0$ if and only if $X = 0$,

and

(5) $g(aX + bY, Z) = ag(X,Z) + bg(Y,Z)$; $a, b \in \mathbb{R}$,

then g is said to be Riemannian metric tensor or fundamental tensor of type (0,2).

Then, the manifold M^n with a Riemannian metric g is called a Riemannian manifold and its geometry is called a Riemannian geometry denoted by (M^n, g) or (M, g) or simply by M.

1.11 Riemannian Connection

Let (M^n, g) is an n-dimensional manifold and ∇ is an affine connection on M^n. Then the affine connection ∇ on M^n is said to be Riemannian connection (or Levi-Civita connection) if it satisfies:

(1) ∇ is symmetric or torsion free i.e.,

$$\nabla_X Y - \nabla_Y X = [X, Y], \tag{1.11.1}$$

and

(2) ∇ is a metric compatible or metric connection i.e.,

$$(\nabla_X g)(Y, Z) = 0. \tag{1.11.2}$$

Thus a Riemannian connection on a Riemannian manifold is a linear connection which is torsion free and metric competible.

1.12 Quarter Symmetric Metric Connection

A linear connection ∇ on an n-dimensional Riemannian manifold (M^n, g) is called a quarter symmetric connection if its torsion tensor T of the connection ∇

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{1.12.1}$$

satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.12.2}$$

where η is 1-form and ϕ is a $(1,1)$ tensor field.
In particular, if $\phi(X) = X$, then the quarter symmetric connection reduces to a semi-symmetric connection. Thus the notion of quarter symmetric connection generalizes the notion of semi symmetric connection.

Moreover, if a quarter symmetric connection ∇ satisfies the condition

$$(\nabla_X g)(Y,Z) = 0,$$

for all $X, Y, Z \in T_p(M^n)$, where $T_p(M^n)$ is the Lie algebra of vector fields of the manifold M^n, then ∇ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter symmetric non-metric connection.

1.13 Torsion Tensor

The mapping $T : \chi(M^n) \otimes \chi(M^n) \to \chi(M^n)$ given by

$$T(X,Y) \overset{def}{=} \nabla_X Y - \nabla_Y X - [X,Y]. \tag{1.13.1}$$

Then the vector field $T(X,Y) \in \chi(M^n)$ is called a torsion tensor field of the connection ∇ for all $X, Y \in \chi M^n$.

Torsion tensor is also a vector valued, skew-symmetry, bilinear function T of the type

$(1,2)$ tensor. Torsion tensor is said to be symmetric or torsion free, if the torsion tensor of a connection $\nabla = 0$.

1.14 Curvature Tensor

The curvature tensor K of type $(1,3)$ with respect to the Riemannian connection ∇ is given by

$$K(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.14.1}$$

for all $X, Y, Z \in T_p(M^n)$.

Let $'K$ be the associative curvature tensor of the type $(0,4)$ of the curvature tensor K. Then

$$'K(X,Y,Z,U) = g(K(X,Y,Z)U), \tag{1.14.2}$$

$'K$ is called the Riemannian-Christoffel curvature tensor of first kind.

The associative curvature tensor $'K$ satisfied the following properties:

$'K$ is skew-symmetric in first two slot

$$i.e., \quad 'K(X,Y,Z,U) = -\,'K(Y,X,Z,U). \tag{1.14.3}$$

$'K$ is skew-symmetric in last two slot

$$i.e., \quad 'K(X,Y,Z,U) = -\,'K(X,Y,U,Z). \tag{1.14.4}$$

$'K$ is symmetric in two pair of slot

$$i.e., \quad 'K(X,Y,Z,U) = \quad 'K(Z,U,X,Y). \tag{1.14.5}$$

$'K$ satisfies Bianchi's first identities

$$i.e., \quad 'K(X,Y,Z,U) + \,'K(Y,Z,X,U) + \,'K(Z,X,Y,U) = 0. \tag{1.14.6}$$

and $'K$ satisfies Bianchi's second identities

$$i.e., \ (\nabla_X\, 'K)(Y,Z,U,V) + (\nabla_Y\, 'K)(Z,X,U,V) + (\nabla_Z\, 'K)(X,Y,U,V) = 0. \tag{1.14.7}$$

1.15 Ricci-Tensor

Let M^n be a Riemannian manifold with a Riemannian connection ∇. Then the Ricci tensor field S is the covariant tensor field of degree 2 defined as $Ric(Y,Z) = S(Y,Z) =$ Trace of the linear map $X \to K(X,Y)Z$ for all $X, Y, Z \in T_p(M^n)$.

If $\{e_1, ..., e_n\}$ is an orthonormal basis of the tangent space $T_p, p \in M^n$ and K is the Riemannian curvature tensor of the Riemannian manifold (M^n, g), then

$$S(X,Y) = \sum_{i=1}^{n} g(K(e_i, X)Y, e_i), \tag{1.15.1}$$

$$= \sum_{i=1}^{n} {}'K(e_i, X, Y, e_i), \tag{1.15.2}$$

$$= \sum_{i=1}^{n} {}'K(X, e_i, e_i, Y),$$

$$= \sum_{i=1}^{n} g(K(X, e_i)e_i, Y),$$

where $'K$ is the Riemannian curvature tensor of the manifold of type $(0,4)$.
Ricci tensor is also symmetric

$$i.e., \quad S(X,Y) = S(Y,X). \tag{1.15.3}$$

The linear map L of the type $(1,1)$ defined by

$$g(LX,Y) \overset{def}{=} S(X,Y) \tag{1.15.4}$$

is called a Ricci-map. It is self-adjoint,

$$i.e., \quad g(LX, Y) = g(X, LY). \tag{1.15.5}$$

The scalar r defined by

$$r \overset{def}{=} (C_1^1 K), \tag{1.15.6}$$

is called the scalar curvature of M^n at the point p.

A Riemannian manifold M^n is said to be Einstein manifold, if

$$S(X, Y) = \frac{r}{n} g(X, Y). \tag{1.15.7}$$

A Riemannian manifold M^n is said to be η-Einstein manifold, if

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{1.15.8}$$

where a and b are smooth functions.

A Riemannian manifold M^n is said to be flat if

$$K(X, Y)Z = 0. \tag{1.15.9}$$

1.16 Certain Curvature Tensors

(A) Conharmonic curvature tensor:

The conharmonic curvature tensor $'C$ is defined as (Ishii, 1957)

$$\begin{aligned} 'C(X, Y, Z, U) &= \; 'K(X, Y, Z, U) - \frac{1}{n-2}\Big\{S(Y, Z)g(X, U) - S(X, Z)g(Y, U) \\ &+ \; S(X, U)g(Y, Z) - S(Y, U)g(X, Z)\Big\}. \end{aligned} \tag{1.16.1}$$

It satisfies the following properties:

$$(a)\, 'C(X,Y,Z,U) = -'C(Y,X,Z,U),$$

$$(b)\, 'C(X,Y,Z,U) = 'C(X,Y,U,Z),$$

$$(c)\, 'C(X,Y,Z,U) = 'C(Z,U,X,Y),$$

$$(d)\, 'C(X,Y,Z,U) + 'C(Y,Z,X,U) + 'C(Z,X,Y,U) = 0,$$

where

$$'C(X,Y,Z,U) = g(C(X,Y,Z),U).$$

(B) Concircular curvature tensor:

The concircular curvature tensor $'V$ of type $(0,4)$, is given by (Yano, 1940)

$$\begin{aligned} 'V(X,Y,Z,U) &= {}'K(X,Y,Z,U) - \frac{r}{n(n-1)}\Big\{g(Y,Z)g(X,U) \\ &- g(X,Z)g(Y,U)\Big\}. \end{aligned} \tag{1.16.2}$$

It satisfies the following algebraic properties:

$$(a)\, 'V(X,Y,Z,U) = -'V(Y,X,Z,U),$$

$$(b)\, 'V(X,Y,Z,U) = -'V(X,Y,U,Z),$$

$$(c)\, 'V(X,Y,Z,U) = {}'V(Z,U,X,Y),$$

$$(d)\, 'V(X,Y,Z,U) + {}'V(Y,Z,X,U) + {}'V(Z,X,Y,U) = 0,$$

where

$$'V(X,Y,Z,U) = g(V(X,Y,Z),U).$$

(C) Projective curvature tensor:

The projective curvature tensor $'P$ of the type $(0,4)$, is defined by (Yano and Bochner, 1953)

$$
\begin{aligned}
'P(X,Y,Z,U) &= {}'K(X,Y,Z,U) - \frac{1}{n-1}\Big\{S(Y,Z)g(X,U) \\
&- S(X,Z)g(Y,U)\Big\}. \qquad (1.16.3)
\end{aligned}
$$

The projective curvature tensor $'P$ satisfies

$$(a)'P(X,Y,Z,U) = -'P(Y,X,Z,U),$$

$$(b)C_1^1 P = C_2^1 P = C_3^1 P = 0,$$

$$(c)'P(X,Y,Z,U) + 'P(Y,Z,X,U) + 'P(Z,X,Y,U) = 0,$$

where

$$'P(X,Y,Z,U) = g(P(X,Y,Z),U).$$

(D) m-projective curvature tensor:

Pokhariyal and Mishra defined m-projective curvature tensor $'W^*$ of the type $(0,4)$ by (Pokhariyal and Mishra, 1971)

$$
\begin{aligned}
'W^*(X,Y,Z,U) &= {}'K(X,Y,Z,U) - \frac{1}{2(n-1)}\Big\{g(X,U)S(Y,Z) - g(Y,U)S(X,Z) \\
&+ S(X,U)g(Y,Z) - S(Y,U)g(X,Z)\Big\}. \qquad (1.16.4)
\end{aligned}
$$

It satisfies the following algebraic properties:

$$(a)\, 'W^*(X,Y,Z,U) = {}'W^*(Z,U,X,Y),$$

$$(b)\, 'W^*(X,Y,Z,U) = -\, 'W^*(Y,X,U,Z),$$

$$(c)\, 'W^*(X,Y,Z,U) = -\, 'W^*(X,Y,U,Z),$$

$$(d)\, 'W^*(X,Y,Z,U) + {}'W^*(Y,Z,X,U) + {}'W^*(Z,X,Y,U) = 0,$$

where

$$'W^*(X,Y,Z,U) = g(W^*(X,Y,Z),U).$$

(E) W_1 **curvature tensor**:

Pokhariyal and Mishra also defined W_1 curvature tensor $'W_1$ of the type $(0,4)$ by (Pokhariyal and Mishra, 1971)

$$'W_1(X,Y,Z,U) = {}'K(X,Y,Z,U) + \frac{1}{(n-1)}\Big\{g(X,U)S(Y,Z) - g(Y,U)S(X,Z)\Big\}. \tag{1.16.5}$$

It satisfies the following properties:

$$(a)\ 'W_1(X,Y,Z,U) = {}'W_1(Z,U,X,Y),$$
$$(b)\ 'W_1(X,Y,Z,U) = -\,'W_1(Y,X,U,Z),$$
$$(c)\ 'W_1(X,Y,Z,U) = -\,'W_1(X,Y,U,Z),$$
$$(d)\ 'W_1(X,Y,Z,U) + {}'W_1(Y,Z,X,U) + {}'W_1(Z,X,Y,U) = 0,$$

where

$$'W_1(X,Y,Z,U) = g(W_1(X,Y,Z),U).$$

1.17 Almost Contact Metric Manifold

If $M^{n\ (=2m+1)}$ be an odd-dimensional differentiable manifold and ϕ, ξ, η be a tensor field of type (1,1), a vector field, a 1-form on M^n satisfying for arbitrary vectors $X, Y, Z, ...$

$$\phi^2 X = -X + \eta(X)\xi, \tag{1.17.1}$$

$$\eta(\xi) = 1, \tag{1.17.2}$$

$$\phi(\xi) = 0, \tag{1.17.3}$$

$$\eta(\phi X) = 0, \tag{1.17.4}$$

and

$$rank(\phi) = n - 1, \tag{1.17.5}$$

is called an almost contact manifold (Sasaki, 1965) and the structure (ϕ, η, ξ) is called an almost contact structure (Hatakeyama *et al.*, 1963; Sasaki and Hatakeyama (1960, 1961)).

An almost contact manifold M^n on which a Riemannian metric tensor g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.17.6}$$

and

$$g(X, \xi) = \eta(X), \tag{1.17.7}$$

is called an almost contact metric manifold (or an almost grayan manifold) and the structure (ϕ, ξ, η, g) is called an almost contact metric structure (Sasaki, 1960).

The fundamental 2-form $'F$ of an almost contact metric manifold M^n is defined by

$$'F(X, Y) = g(\phi X, Y). \tag{1.17.8}$$

From the equations (1.17.6) and (1.17.8), we have

$$'F(X, Y) = -'F(Y, X). \tag{1.17.9}$$

If in an almost contact metric manifold

$$2'F(X, Y) = (\nabla_X \eta)(Y) - (\nabla_Y \eta)(X), \tag{1.17.10}$$

then M^n is called an almost Sasakian manifold.

An almost Sasakian manifold M^n is said to be K-contact Riemannian manifold, if ξ is killing vector(Okumura, 1962; Miyazawa and Yamaguchi, 1966) i.e.

$$(\nabla_X \eta)(Y) + (\nabla_Y \eta)(X) = 0. \tag{1.17.11}$$

If on a K-contact Riemannian manifold

$$(\nabla_X \phi)(Y) = \eta(Y)X - g(X, Y)\xi, \tag{1.17.12}$$

hold, then the manifold is known as a Sasakian manifold (Sasaki, (1965, 1967); Mishra, 1984).

In a Sasakian manifold

$$'F(X, Y) = (\nabla_X \phi)(Y), \tag{1.17.13}$$

and

$$\nabla_X \xi = -\phi X, \tag{1.17.14}$$

also holds.

1.18 Almost Para-Contact Metric Manifold

Let M^n be an n-dimensional C^∞-manifold. If there exist in M^n a tensor field ϕ of the type $(1, 1)$, consisting of a vector field ξ and a 1-form η in M^n satisfying

$$\phi^2 X = X - \eta(X)\xi, \tag{1.18.1}$$

$$\phi(\xi) = 0, \qquad \eta(\xi) = 1, \tag{1.18.2}$$

then M^n is called an almost Para-contact manifold.

Let g be a Riemannian metric satisfying

$$\eta(X) = g(X, \xi), \qquad \eta(\phi X) = 0, \tag{1.18.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.18.4}$$

then the structure (ϕ, ξ, η, g) satisfying (1.17.1) - (1.17.4) is called an almost Para-contact Riemannian structure. The manifold with such structure is called an almost Para-contact Riemannian manifold (Sato and Matsumoto, 1976).

If we defined $F'(X, Y) = g(\phi X, Y)$, then the following relations are satisfied

$$F'(X, Y) = F'(Y, X), \tag{1.18.5}$$

and

$$F'(\phi X, \phi Y) = F'(X, Y). \tag{1.18.6}$$

If in M^n the relation

$$(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0, \tag{1.18.7}$$

$$d\eta(X, Y) = 0, \quad i.e., \ \ \eta \, is \, closed. \tag{1.18.8}$$

$$\begin{aligned} (\nabla_X F')(Y, Z) &= -g(X, Z)\eta(Y) - g(X, Y)\eta(Z) \\ &+ \ 2\,\eta(X)\eta(Y)\eta(Z), \end{aligned} \tag{1.18.9}$$

$$(\nabla_X \eta)(Y) + (\nabla_X \eta)(X) = 2\,F'(X, Y), \tag{1.18.10}$$

and

$$\nabla_X \xi = \phi X, \tag{1.18.11}$$

holds.

1.19 Para-Sasakian Manifold

An n-dimensional differentiable manifold M^n is said to be Para-Sasakian or briefly P-Sasakian manifold if it admits a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Riemannian metric g, which satisfy (Matsumoto, 1977; Miyazawa, 1979)

$$\phi^2(X) = X - \eta(X)\xi, \tag{1.19.1}$$

$$\phi\xi = 0, \tag{1.19.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.19.3}$$

$$g(X, \xi) = \eta(X), \tag{1.19.4}$$

$$(\nabla_X \phi)(Y) = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{1.19.5}$$

$$\nabla_X \xi = \phi X, \tag{1.19.6}$$

$$(a)\ \eta(\xi) = 1, \qquad (b)\ \eta(\phi X) = 0, \tag{1.19.7}$$

$$rank(\phi) = (n-1), \tag{1.19.8}$$

$$(\nabla_X \eta)(Y) = g(\phi X, Y) = g(\phi Y, X), \tag{1.19.9}$$

for any vector fields X, Y, where ∇ denotes covariant differentiation with respect to g.

1.20 Lorentzian Para-Contact Metric Manifold

Let M^n be an n-dimensional differentiable manifold endowed with a tensor field ϕ of the type $(1, 1)$, a vector field ξ, a 1-form η and a Lorentzian metric g satisfying

$$\phi^2 X = X + \eta(X)\xi, \tag{1.20.1}$$

$$\eta(\xi) = -1, \tag{1.20.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{1.20.3}$$

$$g(X, \xi) = \eta(X), \tag{1.20.4}$$

for an arbitrary vector fields X and Y, then M^n is called a Lorentzian Para (or LP)-contact manifold and the structure (ϕ, ξ, η, g) is called the Lorentzian Para-contact structure (Matsumoto, 1989).

Let M^n be a Lorentzian Para-contact manifold with stucture (ϕ, ξ, η, g). Then it satisfying

$$(a)\ \phi(\xi) = 0, \qquad (b)\ \eta(\phi X) = 0, \qquad (c)\ rank(\phi) = n - 1. \tag{1.20.5}$$

A Lorentzian Para-contact manifold is called a Lorentzian Para-Sasakian manifold if (Matsumoto and Mihai, 1988)

$$\nabla_X \xi = \phi X, \tag{1.20.6}$$

$$(\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\,\eta(X)\eta(Y)\xi, \tag{1.20.7}$$

where ∇ denotes the covariant differentiation with respect to g.

Let us put $F'(X,Y) = g(\phi X, Y)$. Then the tensor field F' is symmetric.

$$i.e., \;\; F'(X,Y) = F'(Y,X), \tag{1.20.8}$$

and

$$F'(X,Y) = (\nabla_X \eta)(Y). \tag{1.20.9}$$

Also, in an LP-Sasakian manifold the following relation holds

$$'K(X,Y,Z,\xi) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{1.20.10}$$

and

$$S(X,\xi) = (n-1)\eta(X). \tag{1.20.11}$$

1.21 Recurrent Manifold

Let M^n be an n-dimensional smooth Riemannian manifold and $T_p(M^n)$ denotes the set of differentiable vector fields on M^n. Let $X, Y \in T_p(M^n)$; $\nabla_X Y$ denotes the covariant derivative of Y with respect to X and K be the Riemannian curvature tensor of type $(1,3)$.

A Riemannian manifold M^n is said be recurrent (Kobayashi and Nomizu, 1963) if

$$(\nabla_U K)(X,Y,Z) = \alpha(U)K(X,Y,Z), \tag{1.21.1}$$

where $X, Y, Z \in T_p(M^n)$ and α is a non-zero 1-form known as recurrence parameter. If the 1-form α is zero in (1.21.1), then the manifold reduces to symmetric manifold (Singh and Khan, 1999).

A Riemannian manifold (M^n, g) is said to be semi-symmetric if it satisfies the relation

(Szabo, 1982)

$$(K(X,Y).K)(U,V)W = 0, \tag{1.21.2}$$

where $K(X,Y)$ is considered as the tensor algebra at each point of the manifold i.e., $K(X,Y)$ is curvature transformation or curvature operator.

A Riemannian manifold (M^n, g) is said to be Ricci-recurrent if it satisfies the relation

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z), \tag{1.21.3}$$

for all $X, Y, Z \in T_p(M^n)$, where ∇ denotes the Riemannian connection (or Levi-Civita connection) and A is a 1-form on M^n. If the 1-form A vanishes identically on M^n, then a Ricci-recurrent manifold becomes a Ricci-symmetric manifold.

A Riemannian manifold (M^n, g) is called a generalized recurrent manifold (De and Guha, 1991) if its curvature tensor K satisfies the following condition:

$$(\nabla_X K)(Y,Z)U = A(X)K(Y,Z)U + B(X)\big[g(Z,U)Y - g(Y,U)Z\big], \tag{1.21.4}$$

where A and B are 1-form, B is non-zero and these are defined by

$$A(X) = g(X, \rho_1), \qquad B(X) = g(X, \rho_2), \tag{1.21.5}$$

ρ_1 and ρ_2 are vector fields associated with 1-forms A and B, respectively.

A Riemannian manifold (M^n, g) is said to be ϕ-recurrent if there exists non-zero 1-form A such that

$$\phi^2((\nabla_W K)(Y,Z)U) = A(W)K(Y,Z)U, \tag{1.21.6}$$

for arbitrary vector fields Y, Z, U, W.

A Riemannian manifold (M^n, g) is called generalized ϕ-recurrent if its curvature tensor

K satisfies

$$\begin{aligned} \phi^2((\nabla_W K)(Y,Z)U) &= A(W)K(Y,Z)U \\ &+ B(W)[g(Z,U)Y - g(Y,U)Z], \end{aligned} \tag{1.21.7}$$

where A and B are 1-forms and B is non-zero.

1.22 Ricci Solitons

During 1982, Hamilton made the fundamental observation that Ricci flow is an excellent tool for simplifying the structure of the manifold. It is the process which deforms the metric of a Riemannian manifold analogous to the diffusion of heat there by smooting out the regularity in the metric. It is given by (Hamilton, 1982)

$$\frac{\partial g}{\partial t} = -2Ric(g),$$

where g is Riemannian metric, $Ric(g)$ is the Ricci curvature tensor, t is time.
A Ricci soliton (g, V, λ) is a generalization of an Einstein metric and is defined on a Riemannian manifold (M^n, g) by (Hamilton, 1988)

$$\mathcal{L}_V g + 2S + 2\lambda g = 0, \tag{1.22.1}$$

where $\mathcal{L}$ is Lie-derivative, V is a complete vector field on M^n, λ is constant and S is Ricci tensor. The Ricci soliton is said to be shrinking, steady or expanding according as λ is negative, zero and positive respectively. Long-existing solutions, that is, solutions which exist on an infinite time interval are the self-similar solutions, which in Ricci flow are called Ricci soliton.
If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and (1.22.1) assumes the form

$$\nabla\nabla f = S + \lambda g. \tag{1.22.2}$$

A Ricci Soliton on a compact manifold is a gradient Ricci Soliton. A Ricci soliton on a compact manifold has constant curvature in dimension 2 (Hamilton, 1988) and also in dimension 3 (Ivey, 1993).

1.23 Some Mathematical Tools

The notion of a differentiable manifold is necessary for extending the methods of differential calculus to spaces more general than $\mathbb{R}^n$. Differentiable manifold was defined on the basis of differential calculus, topology and real analysis. With the help of differentiable manifold, we can study curves and surfaces in n-dimensional Euclidean space. Riemannian manifold is a part of differentiable manifold which we study by index free notation and tensor notation. The fundamental theorem of Riemannian Geometry, Lie algebra, Ricci Identity, Jacobi Identity, Bianchi first Identity, Bianchi second Identity, Contraction method, Koszul's formula and Levi-Civita connection are used in our study.

(i) **Contraction**:

The linear mapping

$C_k^h : T_s^r \to T_{s-1}^{r-1}; \quad (i \leq h \leq r) \quad , (i \leq k \leq s)$

such that

$$C_k^h(\lambda_1 \otimes \lambda_2 \otimes ... \otimes \lambda_r \otimes \alpha_1 \otimes ...\alpha_s) = \alpha_k(\lambda_1 \otimes .. \otimes \lambda_{h-1} \otimes \lambda_{h+1}...$$
$$\otimes \lambda_r \otimes \alpha_1 \otimes \alpha_2 \otimes ..\alpha_{k-1} \otimes \lambda k + 1 \otimes \alpha_s),$$

where $\lambda_1, \lambda_2...\lambda_r \in T_pM^n$ and $\alpha_1, \alpha_2...\alpha_s \in T_p\bar{M}^n$ and $\otimes$ denote tensor product, is called contraction with respect to h^{th} contravariant and k^{th} covariant places.

(ii) **Ricci identity**:

For a tensor field K of type $(0, 1)$ on a Riemannian manifold (M^n, g), then

$(\nabla_X \nabla_Y K)(Z) - (\nabla_Y \nabla_X K)(Z) - (\nabla_{[X,Y]} K)Z = -K(R(X, Y)Z).$

(iii) **Jacobi identity**:

If X, Y, Z are vector fields, then

$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$

(iv) **Bianchi's First identity**:

For a tensor field K of type $(0, 1)$ on a Riemannian manifold (M^n, g), then

$K(X, Y, Z) + K(Y, Z, X) + K(Z, X, Y) = 0,$

where X, Y, Z are vector fields.

(v) **Bianchi's Second identity**:

For a Riemannian connection ∇, we have

$(\nabla_X K)(Y, Z, W) + (\nabla_Y K)(Z, X, W) + (\nabla_Z K)(Y, Y, W) = 0,$

where K is a curvature tensor.

(vi) **Koszul's Formula**:

$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y)$

for all $X, Y, Z \in \chi(M^n)$.

(vii) **Inner product**:

If (U, u^i) is a local coordinate system on M^n, then the tensor field g can be expressed as

$$g = g_{ij} du^i \otimes du^j$$

on U, where $g_{ij} = g_{ji}$ is a smooth function on U. Then g provides a bilinear function on T_pM and hence g gives rise to an inner product on T_pM^n for every point $p \in M^n$. If

$$X = X^i \frac{\partial}{\partial u^i}, \qquad Y = Y^j \frac{\partial}{\partial u^j},$$

then we write

$$\begin{aligned} g(X,Y) &= g\left(X^i\frac{\partial}{\partial u^i}, Y^j\frac{\partial}{\partial u^j}\right) \\ &= X^iY^jg\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) \\ &= g_{ij}X^iY^j, \end{aligned}$$

where $g_{ij} = g\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right)$, $i, j = 1, 2, ..., n$.

(viii) **Fundamental Theorem of Riemannian Geometry**:

Every Riemannian manifold (M^n, g) of dimension n admits a unique torsion free connection.

1.24 Review of Literature

Some global properties of contact structure was studied by Gray (1959). Hatakeyama (1960) introduced the idea of an almost contact manifold structure. Sasaki (1960) studied differentiable manifolds with certain structures which are closely related to almost contact structure. Some remarks on spaces with certain contact structure were given by Okumura (1962). In 1963, Tashiro showed that an almost Grayan structure is introduced on a hyper-surface of an almost complex manifold. Hatakeyama (1963), Hatakeyama *et al.* (1963), Sasaki (1960, 1965, 1967, 1968), Sasaki and Hatakeyama (1960, 1961) defined and deeply studied some properties of an almost contact manifold. In the meantime, Sasaki (1960), Hatakeyama *et al.* (1963) defined an almost contact metric manifold or an almost Grayan manifold. In 1971, Tanno classified connected almost contact metric manifolds whose automorphism group possesses the maximum dimension. In 1980, Sinha and Yadava also defined a structure connection in a Riemannian manifold and studied its properties in an almost contact metric manifold.

A semi-symmetric metric connection was defined in an almost contact manifold by Sharfuddin and Hussain (1976). A semi-symmetric metric connections on a Riemannian manifold have been studied by Amur and Pujar (1978), Bihn (1990), Jun *et al.* (2005), Barmen and De (2013), Chaubey and Ojha (2012), Singh and Pandey (2008), Singh *et al.* (2012, 2013) and many other geometers. De and Sengupta (2001) investigated the curvature tensor of an almost contact metric manifold admitting a type of semi-symmetric metric connection and studied the curvature properties of conformal curvature tensor and projective curvature tensor. This was also studied by many geometers like Hatakeyama (1963), Hatakeyama *et al.* (1963), Sato (1976), Sasaki and Hatakeyama (1961), Oubina (1985). Agashe and Chafle (1992) introduced a semi symmetric non-metric connection on a Riemannian manifold and this was further studied by De and Kamilya (1995), Pandey and Ojha (2001), Prasad and Kumar (2002), Chaturvedi and Pandey (2008), Chaubey (2011), Singh (2014a) and others.

The idea of semi-symmetric linear connection on a differentiable manifold was introduced by Friedman and Schouten (1924). Hayden (1932) defined a metric connection with torsion on a Riemannian manifold. Yano (1970) studied some curvature and derivational conditions for semi-symmetric connections in Riemannian manifolds. As a generalization of this, Golab (1975) introduced and studied the notion of quarter-symmetric connection on a differentiable manifold. A linear connection ∇ on an n-dimensional Riemannian manifold (M^n, g) is called a quarter symmetric connection if its torsion tensor T satisfies

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = \eta(Y)\phi X - \eta(X)\phi Y$$

where η is a 1-form and ϕ is a $(1,1)$ tensor field.

In particular, if $\phi X = X$ and $\phi Y = Y$, then the connection ∇ is called a semi-symmetric metric connection according to Yano (1970). Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection. Further, if ∇ satisfies $(\nabla_X g)(Y,Z) = 0$ for all $X, Y, Z \in \chi(M^n)$, where $\chi(M^n)$ is the Lie algebra of vector fields on M^n, then ∇ is said to be a quarter symmetric metric connection, otherwise it is a quarter symmetric non-metric connection. Further this was developed by Yano and Imai (1982), Rastogi (1978, 1987), Mishra and Pandey (1980), Mukhopadhyay *et al.* (1991) , Biswas and De (1997), Nivas and Verma (2005), Sengupta and Biswas (2003), Singh and Pandey (2007) and many other geometers. De and Sengupta (2000) proved the existence of a quarter-symmetric metric connection on a Riemannian manifold and studied some properties of a quarter-symmetric metric connection on a Sasakian manifold. Sular *et al.* (2008) investigated the curvature tensor and the Ricci tensor of a Kenmotsu manifold with respect to the quarter-symmetric metric connection. Prakash and Narain defined and studied quarter-symmetric non-metric connection on an LP-Sasakian manifolds and proved its existence in 2011. They also found some properties of the curvature tensor and the Ricci tensor of quarter-symmetric non-metric connection. Prakash and Pandey (2013) studied a quarter-symmetric non-metric connection in a Kenmotsu manifold. Recently, in 2013, Singh studied weakly symmetric, weakly Ricci symmetric, generalized recurrent LP-Sasakian manifolds admitting a quarter symmetric non-metric connections. Yadav and Dhruwanarain also studied a quarter symmetric non-metric connection in a P-Sasakian manifold in 2014. Singh (2014c) studied some properties of LP-Sasakian manifolds admitting a quarter symmetric non-metric connections. Singh and Singh (2014) also studied quarter symmetric non-metric connection on LP-Sasakian manifolds. In 2015, Singh and Devi (2015) studied and examined a type of quarter symmetric non-metric connection in an LP-Sasakian manifold. In the same year 2015, Singh *et al.* studied some curvature properties of LP-Sasakian manifolds and obtained some interesting results.

In 1989, Tamassy and Binh introduced the notion of weakly symmetric Riemannian manifolds. Binh (1993) also studied weakly symmetric Riemannian spaces. De *et al.* (2000) studied on weakly symmetric and weakly Ricci-symmetric K-contact manifolds. In 2005, De and Ghosh also studied some global properties of weakly Ricci-symmetric manifolds. Shaikh *et al.* (2007) introduced a type of non-flat Riemannian called weakly W_2-symmetric manifolds and studied their geometric properties. In 2007, Jana and Shaikh studied quasi-conformally flat weakly Ricci symmetric manifolds. Jaiswal and Ojha (2010) studied weakly Pseudo-projectively symmetric manifolds and Pseudo-projectively flat weakly

Ricci-symmetric manifolds. Shaikh and Hui (2010) also studied quasi-conformally flat almost pseudo Ricci-symmetric manifolds. Chaubey (2012) studied weakly m-projectively symmetric and m-projectively flat weakly Ricci-symmetric manifolds and gives some example for that. Singh (2017) studied m-projectively flat almost pseudo Ricci symmetric manifolds.

Pokhariyal and Mishra (1971) introduced new curvature tensor called m-projective curvature tensor in a Riemannian manifold and studied its properties. Ojha (1975) studied a note on the m-projective curvature tensor. Later, Pokhariyal (1982) studied some properties of this curvature tensor in a Sasakian manifold. Ojha (1986), Chaubey (2012), Singh (2009, 2012, 2015b, 2016) and many other geometers studied this curvature tensor in different manifolds.

In 1950, Walker introduced the idea of recurrent manifolds. Dubey (1979) introduced the notion of generalized recurrent manifold and then such a manifold was studied by De and Guha (1991). De *et al.* (1994) defined the generalized recurrent Riemannian manifold and generalized Ricci-recurrent Riemannian manifold. Khan (2004) introduced the notion of generalized recurrent Sasakian manifolds to generalize the notion of recurrency. Generalized recurrent and generalized Ricci recurrent manifolds have been studied by several authors such as Ozgur (2007), Arslan *et al.* (2009), Mallick *et al.* (2013) and many others. Devi and Singh (2015) also studied a type of m-projective curvature tensor on Kenmotsu manifolds.

In 2009, Jaiswal and Ojha studied on generalized ϕ recurrent LP-Sasakian manifolds. Shukla and Shukla (2009) studied ϕ-Ricci symmetric Kenmotsu manifolds. Prasad (2009) studied certain classes of almost contact Riemannian manifolds. The notion of generalized ϕ-recurrency to Sasakian manifolds and Lorentzian α-Sasakian manifolds are respectively studied by Patil *et al.* (2009) and Prakasha and Yildiz (2010). Venkatesha and Bagewadi (2010) studied the pseudo-projective ϕ-recurrent Kenmotsu manifold and showed the pseudo-projective ϕ-recurrent Kenmotsu manifold is an Einstein manifold and also a space of constant curvature. By extending the notion of generalized ϕ-recurrency, Shaikh and Hui (2011), introduced the notion of extended generalized ϕ-recurrent manifolds. Prakasha (2013) considered the extended generalized ϕ-recurrent in Sasakian manifold. Further Shaikh *et al.* (2013) studied this notion for LP-Sasakian manifolds. Singh (2014b) studied generalized recurrent and generalized concircularly recurrent P-Sasakian manifolds. Prasad (2000) introduced the notion of semi-generalized recurrent manifold and obtained some interesting results. Jaiswal and Ojha (2009) studied generalized ϕ-recurrent and generalized concircular ϕ-recurrent LP-Sasakian manifolds. In 2013, Debnath and Bhattacharya also studied the generalized ϕ-recurrent trans-Sasakian manifolds. Singh (2016) studied generalized Sasakian space forms with m-projective curvature tensor. In 2018, Singh and Lalmalsawma studied and examined generalized pseudo projectively recurrent manifolds.

Takahashi (1977) introduced the notion of ϕ-symmetric Sasakian manifold and obtained some interesting properties. De and Kamilya (1994) studied the generalized concircular recurrent manifolds. The notion of Lorentzian Para Sasakian manifold was introduced by Matsumoto (1989). Mihai and Rosca (1992) also introduced the same notion independently and they obtained several results on this manifold. Lorentzian Para-Sasakian manifolds had also been studied by Matsumoto and Mihai (1988), Mihai *et al.* (1999a, 1999b), De *et al.* (1999), Shaikh and De (2000), Shaikh and Biswas (2004), Venkatesha and Bagewadi (2008), Perktas and Tripathi (2010), Taleshian and Asghari (2010), Venkatesha *et al.* (2011), and Singh (2013, 2015) obtained some results on Lorentzian Para-Sasakian manifolds.

Miyazawa and Yamaguchi (1966) studied and proved some theorems on K-contact metric manifolds and Sasakian manifolds. Mishra (1982) studied on K-contact Riemannian and Sasakian manifolds and also studied some of the consequences and application of these. Tripathi and Dwivedi (2008) studied projective curvature tensor in K-contact and Sasakian manifolds and they proved that

(i) if a K-contact manifold is quasi-projectively flat then it is Einstein and

(ii) a K-contact manifold is ξ-projectively flat if and only if it is Einstein Sasakian.

In 2000, Chaki and Maity introduced the notion of quasi Einstein manifold. The study of quasi Einstein manifolds was continued by several authors such as Chaki (2001), Guha (2003), Ghosh *et al.* (2006), Ozgur and Sular (2008), De and De (2008), Ozgur (2008) and many others. Quasi Einstein manifolds have been generalized by several authors in different ways such as super quasi Einstein manifolds by Chaki (2004), generalized quasi-Einstein manifolds by De and Ghosh (2004), $N(k)$-quasi Einstein manifolds by Tripathi and Kim (2007), nearly quasi-Einstein manifolds by De and Gazi (2008) and others. De and Ghosh (2004) gives some example of a quasi Eintein manifold $(QE)_n$ and also proved that the existence of $(QE)_n$ manifolds. De and Ghosh (2004) also studied a type of Riemannian manifold called generalized quasi Einstein manifolds. De and Mallick (2011) studied and proved the existence of a generalized quasi Einstein manifold by non-trivial examples. Recently, Shaikh, Kim and Hui (2011) studied Lorentzian quasi Einstein manifolds. De *et al.* (2014) studied some geometric properties of generalized quasi Einstein manifolds and constructed two non-trivial examples for proving the existence of a generalized quasi Einstein manifold. De and Mallick (2016) studied and discussed generalized quasi Einstein manifolds with space matter tensor and some properties related to it.

In 1982, Hamilton introduced the concept of Ricci flow geometric evolution equation in which one starts with a smooth n-dimensional Riemannian manifold. Nagaraja and Premalatha (2012) studied Ricci solitons in Kenmotsu manifolds. They also studied quasi conformal, conharmonic and projective curvature tensors in a Kenmotsu manifold admitting Ricci solitons and proved the conditions for the Ricci solitons to be shrinking, steady and expanding. Ashok *et al.* (2013) studied Ricci solitons in α-Sasakian manifolds and showed that it is a shrinking or expanding soliton and the manifold is Einstein with killing vector field. Adigond and Bagewadi (2017) studied Ricci solitons in Para-Kenmotsu manifolds when the weyl-conformal curvature tensor satisfied some geometric properties like flatness, semi-symmetry, pseudo-symmetry, Ricci pseudo-symmetry and Einstein semi-symmetry.

1.25 Applications

Differential Geometry has wide scope of functioning. Recurrent Geometry is a mathematical discipline that uses the techniques of differentiable manifold, differential calculus and integral calculus as well as linear algebra and multilinear algebra to study problems in geometry. Riemannian Geometry is a special geometry associated with differentiable manifolds and has many applications to several branches of mathematics. The theory of plane and space, curves and surfaces in the 3-dimensional Euclidean space formed the basis for development of differential geometry during the 18^{th} century and the 19^{th} century.

Since the late 19^{th} century, differential geometry has grown into a field concerned more generally with the geometric structures on differentiable manifolds. Differential geometry is closely related to differential topology and the geometric aspects of the theory of differential equations. The differential geometry of surfaces captures many of the key ideas and techniques endemic to this field.

The importance of Differential Geometry may be seen from Einstein's general theory of relativity. According to the theory, the Universe is a smooth manifold equipped with pseudo-Riemannian metric, which describes the curvature space-time, which is essential for the positioning of the satellites into orbit around the earth. It is also indispensable in the study of gravitational Lansing and black holes.

The Differentiable manifolds and their geometry are very useful in studying different areas of mathematics including Lie group theory, local and global differential geometry, homogeneous spaces, electromagnetism, probability theory, differential equations, algebraic geometry, classical mechanics, relativity theory and the theory of elementary particles of physics. It is also useful in studying computer graphics and computer aided designs, digital signal processing and econometrics.

Differential Geometry has many applications in Chemistry, Biophysics, Structural Geology, Engineering and Physics. In Engineering, it can be applied to solve problems in digital signal processing. In Economics, Differential Geometry has applications to the field of Econometrics. Nowadays, it is very useful in Bioinformatics as well.

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Chapter 2

Quarter Symmetric Metric Connection in Sasakian Manifolds

In this chapter, we studied an Einstein manifold admitting a Ricci quarter symmetric metric connection in Sasakian manifolds and obtained some geometrical properties.

2.1 Introduction

Let (M^n, g) be a Riemannian manifold of dimension n and let ∇ be the Levi-Civita connection of (M^n, g). A Riemannian manifold is called locally symmetric if $\nabla K = 0$, where K is the Riemannian curvature tensor of (M^n, g) and U is a vector field.

A linear connection ∇ in a Riemannian manifold M^n is said to be Ricci quarter symmetric connection if the torsion tensor T satisfies (1.12.2)(Mishra and Pandey, 1980), where η is a 1-form and L is the $(1,1)$ Ricci tensor defined in (1.15.4), S is the Ricci tensor of M^n and X, Y are vector fields.

A linear connection ∇ is called a metric connection if it satisfies (1.11.2).

If D is the Riemannian connection of the manifold (M^n, g). Then the Ricci quarter

[1]***Science and Technology Journals,* 3(1)**, 63-67(2015).

symmetric metric connection ∇ is given by (Mishra and Pandey, 1980)

$$\nabla_X Y = D_X LY + \eta(Y)LX - S(X,Y)\rho, \tag{2.1.1}$$

where $\eta(X) = g(X, \rho)$.

This chapter is organized as follows:

After the introduction, section 2.2 explains preliminaries and the relation between curvature tensors of a Ricci quarter symmetric metric connection ∇ and Riemannian connection D. In section 2.3, we obtained an equivalent relation between the locally symmetric, conharmonically symmetric and m-projectively symmetric manifolds. In section 2.4, we have studied an equivalency relation between the locally bi-symmetric, conharmonically bi-symmetric and m-projectively bi-symmetric manifolds. In section 2.5, we have shown that a generalized conharmonically 2-recurrent Einstein manifold admitting a Ricci quarter symmetric metric connection is conharmonically flat and a generalized conharmonically 2-recurrent Einstein manifold admitting a Ricci quarter symmetric metric connection is m-projectively flat.

In section 2.6, we obtained an equivalent relation between the locally symmetric, conharmonically symmetric and concircularly symmetric manifolds. Finally, we have shown that a generalized concircularly 2-recurrent Einstein manifold equipped with Ricci quarter symmetric metric connection is concircularly flat.

2.2 Preliminaries

Let $\overline{K}$ and K be the curvature tensors of the connection ∇ and D respectively. Then it can be shown that (Mishra and Pandey, 1980)

$$\begin{aligned} \overline{K}(X,Y)Z &= K(X,Y)Z - M(Y,Z)LX + M(X,Z)LY \\ &- S(Y,Z)QX + S(X,Z)QY \\ &+ \eta(Z)[(D_X L)Y - (D_Y L)X] \\ &- [(D_X S)(Y,Z) - (D_Y S)(X,Z)]\rho, \end{aligned} \tag{2.2.1}$$

where M is a tensor field of type (0,2) defined by

$$\begin{aligned} M(X,Y) = g(QX,Y) &= (D_X\eta)Y - \eta(Y)\eta(LX) \\ &+ \frac{1}{2}\eta(\rho)S(X,Y), \end{aligned} \tag{2.2.2}$$

and Q is a tensor field of type $(1,1)$ defined by

$$QX = D_X\rho - \eta(LX)\rho + \frac{1}{2}\eta(\rho)LX. \tag{2.2.3}$$

Here, we shall consider M^n to be an Einstein manifold given in (1.15.7) where r is the scalar curvature of the manifold.

Considering (2.2.1),(1.15.7) and (1.15.4), we get

$$\begin{aligned} \overline{K}(X,Y)Z &= K(X,Y)Z - \frac{r}{n}\Big\{M(Y,Z)X - M(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\Big\}. \end{aligned} \tag{2.2.4}$$

Contracting (2.2.4) with respect to X, we get

$$\overline{S}(Y,Z) = \frac{r}{n}\Big[g(Y,Z) - \{(n-2)M(Y,Z) + m\, g(Y,Z)\}\Big], \tag{2.2.5}$$

where $\overline{S}$ is the Ricci tensor of ∇ and m is the trace of M^n. Now, putting $Y = Z = e_i$, where $\{e_i\,; i = 1, 2, 3, ..., n\}$ is an orthonormal basis of the tangent space at any point, we get by taking the sum for $1 \leq i \leq n$ in the relation (2.2.5)

$$\overline{r} = \frac{r}{n}\Big\{n - 2(n-1)m\Big\}, \tag{2.2.6}$$

where $\overline{r}$ is the scalar curvature of ∇.

2.3 Conharmonic and m-Projective Curvature Tensors of ∇

The conharmonic curvature tensors (Ishii, 1957) and m-projective curvature tensors (Pokhariyal and Mishra, 1971) on a Riemannian manifold are defined in equation (1.16.1) and (1.16.4) respectively.

Let $'\overline{C}^*$ be the conharmonic curvature tensor of the connection ∇. Then from (1.16.1), we have

$$\begin{aligned} '\overline{C}^*(X,Y,Z,U) &= {}'\overline{K}(X,Y,Z,U) \\ &- \frac{1}{n-2}\Big\{g(Y,Z)\overline{S}(X,U) - \overline{S}(X,Z)g(Y,U) \\ &+ \overline{S}(Y,Z)g(X,U) - g(X,Z)\overline{S}(Y,U)\Big\}. \end{aligned} \tag{2.3.1}$$

Using (2.2.4) and (2.2.5) in (2.3.1), we obtain

$$\begin{aligned} '\overline{C}^*(X,Y,Z,U) &= {}'K(X,Y,Z,U) \\ &+ \frac{2r(m-1)}{n(n-2)}\Big\{g(Y,Z)g(X,U) \\ &- g(X,Z)g(Y,U)\Big\}. \end{aligned} \tag{2.3.2}$$

From the above, we get

$$\begin{aligned} '\overline{C}^*(X,Y)Z &= K(X,Y)Z \\ &+ \frac{2r(m-1)}{n(n-2)}\Big\{g(Y,Z)X - g(X,Z)Y\Big\}. \end{aligned} \tag{2.3.3}$$

Again, let $'\overline{W}^*$ be the m-projective curvature tensor of the connection ∇.
Then from (1.16.4), we have

$$\begin{aligned} '\overline{W}^*(X,Y,Z,U) &= {}'\overline{K}(X,Y,Z,U) - \frac{1}{2(n-1)}\Big\{g(Y,Z)\overline{Ric}(X,U) \\ &- g(Y,U)\overline{S}(X,Z) + g(X,U)\overline{S}(Y,Z) \\ &- g(X,Z)\overline{Ric}(Y,U)\Big\}. \end{aligned} \tag{2.3.4}$$

Applying (2.2.5), (2.2.6) in (2.3.4), we have

$$\begin{aligned} '\overline{W}^*(X,Y,Z,U) &= K(X,Y,Z,U) \\ &+ \frac{r(r-m-1)}{n(n-1)}\Big\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\Big\}. \end{aligned} \tag{2.3.5}$$

From which we get

$$\begin{aligned} '\overline{W}^*(X,Y)Z &= K(X,Y)Z \\ &+ \frac{r(r-m-1)}{n(n-1)}\Big\{g(Y,Z)X - g(X,Z)Y\Big\}. \end{aligned} \tag{2.3.6}$$

Taking covariant derivative of (2.3.3) and (2.3.6) respectively, we get

$$(D_U{'\overline{C}^*})(X,Y)Z = (D_U K)(X,Y)Z, \tag{2.3.7}$$

and

$$(D_U{'\overline{W}^*})(X,Y)Z = (D_U K)(X,Y)Z. \tag{2.3.8}$$

Hence we can state the following:

Theorem 2.3.1 *In an Einstein manifold (M^n, g) equipped with Ricci quarter symmetric metric connection, the following conditions are equivalent*

a) M^n is locally symmetric.

b) M^n is conharmonically symmetric.

c) M^n is m-projectively symmetric.

2.4 Conharmonic bi-Symmetric and m-Projective bi-Symmetric Manifolds

Definition 2.4.1 *An Einstein manifold is said to be bi-symmetric if it satisfies*

$$(D_V D_U K)(X,Y)Z = 0. \tag{2.4.1}$$

Definition 2.4.2 *An Einstein manifold is said to be conharmonic bi-symmetric if it satisfies*

$$(D_V D_U C)(X, Y)Z = 0. \tag{2.4.2}$$

Definition 2.4.3 *An Einstein manifold is said to be m-projective bi-symmetric if it satisfies*

$$(D_V D_U W^*)(X, Y)Z = 0. \tag{2.4.3}$$

Taking the covariant differentiation on both sides of (2.3.7) and (2.3.8), we obtain

$$(D_V D_U \overline{C}(X, Y)Z = (D_V D_U K)(X, Y)Z, \tag{2.4.4}$$

and

$$(D_V D_U \overline{W}^*)(X, Y)Z = (D_V D_U K)(X, Y)Z. \tag{2.4.5}$$

Thus we can state:

Theorem 2.4.1 *In an Einstein manifold (M^n, g) equipped with Ricci quarter symmetric metric connection, the following conditions are equivalent*
a) M^n is bi-symmetric.
b) M^n is conharmonically bi-symmetric.
c) M^n is m-projectively bi-symmetric.

2.5 Generalized 2-Recurrent Riemannian Manifolds

A non-flat Riemannian manifold of dimension n is called generalized 2-recurrent Riemannian manifold (De and Pathak, 2003) when the Riemannian curvature tensor K satisfies the condition

$$\begin{aligned}(D_V D_U K)(X, Y)Z &= A(V)(D_U K)(X, Y)Z \\ &+ B(U, V)K(X, Y)Z,\end{aligned} \tag{2.5.1}$$

where A is a 1-form, B is a non-zero $(0, 2)$ tensor. The tensor B is defined by

$$B(X, Y) = g(X, Q^l Y), \tag{2.5.2}$$

where Q^l is a linear transformation from the tangent space at

$$(p \in \ M^n) : T_p(M^n) \to T_p(M^n).$$

When the conharmonic curvature tensor satisfy the condition:

$$\begin{aligned}(D_V D_U \overline{C}^*)(X, Y)Z &= A(V)(D_U \overline{C}^*)(X, Y)Z \\ &+ B(U, V)\overline{C}^*(X, Y)Z, \end{aligned} \tag{2.5.3}$$

then the manifold is called generalized conharmonically 2-recurrent manifold.
And when the m-projective curvature tensor satisfy the condition

$$\begin{aligned}(D_V D_U \overline{W}^*)(X, Y)Z &= A(V)(D_U \overline{W}^*)(X, Y)Z \\ &+ B(U, V)\overline{W}^*(X, Y)Z, \end{aligned} \tag{2.5.4}$$

then the manifold is called generalized m-projectively 2-recurrent manifold, where A, B are stated earlier.
Using Bianchi's second identity given in (1.14.7), we find from (2.3.7) that

$$\begin{aligned}(D_U \overline{C})(X, Y)Z + (D_Y \overline{C})(U, X)Z &+ (D_X \overline{C})(Y, U)Z) \\ &= 0. \end{aligned} \tag{2.5.5}$$

Again from (2.5.5), we find that

$$\begin{aligned}(D_V D_U \overline{C})(X, Y)Z &+ (D_V D_Y \overline{C})(U, X)Z \\ &+ (D_V D_X \overline{C}(Y, U)Z) = 0. \end{aligned} \tag{2.5.6}$$

In consequence of (2.5.3) and (2.5.5), the equation (2.5.6) yields

$$\begin{aligned} B(U,V)\overline{C}(X,Y)Z &+ B(Y,V)\overline{C}(U,X)Z \\ &+ B(X,V)\overline{C}(Y,U)Z = 0. \end{aligned} \tag{2.5.7}$$

Now, contracting (2.5.7), we get

$$B(\overline{C}(X,Y)Z, V) = 0. \tag{2.5.8}$$

From (2.3.2), we get

$$\begin{aligned} '\overline{C}(X,Y,Z,W) &= -'\overline{C}(X,Y,W,Z) \\ &= -'\overline{C}(Y,X,Z,W) \\ &= -'\overline{C}(Z,W,X,Y). \end{aligned} \tag{2.5.9}$$

Now, putting $U = Q^l V$ and using (2.5.2), the expression (2.5.7) takes the form

$$\begin{aligned} g(Q^lV, Q^lV)\overline{C}(X,Y)Z &+ g(X, Q^lV)\overline{C}^*(Y, Q^lV)Z \\ &+ g(Y, Q^lV)\overline{C}(Q^lV, X)Z = 0. \end{aligned} \tag{2.5.10}$$

Using (2.5.8) and (2.5.9) in (2.5.10), we have

$$g(Q^lV, Q^lV)\overline{C}(X,Y)Z = 0. \tag{2.5.11}$$

From which we obtain

$$\overline{C}(X,Y)Z = 0. \tag{2.5.12}$$

Thus we can state:

Theorem 2.5.1 *A generalized conharmonically* 2*-recurrent Einstein manifold equipped with Ricci quarter symmetric metric connection is conharmonically flat.*

Next, we assume that the manifold be generalized m-projectively 2-recurrent. Then it follows from (2.3.8) and Bianchi's second identity given in (1.14.7) that

$$\begin{aligned}(D_U\overline{W}^*)(X,Y)Z &+ (D_Y\overline{W}^*)(U,X)Z \\ &+ (D_X\overline{W}^*)(Y,U)Z = 0. \qquad (2.5.13)\end{aligned}$$

After covariant differentiation of (2.5.13) that

$$\begin{aligned}(D_V D_U\overline{W}^*)(X,Y)Z &+ (D_V D_Y\overline{W}^*)(U,X)Z \\ &+ (D_V D_X\overline{W}^*)(Y,U)Z = 0. \qquad (2.5.14)\end{aligned}$$

Using (2.5.4) and (2.5.13) in (2.5.14), we get

$$\begin{aligned}B(U,V)\overline{W}^*(X,Y)Z &+ B(Y,V)\overline{W}^*(U,X)Z \\ &+ B(X,V)\overline{W}^*(Y,U)Z = 0. \qquad (2.5.15)\end{aligned}$$

Contracting (2.5.15), we get

$$B(\overline{W}^*(X,Y)Z,V) = 0. \qquad (2.5.16)$$

From (2.3.4), we get

$$\begin{aligned}\overline{W}^*(X,Y,Z,W) &= -\overline{W}^*(X,Y,W,Z) \\ &= -\overline{W}^*(Y,X,Z,W) \\ &= -\overline{W}^*(Z,W,X,Y). \qquad (2.5.17)\end{aligned}$$

Now, putting $U = Q^l V$ and using (2.5.2), the expression (2.5.17) takes the form

$$\begin{aligned}g(Q^lV,Q^lV)\overline{W}^*(X,Y)Z &+ g(X,Q^lV)\overline{W}^*(Y,Q^lV)Z \\ &+ g(Y,Q^lV)\overline{W}^*(Q^lV,X)Z = 0. \qquad (2.5.18)\end{aligned}$$

Using (2.5.16) and (2.5.17) in (2.5.18), we get

$$g(Q^l V, Q^l V)\overline{W}^*(X,Y)Z = 0. \tag{2.5.19}$$

From which we obtain

$$\overline{W}^*(X,Y)Z = 0. \tag{2.5.20}$$

Hence we can state that:

Theorem 2.5.2 *An Einstein manifold equipped with a Ricci quarter symmetric metric connection is a generalized m-projectively 2-recurrent if and only if it is an m-projectively flat.*

2.6 Conharmonic and Concircular Tensors of ∇

The concircular curvature tensors on Riemannian manifold is defined in equation (1.16.2).

Let $'\overline{V}$ denote the concircular curvature tensor of the connection ∇. Then,

$$\begin{aligned} '\overline{V}(X,Y,Z,U) &= {}'\overline{K}(X,Y,Z,U) \\ &- \frac{\overline{r}}{n(n-1)}\Big\{g(Y,Z)g(X,U) \\ &- g(X,Z)g(Y,U)\Big\}. \end{aligned} \tag{2.6.1}$$

Applying (2.2.4), (2.2.5) in (2.6.1), we have

$$\begin{aligned} '\overline{V}(X,Y,Z,U) &= {}'K(X,Y,Z,U) \\ &- \frac{r(2r+n-2nm+2m)}{n^2(n-1)}\Big\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\Big\}. \end{aligned} \tag{2.6.2}$$

From (2.6.2), we have

$$\begin{aligned} \overline{V}(X,Y)Z &= K(X,Y)Z \\ &- \frac{r(2r+n-2nm+2m)}{n^2(n-1)}\Big\{g(Y,Z)X - g(X,Z)Y\Big\}. \end{aligned} \tag{2.6.3}$$

Taking covariant derivative of (2.6.3), we obtain

$$(D_U\overline{V}(X,Y)Z = (D_U K)(X,Y)Z. \tag{2.6.4}$$

Thus from (2.3.7) and (2.6.4), we can state the following:

Theorem 2.6.1 *In an Einstein manifold (M^n, g) equipped with Ricci quarter symmetric metric connection, then the following conditions are equivalent*

a) M^n is locally symmetric.

b) M^n is conharmonically symmetric.

c) M^n is concircularly symmetric.

An Einstein manifold will be called conharmonic and concircular bi-symmetric manifold if it satisfies

$$(D_V D_U V)(X,Y)Z = 0. \tag{2.6.5}$$

Taking the covariant differentiation on both sides of (2.6.5) and we get

$$(D_V D_U V)(X,Y)Z = (D_V D_U K)(X,Y)Z. \tag{2.6.6}$$

Hence from (2.4.4) and (2.6.6), we conclude the following:

Theorem 2.6.2 *In an Einstein manifold (M^n, g) equipped with Ricci quarter symmetric metric connection, the following conditions are equivalent*

a) M^n is bi-symmetric.

b) M^n is conharmonically bi-symmetric.

c) M^n is concircularly bi-symmetric.

A non-flat Riemannian manifold of dimension n is called generalized 2-recurrent Riemannian manifold (De and Pathak, 2003) when the concircular curvature tensor V satisfies the condition

$$
\begin{aligned}
(D_U D_W \overline{V})(X,Y)Z &= A(U)(D_W\overline{V})(X,Y)Z \\
&+ B(W,U)\overline{V}(X,Y)Z, \qquad (2.6.7)
\end{aligned}
$$

where A and B are stated earlier. Assume that the manifold be generalized concircularly 2-recurrent. Then, it follows from (2.6.3) and Bianchi's identity given in (1.14.7) that

$$
\begin{aligned}
(D_W\overline{V})(X,Y)Z &+ (D_Y\overline{V})(W,X)Z \\
&+ (D_X\overline{V})(Y,W)Z = 0. \qquad (2.6.8)
\end{aligned}
$$

After covariant differentiation, we have

$$
\begin{aligned}
(D_U D_W\overline{V})(X,Y)Z &+ (D_U D_Y\overline{V})(W,X)Z \\
&+ (D_U D_X\overline{V})(Y,W)Z = 0. \qquad (2.6.9)
\end{aligned}
$$

Using (2.6.7) and (2.6.8) in (2.6.9), we get

$$
\begin{aligned}
B(W,U)\overline{V}(X,Y)Z &+ B(Y,U)\overline{V}(W,X)Z \\
&+ B(X,U)\overline{V}(Y,W)Z = 0. \qquad (2.6.10)
\end{aligned}
$$

Contracting (2.6.10), we have

$$
B(\overline{V}(X,Y)Z,U) = 0. \qquad (2.6.11)
$$

From (2.6.1), we have

$$
\begin{aligned}
'\overline{V}(X,Y,Z,W) &= -'\overline{V}(X,Y,W,Z) \\
&= -'\overline{V}(Y,X,Z,W) \\
&= -'\overline{V}(Z,W,X,Y). \qquad (2.6.12)
\end{aligned}
$$

Now, putting $W = Q^l U$ and using (2.5.2) and the expression (2.6.12) takes the form

$$\begin{aligned} g(Q^lU, Q^lU)\overline{V}(X,Y)Z &+ g(X, Q^lU)\overline{V}(Y, Q^lU)Z \\ &+ g(Y, Q^lU)\overline{V}(Q^lU, X)Z = 0. \end{aligned} \tag{2.6.13}$$

Using (2.6.11) and (2.6.12) in (2.6.13), we have

$$g(Q^lU, Q^lU)\overline{V}(X,Y)Z = 0. \tag{2.6.14}$$

From which it follows that

$$\overline{V}(X,Y)Z = 0. \tag{2.6.15}$$

Hence we can state:

Theorem 2.6.3 *A generalized concircular 2-recurrent Einstein manifold equipped with Ricci quarter symmetric metric connection is concircularly flat.*

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Chapter 3

W_1 Flat Weakly Ricci-Symmetric Manifolds

In this chapter, we have studied curvatute tensor W_1. We concentrated on weakly W_1 symmetric manifolds and W_1 flat weakly Ricci-symmetric manifolds and obtained some interesting results.

3.1 Introduction

A Riemannian or a semi-Riemannian manifold (M^n, g), where $(n \geq 2)$, is said to be weakly symmetric manifold if its curvature tensor K of type (0, 4) satisfies the relation

$$\begin{aligned}(\nabla_X K)(Y, Z, U, V) &= A(X)K(Y, Z, U, V) + B(Y)K(X, Z, U, V) \\ &+ C(Z)K(Y, X, U, V) + D(U)K(Y, Z, X, V) \\ &+ E(V)K(Y, Z, U, X), \qquad (3.1.1)\end{aligned}$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where A, B, C, D, E are 1-form (not similtaneously zero) and are called the associated 1-forms of the manifold. ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g and an n-dimensional manifold of this kind is denoted by $(WS)_n$.

Tamassy and Binh (1988) studied weakly symmetric Sasakian manifolds and also proved that such a manifold does not always exist. De and Bandyopadhyay (1999) established the existence of $(WS)_n$ with an example and also proved that in $(WS)_n$, the associated 1-forms $B = C$ and $D = E$. So the condition (3.1.1) of $(WS)_n$ reduces to

$$\begin{aligned}(\nabla_X K)(Y,Z,U,V) &= A(X)K(Y,Z,U,V) + B(Y)K(X,Z,U,V)\\ &+ B(Z)K(Y,X,U,V) + D(U)K(Y,Z,X,V)\\ &+ D(V)K(Y,Z,U,X). \qquad (3.1.2)\end{aligned}$$

Some authors like De and Bandyopadhyay (2000), Shaikh and Baishya (2005) studied and extended this notion for conformal curvature tensor and quasi conformal curvature tensor respectively. In 2008, Malek and Samawaki also studied weakly symmetric Riemannian manifolds.

The curvature tensor W_1 is defined in (1.16.5) (Pokhariyal and Mishra, 1971), where K is curvature tensor and S is Ricci-tensor.

A non-flat W_1 Riemannian manifold (M^n, g), $(n > 2)$ is said to be weakly W_1 symmetric manifold if the W_1 curvature tensor of type $(0, 4)$ satisfies the following condition:

$$\begin{aligned}(\nabla_X W_1)(Y,Z,U,V) &= A(X)W_1(Y,Z,U,V) + B(Y)W_1(X,Z,U,V)\\ &+ C(Z)W_1(Y,X,U,V) + D(U)W_1(Y,Z,X,V)\\ &+ E(V)W_1(Y,Z,U,X), \qquad (3.1.3)\end{aligned}$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where A, B, C, D, E are already defined as before and an n-dimensional manifold is denoted as $(WW_1S)_n$. If $B = C$, $D = E$ and hence the above equation can be reduced to the form

$$\begin{aligned}(\nabla_X W_1)(Y,Z,U,V) &= A(X)W_1(Y,Z,U,V) + B(Y)W_1(X,Z,U,V)\\ &+ B(Z)W_1(Y,X,U,V) + D(U)W_1(Y,Z,X,V)\\ &+ D(V)W_1(Y,Z,U,X), \qquad (3.1.4)\end{aligned}$$

for arbitrary vector fields $X, Y, Z, U, V \in \chi(M^n)$ and A, B, D are non-vanishing 1-forms. This chapter is structured as follows:

Section 3.2 is concerned with preliminaries of W_1 curvature tensor and $(WW_1S)_n$. In section 3.3, we study and investigate the nature of the scalar curvature of a $(WW_1S)_n$. We proved that if in a $(WW_1S)_n$ the Ricci tensor S is of Codazzi type or constant scalar curvature, then $-r$ is an eigenvalue corresponding to the eigenvector ξ defined by $g(X,\xi) = \lambda(X)$, for all X.

The last Section devoted to the study of W_1 flat $(WW_1S)_n$ proved that W_1 flat $(WRS)_n$ $(n > 2)$ is a quasi Einstein manifold.

3.2 Preliminaries

In this section, we obtained some formulas which will be required for the study of $(WW_1S)_n$. Let us consider $\{e_i\}$, $i = 1, 2, 3, ..., n$ be an orthonormal basis of the tangent at any point of the manifold. Then from equation (1.16.5), we have

$$\sum_{i=1}^{n} W_1(e_i, Y, Z, e_i) = 2S(Y,Z), \qquad (3.2.1)$$

and

$$r = \sum_{i=1}^{n} S(X,Y) = \sum_{i=1}^{n} g(Le_i, e_i),$$

where L is the Ricci operator which is given in (1.15.4).

$$\sum_{i=1}^{n} W_1(X, Y, e_i, e_i) = 0 = \sum_{i=1}^{n} W_1(e_i, e_i, Z, U), \tag{3.2.2}$$

Proposition 3.2.1 *In a Riemannian manifold* (M^n, g) $(n > 2)$*, the* W_1 *curvature tensor satisfies the relations:*

$$\begin{aligned} (i)\ \ & W_1(X, Y, Z, U) + W_1(Y, Z, X, U) + W_1(Z, X, Y, U) = 0, \\ (ii)\ \ & W_1(X, Y, U, Z) + W_1(Y, Z, U, X) + W_1(Z, X, U, X) = 0. \end{aligned} \tag{3.2.3}$$

Proposition 3.2.2 *The defining condition of* $(WW_1S)_n$ *can always be expressed in the form of equation* (3.1.4)

Proof: Using (3.1.3), interchanging Y and Z, we get

$$\begin{aligned} (\nabla_X W_1)(Z, Y, U, V) &= A(X)W_1(Z, Y, U, V) + B(Z)W_1(X, Y, U, V) \\ &+ C(Y)W_1(Z, X, U, V) + D(U)W_1(Z, Y, X, V) \\ &+ E(V)W_1(Z, Y, U, X). \end{aligned} \tag{3.2.4}$$

Adding equation (3.1.3) and equation (3.2.4) and then using skew-symmetric property given in (1.14.4) of W_1, we get

$$\{B(Y) - C(Y)\}W_1(X, Z, U, V) + \{B(Z) - C(Z)\}W_1(X, Y, U, V) = 0,$$

which can be written as

$$\mu(Y)W_1(X, Z, U, V) + \mu(Z)W_1(X, Y, U, V) = 0, \tag{3.2.5}$$

where $\mu(X) = B(X) - C(X)$, for all $X \in \chi(M^n)$.

Now we choose a particular vector fields ρ such that $\mu(\rho) \neq 0$. Putting $Y = Z = \rho$ in equation (3.2.5), we have

$$\mu(\rho)W_1(X, \rho, U, V) + \mu(\rho)W_1(X, \rho, U, V) = 0,$$

which implies that

$$W_1(X, \rho, U, V) = 0.$$

Again, putting $Z = \rho$ in (3.2.5), we get

$$W_1(X, Y, U, V) = 0,$$

for all $X, Y, U, V \in \chi(M^n)$, which is contradicts because in our assumption the manifold is not W_1 flat. Hence we will have $\mu(X) = 0$, for all $X \in \chi(M^n)$ and $B = C$. In the same manner, by interchanging U and V in the equation (3.1.3) and proceeding as above, we have the relation $D = E$. Thus all the associated 1-forms A, B, C, D and E coincide, because $B = C$ and $D = E$. Therefore equation (3.1.3) can be written as equation (3.1.4).

3.3 The Nature of the Scalar Curvature of $(WW_1S)_n$

Let L be the symmetric endomorphism of the tangent (bundle) space at any point of the manifold corresponding to the Ricci tensor S given in (1.15.4), for all $X, Y \in \chi(M^n)$.

Theorem 3.3.1 *The Ricci tensor S in a Riemannian manifold (M^n, g), $(n > 2)$ is Codazzi type if and only if the relation* (3.3.3) *holds.*

Proof: From (1.16.5), it follows by virtue of Bianchi's identity given in (1.14.7) that

$$\begin{aligned}
(\nabla_X W_1)(Y, Z, U, V) &+ (\nabla_Y W_1)(Z, X, U, V) + (\nabla_Z W_1)(X, Y, U, V) \\
&= (\nabla_X R)(Y, Z, U, V) + (\nabla_Y R)(Z, X, U, V) + (\nabla_Z R)(X, Y, U, V) \\
&+ \frac{1}{(n-1)}\Big[\{(\nabla_X S)(Z, U) - (\nabla_Z S)(X, U)\}\, g(Y, V) \\
&+ \{(\nabla_Y S)(X, U) - (\nabla_X S)(Y, U)\}\, g(Z, V) \\
&+ \{(\nabla_Z S)(Y, U) - (\nabla_Y S)(Z, U)\}\, g(X, V)\Big],
\end{aligned}$$

which is equivalent to

$$
\begin{aligned}
(\nabla_X W_1)(Y,Z,U,V) &+ (\nabla_Y W_1)(Z,X,U,V) + (\nabla_Z W_1)(X,Y,U,V) \\
&= \frac{1}{(n-1)}\Big[\{(\nabla_X S)(Z,U) - (\nabla_Z S)(X,U)\}\, g(Y,V) \\
&+ \{(\nabla_Y S)(X,U) - (\nabla_X S)(Y,U)\}\, g(Z,V) \\
&+ \{(\nabla_Z S)(Y,U) - (\nabla_Y S)(Z,U)\}\, g(X,V)\Big]. \qquad (3.3.1)
\end{aligned}
$$

If the Ricci tensor is of Codazzi type (Ferus, 1981), i.e.

$$(\nabla_X S)(Z,U) = (\nabla_Z S)(X,U), \qquad (3.3.2)$$

then using (3.3.2) in (3.3.1), we get

$$(\nabla_X W_1)(Y,Z,U,V) + (\nabla_Y W_1)(Z,X,U,V) + (\nabla_Z W_1)(X,Y,U,V) = 0. \qquad (3.3.3)$$

Conversely suppose that in a Riemannian manifold (3.3.3) holds, then (3.3.1) becomes

$$
\begin{aligned}
&\frac{1}{(n-1)}\Big[\{(\nabla_X S)(Z,U) - (\nabla_X S)(Z,U)\}g(Y,V) + \{(\nabla_Y S)(X,U) \\
&-(\nabla_X S)(Y,U)\}g(Z,V) + \{(\nabla_Z S)(Y,U) - (\nabla_Y S)(Z,U)\}g(X,V)\Big] = 0. \qquad (3.3.4)
\end{aligned}
$$

Putting $Y = V = e_i$ in equation (3.3.4) and then taking summation over i, $1 \leq i \leq n$, we get

$$(n-2)\big[(\nabla_X S)(Z,U) - (\nabla_Z S)(X,U)\big] = 0,$$

which implies that

$$(\nabla_X S)(Z,U) - (\nabla_Z S)(X,U) = 0, \qquad (3.3.5)$$

i.e.,

$$(\nabla_X S)(Z,U) = (\nabla_Z S)(X,U).$$

Then the above equation (3.3.5) shows that the Ricci tensor is of Codazzi type given in (3.3.2).

Now, we suppose that the Ricci tensor is of Codazzi type, so using equation (3.1.4), (3.2.3) and (3.3.1), we have

$$\begin{aligned}\{A(X)-2B(X)\}W_1(Y,Z,U,V) &+ \{A(Y)-2B(Y)\}W_1(Z,X,U,V)\\ &+ \{A(Z)-2B(Z)\}W_1(X,Y,U,V)=0,\end{aligned}$$

which is equivalent to

$$\nu(X)W_1(Y,Z,U,V)+\nu(Y)W_1(Z,X,U,V)+\nu(Z)W_1(X,Y,U,V)=0, \tag{3.3.6}$$

where $\nu(X)=A(X)-2B(X)$ for all $X\in\chi(M^n)$.
Setting $Y=V=e_i$ in (3.3.6) and then taking summation over i, $1\leq i\leq n$ and then using (3.2.1), we get

$$2\big[\nu(X)S(Z,U)+\nu(Z)S(X,U)\big]+\nu(W_1(Z,X,U))=0. \tag{3.3.7}$$

Again, putting $X=U=e_i$ in (3.3.7) and taking summation over i, $1\leq i\leq n$ and then using (3.2.2), we get

$$\nu(LZ)=-r\nu(Z), \tag{3.3.8}$$

which gives

$$S(Z,\xi)=-rg(Z,\xi). \tag{3.3.9}$$

Hence from the above, we can state:

Theorem 3.3.2 *If in a $(WW_1S)_n$ the Ricci tensor is of Codazzi type, $-r$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ξ defined by* (1.17.7) *for all X.*

Now, again setting $Y = V = e_i$ in equation (3.1.4) and then taking summation over i, $1 \leq 1 \leq n$, we have

$$\begin{aligned} 2\left(\nabla_X S\right)(Z,U) &= 2\big[A(X)S(Z,U) + B(Z)S(X,U) + D(U)S(Z,X)\big] \\ &+ \{B(W_1(X,Z)U) + D(W_1(Z,U)X)\}. \end{aligned} \tag{3.3.10}$$

Let ρ_1, ρ_2, ρ_3 be the vector fields associated to the 1-forms A, B and D respectively. Therefore

$$i.e.\ \ A(X) = g(X,\rho_1), \qquad B(X) = g(X,\rho_2), \qquad D(X) = g(X,\rho_3).$$

Putting $Z = U = e_i$ in equation (3.3.10) and then taking summation over i, $1 \leq 1 \leq n$, we get

$$S(X,\rho_2) + S(X,\rho_3) = -rg(X,\rho_1). \tag{3.3.11}$$

From equation (3.3.11),we have

$$S(X,\widehat{\rho}) = -rg(X,\rho_1), \tag{3.3.12}$$

where $\widehat{\rho} = \rho_2 + \rho_3$ and $g(X,\rho) = T(X) = B(X) + D(X)$.

From equation (3.3.12), it is clear that $-r$ is an eigenvalue of S corresponding to the eigenvector $\widehat{\rho}$.

Thus we can state:

Theorem 3.3.3 *The Ricci tensor S in $(WW_1S)_n$ has eigenvalue $-r$ corresponding to the eigenvector $\widehat{\rho}$.*

If the scalar curvature r of $(WW_1S)_n$ is zero, then equation (3.3.11) will be $S(X,\widehat{\rho}) = 0$ and hence using (1.16.5), we obtain

$$W_1(X,Y,\widehat{\rho},U) = R(X,Y,\widehat{\rho},U). \tag{3.3.13}$$

Also, if equation (3.3.13) holds in $(WW_1S)_n$, then by virtue of (3.3.12) it follows from

(1.16.5) that $r = 0$ for $T(X) \neq 0$ for all $X \in \chi(M^n)$.

Thus we have the following corollary

Corollary 3.3.1 *If the scalar curvature of a* $(WW_1S)_n$ *vanishes, then we have the relation* $W_1(X, Y, \widehat{\rho}, U) = R(X, Y, \widehat{\mu}, U)$.

3.4 W_1 Flat Weakly Ricci-Symmetric Manifolds

A non-flat Riemannian manifold (M^n, g), $(n > 2)$ is called weakly Ricci symmetric if its Ricci tensor of type $(0, 2)$ is not identically zero and satisfies the condition

$$(\nabla_X K)(Y, Z) = A(X)K(Y, Z) + B(Y)K(X, Z) + D(Z)K(Y, X), \tag{3.4.1}$$

where A, B, D are 1-form and ∇ is the operator of covariant differentiation with respect to g. Such an n-dimensional manifold is denoted by $(WRS)_n$.

Proposition 3.4.1 *In a* $(WRS)_n$ *with* $\rho(X) \neq 0$*, the scalar curvature cannot be zero and the Ricci tensor will be of the form* $S(X, Y) = rH(X)H(Y)$*, where the vector field* ρ *associated with the* 1*-form* H *is a unit vector field.*

proof: From the above equation (3.4.1)

$$\begin{aligned}(\nabla_X S)(Y, Z) - (\nabla_X S)(Z, Y) &= \{B(Y) - D(Y)\}S(X, Z) \\ &+ \{D(Z) - B(Z)\}S(X, Y).\end{aligned} \tag{3.4.2}$$

Since S is symmetric, then (3.4.2) will be

$$\{B(Y) - D(Y)\}S(X, Z) = \{B(Z) - D(Z)\}S(X, Y). \tag{3.4.3}$$

Let $\mu(X) = B(X) - D(X)$ for any vector field X. Then equation (3.4.3) can be written as

$$\mu(Y)S(X,Z) = \mu(Z)S(X,Y). \tag{3.4.4}$$

Now, Let $\{e_i\}$, $i = 1, 2, ..., n$, be an orthonormal basis of the tangent space at any point of the manifold. Setting $X = Z = e_i$ in (3.4.4) and then taking summation over i, $1 \leq i \leq n$, we get

$$\mu(Y)S(e_i, e_i) = \mu(e_i)S(e_i, Y),$$

which implies that

$$r\mu(Y) = \mu(LY), \tag{3.4.5}$$

where $\mu(X) = g(X, \xi)$ given in (1.17.7) for any vector field X and r is the scalar curvature. From equation (3.4.4), we have

$$\mu(\delta)S(X,Z) = \mu(Z)S(X,\delta) = \mu(Z)\mu(LY). \tag{3.4.6}$$

Using (3.4.5) and (3.4.6), we have

$$S(X,Z) = r\,\frac{\mu(X)\mu(Z)}{\mu(\delta)},$$

$$S(X,Z) = rH(X)H(Z), \tag{3.4.7}$$

where $H(X) = \frac{\mu(X)}{\sqrt{\mu(\delta)}}$ and $g(X,\rho) = H(X)$, ρ is a unit vector field. Now, from (3.4.7), if $r = 0$, then $S(X,Z) = 0$ which is contradicts or inadmissible by the definition of $(WRS)_n$. So, $r \neq 0$.

Proposition 3.4.2 *In a $(WRS)_n$ with $\mu(X) \neq 0$, r is an eigenvalue of the Ricci tensor corresponding to the eigenvector δ.*

Proof: From equation (3.4.5), we have

$$rg(Y,\delta) = S(Y,\delta),$$

that means r is an eigenvalue of the Ricci tensor corresponding to the eigenvector δ.

Theorem 3.4.1 *In a W_1 flat $(WRS)_n$, $(n>2)$ with $\mu(X)\neq 0$, the vector field ρ defined by $g(X,\rho) = H(X)$ is not a proper concircular vector field.*

Proof: Differentiating (1.16.5) covariantly with respect to U, we have

$$\begin{aligned}(\nabla_U W_1)(X,Y)Z &= (\nabla_U R)(X,Y)Z \\ &+ \frac{1}{(n-1)}\Big\{(\nabla_U S)(Y,Z)X - (\nabla_U S)(X,Z)Y\Big\}.\end{aligned} \tag{3.4.8}$$

Now, contracting (3.4.8) with respect to U, we get

$$\begin{aligned}(div\ \ W_1)(X,Y)Z &= (div\ \ R)(X,Y)Z \\ &+ \frac{1}{(n-1)}\Big\{(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\Big\}.\end{aligned} \tag{3.4.9}$$

We know that a Riemannian manifold

$$(div\ \ R)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z). \tag{3.4.10}$$

Using equation (3.4.10), (3.4.9) will be

$$\begin{aligned}(div\ \ W_1)(X,Y)Z &= \{(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\} \\ &+ \frac{1}{(n-1)}\Big\{(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\Big\}.\end{aligned} \tag{3.4.11}$$

Since the manifold W_1 is flat i.e. $(div\ \ W_1) = 0$ and hence equation (3.4.11) gives

$$\begin{aligned}(div\ \ W_1)(X,Y)Z &= \{(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\} \\ &+ \frac{1}{(n-1)}\Big\{(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\Big\} \\ &= 0.\end{aligned} \tag{3.4.12}$$

Now, (3.4.7) implies

$$\begin{aligned}(\nabla_Y S)(X,Z) &= dr(Y)H(X)H(Z) \\ &+ r\{(\nabla_Y H)(X)H(Z) + (\nabla_Y H)(Z)H(X)\}.\end{aligned} \tag{3.4.13}$$

In view of (3.4.13), (3.4.12) becomes

$$\begin{aligned}dr(X)H(Y)H(Z) - dr(Y)H(X)H(Z) + r[(\nabla_X H)(Y)H(Z) \\ +(\nabla_X H)(Z)H(Y) - (\nabla_Y H)(X)H(Z) - (\nabla_Y H)(Z)H(X)] = 0.\end{aligned} \tag{3.4.14}$$

Putting $Y = Z = e_i$ in the above equation and then taking summation over i, $1 \leq i \leq n$, we obtain

$$dr(X) - dr(\rho)H(X) - r\{(\nabla_\rho H)(X) + H(X)\sum_{i=1}^{n}(\nabla_{e_i} H)(e_i)\} = 0,$$

which is equivalent to

$$dr(\rho)H(X) + r\{(\nabla_\rho H)(X) + H(X)\sum_{i=1}^{n}(\nabla_{e_i} H)(e_i)\} = dr(X). \tag{3.4.15}$$

Now, putting $Y = Z = \rho$ in (3.4.14), we get

$$r(\nabla_\rho H)(X) = dr(X) - dr(\rho)H(X). \tag{3.4.16}$$

By virtue of (3.4.16), (3.4.15) reduces to the form

$$dr(\rho)H(X)dr(X) - dr(\rho)H(X) + rH(X)\sum_{i=1}^{n}(\nabla_{e_i}H)(e_i) = dr(X),$$

which gives

$$rH(X)\sum_{i=1}^{n}(\nabla_{e_i}H)(e_i) = 0. \tag{3.4.17}$$

Again, putting $X = \rho$ in equation (3.4.17), we have

$$r\sum_{i=1}^{n}(\nabla_{e_i}H)(e_i) = 0. \tag{3.4.18}$$

From equation (3.4.17) and (3.4.18), we have

$$(\nabla_{e_i}H)(e_i) = 0. \tag{3.4.19}$$

which shows that the vector field ρ defined by $g(X,\rho) = H(X)$ is not a proper concircular vector field when a W_1 flat $(WRS)_n$, $(n > 2)$ with $\mu(X) \neq 0$.

Theorem 3.4.2 *A W_1 flat $(WRS)_n$, $(n > 2)$ a quasi Einstein manifold.*

Proof: Let us consider a W_1 flat $(WRS)_n$ manifold, then equation (1.16.5) gives

$$R(X,Y,Z,U) = -\frac{1}{(n-1)}\Big\{S(Y,Z)g(X,U) - S(X,Z)g(Y,U)\Big\}. \tag{3.4.20}$$

Using (3.4.7) in (3.4.20), we have

$$R(X,Y,Z,U) = \frac{r}{(n-1)}\Big\{H(X)H(Z)g(Y,U) - H(Y)H(Z)g(X,U)\Big\}. \tag{3.4.21}$$

Substituting $X = U = e_i$ in (3.4.21) and then taking summation over i, $1 \leq i \leq n$, we obtain

$$S(Y,Z) = \frac{r}{(n-1)}\,g(Y,Z) + \frac{nr}{1-n}\,H(Y)H(Z), \tag{3.4.22}$$

which implies that

$$S(Y,Z) = ag(Y,Z) + bH(Y)H(Z), \tag{3.4.23}$$

where $a = \frac{r}{(n-1)}$ and $b = \frac{rn}{(1-n)}$,

which shows that the manifold is quasi Einstein.

Hence the proof is complete.

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Chapter 4

ϕ-Recurrent and Generalized Recurrent Curvature Tensor in LP-Sasakian Manifolds

This Chapter deals with the study of ϕ-symmetric LP-Sasakian manifolds and ϕ-Ricci symmetric LP-Sasakian manifolds. Here, we show that ϕ-symmetric LP-Sasakian manifold and ϕ-Ricci symmetric LP-Sasakian manifold is an Einstein manifold. We also constructed an example of 3-dimensional ϕ-Ricci symmetric LP-Sasakian manifolds.

In this chapter, we also study m-projective ϕ-recurrent LP-Sasakian manifolds. Here we show that m-projective ϕ-recurrent LP-Sasakian manifold is an η-Einstein manifold.

4.1 Introduction

In 1989, K. Matsumoto introduced the notion of Lorentzian para-contact manifolds. The properties of Lorentzian para contact manifolds and their different classes, viz. Lorentzian Para-Sasakian (LP-Sasakian) studied by several authors. Also K. Matsumoto (1989), Taleshian and Asghari (2011) studied an LP-Sasakian manifolds. ϕ-Ricci symmetric Kenmotsu manifolds and ϕ-symmetric Para-Sasakian manifolds are also studied by A. A. Shukla and M. K.

Shukla (2009). This chapter is clarified as follows:

After introduction, Section 4.2 is equipped with some prerequisites about Lorentzian Para-Sasakian manifolds. In section 4.3, we examined ϕ-symmetric LP-Sasakian manifolds. In section 4.4, 3-dimensional locally ϕ-symmetric LP-Sasakian manifolds are studied. In section 4.5, we study ϕ-Ricci symmetric LP-Sasakian manifolds. 3-dimensional ϕ-Ricci symmetric LP-Sasakian manifolds have studied in section 4.6. In the next section, we have constructed an example of 3-dimensional LP-Sasakian manifold which supports the results obtained in section 4.5 and section 4.6.

In the last section, we studied m-projective ϕ-recurrent LP-Sasakian manifolds (M^n, g) and proved some interesting results.

4.2 Preliminaries

A differential manifold of dimension n is called Lorentzian Para-Sasakian (LP-Sasakian) manifold (Matsumoto (1989), Mihai and Rosca (1992)) if it admits a $(1, 1)$ tensor field ϕ, a unit time like contravariant vector field ξ, a 1-form η and a Lorentzian metric g satisfying the equation (1.20.1) - (1.20.6) and

$$(\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta X)\eta Y)\xi, \tag{4.2.1}$$

where X, Y and ∇ are defined earlier.

Again, if we put

$$\Omega(X, Y) = g(X, \phi Y), \tag{4.2.2}$$

for any vector fields X, Y, then, the tensor field $\Omega(X, Y)$ is a symmetric $(0, 2)$ tensor field. Since the vector field η is closed in an LP-Sasakian manifold, we have

$$(\nabla_X \eta)(Y) = \Omega(X, Y), \qquad \Omega(X, \xi) = 0, \tag{4.2.3}$$

for any vector fields X, Y.

Let M^n be an n-dimensional LP-Sasakian manifold with structure (ϕ, ξ, η, g). Then the following relations hold (Matsumoto and Mihai, 1988) equation (1.20.10), (1.20.11) and

$$K(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{4.2.4}$$

$$K(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{4.2.5}$$

$$K(X,\xi)\xi = -X - \eta(X)\xi, \tag{4.2.6}$$

$$Q\xi = (n-1)\xi, \tag{4.2.7}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{4.2.8}$$

$$(\nabla_W K)(X,Y)\xi = \Omega(Y,W)X - \Omega(X,W)Y - K(X,Y)\phi W, \tag{4.2.9}$$

$$(\nabla_W K)(X,\xi)Y = \Omega(W,Z)X - \Omega(X,Z)\phi W - K(X,\phi W)Z, \tag{4.2.10}$$

for any vector fields X, Y, Z, where K is the curvature tensor and S is the Ricci tensor of the manifold.

4.3 ϕ-Symmetric LP-Sasakian Manifolds

Definition 4.3.1 *An LP-Sasakian manifold M^n is said to be locally ϕ-symmetric if (Venkatesha and Bagewadi, 2006)*

$$\phi^2((\nabla_W K)(X,Y)Z) = 0, \tag{4.3.1}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 4.3.2 *An LP-Sasakian manifold M^n is said to be ϕ-symmetric if (Venkatesha and Bagewadi, 2006)*

$$\phi^2((\nabla_W K)(X,Y)Z) = 0, \tag{4.3.2}$$

for arbitrary vector fields X, Y, Z, W on M^n.

Theorem 4.3.1 *A ϕ-symmetric LP-Sasakian manifold is an Einstein manifold.*

Let us assume that the manifold is ϕ-symmetric. Then using (4.3.2) and (1.20.1), we have

$$(\nabla_W K)(X,Y)Z + \eta((\nabla_W K)(X,Y)Z)\xi = 0. \tag{4.3.3}$$

Taking inner product of (4.3.3) with U, we get

$$g((\nabla_W K)(X,Y)Z, U) + \eta((\nabla_W K)(X,Y)Z)g(\xi, U) = 0. \tag{4.3.4}$$

Let $\{e_i\}$, $i = 1, 2, 3, ..., n$ be an orthogonal basis of the tangent space at any point of the manifold. Then by putting $X = U = e_i$ in (4.3.4) and taking summation over i, $1 \leq i \leq n$, we get

$$(\nabla_W S)(Y,Z) + \sum_{i=1}^{n} \eta((\nabla_W K)(e_i,Y)Z)g(\xi, e_i). \tag{4.3.5}$$

Replacing Z by ξ in (4.3.5), we have

$$(\nabla_W S)(Y,\xi) + \sum_{i=1}^{n} \eta((\nabla_W K)(e_i,Y)\xi)\eta(e_i). \tag{4.3.6}$$

The second term of (4.3.6), takes the form

$$\begin{aligned} \eta((\nabla_W K)(e_i,Y)\xi) &= g((\nabla_W K)(e_i,Y)\xi, \xi) \\ &= g(\nabla_W K(e_i,Y)\xi, \xi) + g(K(\nabla_W e_i, Y)\xi, \xi) \\ &- g(K(e_i, \nabla_W Y)\xi, \xi) - g(K(e_i,Y)\nabla_W \xi, \xi). \end{aligned} \tag{4.3.7}$$

Since $\{e_i\}$ is an orthonormal basis, $\nabla_W e_i = 0$ at p. Also using (1.20.10), we have

$$g(K(e_i, \nabla_W Y)\xi, \xi) = 0. \tag{4.3.8}$$

Putting (4.3.8) in (4.3.7), we have

$$\begin{aligned} \eta((\nabla_W K)(e_i, Y)\xi) &= g((\nabla_W K)(e_i, Y)\xi, \xi) \\ &= g(\nabla_W K(e_i, Y)\xi, \xi) - g(K(e_i, Y)\nabla_W \xi, \xi). \end{aligned} \tag{4.3.9}$$

Now, since $g(K(e_i, Y)\xi, \xi) = -g(K(\xi, \xi)Y, e_i) = 0$, we have

$$g(\nabla_W K(e_i, Y)\xi, \xi) + g(K(e_i, Y)\xi, \nabla_W \xi) = 0. \tag{4.3.10}$$

Putting (4.3.10) in (4.3.9), we get

$$g((\nabla_W K)(e_i, Y)\xi, \xi) = -g(K(e_i, Y)\xi, \nabla_W \xi) - g(K(e_i, Y)\nabla_W \xi, \xi).$$

Using (1.20.6) in the above equation, we get

$$g(\nabla_W K)(e_i, Y)\xi, \xi) = -g(K(e_i, Y)\xi, \phi W) - g(K(e_i, Y)\phi W, \xi). \tag{4.3.11}$$

Using (4.2.3), (1.20.10) and (4.3.11), we obtain

$$g((\nabla_W K(e_i, Y)\xi, \xi) \;=\; -g(K(e_i, Y)\xi, \phi W) - g(K(e_i, Y)\phi W, \xi),$$

which implies that

$$g((\nabla_W K(e_i, Y)\xi, \xi) \;=\; 0. \tag{4.3.12}$$

The equations (4.3.12) and (4.3.6) imply that

$$(\nabla_W S)(Y, \xi) = 0, \tag{4.3.13}$$

which gives

$$\nabla_W (S(Y, \xi)) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi) = 0. \tag{4.3.14}$$

Using (1.20.6) and (1.20.10) in (4.3.14), we get

$$(n-1)(\nabla_W \eta(Y)) - (n-1)\eta(\nabla_W Y) - S(Y, \phi W) = 0.$$

Replacing Y by ϕY in the above, we get

$$(n-1)(\nabla_W \eta(\phi Y)) - (n-1)\eta(\nabla_W \phi Y) - S(\phi Y, \phi W) = 0,$$

which implies that

$$S(\phi Y, \phi W) = -(n-1)((\nabla_W \phi)Y).$$

Then,

$$S(\phi Y, \phi W) = -(n-1)g((\nabla_W \phi)Y, \xi). \tag{4.3.15}$$

In view of (4.2.1) and (4.2.7), the equation (4.3.15) yields

$$\begin{aligned} S(Y, W) &= (n-1)g(Y, W), \\ \Rightarrow S(Y, W) &= \lambda g(Y, W), \end{aligned}$$

where $\lambda = (n-1)$ and λ is constant,

which proves the theorem.

4.4 3-Dimensional Locally ϕ-Symmetric LP-Sasakian Manifolds

In a 3-dimensional Riemannian manifold, we have

$$\begin{aligned} K(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &- \frac{r}{2}\Big[g(Y,Z)X - g(X,Z)Y\Big], \end{aligned} \tag{4.4.1}$$

where Q is the Ricci operator i.e., $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold. Replacing Z by ξ in (4.4.1) and using (1.20.10), (4.2.6) $(for\ n = 3)$, we have

$$\eta(Y)QX - \eta(X)QY = \frac{(r+10)}{2}\Big[\eta(Y)X - \eta(X)Y\Big]. \tag{4.4.2}$$

Replacing Y by ξ in (4.4.2) and using (1.20.11) $(for\ n = 3)$, we get

$$QX = \frac{1}{2}\Big[(r+10)X + (r+6)\eta(X)\xi\Big], \tag{4.4.3}$$

and

$$S(X,Y) = \frac{1}{2}\Big[(r+10)g(X,Y) + (r+6)\eta(X)\eta(Y)\Big]. \tag{4.4.4}$$

Using (4.4.4) and (4.4.3) in (4.4.1), we get the curvature tensor of 3-dimensional LP-Sasakian manifold as

$$\begin{aligned} K(X,Y)Z &= \frac{(r+20)}{2}\Big[g(Y,Z)X - g(X,Z)Y\Big] - \frac{(r+6)}{2}\Big[g(Y,Z)\eta(X)\xi \\ &+ g(X,Z)\eta(Y)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y\Big]. \end{aligned} \tag{4.4.5}$$

Theorem 4.4.1 *The necessary and sufficient condition of a 3-dimensional LP-Sasakian manifold to be locally ϕ-symmetric if the scalar curvature r is constant.*

The curvature tensor of 3-dimensional LP-Sasakian manifold is of the form

$$\begin{aligned} K(X,Y)Z &= \frac{(r+20)}{2}\Big[g(Y,Z)X - g(X,Z)Y\Big] - \frac{(r+6)}{2}\Big[g(Y,Z)\eta(X)\xi \\ &+ g(X,Z)\eta(Y)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y\Big]. \end{aligned}$$

Taking covariant differentiation of (4.4.5) with respect to W, we get

$$\begin{aligned} (\nabla_W K)(X,Y)Z &= \frac{dr(W)}{2}\Big[g(Y,Z)X - g(X,Z)Y\Big] - \frac{dr(W)}{2}\Big[g(Y,Z)\eta(X)\xi \\ &+ g(X,Z)\eta(Y)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y\Big] \\ &- \frac{(r+6)}{2}\Big[g(Y,Z)(\nabla_W\eta)(X)\xi + g(X,Z)(\nabla_W\eta)(Y)\xi \\ &- (\nabla_W\eta)(Y)\eta(Z)X - \eta(Y)(\nabla_W\eta)(Z\eta) \\ &+ (\nabla_W\eta)(X)\eta(Z)Y + \eta(X)(\nabla_W\eta)(Z)Y\Big]. \end{aligned} \tag{4.4.6}$$

Now, applying both sides of (4.4.6) and using (1.20.1) and (1.20.5), we obtain

$$\begin{aligned} \phi^2(\nabla_W K)(X,Y)Z &= \frac{dr(W)}{2}\Big[g(Y,Z)X + g(Y,Z)\eta(X)\xi - g(X,Z)Y \\ &- g(X,Z)\eta(Y)\xi - \eta(X)\eta(Z)Y - \eta(X)\eta(Y)\eta(Z)\xi \\ &+ \eta(Y)\eta(Z)X + \eta(Y)\eta(Z)\eta(X)\xi\Big] \\ &- \frac{(r+6)}{2}\Big[g(Y,Z)\eta(X)(\nabla_W\xi) + g(X,Z)\eta(Y)(\nabla_W\xi) \\ &- (\nabla_W\eta)(Y)\eta(Z)X - (\nabla_W\eta)(Y)\eta(Z)\eta(X)\xi \\ &- \eta(Y)(\nabla_W\eta)(Z)X - \eta(Y)(\nabla_W\eta)(Z)\eta(X)\xi \\ &+ (\nabla_W\eta)(X)\eta(Z)Y + (\nabla_W\eta)(X)\eta(Z)\eta(Y)\xi \\ &+ \eta(X)(\nabla_W\eta)(Z)Y + \eta(X)\eta(Y)(\nabla_W\eta)(Z)\xi\Big]. \end{aligned} \tag{4.4.7}$$

Now, taking X, Y, Z orthogonal to ξ, the equation (4.4.7) gives

$$\phi^2(\nabla_W K)(X,Y)Z = \frac{dr(W)}{2}\Big[g(Y,Z)X - g(X,Z)Y\Big]. \tag{4.4.8}$$

This proof follows from (4.4.8).

4.5 ϕ-Ricci Symmetric LP-Sasakian Manifolds

Definition 4.5.1 *An LP-Sasakian manifold M^n is said to be ϕ-Ricci symmetric if the Ricci operator satisfies*

$$\phi^2((\nabla_X Q)(Y)) = 0,$$

for all vector fields X and Y on M^n and $S(X,Y) = g(QX,Y)$.

If X and Y are orthogonal to ξ, then the manifold is said to be locally ϕ-Ricci symmetric (Shukla and Shukla, 2009).

Theorem 4.5.1 *The neccessary and sufficient condition of an n-dimensional LP-Sasakian manifold is to be ϕ-Ricci symmetric if the manifold is an Einstein manifold.*

Let us assume that the manifold is ϕ-Ricci symmetric. Then we have

$$\phi^2(\nabla_X Q)(Y) = 0, \tag{4.5.1}$$

Using (1.20.1) in above, we get

$$(\nabla_X Q)(Y) + \eta((\nabla_X Q)(Y))\xi = 0. \tag{4.5.2}$$

Taking inner product of (4.5.2) with Z, we have

$$g((\nabla_X Q)(Y), Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0, \tag{4.5.3}$$

which on simplifying gives

$$g(\nabla_X Q(Y), Z) - S(\nabla_X Y, Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0. \tag{4.5.4}$$

Replacing Y by ξ in (4.5.4), we get

$$g(\nabla_X Q(\xi), Z) - S(\nabla_X X\xi, Z) + \eta((\nabla_X Q)(\xi))\eta(Z) = 0. \tag{4.5.5}$$

By using (1.20.6), (1.20.11) and (4.2.7) in (4.5.5), we obtain

$$(n-1)g(\phi X, Z) - S(\phi X, Z) + \eta((\nabla_X Q)(\xi))\eta(Z) = 0. \tag{4.5.6}$$

Replacing Z by ϕZ in (4.5.6), we have

$$S(\phi X, \phi Z) = (n-1)g(\phi X, \phi Z). \tag{4.5.7}$$

Using (1.20.6) and (4.2.8) in the above, we obtain

$$S(X, Z) = (n-1)g(X, Z), \tag{4.5.8}$$

which implies that

$$g(QX, Z) = \lambda g(X, Z),$$

where $S(X, Y) = g(QX, Y)$ and λ is a constant.

Hence $QX = \lambda X$.

Therefore, we have

$$\phi^2((\nabla_Y Q)(X)) = 0.$$

This completes the proof.

4.6 3-Dimensional ϕ-Ricci Symmetric LP-Sasakian Manifolds

Theorem 4.6.1 *The LP-Sasakian manifold is ϕ-Ricci symmetric if the scalar curvature r of a 3-dimensional LP-Sasakian manifold is equal to -6.*

The curvature tensor of a 3-dimensional LP-Sasakian manifold is of the form

$$\begin{aligned} K(X,Y)Z &= \frac{(r+20)}{2}\Big[g(Y,Z)X - g(X,Z)Y\Big] - \frac{(r+6)}{2}\Big[g(Y,Z)\eta(X)\xi \\ &+ g(X,Z)\eta(Y)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y\Big]. \end{aligned}$$

From the above, we obtain the Ricci tensor

$$S(X,Y) = \frac{(r+10)}{2}g(X,Y) + \frac{(r+6)}{2}\eta(X)\eta(Y),$$

which implies that

$$QX = \frac{(r+10)}{2}X + \frac{(r+6)}{2}\eta(X)\xi.$$

Taking covariant differentiation of the above equation with respect to W, we get

$$\begin{aligned} (\nabla_W Q)X &= \frac{1}{2}\Big[dr(W)X + dr(W)\eta(X)\xi + (r+6)g(X,\phi X)\xi \\ &+ (r+6)\eta(X)\nabla_W\xi\Big]. \end{aligned} \tag{4.6.1}$$

Now, applying ϕ^2 on both sides of (4.6.1) and using (1.20.1), we have

$$\begin{aligned} \phi^2((\nabla_W Q)(X)) &= \frac{1}{2}\Big[dr(W)X + dr(W)\eta(X)\xi \\ &+ (r+6)\eta(X)\phi^2(\nabla_W\xi)\Big]. \end{aligned} \tag{4.6.2}$$

The proof follows from (4.6.2).

Taking X orthogonal to ξ in (4.6.2), we obtain

$$\phi^2((\nabla_W Q)(X)) = \frac{1}{2}dr(W). \tag{4.6.3}$$

In view of (4.6.3), we have the following

Corollary 4.6.1 *A 3-dimensional LP-Sasakian manifold is locally ϕ-Ricci symetric if and only if scalar curvature r is constant.*

4.7 Example of 3-Dimensional ϕ-Ricci Symmetric LP-Sasakian Manifolds

In this section, we construct an example of ϕ-Ricci symmetric LP-Sasakian manifold which supports Theorem $4.6.1$.

Consider 3-dimensional manifold $M = \{(x, y, z) : (x, y, z) \in \mathbb{R}^3\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$ (Shaikh and De, 2000). The vector fields

$$e_1 = e^x \frac{\partial}{\partial y}, \qquad e_2 = e^x (\frac{\partial}{\partial y} + \frac{\partial}{\partial z}), \qquad e_3 = \frac{\partial}{\partial x},$$

which are linearly independent at each point of M. Let g be the Lorentzian metric defined by

$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0,$$
$$g(e_1, e_1) = g(e_2, e_2) = 1, \qquad g(e_3, e_3) = -1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \epsilon M$. Let ϕ be the $(1, 1)$ tensor field defined by $\phi e_1 = -e_1$, $\phi e_2 = -e_2$, $\phi e_3 = 0$.
Then using the linearity of ϕ and g, we have

$$\phi(e_3) = -1, \qquad \phi^2 X = X + \eta(X) e_3,$$
$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

for any vector fields Y_1, Y_2 on M. Thus for $e_3' = \xi$, (ϕ, ξ, η, g) defines a Lorentzian Para-contact structure on M.

Let ∇ be the Levi-Civita connection with respect to Lorentzian metric g and K be the curvature tensor of g. Then we have

$$[e_1, e_2] = 0, \qquad [e_1, e_3] = -e_1, \qquad [e_2, e_3] = -e_2.$$

Taking $e_3' = \xi$ and using Koszul formula for the Lorentzian metric g, we can easily calculate

$$\nabla_{e_1} e_3 = -e_1, \qquad \nabla_{e_1} e_2 = 0, \qquad \nabla_{e_1} e_1 = -e_3,$$

$$\nabla_{e_2} e_3 = -e_2, \qquad \nabla_{e_2} e_2 = -e_3, \qquad \nabla_{e_2} e_1 = 0,$$

$$\nabla_{e_3} e_3 = 0, \qquad \nabla_{e_3} e_2 = 0, \qquad \nabla_{e_3} e_1 = 0.$$

From the above it can be easily seen that (ϕ, ξ, η, g) is an LP-Sasakian structure on M. Consequently $M^3(\phi, \xi, \eta, g)$ is a 3-dimensional LP-Sasakian manifold.

Using the above relations, it can be easily seen that

$$R(e_1, e_2)e_3 = 0, \qquad R(e_2, e_3)e_3 = -e_2, \qquad R(e_1, e_3)e_3 = -e_1,$$

$$R(e_1, e_2)e_2 = -e_1, \qquad R(e_2, e_3)e_2 = -e_3, \qquad R(e_1, e_3)e_2 = 0,$$

$$R(e_1, e_2)e_1 = -e_2, \qquad R(e_2, e_3)e_1 = 0, \qquad R(e_1, e_3)e_1 = -e_3.$$

The definition of Ricci tensor in 3-dimensional manifold gives

$$S(X, Y) = \sum_{i=1}^{3} g(R(e_i, X)Y, e_i). \tag{4.7.1}$$

Using the components of the curvature tensor in (4.7.1), we get the following:

$$S(e_1, e_1) = -2, \qquad S(e_2, e_2) = -2, \qquad S(e_3, e_3) = -2,$$

$$S(e_1, e_2) = 0, \qquad S(e_1, e_3) = 0, \qquad S(e_2, e_3) = 0,$$

and the scalar curvature

$$r = -6.$$

From the above relation, we have the scalar curvature r of the manifold is equal to -6 and the Ricci tensor $S(X,Y) = 2g(X,Y)$. Hence $QX = 2X$ which implies that $\phi^2((\nabla_W Q)(X)) = 0$. Thus we observe that the scalar curvature of the manifold is -6 and it is ϕ-Ricci symmetric. So this example supports Theorem $4.6.1$ and also agrees with Theorem $4.5.1$, for 3-dimensional case.

4.8 m-Projective ϕ-Recurrent LP-Sasakian Manifolds

Definition 4.8.1 *An LP-Sasakian manifold M^n is said to be m-projective ϕ-recurrent manifold if there exists a non-zero 1- form A such that*

$$\phi^2((\nabla_U W^*)(X,Y)Z) = A(W)W^*(X,Y)Z, \tag{4.8.1}$$

for arbitrary vector fields X, Y, Z, U, where W^* is an m-projective curvature tensor given by

$$W^*(X,Y)Z = K(X,Y)Z - \frac{1}{2(n-1)}\Big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\Big], \tag{4.8.2}$$

where K is the curvature tensor and S is the Ricci tensor.

Let us consider an m-projective ϕ-recurrent LP-Sasakian manifold. Then by virtue of (1.20.4) and (1.20.11), we have

$$(\nabla_U W^*)(X,Y)Z + \eta((\nabla_U W^*)(X,Y)Z)\xi = A(U)W^*(X,Y)Z, \tag{4.8.3}$$

from which it follow that

$$g((\nabla_U W^*)(X,Y)Z, V) + \eta((\nabla_U W^*)(X,Y)Z)\xi = A(U)W^*(X,Y)Z, \tag{4.8.4}$$

Let $\{e_i\}$, $i = 1, 2, ..., n$, be an orthonormal basis of the tangent space at any point of the manifold. Then, putting $X = V = e_i$ in equation (4.8.4) and taking summation over i,

$1 \leq i \leq n$, we obtain

$$\frac{\nabla_U S(Y,Z)}{2} = \frac{1}{2(n-1)}\Big\{S(Y,Z)(nA(U)-1) - g(Y,Z)(rA(U)-r)\Big\}. \tag{4.8.5}$$

Now, substituting Z by ξ in (4.8.5) and using (1.20.4) and (1.20.11), we get

$$(\nabla_U S)(Y,\xi) = \frac{\eta(Y)}{(n-1)}\Big[A(U)(n^2-n-r) - (n-1-r)\Big]. \tag{4.8.6}$$

Now, we have the relation

$$(\nabla_U S)(Y,\xi) = \nabla_U S(Y,\xi) - S(\nabla_U Y,\xi) - S(Y,\nabla_U \xi),$$

and using equation (4.2.1), (4.2.4) and (4.2.8) in (4.8.6), we have

$$(\nabla_U S)(Y,\xi) = (n-1)g(U,\phi Y) - S(Y,\phi U). \tag{4.8.7}$$

Using (4.8.6) in (4.8.7), we get

$$\frac{\eta(Y)}{(n-1)}\Big[A(U)(n^2-n-r) - (n-1-r)\Big] = (n-1)g(U,\phi Y) - S(Y,\phi U). \tag{4.8.8}$$

Substituting Y by ϕY in (4.8.8) and using (1.20.3), (1.20.6), (1.20.10) and (4.2.3), we get

$$S(Y,U) = (n-1)g(U,Y) + 2(n-1)\eta(U)\eta(Y). \tag{4.8.9}$$

From the above equation, we can state

Theorem 4.8.1 *An m-projective ϕ-recurrent LP-Sasakian manifold (M^n,g) is an η-Einstein manifold.*

Now, from equation (4.8.3), we have

$$(\nabla_U W^*)(X,Y)Z = A(U)W^*(X,Y)Z - \eta((\nabla_U W^*)(X,Y)Z)\xi. \tag{4.8.10}$$

Using (4.8.2) in the above equation (4.8.10), we obtain

$$\begin{aligned} A(U)\eta(R(X,Y)Z) &= \frac{A(U)}{2(n-1)}\Big[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)) \\ &+ g(Y,Z)\eta(QX) - g(X,Z)\eta(QY)\Big]. \end{aligned} \tag{4.8.11}$$

From the above equation and using Bianchi's identity, we get

$$\begin{aligned} & A(U)\eta(R(X,Y)Z) + A(X)\eta(R(Y,U)Z) + A(Y)\eta(R(U,X)Z) \\ = & \frac{A(U)}{2(n-1)}\Big[S(Y,Z)\eta(X) - S(X,Z)\eta(Y) + g(Y,Z)\eta(QX) - g(X,Z)\eta(QY)\Big] \\ & + \frac{A(X)}{2(n-1)}\Big[S(U,Z)\eta(Y) - S(Y,Z)\eta(U) + g(U,Z)\eta(QY) - g(Y,Z)\eta(QU)\Big] \\ & + \frac{A(Y)}{2(n-1)}\Big[S(X,Z)\eta(U) - S(U,Z)\eta(X) + g(X,Z)\eta(QU) - g(U,Z)\eta(QX)\Big], \end{aligned} \tag{4.8.12}$$

using (4.2.9) in (4.8.12), we obtain

$$\begin{aligned} & A(U)\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\} + A(X)\{\eta(Y)g(U,Z) - \eta(U)g(Y,Z)\} \\ & + A(Y)\{\eta(U)g(X,Z) - \eta(X)g(U,Z)\} = \frac{A(U)}{2(n-1)}\Big[S(Y,Z)\eta(X) - S(X,Z)\eta(Y) \\ & + g(Y,Z)\eta(QX) - g(X,Z)\eta(QY)\Big] + \frac{A(X)}{2(n-1)}\Big[S(U,Z)\eta(Y) - S(Y,Z)\eta(U) \\ & + g(U,Z)\eta(QY) - g(Y,Z)\eta(QU)\Big] + \frac{A(Y)}{2(n-1)}\Big[S(X,Z)\eta(U) - S(U,Z)\eta(X) \\ & + g(X,Z)\eta(QU) - g(U,Z)\eta(QX)\Big]. \end{aligned} \tag{4.8.13}$$

Substituting $Y = Z = e_i$ in (4.8.13) and taking summation over i, $1 \leq i \leq n$, we get

$$\{A(X)\eta(U) - A(U)\eta(X)\}\{-\big(n^2 - 2n + 1\big) + r\} = A(QX)\eta(U) - A(QU)\eta(X), \tag{4.8.14}$$

for all vector fields X, U.

Replacing X by ξ in (4.8.14) and we obtain

$$A(U) = -\eta(\rho)\eta(U) - \Big\{\frac{A(Q\xi)\eta(U) + A(QU)}{(n^2 - 2n + 1) + r}\Big\}, \tag{4.8.15}$$

for all vector fields U, where $A(\xi) = g(\xi, \rho) = \eta(\rho)$, ρ is the vector field associated to the 1-form A.

i.e., $g(X, \rho) = A(X)$.

From (4.8.14) and (4.8.15), we have the following statement:

Theorem 4.8.2 *The vector field ρ associated to the 1-form A and the characteristic vector field ξ are opposite to each other in an m-projective ϕ-recurrent LP-Sasakian manifold (M^n, g) which is given by* (4.8.15)

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Chapter 5

On a Quasi Einstein K-Contact Manifolds

This chapter deals with the study of some properties of K-contact quasi Einstein manifolds. We obtain some conditions which satisfies semi-symmetric, Ricci symmetric and Ricci-recurrent. We also study and obtain some results on Ricci Solitons in K-contact quasi Einstein manifolds.

5.1 Introduction

A Riemannian or a semi-Riemannian manifold (M^n, g), where $n \geq 2$, is said to be an Einstein manifold (Avik *et al.*, 2014) if it satisfies (1.15.7) holds on M^n, where S is Ricci tensor and r is scalar curvature of (M^n, g). According to Besse (1987), (1.15.7) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry as well as in general theory of relativity.

A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is defined to be a quasi Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies (De and Ghosh, 2004)

$$S(X,Y) = ag(X,Y) + bA(X)A(Y), \tag{5.1.1}$$

where a, b are scalars, b $\neq 0$ and A is a non-zero 1-form such that

$$g(X,U) = A(X), \tag{5.1.2}$$

for all vector field X and U being a unit vector field. In this case, a, b are associated scalars. A is associated 1-form and a unit vector U is the generator of the manifold. If $b = 0$, then the manifold reduces to an Einstein manifold. An n-dimensional manifold of a quasi Einstein manifold is denoted by the symbol $(QE)_n$.

An emerging branch of modern mathematics is the geometry of contact manifold. The notion of contact geometry has evolved from the mathematical formalism of classical mechanics (Geiges, 2001). A contact manifold is a smooth n $(= 2m+1)$-dimensional C^∞ manifold M^n equipped with a global 1-form η called a contact form of M^n such that

$$\eta \wedge (d\eta)^n \neq 0,$$

everywhere on M^n. In particular, $\eta \wedge (d\eta)^n \neq 0$ is a volume element on M^n so that a contact manifold is orientable.

The two important classes of contact manifolds are K-contact manifolds and Sasakian manifolds (Blair, 1976). K-contact manifolds have been studied by several authors such as De and Biswas (2005), De and De (2017), De and Mandal (2016) and many others.

Let M^n be an n $(= 2m+1)$-dimensional contact metric manifold with contact metric stucture (ϕ, ξ, η, g) consisting of a $(1,1)$-tensor field ϕ, an associated vector field ξ, a 1-form η and associated Riemannian metric g satisfies equation (1.17.1) - (1.17.7) (Blair (1976), Sasaki (1965)) where X, Y are smooth vector fields on M^n. In addition, we have

$$(\phi X, Y) = -g(X, \phi Y). \tag{5.1.3}$$

If the characteristic vector field ξ is a killing vector field, then the contact metric structure on M^n is called a K-contact metric structure and the manifold M^n is called a K-contact metric manifold or K-contact Riemannian manifold or simply a K-contact manifold.
A vector field X on a Riemannian manifold M^n is called a killing vector field if and only if

$$\mathcal{L}_X g = 0,$$

where $\mathcal{L}$ denotes the operator of Lie differentiation.

This chapter is arranged as follows:
Section 5.2 is equipped with some prerequisites about K-contact manifolds. In section 5.3, we have proved that a conformally flat K-contact quasi Einstein manifold is of quasi-constant curvature. We study semi-symmetric K-contact quasi Einstein manifold which satisfies the condition $K(\xi, X)\xi$=0 in section 5.4.

Section 5.5 is devoted to study of Ricci semi-symmetric property. It also proved that Ricci semi-symmetric K-contact quasi Einstein manifold is an Einstein manifold. In section 5.6, we considered Ricci-recurrent K-contact quasi Eistein manifolds. In section 5.7, we have proved that K-contact Einstein manifold satisfying the curvature condition $C.S = 0$. In the last section, we study Ricci solitons in K-contact quasi Einstein manifolds and show that it could not be steady.

5.2 Preliminaries

From the above, in an $n\ (= 2m+1)$-dimensional K-contact manifold the following relations hold (Blair (1976), Sasaki (1965), Mandal (2016)):

$$g(K(\xi, X)Y, \xi) = \eta(K(\xi, X)Y) = g(X, Y) - \eta(X)\eta(Y), \tag{5.2.1}$$

$$K(\xi, X)\xi = -X + \eta(X)\xi, \tag{5.2.2}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{5.2.3}$$

$$(\nabla_X \phi)Y = K(\xi, X)Y, \tag{5.2.4}$$

and equation (1.17.14) for all any vector fields X and Y.

Now, using (5.1.1) and putting $Y = \xi$, we get

$$S(X,\xi) = (a+b)\eta(X). \tag{5.2.5}$$

From (5.2.5), we have

$$g(QX,\xi) = (a+b)g(X,\xi),$$

which implies that

$$QX = (a+b)X, \tag{5.2.6}$$

putting $X = \xi$ in (5.2.6), we have

$$Q\xi = (a+b)\xi. \tag{5.2.7}$$

Again, putting $X = Y = \xi$ in (5.1.1), we have

$$S(\xi,\xi) = ag(\xi,\xi) + bA(\xi)A(\xi),$$

which implies that

$$(n-1) = a+b, \tag{5.2.8}$$

and contracting (5.1.1) over X and Y, we get

$$r = na + b, \tag{5.2.9}$$

where r denotes the scalar curvature of the manifold M^n.

5.3 Conformally Flat K-Contact Quasi Einstein Manifolds

Definition 5.3.1 *Let M^n be an n-dimentional contact metric manifold with contact metric structure (ϕ, ξ, η, g). Then, the Weyl conformally curvature tensor C is defined by (De and Biswas, 2006)*

$$\begin{aligned} C(X,Y)Z &= K(X,Y)Z - \frac{1}{n-2}\Big[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\Big] \\ &+ \frac{r}{(n-1)(n-2)}\Big[g(Y,Z)X - g(X,Z)Y\Big], \end{aligned} \tag{5.3.1}$$

for $X, Y, Z \in T(M^n)$, where K is Riemannian curvature tensor, r is a scalar curvature and Q is the Ricci-operator of M^n.

For conformally flat manifold (De and Biswas, 2006), we have the relation

$$C(X,Y)Z = 0.$$

Then, using the above equation in (5.3.1), we get

$$\begin{aligned} K(X,Y)Z &= \frac{1}{(n-2)}\Big[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\Big] \\ &- \frac{r}{(n-1)(n-2)}\Big[g(Y,Z)X - g(X,Z)Y\Big]. \end{aligned} \tag{5.3.2}$$

Taking the inner product with W in (5.3.2), we have

$$\begin{aligned} \widehat{K}(X,Y,Z,W) &= \frac{1}{(n-2)}\Big[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &+ g(Y,Z)S(X,W) - g(X,Z)S(Y,W)\Big] \\ &- \frac{r}{(n-1)(n-2)}\Big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\Big]. \end{aligned} \tag{5.3.3}$$

Putting the value of (5.1.1) in the above equation, we get

$$\begin{aligned}\widehat{K}(X,Y,Z,W) &= \Big[\frac{2a}{(n-2)} - \frac{r}{(n-1)(n-2)}\Big]\Big[g(Y,Z)g(X,W) \\ &- g(X,Z)g(Y,W)\Big] - b\Big[\eta(Y)\eta(Z)g(X,W) \\ &- \eta(X)\eta(Z)g(Y,W) + \eta(X)\eta(W)g(Y,Z) \\ &- \eta(Y)\eta(W)g(X,Z)\Big]. \end{aligned} \tag{5.3.4}$$

Therefore, it is a quasi constant curvature.

From the above equation, we can state:

Theorem 5.3.1 *Conformally flat K-contact quasi Einstein manifold is of quasi constant curvature.*

5.4 Semi-Symmetric K-Contact Quasi Einstein Manifolds

Definition 5.4.1 *A K-contact quasi Einstein manifold is said to be semi-symmetric if it satisfies the condition (Tripathi and Kim, 2007):*

$$K(X,Y).K = 0, \tag{5.4.1}$$

where $K(X,Y)$ acts on K as a derivation and X, Y are vector fields.

From (5.4.1), we have equation (1.21.2).

Also we have

$$\begin{aligned} K(X,Y)K(Z,U)V &- K(K(X,Y)Z,U)V - K(Z,K(X,Y)U)V \\ &- K(Z,U)K(X,Y)V = 0. \end{aligned} \tag{5.4.2}$$

Putting $X = Z = V = \xi$ in the above equation and using (5.2.2), we get

$$2K(Y,U)\xi - \eta(U)Y + 2U\eta(Y) + K(\xi,U)Y - g(U,Y)\xi = 0, \tag{5.4.3}$$

putting $Y = \xi$ in (5.4.3), we obtain

$$2K(\xi, U)\xi - \eta(U)\xi + 2U\eta(\xi) + K(\xi, U)\xi - g(U, \xi)\xi = 0,$$

which implies that

$$K(\xi, U)\xi = 0. \tag{5.4.4}$$

From the above result, we can state:

Theorem 5.4.1 *In a semi-symmetric K-contact quasi Einstein manifold $K(\xi, X)\xi = 0$.*

5.5 K-Contact Quasi Einstein Manifold Satisfying Ricci Semi-Symmetric

Definition 5.5.1 *A K-contact quasi Einstein manifold is said to be Ricci-semi symmetric if it satisfies the condition (Tripathi and Kim, 2007):*

$$K(X, Y).S = 0, \tag{5.5.1}$$

where $K(X, Y)$ acts on K as a derivation and X, Y are vector fields.

Then from (5.5.1)

$$(K(X, Y).S)(Z, W) = 0, \tag{5.5.2}$$

it follows that

$$S(K(X, Y)Z, W) + S(Z, K(X, Y)W) = 0. \tag{5.5.3}$$

Putting $X = W = \xi$ in (5.5.3), we have

$$S(K(\xi, Y)Z, \xi) + S(Z, K(\xi, Y)\xi) = 0,$$

and using (5.2.2) and (5.2.1), we get

$$S(Z,Y) = (a+b)g(Z,Y), \tag{5.5.4}$$

which gives the following theorem;

Theorem 5.5.1 *Ricci semi-symmetric K-contact quasi Einstein manifold is Einstein manifold.*

5.6 Ricci-Recurrent K-Contact Quasi Einstein Manifolds

Definition 5.6.1 *A non-flat Riemannian manifold M^n is called a Ricci-recurrent K-contact quasi Einstein manifolds if its Ricci tensor S satisfies equation* (1.21.3)*, where ∇ is Levi-Civita connection of the Riemannian metric g and A is a 1-form and X, Y are vector fields on M^n.*

We have the relation

$$(\nabla_X S)(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \tag{5.6.1}$$

Using (1.21.3) and (5.6.1), we get

$$A(X)S(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \tag{5.6.2}$$

Setting $Y = Z = \xi$ in the (5.6.2), we get

$$A(X)S(\xi,\xi) = XS(\xi,\xi) - 2S(\nabla_X \xi, \xi). \tag{5.6.3}$$

Using (5.2.3) in (5.6.3), we have

$$(n-1)A(X) = (n-1)X, \tag{5.6.4}$$

where X is a vector field.

A Ricci-recurrent K-contact quasi Einstein manifold is Ricci-symmetric K-contact quasi

Einstein if and only if the 1-form A is zero. Thus we have the following theorem:

Theorem 5.6.1 *If M^n is a Ricci-recurrent K-contact quasi Einstein manifold, then $(n-1)A(X) = X(n-1)$ where X is a vector field.*

5.7 K-Contact Quasi Einstein Manifolds

Let us consider a K-contact quasi Einstein manifold (De and Mandal, 2016).

Using (5.1.1) and (5.2.6) in (5.3.1), we get

$$C(X,Y)Z = K(X,Y)Z - \left(\frac{2(a+b)(n-1)-r}{(n-1)(n-2)}\right)\Big[g(Y,Z)X - g(X,Z)Y\Big]. \quad (5.7.1)$$

Putting $Z = U$ in (5.7.1), we have

$$C(X,Y)U = K(X,Y)U - \left(\frac{2(a+b)(n-1)-r}{(n-1)(n-2)}\right)\Big[g(Y,U)X - g(X,U)Y\Big].$$

Taking the inner product with V, we have

$$\begin{aligned} g(C(X,Y)U,V) &= g(K(X,Y)U,V) \\ &- \left(\frac{2(a+b)(n-1)-r}{(n-1)(n-2)}\right)\Big[g(Y,Z)g(X,V) - g(X,Z)g(Y,V)\Big]. \end{aligned} \quad (5.7.2)$$

Interchanging U and V, we get

$$\begin{aligned} g(C(X,Y)V,U) &= g(K(X,Y)V,U) \\ &- \left(\frac{2(a+b)(n-1)-r}{(n-1)(n-2)}\right)[g(Y,Z)g(X,V) - g(X,Z)g(Y,U)]. \end{aligned} \quad (5.7.3)$$

Adding (5.7.2) and (5.7.3), we obtain

$$g(C(X,Y)U,V) + g(C(X,Y)V,U) = 0. \quad (5.7.4)$$

Now, we have the relation

$$(C(X,Y).S)(U,V) = -S(C(X,Y)U,V) - S(U,C(X,Y)V). \tag{5.7.5}$$

Using (5.1.1) in (5.7.5), we get

$$(C(X,Y).S)(U,V) = 0. \tag{5.7.6}$$

Theorem 5.7.1 *In K-contact quasi Einstein manifold the Weyl curvature tensor satisfies the condition $C(X,Y).S = 0$.*

5.8 Ricci Solitons in K-Contact Quasi Einstein Manifolds

A Ricci soliton (g, V, λ) is a generalization of an Einstein metric and is defined on a Riemannian manifold (M^n, g) is given by equation (1.22.1) (Hamilton (1988), Ashoka *et al.* (2013)), where $\mathcal{L}$, V , λ and S are already defined as earlier. The Ricci soliton is said to be shrinking, steady or expanding according as λ is negative, zero and positive respectively. Long-existing solutions, that is, solutions which exist on an infinite time interval are the self-similar solutions, which in Ricci flow are called Ricci soliton.
If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and (1.22.1) assumes the form of equation (1.22.2).
Then, the above equation (1.22.2) can be written as

$$\nabla_Y Df = QY + \lambda Y, \tag{5.8.1}$$

where D is gradient operator of g and Y denotes an arbitrary vector field on M^n. Using this, we derive

$$K(X,Y)Df = (\nabla_X Q)Y + (\nabla_Y Q)X. \tag{5.8.2}$$

Taking inner product with ξ and putting $X = \xi$, we have

$$g(K(\xi, Y)Df, \xi) = g((\nabla_\xi Q)Y, \xi) + g((\nabla_Y Q)\xi, \xi). \tag{5.8.3}$$

Using (1.17.14) and (5.2.6), we get

$$Df = (\xi f)\xi, \tag{5.8.4}$$

and

$$Y(\xi f)\xi + (\xi f)[-\phi Y] = QY + \lambda Y. \tag{5.8.5}$$

Again, taking the inner product in (5.8.5) with X and using (5.8.4) in (5.8.1), we get

$$Y(\xi f)g(X, \xi) - (\xi f)g(X, \phi Y) = g(QY, X) + \lambda g(X, Y),$$

which is equivalent to

$$Y(\xi f)\eta(X) - (\xi f)g(X, \phi Y) = S(X, Y) + \lambda g(X, Y). \tag{5.8.6}$$

Now, from (1.22.1) and putting $V = \xi$, we get

$$\mathcal{L}_\xi g + 2S + 2\lambda g = 0,$$

which gives

$$S + \lambda g = 0. \tag{5.8.7}$$

Using (5.8.7) in (5.8.6), we have

$$Y(\xi f)\xi f\eta(X) = (\xi f)g(X, \phi Y). \tag{5.8.8}$$

Again, putting $X = \xi$ in the above equation, we get

$$Y(\xi f) = 0. \tag{5.8.9}$$

Here, we have $Y(\xi f) = 0$, that is, ξf is constant or $\xi f = c$.
By virtue of the equation (5.8.9), the equation (5.8.8) reduces to

$$(\xi f)g(X, \phi Y) = 0. \tag{5.8.10}$$

From the above relation, we can state:

Theorem 5.8.1 *In K-contact manifold admitting Ricci solitons $g(X, \phi Y) = 0$.*

Now, using (5.8.8) in (5.8.5), we obtain

$$-(\xi f)\phi\xi = QY + \lambda Y,$$

which implies that

$$-(\xi f)\phi\xi = aY + b\eta(Y)\xi + \lambda Y, \tag{5.8.11}$$

since, $QX = aX + b\eta(Y)\xi$.
Putting $Y = \xi$ in the above equation, we get

$$\lambda = -(a + b). \tag{5.8.12}$$

As we know that the Ricci soliton is steady if

$$\begin{aligned} \lambda &= 0, \\ i.e., \ (a + b) &= 0. \end{aligned}$$

From (5.2.8), we have $n = 1$, which is not possible. Therefore, Ricci soliton cannot be steady.
Thus we have the following result:

Theorem 5.8.2 *Ricci soliton in K-contact quasi Einstein manifold cannot be steady.*

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Chapter 6

Summary and Conclusion

Chapter 1 contains the general introduction which includes some basic definitions of differentiable manifolds, tangent vector, tangent space, vector field, Lie-bracket, Lie derivative, covariant derivative, exterior derivative, connection, Riemannian manifold, Riemannian connection, quarter symmetric metric connection, torsion tensor, curvature tensors on Riemannian manifold, Ricci tensor, certain curvature tensors. An almost contact metric manifold, almost para-contact metric manifold, para-Sasakian manifold, Lorentzian para-contact metric manifold, recurrent manifold and Ricci solitons are also defined. Some mathematical tools used for solving problems, applications and the literature review are also included in this chapter.

In chapter 2, we studied an Einstein manifold admitting a Ricci quarter symmetric metric connection in Riemannian manifolds and obtained some geometrical properties. We have discussed and obtained an equivalent relation between the locally symmetric, conharmonically symmetric and m-projectively symmetric manifolds. We also examined an equivalency relation between the locally bi-symmetric, conharmonically bi-symmetric and m-projectively bi-symmetric manifolds. Here, we showed that a generalized conharmonically 2-recurrent Einstein manifold admitting a Ricci quarter symmetric metric connection is conharmonically flat and a generalized conharmonically 2-recurrent Einstein manifold admitting a Ricci quarter symmetric metric connection is m-projectively flat. We also obtained an equivalent relation between the locally symmetric, conharmonically symmetric and concircularly symmetric manifolds. Finally, we have shown that a generalized concircularly 2-recurrent Einstein manifold equipped with Ricci quarter symmetric metric connection is concircularly flat.

Chapter 3 deals with the study of weakly W_1 symmetric manifolds. We examined and investigated the nature of the scalar curvature of $(WW_1S)_n$. Here, we have proved that the Ricci tensor S in a Riemannian manifold (M^n, g), $(n > 2)$ is codazzi type and the Ricci tensor S in $(WW_1S)_n$ has an eigenvalue $-r$ corresponding to the eigenvector $\widehat{\rho}$. We also proved that if in a $(WW_1S)_n$ the Ricci tensor is of Codazzi type, $-r$ is an eigenvalue of the Ricci tensor S corresponding to the eigen vector ξ defined by $g(X, \xi) = \nu(X)$ for all X. W_1 flat weakly Ricci-symmetric manifolds are also discussed in this chapter. We also examined and proved that if the scalar curvature of a $(WW_1S)_n$ vanishes, then $W_1(X, Y, \widehat{\rho}, U) = K(X, Y, \widehat{\mu}, U)$. We also proved that in a W_1 flat $(WRS)_n$, $(n > 2)$ with $\mu(X) \neq 0$, the vector field ρ defined by $g(X, \rho) = H(X)$ is not a proper concircular vector field and W_1 flat $(WRS)_n$, $(n > 2)$ is a quasi Einstein manifold.

In chapter 4, we have discussed ϕ-recurrent and generalized recurrent curvature tensor in LP-Sasakian manifolds and showed some interesting results. Here, we proved that ϕ-symmetric LP-Sasakian manifold is an Einstein manifold. We also studied 3-dimensional locally ϕ-symmetric LP-Sasakian manifolds. It is shown that if the scalar curvature r is constant, then the necessary and sufficient condition of a 3-dimensional LP-Sasakian manifold to be locally ϕ-symmetric. We obtained if the manifold is an Einstein manifold, then the necessary and sufficient condition of an n-dimensional manifold is to be ϕ-Ricci symmetric. We also proved that if the scalar curvature r of a 3-dimensional LP-Sasakian manifold is equal to -6, then we can say that the manifold is ϕ-Ricci symmetric. We constructed an example of 3-dimensional ϕ-Ricci symmetric LP-Sasakian manifolds. In this chapter, we also studied and discussed m-projective ϕ-recurrent LP-Sasakian manifolds. We showed that m-projective ϕ-recurrent LP-Sasakian manifold is an η-Einstein manifold.

In the fifth chapter, we explored some properties of K-contact quasi Einstein manifolds. We discussed conformally flat K-contact quasi Einstein manifolds. Here,it is proved that conformally flat K-contact quasi Einstein manifold is of quasi constant curvature. We obtained in a semi-symmetric K-contact quasi Einstein manifold $K(\xi, X)\xi = 0$ and Ricci semi-symmetric K-contact quasi Einstein manifold is Einstein manifold. We also studied and obtained in K-contact manifold admitting Ricci solitons $g(X, \phi Y) = 0$. We also showed that Ricci soliton in K-contact quasi Einstein manifold cannot be steady.

Finally, we conclude that the whole work of this thesis gives the properties and geometrical structure of Sasakian manifolds equipped with Ricci quarter symmetric metric connection, W_1 flat weakly Ricci-symmetric manifolds, ϕ-symmetric and m-projective ϕ-recurrent LP-Sasakian manifolds and K-contact manifolds.

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Bibliography

Adigond, S. and Bagewadi, C. S. (2017). Ricci solitons on Para-Kenmotsu manifolds, *Gulf Journal of Mathematics*, **5(1)**, 84-95.

Agashe, N. S. and Chafle, M. R. (1992). A semi-symmetric non-metric connection in a Riemannian manifold, *Indian J. Pure Appl. Math.*, **23**, 399-409.

Amur, K. and Pujar, S. S. (1978). On submanifolds of a Riemannian manifold admitting a metric semi-symmetric connection, *Tensor N. S.*, **32(1)**, 35-38.

Arslan, K., De, U. C., Murathan and Yildiz A. (2009). On generalized recurrent Riemannian manifolds, *Acta Math. Hungar.*, **123(1-2)**, 27-39.

Ashoka, S. R., Bagewadi, C. S. and Ingalahalli, G. (2013). Research Article-Certain results on Ricci Solitons in α-Sasakian manifolds, *Hindawi Publishing Corporation - Geometry*, http://dx.doi.org/10.1155/2013/573925.

Besse, A. L. (1987). Einstein manifolds, *Classics in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York.*

Barman, A. and De, U. C. (2013). Projective curvature tensor of a semi-symmetric metric connection in a Kenmotsu manifold, *Int. Elect. J. Geom.,* **6(1)**, 159-169.

Binh, T. Q. (1990). On semi-symmetric connection, *Period. Math. Hungar.*, **21(2)**, 101-107.

Binh, T. Q.(1993). On weakly symmetric Riemannian spaces, *Publ. Math. Debrecen*, **42**, 103-107.

Biswas, S. C. and De. U. C. (1997). Quarter symmetric metric connection in an SP-Sasakian manifold, *Commun. Fac. Sci. Univ. Ank. Series*, **46**, 49-56.

Blair, D. E. (1976). Contact manifolds in Riemannian geometry, *Lecture notes in Math. 509, Springer-Verlag*.

Boothby, M. M. and Wong, R. C.(1958). On contact manifolds,*Anna. Math.*,, **68**, 421-450.

Chaki, M. C. (2001). On generalized quasi-Einstein manifolds, *Publ. Math. Debrecen,* **58**, 683-691.

Chaki, M. C. (2004). On super quasi-Einstein manifolds, *Publ. Math. Debrecen*, **64(3-4)**, 481-488.

Chaki, M. C. and Maity, R. K. (2000). On quasi Einstein manifold, *Publ. Math. Debrecen*, **57**, 297-306.

Chaki, M. C. and Saha, S. K. (1994). On Pseudo projective Ricci-symmetric manifolds, *Bulg. Journ. Phys.*, **24**, 95-102.

Chaturvedi, B. B. and Pandey, P. N. (2008). Semi-symmetric non-metric connection on a Kahler manifold, *Diff. Geom. Dyn. Syst.*, **10**, 86-90.

Chaubey, S. K. (2011). Some properties of LP-Sasakian manifolds equipped with m-projective curvature tensor, *Bull. Math. Anal. and Apll.*, **3(4)**, 50-58.

Chaubey, S. K. (2012). On Weakly m-projectively symmetric manifolds, *Novi Sad J. Math.*, **42(1)**, 67-79.

Chaubey, S. K. and Ojha, R. H. (2010). On the m-projective curvature tensor of a Kenmotsu manifolds, *Diff. Geom. Dyn. Syst.*, **12**, 52-60.

Chaubey, S. K. and Ojha, R. H. (2012). On a semi-symmetric non-metric connection, *Filomat*, **26(2)**, 269-275.

De, A., Yildiz, A. and De, U. C. (2014). On generalized quasi Einstein manifolds, *Filomat*, **28(4)**, 811-820.

De, U. C. and Bandyopadhyay, S. (1999). On weakly symmetric Riemannian spaces, *Publ. Math. Debrecen*, **57(3-4)**, 377-381.

De, U. C. and Bandyopadhyay, S. (2000). On weakly conformally symmetric spaces, *Publ. Math. Debrecen*, **57(1-2)**, 71-78.

De, U. C. and Biswas, S. (2005). On K-contact η-Einstein manifolds, *Bull. Math. Soc. Sc. Math. Roumanie*, **48(3)**, 295-301.

De, U. C. and Biswas, S. (2006). A Note on ξ-conformally flat contact manifolds, *Bull. Malays. Math. Sci. Soc.(2)*, **29(1)**, 51-57.

De, U. C. and De, A. (2017). On some curvature properties of K-contact manifolds, *Extracta Math.*, **27**, 125-134.

De, U. C. and De, B. K. (2008). On quasi-Einstein manifolds, *Commun. Korean Math. Soc.*, **23**, 413-420.

De, U. C. and De, K.(2014). On η-Einstein LP-Sasakian manifolds, *Acta Universitatis Apulensis.*, **38**, 143-151.

De, U. C. and De, K. (2011). On three dimensional Kenmotsu manifolds admitting a quarter-symmetric metric connection, *Azerbaijan J. Math.*, **1**, 132-142.

De, U. C. and Gazi, A. K. (2008). On nearly quasi-Einstein manifolds, *Novi Sad J. Math.*, **38(2)**, 115-121.

De, U. C. and Ghosh, G. C.(2004a). On quasi-Einstein Manifolds, *Period. Math. Hungar.*, **48(1-2)**, 223-231.

De, U. C. and Ghosh, G. C. (2004b). On generalized quasi Einstein manifolds *Kyungpook Math. J.*, **44**, 607-615.

De, U. C. and Guha, N. (1991). On generalized recurrent manifold, *J. Nat. Acad. Math.*, **9**, 85-92.

De, U. C., Guha, N. and Kamilya, D.(1995). On generalized Ricci-recurrent manifolds, *Tensor (N.S.)*, **56**, 312-317.

De, U. C. and Kamilya, D. (1994). On generalized concircular recurrent manifolds, *Bull. Cal. Math. Soc.*, **86**, 69-72.

De, U. C. and Kamilya, D. (1995). Hyper surfaces of a Riemannian manifold with semi-symmetric non-metric connection, *J. Indian Inst. Sci.*, **75**, 707-710.

De, U. C. and Mallick, S. (2011).On the existence of generalized quasi Einstein manifolds, *Archivum Mathematicum (BRNO)*, **47**, 279-291.

De, U. C. and Mandal, K. (2016). On K-contact Einstein manifolds, *Novi Sad J. Math.*, **46(1)**, 105-114.

De, U. C., Matsumoto, K. and Shaikh, A.A. (1999). On Lorentzian para-Sasakian manifolds, *Rensicontidel Seminario Matematico di Messina, Serie II, Supplemento al.*, **3**, 149-158.

De, U. C. and Mondal, A. K. (2010). Quarter-symmetric metric connection on 3-dimensional quasi-Sasakian manifolds, *Sut J. Math.*, **46**, 35-52.

De, U. C. and Pathak, G. (2003). On Riemannian manifold with certain curvature conditions, *Mathematica Pannonica*, **14(2)**, 227-235.

De, U. C. and Sarkar, A. (2009). On a type of P-Sasakian manifolds, *Math. Reports*, **11(61)**, 139-144.

De, U. C. and Sengupta, J. (2000). Quarter-symmetric metric connection on a Sasakian manifold, *Comm. Fac. Sci. Univ. Ank. Series. Al* **49**, 7-13.

De, U. C. and Sengupta, J. (2001). On a type of semi-symmetric metric connection on an almost contact metric manifold, *Facta Universitatis, Ser. Math. Inform.*, **16**, 87-96.

Devi, M. S. and Singh, J. P (2015). On a type of m-projective curvature tensor on Kenmotsu manifolds, *Int. J. Math. Sci. Eng. Appls.*, **9(3)**, 37-49.

Ferus, D. (1981). A remark on Codazzi tensors on constant curvature spaces, Lecture Notes Math. 838,Global differential Geometry and Global Analysis, Springer-verlag, New York.

Friedmann, A. and Schouten, J. A. (1924). Uber die geometric der halbsymmetrischen ubertragung, *Math. Zeitschr*, **21**, 211-223.

Geiges, H. (2001). A brief history of contact geometry and topology, *Expo. Math.*, **19**, 25-53.

Ghosh, G. C., De, U. C. and Binh, T. Q. (2006). Certain curvature restrictions on a quasi-Einstein manifold, *Publ. Math. Debrecen*, **69**, 209-217.

Golab, S. (1975). On semi-symmetric and quarter-symmetric linear connections, *Tensor N.S.*, **29**, 249-254.

Gray, J. W. (1959). Annals of Mathematics, *Mathematics Department, Princeton University*, **69(2)**, 421-450.

Guha, N. (2000). On generalized Ricci-recurrent Sasakian manifolds, *Bull. Cal. Math. Soc.*, **92(5)**, 361-364.

Guha, S. (2003). On quasi Einstein and generalized quasi Einstein Manifolds, *Facta Univ. Series, Mech., Auto. Contr. and Robotics*, **3(14)**, 821-842.

Hamilton, R. S. (1988). The Ricci flow on surfaces, *Mathematics and general relativity*, *Contemp. Math.*, *American Math. Soc.*, **71**, 237-262.

Hatakeyama, Y. (1963). Some notes on differentiable manifolds with almost contact structure, *Tohoku Math. J.*, **15**, 176-181.

Hatakeyama, Y., Ogawa, Y. and Tanano, S. (1963). Some properties of manifolds with contact metric structures, *Tohoku Math. J.*, **15**, 42-48.

Hayden, H. A. (1932). Subspaces of a space with torsion tensor, *Proc. London Math. Soc.*, **34**, 27-50.

Ishii, Y.(1957). On conharmonic transformations, *Tensor N. S.* **7**, 73-80.

Jaiswal, J. P.(2011). The existence of weakly symmetric and weakly Ricci symmetric Sasakian manifold admitting a quarter symmetric metric connection, *Acta Math. Hunger.*, **132(4)**, 358-366.

Jaiswal, J. P. and Ojha, R.H.(2009). On generalized ϕ-recurrent LP-Sasakian manifolds, *Kyungpook Math. J.*, **49**, 779-788.

Jaiswal, J. P. and Ojha, R. H. (2010). On weakly pseudo-projectively symmetric manifolds, *Diff. Geom. Dyn. Syst.*, **12** , 83-94.

Jana, S. K. and Shaikh, A. A. (2007). On quasi conformally flat weakly symmetric manifolds, *Acta Math. Hungar.*, **115(3)** , 197-214.

Kenmotsu, K.(1972). A class of almost contact Riemannian manifolds, *J. Korean Math. Soc.*, **42**, 92-103.

Khan Quddus, (2004). On generalized recurrent Sasakian manifolds, *Kyungpook Math. J.*, **44**, 167-172.

Kobayashi, S. and Nomiza, K. (1963). Foundations of differential geometry, *Interscience Publishers, New York*, **1**, 574.

Malek, F. and Samawaki, M. (2008). On weakly symmetric Riemannian manifolds, *Diff. Geom. Dyn. Syst.*, **10**, 215-220.

Mallick, S., De, A. and De, U. C. (2013). On generalized Ricci recurrent manifolds with applications to relativity, *Proc. Nat. Acad. Sci. India Sect. A*, **83(2)**, 143-152.

Mallick, S. and De, U. C. (2016). On a class of generalized quasi Einstein manifolds with Application to Relativity *Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica*, **55**, 111-127.

Matsumoto, K. (1989). On Lorentzian para-contact manifolds, *Bull. of Yamagata Univ. Nat. Sci*, **12(2)**, 151-156.

Matsumoto, K. and Mihai, I. (1988). On a certain transformations in Lorentzian para Sasakian manifold, *Tensor N.S.*, **47**, 189-197.

Mihai, I., De, U. C. and Shaikh, A. A. (1999a). On Lorentzian Para-Sasakian manifolds. *Korean J. Math. Sci.*, **6**, 1-13.

Mihai, I. and Rosca, R. (1992). On Lirentzian P-Sasakian manifolds, Classical Analysis, *World Sci. Publ.*, 155-169.

Mihai, I., Shaikh, A. A., and De, U. C.(1999b). On Lorentzian Para-Sasakian manifolds, *Rendicontidel Seminario Matematico Di Messina. Serie II, supplemento al.*, **3** 149-158.

Mishra, R. S. and Pandey, S. N. (1980). On quarter symmetric metric F-connections, *Tensor N.S.*, **34**, 1-7.

Miyazawa, T. and Yamaguchi, S.(1966). Some theorems on K-contact metric manifolds and Sasakian manifolds, *T.R.U. Math. Japan*, **2**, 46-52.

Mondal, A. K. and De, U. C. (2009). Some properties of a quarter symmetric metric connection on a Sasakian manifold, *Bull. Math. Anal. and Appl.*, **1(3)**, 99-108.

Mukhopadhyay, S., Roy, A. K. and Barua, B. (1991). Some properties of a quarter symmetric metric connection on a Riemannian manifold, *Soochow J. Math.*, **17(2)**, 205-211.

Murathan, C. and Ozgur, C. (2008). Riemannian manifolds with a semi-symmetric metric connection satisfying some semisymmetry conditions, *Proc. Est. Acad. Sci.*, **57(4)**, 210-216.

Nagaraja, H. G. and Premalatha, C. R. (2012). Ricci solitons in Kenmotsu manifolds, *Journals of Mathematical Analysis*, **3(2)**, 18-24.

Nivas, R. and Verma, G. (2005). On quarter-symmetric non-metric connection in a Riemannian manifold, *J. Rajasthan Acad. Phys. Sci.*, **4(1)**, 57-68.

Ojha, R. H. (1975). A note on the m-projective curvature tensor, *Ind. J. Pure Appl. Math.*, **8(12)**, 1531-1534.

Ojha, R. H. (1986). m-projectively flat Sasakian manifolds, *Ind. J. Pure Appl. Math.*, **17(4)**, 481-484.

Oubina, J. (1985). New classes of almost contact metric structures, *Publications Mathematicae*, **32**, 187-193.

Ozgur, C. (2007). On generalized recurrent Kenmotsu manifolds, *World Appl. Sci. J.*, **2(1)**, 9-33.

Ozgur, C. and Sular, S. (2008). On some properties of generalized quasi-Einstein manifolds, *Ind. J. Math.*, **50**, 297-302.

Pandey, S. N. and Mishra, R. S. (1980). On quarter symmetric metric F-connections, *Tensor N.S.*, **34**, 1-7.

Pandey, L. K. and Ojha, R. H. (2001). Semi-symmetric metric and non-metric connections in Lorentzian Para contact manifold, *Bull. Cal. Math. Soc.*, **93(6)**, 497-504.

Patterson, E. M. (1952). Some theorems on Ricci-recurrent spaces, *J. London Math. Soc.*, **27**, 287-295.

Patil, D. A., Prakasha, D. G. and Bagewadi, C. S. (2009). On generalized ϕ-recurrent Sasakian manifolds, *Bull. Math. Anal. and Apll.*, **1(3)**, 42-48.

Pokhariyal, G. P. (1982). Study of a new curvature tensor in a Sasakian manifold, *Tensor N.S.*, **36**, 222-225.

Pokhariyal, G. P. and Mishra, R. S. (1970). Curvature tensor and their relativistic significance, *Yokohama Math. J.*, **18**, 105-108.

Pokhariyal, G. P. and Mishra, R. S. (1971). Curvature tensor and their relativistic significance II, *Yokohama Math. J.*, **19(2)**, 97-103.

Prakash, A. and Pandey, V. K. (2013). On a quarter symmetric non-metric connection in a Kenmotsu manifolds, *International Journal of Pure and Applied Mathematics*, **83(2)**, 271-2787.

Prakash, A. and Narain, D. (2011). On a quarter symmetric non-metric connection in an Lorentzian para-Sasakian manifolds, *Int. Electronic J. Geom.*, **4(1)**, 129-137.

Prakasha, D. G.(2013). On extended generalized ϕ-recurrent Sasakian manifolds, *J. of the Egyptian Math. Soc.*, **21**,25-31.

Prakasha, D. G. and Yildiz, A. (2010). Generalized ϕ-recurrent Lorentzian α-Sasakian manifold, *comm. Fac. Sci. Univ. Ank. Series Al.*, **59(1)**, 53-62.

Prasad, B. and Kumar, R. (2002). A semi-symmetric non-metric SP-connection in an SP-Sasakian manifold, *J. Nat. Acad. Math.*, **16**, 37-45.

Rastogi, S. C. (1978). On quarter symmetric metric connection, *C. R. Acad. Sci. Bulgar*, **31**, 811-814.

Rastogi, S. C. (1987). On quarter symmetric metric connection, *Tensor N.S.*, **44(2)**, 133-141.

Sasaki, S. (1965). Almost contact manifold, I, A Lecture Note, *Tohoku University*.

Sasaki, S. (1967). Almost contact manifold, II, A Lecture Note, *Tohoku University*.

Sasaki, S. (1968). Almost contact manifold, III, A Lecture Note, *Tohoku University*.

Sasaki, S. (1960). On differentiable manifolds with certain structures which are closely related to almost contact structure, *Tohoku Math. J.* **12**, 459-476.

Sasaki, S. (1975). Lecture notes on almost contact manifolds, part 1, *Tohoku University*.

Sasaki, S. and Hatakeyama, Y. (1961). On differentiable manifolds with certain structures which are closely related to almost contact structure, *Tohoku Math. J.*, **13**, 281-294.

Sato, I.(1977). On a structure similar to almost contact structures-II, *Tensor N.S.*, **31**,199-205.

Sato, I. and Matsumoto, K. (1976). On P-Sasakian manifolds satisfying certain conditions, *Tensor N.S.*, **33**, 173.

Sengupta, J. and Biswas, B.(2003). Quarter symmetric non-metric connection on a Sasakian manifold, *Bull.Cal.Math.Soc.*, **95(2)**,196-176.

Shaikh, A. A. and Baishya, K. K. (2005). On weakly quasi conformally symmetric manifolds, *Soochow Journal of Mathematics*, **31(1)**, 581-595.

Shaikh, A. A. and Baishya, K. K. (2006). On ϕ-symmetric LP-Sasakian manifolds, *Yakohama Mathematical Journal*, **52**, 581-595.

Shaikh A. A. and Biswas, S. (2004). On LP-sasakian manifolds, *Bull. Malaysian Math. Sci. Soc.*, **27**, 17-26.

Shaikh, A. A. and Hui, S.K.(2011). On extended generalized ϕ-recurrent β-Kenmotsu manifolds,*Publ.De L'Inst. Math.*, **89(103)**,77-88.

Shaikh, A. A., Kim Y. H. and Hui H. K. (2011). On Lorentzian quasi Einstein Manifolds, *J. Korean Math. Soc.*, **48(4)**, 669-689.

Shaikh, A. A., Prakasha, D. G. and Ahmad, H.(2013). On generalized ϕ-recurrent LP-Sasakian manifolds, *Journal of the Egyptian Mathematical Society*, **21(1)**, 25-31.

Sharfuddin, A. and Hussain, S. I. (1976). Semi-symmetric metric connections in almost contact manifolds, *Tensor N.S.*, **30**, 133-139.

Shukla, S. S. and Shukla, M. K. (2009). On ϕ-Ricci symmetric Kenmotsu manifolds, *Novi Sad J. Math.*, **39(2)**, 89-95.

Shukla, S. S. and Shukla, M. K. (2010). On ϕ-symmetric Para-Sasakian manifolds, *Int. Journal of Math. Analysis*, **4(16)**, 761-769.

Singh, H. and Khan, Q. (1999). On symmetric Riemannian manifolds, *Novi Sad J. Math.*, **29(3)**, 301-308.

Singh, J. P. (2009). On an Einstein m-projective P-Sasakian manifolds, *Bull. Cal. Math. Soc.*, **101(2)**, 175-180.

Singh, J. P. (2012). On m-projective recurrent Riemannian manifold, *Int. J. Math. Anal.*, **6(24)**, 1173-1178.

Singh, J. P. (2013). Some properties of LP-Sasakian manifolds admitting a quarter symmetric non-metric connection, *Tamsui Oxford J. of Info. Math. Sci.*, **29(4)**, 505-517.

Singh, J. P. (2014a). On an Einstein quarter symmetric metric F-connection, *Int. J. of advances in Math.*, **1(1)**, 1-8.

Singh, J. P. (2014b). On generalized recurrent and generalized concircularly recurrent P-Sasakian manifolds, *Novi Sad. J. Math.*, **44(1)**, 153-163.

Singh, J. P. (2014c). Some properties of LP-Sasakian manifolds admitting a quarter symmetric non-metric connections, *Tomsui Oxford J. Inf. Math. Sc.*, **29(4)**, 505-517.

Singh, J. P. and Singh, A. (2014). Quarter symmetric non-metric connection on LP-Sasakian manifolds, *Int. J. Appl. Math. Sc.*, **7(2)**, 113-122.

Singh, J. P. (2015a). On a type of LP-Sasakian manifolds admitting a quarter symmetric non-metric connection, *The Mathematical Student*, **84(1-2)**, 57-67.

Singh, J. P. (2015b). On the m-projective curvature tensor of Sasakian manifolds, *Science Vision*, **15(2)**, 76-79.

Singh, J. P. and Devi, M. S. (2015). On a type of quarter symmetric non-metric connection in an LP-Sasakian manifold, *Sciensce and Technology Jour.*, **4(1)**, 65-68.

Singh, J. P. and Singh, A. and Kumar, R. (2015). Some curvature properties of LP-Sasakian manifolds, *Journal of Pure Mathematics*, **31**, 13-28.

Singh, J. P. (2016). Generalized Sasakian space forms with m-projective curvature tensor, *Acta Math. Univ. Comenianae*, **85(1)**, 135-146.

Singh, J. P. (2017). On m-projectively flat almost pseudo Ricci symmetric manifolds, *Acta Math. Univ. Comenianae*, **86(2)**, 335-343.

Singh, J. P. and Lalmalsawma, C. (2018). On generalized pseudo projectively recurrent manifolds, *J. of the Ind. Math. Soc.*, **85(3-4)**, 448-468.

Singh, R. N. and Pandey, M. K. (2007). On a type of non-metric connection in a Kenmotsu manifold, *Bull. Cal. Math. Soc.*, **99(4)**,433-444.

Singh, R. N. and Pandey, M. K. (2008). On a type of non-metric connection on a Riemannian manifold, *Rev. Bull. Cal. Math. Soc.*, **16(2)**, 179-184.

Singh, R. N., Pandey, M. K. and Pandey, G. (2012). Some curvature properties of a semi-symmetric metric connection in a Kenmotsu manifold, *Rev. Bull. Cal. Math. Soc.*, **20(1)**,81-92.

Sinha, B. B. and Yadava, S. L. (1980). Structure connection in an almost contact metric manifold, *Publications De L'Institut Mathematique*, **28(42)**, 195-202.

Sular, S., Ozgur, C. and De, U. C. (2008). Quarter symmetric metric connection in a Kenmotsu manifold, *SUT Journals of Mathematics*, **44(2)**, 297-306.

Szabo, Z. I. (1982). Structure theorems on Riemannian spaces satisfying $R(X,Y).R = 0$, I, The local version, *J. Diff. Geome.*, **17**, 531-582.

Taleshian, A. and Asghari, N. (2011). On LP-Sasakian manifolds, *Bull. of Math. Analysis and Applications*, **3(1)**, 45-51.

Tamassy, L. and Binh, T. Q. (1989). On weakly symmetric and weakly projective symmetric Riemannian manifolds, *Coll. Math. Soc., J. Bolyani*, **50** , 663-670.

Tamassy, L. and Binh, T. Q. (1988). On weakly symmetries of Einstein and Sasakian manifold, *Tensor N.S.*, **53**, 140-148.

Tripathi, M. M. and Kim, J. S. (2007). On $N(k)$ quasi Einstein manifolds. *Comm. Korean Math. Soc.*, **22(3)**, 411-417.

Venkatesha and Bagewadi, C. S. (2006). On pseudo projective ϕ-recurrent Kenmotsu manifolds, *Soochow Journal of Math.*, **32(3)**, 221-229.

Venkatesha and Bagewadi, C. S. (2008). On concircular ϕ-recurrent LP-Sasakian manifold, *Balkan Soc. Geometers*, **10**, 312-319.

Venkatesha, Bagewadi, C. S. and Kumar Pradeep, K. T. (2011). Some results on Lorentzian Para-Sasakian manifolds, *ISRN Geometry*, ID 161523, **11**, 1-9.

Venkatesha and Divyashree, G. (2017). Quarter symmetric metric connection on a Lorentzian α-Sasakian manifold, *New trends in Math. Sci.*, **5(2)**, 69-79.

Venkatesha, Kumar, K. T. P. and Bagewadi, C. S. (2015). On Quarter Symmetric metric connection in a Lorentzian Para-Sasakian manifold, *Azer. J. Math.*, **5(1)**, 2218-6816.

Yadav, V. and Dhruwanarain (2014). On a quarter symmetric non-metric connection in a P-Sasakian manifold, *Journal of Rajasthan Academy of Physical Sciences*, **13(2)**, 203-212.

Yano, K. (1940). Concircular Geometry I, Concircular transformations, *Math. Institute, Tokyo Imperial Univ. Proc.*, **16**, 195-200.

Yano, K. (1970). On semi-symmetric connection, *Rev. Roum. Math. Pures, Appliqu*, 1579-1586.

Yano, K. and Bochner, S. (1953). Curvature and Betti numbers, *Ann. Math. Stud., Princeton University Press*, **32** .

Yano, K. and Imai, T. (1982). Quarter symmetric metric connections and their curvature tensors, *Tensor N.S.*, **38**, 13-18.

LIST OF PUBLICATIONS

I. LIST OF PUBLICATIONS

1. Singh, J. P. and Zosangliani, R. (2015). On an Einstein Ricci-symmetric metric connection in a Riemannian manifold, *Science and Technology Journal*, **3(1)**, 56-63. ISSN: 2321-3388.

2. Singh, J. P. and Zosangliani, R. and Lalmalsawma, C. (2018). On K-contact quasi Einstein manifolds, *Ind. J. of Math.* (Communicated).

3. Singh, J. P. and Zosangliani, R. (2018). On ϕ-symmetric LP-Sasakian manifolds, *J. of Statistics and Math. Eng.* (Communicated).

II. CONFERENCES/SEMINARS/WORKSHOPS

1. Attended ”ISI-MZU School on Soft Computing and Applications” organized jointly by Machine Intelligence Unit, Indian Statistical Institute, Kolkata and Department of Mathematics & Computer Science, Mizoram University, Aizawl - 796 004, Mizoram on $5^{th} - 9^{th}$ November, 2012.

2. Attended ”National workshop on Mathematical Analysis” organized by Department of Mathematics & Computer Science, Mizoram University, Aizawl - 796 004, Mizoram on $7^{th} - 8^{th}$ March, 2013.

3. Attended ”National workshop on Dynamical Systems” organized by Department of Mathematics & Computer Science, Mizoram University, Aizawl - 796 004, Mizoram on $26^{th} - 27^{th}$ November, 2013.

4. Attended ”National Conference on Application of Mathematics” organized by Department of Mathematics & Computer Science, Mizoram University, Aizawl - 796 004, Mizoram on $25^{th} - 26^{th}$ February, 2016.

5. Attended ”North-East ISI-MZU Winter School on Algorithms with special focus on Graphs” organized jointly by Advanced Computing and Microelectronics Unit, Indian Statistical Institute and Department of Mathematics & Computer Science, Mizoram University, Aizawl - 796 004, Mizoram on $6^{th} - 11^{th}$ March, 2017.

Science and Technology Journal, Vol. 3 Issue: 1 ISSN: 2321-3388

On an Einstein Ricci Quarter Symmetric Metric Connection in a Riemannian Manifold

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Abstract— In the present paper we have studied an Einstein manifold admitting a Ricci quarter symmetric metric connection in a Riemannian manifold and obtained some geometrical properties.

Mathematics Subject Classification (2010): 53C15, 35B05.

Keywords: Ricci quarter symmetric metric connection, Einstein manifold, generalized conharmonically 2-recurrent Riemannian manifold, generalized M-Projectively 2-recurrent Riemannian manifold, generalized concircularly 2-recurrent Riemannian manifold.

INTRODUCTION

The idea of metric connection with torsion tensor in a Riemannian manifold was introduced by Hayden [1]. Later, Yano [2] studied some properties of semi-symmetric metric connection on a Riemannian manifold. Golab [3] introduced and studied quarter symmetric connection in a Riemannian manifold with an affine connection which generalizes the idea of semi-symmetric connection. Pandey and Mishra [4], studied quarter symmetric metric connections in a Riemannian, Kahlerian and Sasakian manifolds. In the present paper, we study a Ricci Quarter symmetric metric connection given by the last two authors [4] in an Einstein Riemannian manifold.

Let (M_n, g) be a Riemannian manifold of dimension n and let D be the Riemannian connection of (M_n, g). A Riemannian manifold is called locally symmetric if $D_U R = 0$, where R is the Riemannian curvature tensor of (M_n, g) and U is a vector field.

The Present Paper is Organized as Follows: Section 2 is preliminaries and about the relation between curvature tensors of a Ricci quarter symmetric metric connection ∇ and Riemannian connection D. In section 3, we obtained an equivalent relation between the locally symmetric, conharmonically symmetric and M-projectively symmetric manifold. In section 4, we have studied an equivalency between the locally bi-symmetric, conharmonically bi-symmetric and M-projectively bi-symmetric. In section 5, we have shown that a generalized conharmonically 2-recurrent Einstein manifold admitting a Ricci quarter symmetric metric connection is conharmonically flat and a generalized conharmonically 2-recurrent Einstein manifold admitting a Ricci quarter symmetric metric connection is M-projectively flat. In section 6, we obtained an equivalent relation between the locally symmetric, conharmonically symmetric and concircularly symmetric manifold. Finally, we have shown that a generalized concircularly 2-recurrent Einstein manifold equipped with Ricci quarter symmetric metric connection is concircularly flat.

A linear connection ∇ in a Riemannian manifold M_n is said to be Ricci quarter symmetric connection if the torsion tensor S satisfies [4]

$$S(X,Y) = \eta(Y)LX - \eta(X)LY \tag{1}$$

where, η is a 1-form and L is the (1,1) Ricci tensor defined by

$$g(LX,Y) = Ric(X,Y) \tag{2}$$

Ric is the Ricci tensor of M_n and X, Y are vector fields.

A linear connection ∇ is called a metric connection if

$$\nabla_X g(Y,Z) = 0. \tag{3}$$

If D is the Riemannian connection of the manifold (M_n, g). Then the Ricci quarter symmetric metric connection

∇ is given by [4]

$$\nabla_X Y = D_X LY + \eta(Y)LX - Ric(X,Y)\rho \tag{4}$$

where, $\eta(X) = g(X, \rho)$.

PRELIMINARIES

Let $\bar{R}$ and R be the curvature tensors of the connections ∇ and D respectively, then it can be shown that [4]

$$\bar{R}(X,Y)Z = R(X,Y)Z - M(Y,Z)LX + M(X,Z)LY$$
$$- Ric(Y,Z)QX + Ric(X,Z)QY + \eta(Z)[(D_X L)Y - (D_Y L)X]$$
$$-[(D_X Ric)(Y,Z) - (D_Y Ric)(X,Z)]\rho \tag{5}$$

where, M is a tensor field of type (0, 2) defined by

$$M(X,Y) = g(QX,Y) = (D_X\eta)Y - \eta(Y)\eta(LX)$$
$$+\frac{1}{2}\eta(\rho)Ric(X,Y) \tag{6}$$

and Q is a tensor field of type (1,1) defined by

$$QX = D_X\rho - \eta(LX)\rho + \frac{1}{2}\eta(\rho)LX. \tag{7}$$

Here, we shall consider M_n to be an Einstein manifold i.e.

$$Ric(X,Y) = \frac{r}{n}g(X,Y), \tag{8}$$

where r is the scalar curvature of the manifold.

Considering (5), (8) and (2), we get

$$\bar{R}(X,Y)Z = R(X,Y)Z - \frac{r}{n}\{M(Y,Z)X - M(X,Z)Y$$
$$+g(Y,Z)QX - g(X,Z)QY\}. \tag{9}$$

Contracting (9) with respect to X, we get

$$\overline{Ric}(Y,Z) = \frac{r}{n}[g(Y,Z) - \{(n-2)M(Y,Z) + m\,g(Y,Z)\}, \tag{10}$$

where, $\overline{Ric}$ is the Ricci tensor of ∇ and m is the trace of M_n. Now, putting $Y = Z = e_i$, where $\{e_i;\ i = 1, 2, 3, \ldots, n\}$ is an orthonormal basis of the tangent space at any point, we get by taking the sum for $1 \leqslant i \leqslant n$ in the relation (10)

$$\bar{r} = \frac{r}{n}[n - 2(n-1)m], \tag{11}$$

where, $\bar{r}$ is the scalar curvature of ∇.

CONHARMONIC AND M–PROJECTIVE CURVATURE TENSORS OF ∇

The conharmonic curvature tensors and M-projective curvature tensor [5] on a Riemannian manifold are defined respectively by

$$'C(X,Y,Z,U) = {}'R(X,Y,Z,U)$$
$$-\frac{1}{n-2}\{g(Y,Z)Ric(X,U) - Ric(X,Z)g(Y,U)\}$$
$$+Ric(Y,Z)g(X,U) - g(X,Z)Ric(Y,U)\} \tag{12}$$

and

$$'W^*(X,Y,Z,U) = {}'R(X,Y,Z,U)$$
$$-\frac{1}{2(n-1)}\{g(Y,Z)Ric(X,U) - Ric(X,Z)g(Y,U)\}$$
$$+Ric(Y,Z)g(X,U) - g(X,Z)Ric(Y,U)\}, \tag{13}$$

where

$$'C(X,Y,Z,U) = g(C(X,Y)Z,U)$$
$$'W^*(X,Y,Z,U) = g(W^*(X,Y)Z,U)$$
$$'R(X,Y,Z,U) = g(R(X,Y)Z,U). \tag{14}$$

Let $'\bar{C}$ be the conharmonic curvature tensor of the connection ∇. Then from (12), we have

$$'\bar{C}(X,Y,Z,U) = {}'\bar{R}(X,Y,Z,U)$$
$$-\frac{1}{n-2}\{g(Y,Z)\overline{Ric}(X,U) - \overline{Ric}(X,Z)g(Y,U)\}$$
$$+\overline{Ric}(Y,Z)g(X,U) - g(X,Z)\overline{Ric}(Y,U)\} \tag{15}$$

Using (9) and (10) in (15), we obtain

$$'\bar{C}(X,Y,Z,U) = {}'R(X,Y,Z,U)$$
$$+\frac{2r(m-1)}{n(n-2)}\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\}. \tag{16}$$

From the above equation, we get

$$\bar{C}(X,Y)Z = R(X,Y)Z + \frac{2r(m-1)}{n(n-2)} + g(Y,Z)X - g(X,Z)Y\}. \tag{17}$$

Again, let $'\bar{W}^*$ be the M-projective curvature tensor of the connection ∇.

Then from (13), we have

$$'\overline{W^*}(X,Y,Z,U) = \overline{R'}(X,Y,Z,U)$$
$$-\frac{1}{2(n-1)}\{g(Y,Z)\overline{Ric}(X,U) - \overline{Ric}(X,Z)g(Y,U)\}$$
$$+\overline{Ric}(Y,Z)g(X,U) - g(X,Z)\overline{Ric}(Y,U)\} \tag{18}$$

Applying (9), (10) in (18), we have

$$'\overline{W^*}(X,Y,Z,U) = {}'R(X,Y,Z,U)$$
$$+\frac{r(r-m-1)}{n(n-1)}\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\}. \tag{19}$$

From which we get

$$\overline{W^*}(X,Y)Z = R(X,Y)Z + \frac{r(r-m-1)}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\}. \tag{20}$$

Taking covariant derivative of (17) and (20), we get

$$(D_U\bar{C})(X,Y)Z = (D_U R)(X,Y)Z \tag{21}$$

and

$$(D_U\overline{W^*})(X,Y)Z = (D_U R)(X,Y)Z \tag{22}$$

Hence we can state the following:

Theorem 1: In an Einstein manifold (M_n, g) equipped with Ricci quarter symmetric metric connection, the following conditions are equivalent:

a. M_n is locally symmetric.

b. M_n is conharmonically symmetric.

c. M_n is M-projectively symmetric.

CONHARMONIC BI-SYMMETRIC AND *M*–PROJECTIVE BI- SYMMETRIC MANIFOLDS

An Einstein manifold is said to be bi-symmetric if it satisfies

$$(D_V D_U R)(X,Y)Z = 0. \tag{23}$$

An Einstein manifold is said to be conharmonic and M-projective bisymmetric manifold respectively, if it satisfies

$$(D_V D_U C)(X,Y)Z = 0 \tag{24}$$

and

$$(D_V D_U W^*)(X,Y)Z = 0. \tag{25}$$

Taking the covariant differentiation on both sides of (21) and (22), we obtain

$$(D_V D_U\bar{C})(X,Y)Z = (D_V D_U R)(X,Y)Z \tag{26}$$

and

$$(D_V D_U\overline{W^*})(X,Y)Z = (D_V D_U R)(X,Y)Z \tag{27}$$

Thus we can state that:

Theorem 2: In an Einstein manifold (M_n, g) equipped with Ricci quarter symmetric metric connection, the following conditions are equivalent:

a. M_n is bi-symmetric.

b. M_n is conharmonically bi-symmetric.

c. M_n is M-projectively bi-symmetric.

GENERALIZED 2-RECURRENT RIEMANNIAN MANIFOLDS

A non-flat Riemannian manifold of dimension *n* is called generalised 2- recurrent Riemannian manifold [6] when the Riemannian curvature tensor *R* satisfies the condition

$$(D_V D_U R)(X,Y)Z = A(V)(D_U R)(X,Y)Z + B(U,V)R(X,Y)Z, \tag{28}$$

where, A is a 1-form, *B* is a non-zero (0, 2) tensor. The tensor *B* is defined by

$$B(X,Y) = g(X, Q^I Y), \tag{29}$$

where Q^I is a linear transformation from the tangent space at

$(p\epsilon M_n)$: $T_p(M_n) \to T_p(M_n)$.

When the conharmonic and M-projective curvature tensor satisfy the conditions

$$(D_V D_U\bar{C})(X,Y)Z = A(V)(D_U\bar{C})(X,Y)Z + B(U,V)\bar{C}(X,Y)Z \tag{30}$$

and

$$(D_V D_U\overline{W^*})(X,Y)Z = A(V)(D_U\overline{W^*})(X,Y)Z + B(U,V)\overline{W^*}(X,Y)Z, \tag{31}$$

then the manifold is respectively called generalized conharmonically 2-recurrent manifold and generalized M-projectively 2-recurrent manifold, where A, *B* are stated earlier.

Using Bianchi's second identity, we find from (21) that

$$(D_U\bar{C})(X,Y)Z + (D_Y\bar{C})(U,X)Z + (D_X\bar{C})(Y,U)Z) = 0. \tag{32}$$

Again from (32), we find that

$$(D_V D_U\bar{C})(X,Y)Z + (D_V D_Y\bar{C})(U,X)Z + (D_V D_X\bar{C})(Y,U)Z) = 0. \tag{33}$$

In consequence of (30) and (32), the equation (33) yields

$$B(U,V)\bar{C}(X,Y)Z + B(Y,V)\bar{C}(U,X)Z + B(X,V)\bar{C}(Y,U)Z = 0. \tag{34}$$

Now, contracting (34), we get

$$B(\bar{C}(X,Y)Z, V) = 0. \tag{35}$$

From (16), we get

$$'\bar{C}(X, Y, Z, W) = -'\bar{C}(X, Y, W, Z)$$
$$= -'\bar{C}(Y, X, Z, W)$$
$$= -'\bar{C}(Z, W, X, Y). \quad (36)$$

Now, putting $U = Q^l V$ and using (29), the expression (34) takes the form

$$g(Q^lV, Q^lV)\,\bar{C}(X, Y)Z + g(X, Q^lY)\,\bar{C}(X, Q^lY)Z$$
$$+ g(Y, Q^lV)\,\bar{C}(Q^lV, X)Z = 0. \quad (37)$$

Using (35) and (36) in (37), we have

$$g(Q^lV, Q^lV)\,\bar{C}(X, Y)Z = 0. \quad (38)$$

From which we obtain

$$\bar{C}(X, Y)Z = 0.z \quad (39)$$

Thus we can state

Theorem 3: A generalized conharmonically 2-recurrent Einstein manifold equipped with Ricci quarter symmetric metric connection is conharmonically flat.

Next, we assume that the manifold be generalized m-projectively 2-recurrent. Then it follows from (22) and Bianchi's second identity that

$$(D_U\overline{W^*})(X, Y)Z + (D_Y\overline{W^*})(U, X)Z + (D_X\overline{W^*})(Y, U)Z = 0. \quad (40)$$

After covariant differentiation of (40) that

$$(D_V D_U\overline{W^*})(X, Y)Z + (D_V D_Y\overline{W^*})(U, X)Z$$
$$+ (D_V D_X\overline{W^*})(Y, U)Z = 0. \quad (41)$$

Using (31) and (40) in (41), we get

$$B(U, V)\,\overline{W^*}(X, Y)Z + B(Y, V)\,\overline{W^*}(U, X)Z$$
$$+ B(X, V)\,\overline{W^*}(Y, U)Z = 0. \quad (42)$$

Contracting (42), we get

$$B(\overline{W^*}(X, Y)Z, V) = 0. \quad (43)$$

From (18), we get

$$'\overline{W^*}(X, Y, Z, W) = -'\overline{W^*}(X, Y, W, Z)$$
$$= -'\overline{W^*}(Y, X, Z, W)$$
$$= -'\overline{W^*}(Z, W, X, Y). \quad (44)$$

Now, putting $U = Q^l V$ and using (29), the expression (44) takes the form

$$g(Q^lV, Q^lV)\,\overline{W^*}(X, Y)Z + g(X, Q^lV)\,\overline{W^*}(Y, Q^lV)Z$$
$$+ g(Y, Q^lV)\,\overline{W^*}(Q^lV, X)Z = 0. \quad (45)$$

Using (43) and (44) in (45), we get

$$g(Q^lV, Q^lV)\,\overline{W^*}(X, Y)Z = 0. \quad (46)$$

From which we obtain

$$\overline{W^*}(X, Y)Z = 0. \quad (47)$$

Hence we can state that:

Theorem 4: An Einstein manifold equipped with a Ricci Quarter symmetric metric connection is a generalized M-projectively 2-recurrent if and only if it is an M-projectively flat.

CONHARMONIC AND CONCIRCULAR TENSORS OF $\bar{\nabla}$

The concircular curvature tensor on Riemannian manifold is defined by

$$'V(X, Y, Z, U) = 'R(X, Y, Z, U)$$
$$-\frac{r}{n(n-1)}\{g(Y, Z)\,g(X, U) - g(X, Z)\,g(Y, U)\}, \quad (48)$$

where,

$$'V(X, Y, Z, U) = g(V(X, Y)Z, U) \quad (49)$$

Let $'\bar{V}$ denote the concircular curvature tensor of the connection $\bar{\nabla}$. Then,

$$'\bar{V}(X, Y, Z, U) = '\bar{R}(X, Y, Z, U)$$
$$-\frac{\bar{r}}{n(n-1)}\{g(Y, Z)\,g(X, U) - g(X, Z)\,g(Y, U)\}. \quad (50)$$

Applying (9), (10) in (50), we have

$$'\bar{V}(X, Y, Z, U) = 'R(X, Y, Z, U)$$
$$-\frac{r(2r+n-2nm+2m)}{n^2(n-1)}\{g(Y, Z)g(X,U) - g(X,Z)g(Y,U)\}. \quad (51)$$

From (51), we have

$$\bar{V}(X, Y)Z = R(X, Y)Z$$
$$-\frac{r(2r+n-2nm+2m)}{n^2(n-1)}\{g(Y, Z)X - g(X, Z)Y\}. \quad (52)$$

Taking covariant derivative of (52) we obtain

$$(D_U\bar{V})(X, Y)Z = (D_U R)(X, Y)Z. \quad (53)$$

Thus from (21) and (53) we can state the following:

Theorem 5: In an Einstein manifold (M_n, g) equipped with Ricci quarter symmetric metric connection, the following conditions are equivalent:

a. M_n is locally symmetric.
b. M_n is conharmonically symmetric.
c. M_n is concircularly symmetric.

An Einstein manifold will be called concircular bi-symmetric manifold if it satisfies

$$(D_V D_U V)(X, Y)Z = 0. \tag{54}$$

Taking the covariant differentiation on both sides of (54) and we get

$$(D_V D_U V)(X, Y)Z = (D_V D_U R)(X, Y) Z. \tag{55}$$

Hence, from (26) and (55), we conclude the following:

Theorem 6: In an Einstein manifold (M_n, g) equipped with Ricci quarter symmetric metric connection, the following conditions are equivalent:

a. M_n is bi-symmetric.

b. M_n is conharmonically bi-symmetric.

c. M_n is concircularly bi-symmetric.

A non-flat Riemannian manifold of dimension *n* is called generalized concircularly 2- recurrent Riemannian manifold [6] when the concircular curvature tensor *V* satisfies the condition

$$(D_U D_W \bar{V})(X, Y)Z = A(U)(D_W \bar{V})(X, Y) Z + B(W, U)\bar{V}(X, Y) Z, \tag{56}$$

where, *A* and *B* are stated earlier. Assume that the manifold be generalised concircularly 2-recurrent. Then, it follows from (52) and Bianchi's identity that

$$(D_W \bar{V})(X, Y)Z + (D_Y \bar{V})(W, X) Z + (D_X \bar{V})(Y, W) Z = 0. \tag{57}$$

After covariant differentiation, we have

$$(D_U D_W \bar{V})(X, Y)Z + (D_U D_Y \bar{V})(W, X) Z + (D_U D_X \bar{V})(Y, W) Z = 0. \tag{58}$$

Using (56) and (57) in (58), we get

$$B(W,U)\bar{V}(X, Y)Z + B(Y,U)\bar{V}(W,X)Z + B(X,U)\bar{V}(Y,W)Z = 0. \tag{59}$$

Contracting (59), we have

$$B(\bar{V}(X, Y) Z, U) = 0. \tag{60}$$

From (50), we have

$$\begin{aligned} {}'\bar{V}(X, Y, Z, W) &= -{}'\bar{V}(X, Y, W, Z) \\ &= -{}'\bar{V}(Y, X, Z, W) \\ &= -{}'\bar{V}(Z, W, X, Y). \end{aligned} \tag{61}$$

Now, putting $W = Q^l U$ and using (29) and the expression (61) takes the form

$$g(Q^lU,Q^lU)\bar{V}(X, Y)Z + g(X,Q^lU)\bar{V}(Y,Q^lU)Z + g(Y,Q^lU)\bar{V}(Q^lU, X) Z = 0. \tag{62}$$

Using (60) and (61) in (62), we have

$$g(Q^lU,Q^lU)\bar{V}(X, Y)Z = 0. \tag{63}$$

From which it follows that

$$\bar{V}(X, Y) Z = 0. \tag{64}$$

Hence we can state

Theorem 7: A generalized concircular 2-recurrent Einstein manifold equipped with Ricci quarter symmetric metric connection is concircularly flat.

REFERENCES

1. Hayden, H.A.1932. Subspaces of a space with torsion tensor. Proc. London Math. Soc.34: 27-50.
2. Yano, K.1970. On symmetric metric connection. Rev. Roumaine Math. Pure Appl. Math.15: 1579-1586.
3. Golab, S.1975. On semi-symmetric and quarter symmetric linear connections. Tensor N. S.29: 249-254.
4. Mishra, R.S. and Pandey, S. N.1980. On Quarter symmetric metric F- connection. Tensor N. S. 34: 1-7.
5. Pokhariyal, G.P. and Mishra, R.S.1970. The curvature tensors and their relativistic significance. Yakohama Math. J. 18: 105-108.
6. De, U.C. and Pathak, G. 2003. On Riemannian manifold with certain curvature conditions. Mathematica Pannonica.14 (2): 227-235.