AN ANALYTICAL STUDY OF CERTAIN ALMOST CONTACT MANIFOLDS

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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Ph.D REGN. No.: MZU/Ph.D./946 of 26.10.2016

MZU REGN. No.: 1631 of 2009-10

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DECEMBER, 2020

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Submitted
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CERTIFICATE

This is to certify that the thesis entitled "An Analytical Study of Certain Almost Contact Manifolds" submitted by Mr. C. Lalmalsawma (Registration No: MZU/ Ph. D./946 of 26.10.2016) for the degree of Doctor of Philosophy (Ph. D.) of the Mizoram University, embodies the record of original investigation carried out by him under my supervision. He has been duly registered and the thesis presented is worthy of being considered for the award of the Ph. D. degree. This work has not been submitted for any degree of any other universities.

(Dr. JAY PRAKASH SINGH)
Supervisor

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Month: December **Year: 2020**

CANDIDATE'S DECLARATION

I, C. Lalmalsawma, hereby declare that the subject matter of this thesis entitled "An Analytical Study of Certain Almost Contact Manifolds" is the record of work done by me, that the contents of this thesis do not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in other University/Institute.

This is being submitted to the Mizoram University for the degree of Doctor of Philosophy (Ph. D.) in Mathematics.

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ACKNOWLEDGEMENT

Firstly, I would like to express my profound sense of gratitude to Dr. Jay Prakash Singh, Associate Professor and Head of department, Department of Mathematics and Computer Science, Mizoram University, Aizawl, for offering unstinting co-operation, help and guidance at every stage of accomplishment of this work.

I cannot but acknowledge the purposeful assistance, timely suggestions and valuable opinions extended to me by Prof. Zaithanzauva Pachuau, Department of Physics and Dean, School of physical Sciences, Mizoram University, Aizawl.

I am indeed very grateful to such experts as Prof. Jamal Hussain, Dr. S.S. Singh, Mr. Laltanpuia and Dr. M. Saroja Devi for extending to me their valuable assistance in the completion of this work. I also wish to express my thanks to all my friends Ms. K. Lalnunsiami, Mr. A. Lalchhuangliana, Mr. Lalawmpuia, Mr. Vanlalruata, Ms. Laldinsangi and Ms. H. Lalduhsaki for helping me in various ways.

Finally, I would like to thank everybody who contributed in the successful realization of this thesis, as well as expressing my apology that I could not mention personally one by one. I am ever grateful to God, the merciful and the passionate, for providing me the opportunity to step in the excellent world of science.

Last but not the least, I would like to express my deep sense of gratitude to my beloved mother Mrs. Zirkungi without whom I would not have been able to achieve my goal.

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PREFACE

The present thesis entitled "An Analytical Study of Certain Almost Contact Manifolds" is an outcome of the research carried out by the author under the supervision of Dr. Jay Prakash Singh, Associate professor, Department of Mathematics and Computer science, Mizoram University, Aizawl, Mizoram.

This thesis has been divided into six chapters and each chapter is subdivided into a smaller sections. The first chapter is introductory in which we have defined topological manifolds, differentiable manifolds, vector fields and tangent spaces, Lie-bracket, Covariant derivatives, Lie derivative and exterior derivatives, connection, Riemannian manifolds, Torsion tensor, Ricci tensor and curvature tensors on Riemannian manifolds. We also gives some mathematical tools necessary for the studies.

The second chapter is related with the characterization of Ricci solitons in almost contact manifolds. In this chapter we studied the Ricci solitons and gradient Ricci solitons and we have given the cases in which the Ricci solitons is shrinking, steady or expanding. The studies also includes η-Ricci soliton which is generalization of Ricci soliton. We considered certain geometric condition in Sasakian manifolds for the study of η-Ricci soliton.

The third chapter deals with a study of generalized Sasakian-space-form. In this chapter we studied τ-curvature tensor in generalized Sasakian-spaceform. We obtain results for each particular case of the curvature tensor. We also studied the cases in which the generalized Sasakian-spaceform admits generalized Tanaka-Webster connection. We considered symmetric properties with respect to the generalized Tanaka-Webster connection.

The fourth chapter deals with the study of generalized recurrent manifolds. We considered a manifold which is generalized recurrent with respect to the pseudo-projective curvature tensor. Some curvature condition are considered to obtain various results. We have also given two examples to support the results.

In the fifth chapter we studied quarter-symmetric non-metric connection in trans-Sasakian manifolds. In this chapter we considered weakly symmetries, local symmetries, semisymmetries and recurrency with respect to the quarter-symmetric non-metric connection. All studies in the chapter is considered in trans-Sasakian manifolds.

Chapter 6 is the last chapter. In this chapter we gave the summary of the whole work and we concluded the thesis.

In the end, the references of the papers of the authors have been given with surname of the author and their years of the publication, which are decoded in chronological order in the Bibliography.

Contents

Chapter 1

INTRODUCTION

1.1 Topological Space

A topology on a set S is a collection $\mathcal{T}$ of subsets containing both the empty set $\emptyset$ and the set S such that $\mathcal{T}$ is closed under arbitrary unions and finite intersections: i.e., if $U_\alpha \in \mathcal{T}$ for all α in an index set A, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$ and if $U_1, ..., U_n \in \mathcal{T}$, then $\bigcap_i^n U_i \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called open sets and the pair $(S, \mathcal{T})$ is called a topological space. A topological space is second countable if it has a countable basis. A neighborhood of a point p in S is an open set U containing p.

A topological space M is locally Euclidean of dimension n if every point p in M has a neighborhood U such that there is a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. We call the pair $(U, \phi : U \to \mathbb{R}^n)$ a chart, U a coordinate neighborhood or a coordinate open set, and ϕ a coordinate map or a coordinate system on U. We say that a chart (U, ϕ) is centered at $p \in U$ if $\phi(p) = 0$.

1.2 Topological Manifolds

A topological manifold is a Hausdorff, second countable, locally Euclidean space. It is said to be of dimension n if it is locally Euclidean of dimension n.

Definition 1.2.1 *The Charts $(U, \phi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ are said to be C^∞-compatible if $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ and $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ are C^∞-mappings.*

Definition 1.2.2 *A C^∞ atlas or simply an atlas on a locally Euclidean space M is a collection $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ of pairwise C^∞-compatible charts that cover M, i.e., such* *that* $M = \bigcup_\alpha U_\alpha$.

Definition 1.2.3 *An atlas* $\mathfrak{M}$ *on a locally Euclidean space is said to be maximal if it is not contained in a larger atlas; in other words, if* $\mathfrak{U}$ *is any other atlas containing* $\mathfrak{M}$*, then* $\mathfrak{U} = \mathfrak{M}$.

1.3 Differentiable Manifolds

Definition 1.3.1 *A differentiable or* C^∞ *manifold is a topological manifold* M^n *together with a maximal atlas.*

The maximal atlas is also called a differentiable structure on M^n. A manifold is said to have dimension n if all of its connected components have dimension n. A 1-dimensional manifold is also called a curve, a 2-dimensional manifold a surface.

1.4 Tangent Spaces

Let M^n be an n-dimensional differentiable manifold and $p \in M^n$ and $C^\infty(p)$ be the set of all real valued C^∞ function on some neighbourhood $\cup$ of p. Let us consider a vector X at p such that
(i) $X \in M^n$, $f \in C^\infty(p)$ implies that $Xf \in C^\infty(p)$,
(ii) $X(f+g) = Xf + Xg$, $\quad f, g \in C^\infty(p)$,
(iii) $X(fg) = f(Xg) + (Xf)g$, and
(iv) $X(af) = a(Xf)$, $a \in R$,
then X is called a tangent vector to M^n at p.

The collection of all tangent vectors through the point p forms a tangent space of a manifold M^n to the point p and is denoted by $T_p(M)$ and its elements are called tangent vectors to the manifold at p.

1.5 Connection

Let us consider a C^∞-manifold M^n. Let $p \in M^n$ be a point of M^n. Let $T_{(p)}$ be a tangent space to M^n at the point p. Let T^r_s be a vector space whose elements are the tensors of the type (r, s).A connection ∇ is a type preserving mapping $\nabla : T_{(p)} \otimes T^r_s \to T^r_s$, which assigns to each pair of C^∞-vector field (X, P), $X \in X_p$, $P \in T^r_s$, a C^∞-vector fields $\nabla_X P$, such that

$$\nabla_X f = Xf, \qquad f \quad \text{is} \quad C^\infty\text{-function}$$

$$\nabla_X a = 0, \qquad a \in R,$$

$$\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z,$$

$$\nabla_X (fY) = f(\nabla_X Y) + (Xf)Y,$$

$$\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z,$$

$$\nabla_{fX} Y = f \nabla_X Y,$$

$$(\nabla_X A)Y = X(A(Y)) - A(\nabla_X Y),$$

and

$$\begin{aligned}
(\nabla_X P)(A_1, ..., A_r, X_1, ...X_s) &= X(P(A_1, ...A_r, X_1, ...X_s)) \\
&- P(\nabla_X A_1, A_2, ...A_r, X_1, ...X_s)......... \\
&- P(A_1......A_r, X_1, ...\nabla_X X_s).
\end{aligned}$$

1.6 Lie Brackets And Lie Derivative

Lie Brackets: Let X and Y be arbitrary C^∞ vector field of M^n. Then a mapping $[\quad , \quad] : M^n \times M^n \to M^n$ such that

$$[X, Y]f = X(Yf) - Y(Xf),$$

where f is a C^∞-function on M^n is called the Lie-bracket of C^∞ vector fields X and Y.

The Lie-bracket has the following properties:

$$[X, Y](f + g) = [X, Y]f + [X, Y]g,$$

$$[X,Y](f.g) = f[X,Y]g + g[X,Y]f,$$

$$[X,Y] + [Y,X] = 0, \qquad \text{(skew-symmetry)}$$

$$[X+Y,Z] = [X,Z] + [Y,Z], \qquad \text{(bilinear)}$$

$$[X,[Y+Z]] + [Y,[Z+X]] + [Z,[X,Y]] = 0, \text{(Jacobian identity)}$$

and

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X.$$

Lie Derivative: Let X be a C^∞ vector field on an open set of M^n. An operator L_X is called the Lie derivative along the vector field X if it is a type preserving mapping

$$L_X : T^r_s \to T^r_s \text{ such that}$$

$$L_X f = Xf,$$

$$L_X a = 0, \qquad a \in \mathbb{R},$$

$$L_X Y = [X,Y], \qquad Y \in T_p,$$

$$(L_X A)(Y) = X(A(Y)) - A([X,Y]), \qquad where\ A\ is\ a\ 1-form$$

and

$$\begin{aligned}
(L_X P)(A_1, ...A_r, X_1, ...X_s) &= X(P(A_1, ...A_r, X_1, ...X_s)) \\
&- P(L_X, (A_1, ...A_r, X_1, ...X_s)... \\
&- P((A_1, ...A_r, [X, X_1], X_2...X_s)... \\
&\quad .... \\
&- P(A_1...X_{s-1}, [X, X_s]), P \in T^r_s,
\end{aligned}$$

where f is a C^∞ function, $X_1, ...X_s$ are vector fields, $A_1, ...A_r$ are 1-forms and P is a tensor field of type (r,s) is called Lie differentiation with respect to X and $L_X P$

is called Lie derivative of P with respect to X.

COVARIANT DERIVATIVE: A linear affine connection on M is a function

$$\nabla : \psi(M) * \psi(M) \to \psi(M) \text{ such that}$$

$$\nabla_{fX+gY} Z = f(\nabla_X Z) + g(\nabla_Y Z),$$

$$\nabla_X f = Xf,$$

$$\nabla_X(fY + gZ) = f(\nabla_X Y) + g(\nabla_X Z) + (Xf)Y + (Xg)Z,$$

for arbitrary vector fields X, Y, Z and smooth function f, g on M. ∇_X is called covariant derivative operator and $\nabla_X Y$ is called covariant derivative of Y with respect to X.

The covariant derivative of a 1-form w is given by

$$(\nabla_X w)(Y) = X(w(Y)) - w(\nabla_X Y).$$

1.7 Contraction

The linear mapping
$C^h_k : T^r_s \to T^{r-1}_{s-1}; \qquad (i \leq h \leq r) \qquad , (i \leq k \leq s)$
such that

$$C^h_k(\lambda_1 \otimes \lambda_2 \otimes ... \otimes \lambda_r \otimes \alpha_1 \otimes ...\alpha_s) = \alpha_k(\lambda_1 \otimes .. \otimes \lambda_{h-1} \otimes \lambda_{h+1}... \\ \otimes \lambda_r \otimes \alpha_1 \otimes \alpha_2 \otimes ..\alpha_{k-1} \otimes \alpha_{k+1} \otimes \alpha_s)$$

where $\lambda_1, \lambda_2...\lambda_r \in T_pM^n$ and $\alpha_1, \alpha_2...\alpha_s \in T_p\bar{M}^n$ and $\otimes$ denote tensor product, is called contraction with respect to h^{th} contravariant and k^{th} covariant places.

1.8 Riemannian Manifold

Let us consider an n-dimensional C^∞ with the tangent space T_p at $p \in M^n$. A real valued, bilinear, symmetric, non-singular positive definite function g on the ordered pair X, Y of tangent vectors $T_{(p)}$ at each point p, such that

(1) $g(X, Y)$ is a real number,

(2) g is symmetric $\Rightarrow g(X,Y) = g(Y,X)$,

(3) g is non-singular i.e. $g(X,Y) = 0$, for all $Y \neq 0 \Rightarrow X = 0$,

(4) g is positive definite i.e. $g(X,X) \geqslant 0$, for all $X \in C^\infty$ and $g(X,X) = 0$ if and only if $X = 0$,

and

(5) $g(aX + bY, Z) = ag(X,Z) + bg(Y,Z)$; $a, b \in \mathbb{R}$,

then g is said to be Riemannian metric tensor or fundamental tensor of type (0,2).

Then, the manifold M^n with a Riemannian metric g is called a Riemannian manifold and its geometry is called a Riemannian geometry denoted by (M^n, g) or (M, g) or simply by M.

1.9 Torsion Tensor

A vector valued, skew-symmetry, bilinear function T of the type $(1,2)$ defined by

$$T(X,Y) \stackrel{def}{=} \nabla_X Y - \nabla_Y X - [X,Y] \tag{1.9.1}$$

is called a torsion tensor of the connection ∇ in a C^∞-manifold M^n.
If the torsion tensor of a connection ∇ vanishes, it is said to be symmetric or torsion free.
A connection ∇ is said to be Riemannian, if

$$T(X,Y) = 0 \tag{1.9.2}$$

and

$$\nabla_X g = 0. \tag{1.9.3}$$

1.10 Semi Symmetric and Quarter Symmetric Connection

A linear connection $\tilde{\nabla}$ in a Riemannian manifold is said to be a quarter-symmetric connection (Golab, 1975) if the torsion tensor T of $\tilde{\nabla}$ satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.10.1}$$

where η is a 1-form and ϕ is a $(1,1)$ tensor field. In particular if $\phi X = X$ for all $X \in \chi(M)$, then the quarter-symmetric connection reduces to a semi-symmetric connection. Moreover if a quarter-symmetric linear connection satisfies the condition

$$(\tilde{\nabla}_X g)(Y,Z) = 0, \tag{1.10.2}$$

for all $X, Y, Z \in \chi(M)$, then is said to be a quarter-symmetric metric connection.

1.11 Generalized TanakaWebster connection

The generalized TanakaWebster connection $\tilde{\nabla}$ for contact metric manifolds is given by (Tanno, 1989)

$$\tilde{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi(Y) \tag{1.11.1}$$

where $X, Y \in \chi M$, and ∇ is the Riemannian connection.

1.12 Curvature Tensors

The curvature tensor R with respect to the Riemannian connection ∇ is given by

$$R(X,Y,Z) \overset{def}{=} \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{1.12.1}$$

is called Riemann-Christoffel curvature tensor of the second kind i.e. of the type $(1,3)$.

Let $'R$ be the associate curvature tensor of the type $(0,4)$ of the curvature tensor R. Then

$$'R(X,Y,Z,W) = g(R(X,Y,Z),W), \tag{1.12.2}$$

$'R$ is called the Riemann-Christoffel curvature tensor of first kind.

The following identities are satisfied by associate curvature tensor $'R$:
$'R$ is skew-symmetric in first two slot

$$i.e., \quad 'R(X,Y,Z,W) = -'R(Y,X,Z,W) \tag{1.12.3}$$

$'R$ is skew-symmetric in last two slot

$$i.e., \quad 'R(X,Y,Z,W) = -'R(X,Y,W,Z) \tag{1.12.4}$$

$'R$ is symmetric in two pair of slot

$$i.e., \quad 'R(X,Y,Z,W) =' R(Z,W,X,Y) \tag{1.12.5}$$

$'R$ satisfies Bianchi's first identities

$$i.e., \quad 'R(X,Y,Z,W) +' R(Y,Z,X,W) +' R(Z,X,Y,W) = 0 \tag{1.12.6}$$

and $'R$ satisfies Bianchi's second identities

$$\begin{aligned} i.e., \qquad & (\nabla_X \, 'R)(Y,Z,W,V) + (\nabla_Y \, 'R)(Z,X,W,V) \\ + \quad & (\nabla_Z \, 'R)(X,Y,W,V) = 0. \end{aligned} \tag{1.12.7}$$

1.13 Ricci-Tensor

The tensor defined by

$$S(Y,Z) \stackrel{def}{=} (C_1^1 R)(Y,Z) = -(C_2^1 R)(Z,Y) \tag{1.13.1}$$

is called the Ricci-tensor of type $(0,2)$ where C_1^1 and C_2^1 denote respective contractions. It is symmetric tensor,

$$i.e., \quad S(X,Y) = S(Y,X). \tag{1.13.2}$$

The linear map Q of the type $(1,1)$ defined by

$$g(QX,Y) \stackrel{def}{=} S(X,Y) \tag{1.13.3}$$

is called a Ricci-map. It is self-adjoint,

$$i.e., \quad g(QX,Y) = g(X,QY). \tag{1.13.4}$$

The scalar r defined by

$$r \stackrel{def}{=} (C_1^1 R) \tag{1.13.5}$$

is called the scalar curvature of M^n at the point p.
A Riemannian manifold M^n is said to be Einstein manifold if

$$S(X,Y) = \frac{r}{n} g(X,Y). \tag{1.13.6}$$

A Riemannian manifold M^n is said to be flat manifold if

$$R(X, Y, Z) = 0. \tag{1.13.7}$$

1.14 Z-tensor

The Z-tensor in Riemannian manifolds is given by (Mantica and Suh, 2012)

$$\tilde{\mathbf{Z}}(X, Y) = S(X, Y) + \phi_1 g(X, Y), \tag{1.14.1}$$

where ϕ_1 is an arbitrary scalar function and S and g denotes the Ricci tensor and metric tensor respectively.

1.15 Ricci Soliton

Ricci soliton generalized the Einstein metric and is defined as a triple (g, V, λ) with g a Riemannian metric, V a vector field, and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.15.1}$$

where S is a Ricci tensor and $\pounds$ denotes the Lie derivative operator along the vector field V (Ivey, 1993). The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive respectively (Chow et al., 2006). If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and equation (1.15.1) assumes the form

$$\nabla\nabla f = S + \lambda g. \tag{1.15.2}$$

An η-Ricci soliton is a tuple (g, V, λ, μ), where V is a vector field on M, λ and μ are constants, and g is a Riemannian (or pseudo-Riemannian) metric satisfying the equation

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0. \tag{1.15.3}$$

1.16 Important Curvature Tensors On Riemannian Manifolds

The concircular curvature tensor $'\dot{C}$ of type $(0,4)$, $(n>3)$ is given by (Yano, 1940)

$$\begin{aligned} '\dot{C}(X,Y,Z,W) &= {}'R(X,Y,Z,W) - \frac{r}{n(n-1)}\{g(Y,Z)g(X,W) \\ &- g(X,Z)g(Y,W)\}. \end{aligned} \tag{1.16.1}$$

It satisfies the following algebraic properties

$$\begin{aligned} &(a)\ '\dot{C}(X,Y,Z,W) = -'\dot{C}(Y,X,Z,W), \\ &(b)\ '\dot{C}(X,Y,Z,W) = -'\dot{C}(X,Y,W,Z), \\ &(c)\ '\dot{C}(X,Y,Z,W) = \ '\dot{C}(Z,W,X,Y), \\ &(d)\ '\dot{C}(X,Y,Z,W) +' \dot{C}(Y,Z,X,W) +' \dot{C}(Z,X,Y,W) = 0, \end{aligned}$$

where

$$'\dot{C}(X,Y,Z,W) = g(\dot{C}(X,Y,Z),W).$$

The conharmonic curvature tensor $'\hat{C}$ of the type $(0,4)$, $(n>3)$ is defined as follows (Ishii, 1957)

$$\begin{aligned} '\hat{C}(X,Y,Z,W) &=' R(X,Y,Z,W) - \frac{1}{n-1}\{S(Y,Z)g(X,W) \\ - \ & S(X,Z)g(Y,W) + S(X,W)g(Y,Z) - S(Y,W)g(X,Z)\}. \end{aligned} \tag{1.16.2}$$

It satisfies the following properties

$$\begin{aligned} &(a)\ '\hat{C}(X,Y,Z,W) = -'\hat{C}(Y,X,Z,W), \\ &(b)\ '\hat{C}(X,Y,Z,W) = \ '\hat{C}(X,Y,W,Z), \\ &(c)\ '\hat{C}(X,Y,Z,W) = \ '\hat{C}(Z,W,X,Y), \\ &(d)\ '\hat{C}(X,Y,Z,W) +' \hat{C}(Y,Z,X,W) +' \hat{C}(Z,X,Y,W) = 0 \end{aligned}$$

where

$$'\hat{C}(X,Y,Z,W) = g(\hat{C}(X,Y,Z),W).$$

The pseudo-projective curvature tensor in an n-dimensional Riemannian manifold

(M^n, g), denoted by $\hat{P}$ and is defined by (Prasad, 2002)

$$\begin{aligned} P(Y,Z)U &= aR(Y,Z)U + b[S(Z,U)Y - S(Y,U)Z] \\ &- \frac{r}{n}\Big(\frac{a}{n-1} + b\Big)[g(Z,U)Y - g(Y,U)Z], \end{aligned} \tag{1.16.3}$$

where a and b are non-zero constants. Such a tensor P is known as pseudo-projective curvature tensor. The pseudo-projective curvature tensor P satisfies the following identities

$$\begin{aligned} &(i) \quad 'P(Y,Z,U,V) = -'P(Z,Y,U,V), \\ &(ii) \quad 'P(Y,Z,U,V) \neq \mp' P(Y,Z,V,U). \end{aligned}$$

The projective curvature tensor $'\tilde{P}$ of the type $(0,4)$, $(n > 2)$ is defined by

$$\begin{aligned} '\tilde{P}(X,Y,Z,W) &= 'R(X,Y,Z,W) - \frac{1}{n-1}\{S(Y,Z)g(X,W) \\ &- S(X,Z)g(Y,W)\}. \end{aligned} \tag{1.16.4}$$

The projective curvature tensor $'P$ satisfies the following identities

$$\begin{aligned} &(a)\ '\tilde{P}(X,Y,Z,W) = -'\tilde{P}(Y,X,Z,W), \\ &(b)\ C_1^1 P = C_2^1 P = C_3^1 P = 0, \\ &(c)\ '\tilde{P}(X,Y,Z,W) +' \tilde{P}(Y,Z,X,W) +' \tilde{P}(Z,X,Y,W) = 0, \end{aligned}$$

where

$$'\tilde{P}(X,Y,Z,W) = g(\tilde{P}(X,Y,Z),W).$$

The M-projective curvature tensor $'M$ of the type $(0,4)$, $(n > 2)$ is defined by

$$\begin{aligned} &'M(X,Y,Z,W) =' R(X,Y,Z,W) - \frac{1}{2(n-1)}\{g(X,W)S(Y,Z) \\ &- \quad g(Y,W)S(X,Z) + S(X,W)g(Y,Z) - S(Y,W)g(X,Z)\}. \end{aligned} \tag{1.16.5}$$

The M-procective curvature tensor $'M$ satisfies the following identities

$$(c)\ 'M(X,Y,Z,W) = \ 'M(Z,W,X,Y),$$
$$(a)\ 'M(X,Y,Z,W) = -'M(Y,X,Z,W),$$
$$(b)\ 'M(X,Y,Z,W) = -'M(X,Y,W,Z),$$
$$(d)\ 'M(X,Y,Z,W) +' M(Y,Z,X,W) +' M(Z,X,Y,W) = 0$$

where

$$'M(X,Y,Z,W) = g(M(X,Y,Z),W).$$

The Quasi-conformal curvature tensor $'C$ of the type $(0,4)$, $(n>3)$ is defined as

$$\begin{aligned} 'C(X,Y,Z,W) &= a'R(X,Y,Z,W) + b[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &+ g(Y,Z)S(X,W) - g(X,Z)S(Y,W)] \\ &+ \frac{r}{n}(\frac{a}{n-1} + 2b)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \quad (1.16.6) \end{aligned}$$

It satisfies the following identities

$$(a)\ 'C(X,Y,Z,W) = -'C(Y,X,Z,W),$$
$$(b)\ 'C(X,Y,Z,W) = -'C(X,Y,W,Z),$$
$$(c)\ 'C(X,Y,Z,W) = \ 'C(Z,W,X,Y),$$
$$(d)\ 'C(X,Y,Z,W) +' C(Y,Z,X,W) +' C(Z,X,Y,W) = 0$$

where

$$'C(X,Y,Z,W) = g(C(X,Y,Z),W).$$

Finally the Weyl conformal curvature tensor $'\tilde{C}$ of type $(0,4)$ which is defined as

$$\begin{aligned} '\tilde{C}(X,Y,Z,W) &= \ 'R(X,Y,Z,W) - \frac{1}{n-2}\big[g(Y,Z)S(X,W) \\ &- g(X,Z)S(Y,W) + S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\big] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \quad (1.16.7) \end{aligned}$$

1.17 τ-curvature Tensor

In an n-dimensional semi-Riemannian manifold, a τ-curvature tensor is a tensor of type $(1,3)$, which is defined by (Tripathi and Gupta, 2011)

$$\begin{aligned}\tau(X,Y)Z = a_0R(X,Y)Z + a_1S(Y,Z)X + a_2S(X,Z)Y \\ +a_3S(X,Y)Z + a_4g(Y,Z)QX + a_5g(X,Z)QY \\ +a_6g(X,Y)QZ + a_7r[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{1.17.1}$$

Particularly the τ-curvature tensor reduces to

1. Riemannian curvature tensor R if

$$a_0 = 1,\ a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

2. Quasi conformal curvature tensor C if

$$a_1 = -a_2 = a_4 = -a_5,\ a_3 = a_6 = 0,\ a_7 = -\frac{1}{n}\Big(\frac{a_0}{(n-1)} + 2a_1\Big),$$

3. Conformal curvature tensor $\tilde{C}$ if

$$a_0 = 1,\ a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{n-2},\ a_3 = a_6 = 0,\ a_7 = \frac{1}{(n-1(n-2))},$$

4. Conharmonic curvature tensor $\hat{C}$ if

$$a_0 = 1,\ a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{n-2},\ a_3 = a_6 = a_7 = 0,$$

5. Concircular curvature tensor $\dot{C}$ if

$$a_0 = 1,\ a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0,\ a_7 = -\frac{1}{n(n-1)},$$

6. Pseudo-projective curvature tensor P if

$$a_1 = -a_2,\ a_3 = a_4 = a_5 = a_6 = 0,\ a_7 = -\frac{1}{n}\Big(\frac{a_0}{(n-1)} + a_1\Big),$$

7. Projective curvature tensor $\tilde{P}$ if

$$a_0 = 1,\ a_1 = -a_2 = -\frac{1}{(n-1)},\ a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

8. m-rojective curvature tensor M if

$$a_0 = 1,\ a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{2(n-1)},\ a_3 = a_6 = a_7 = 0,$$

9. W_0 curvature tensor if

$$a_0 = 1,\ a_1 = -a_5 = -\frac{1}{n-1},\ a_2 = a_3 = a_4 = a_6 = a_7 = 0,$$

10. W_0^* curvature tensor if

$$a_0 = 1,\ a_1 = -a_5 = \frac{1}{n-1},\ a_2 = a_3 = a_4 = a_6 = a_7 = 0,$$

11. W_1 curvature tensor if

$$a_0 = 1,\ a_1 = -a_2 = \frac{1}{n-1},\ a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

12. W_1^* curvature tensor if

$$a_0 = 1,\ a_1 = -a_2 = -\frac{1}{n-1},\ a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

13. W_2 curvature tensor if

$$a_0 = 1,\ a_4 = -a_5 = -\frac{1}{n-1},\ a_1 = a_2 = a_3 = a_6 = a_7 = 0,$$

14. W_3 curvature tensor if

$$a_0 = 1,\ a_2 = -a_4 = -\frac{1}{n-1},\ a_1 = a_3 = a_5 = a_6 = a_7 = 0,$$

15. W_4 curvature tensor if

$$a_0 = 1,\ a_5 = -a_6 = \frac{1}{n-1},\ a_1 = a_2 = a_3 = a_4 = a_7 = 0,$$

16. W_5 curvature tensor if

$$a_0 = 1,\ a_2 = -a_5 = \frac{1}{n-1},\ a_1 = a_3 = a_4 = a_6 = a_7 = 0,$$

17. W_6 curvature tensor if

$$a_0 = 1,\ a_1 = -a_6 = -\frac{1}{n-1},\ a_2 = a_3 = a_4 = a_5 = a_7 = 0,$$

18. W_7 curvature tensor if

$$a_0 = 1,\ a_1 = -a_4 = -\frac{1}{n-1},\ a_2 = a_3 = a_5 = a_6 = a_7 = 0,$$

19. W_8 curvature tensor if

$$a_0 = 1,\ a_1 = -a_3 = -\frac{1}{n-1},\ a_2 = a_4 = a_5 = a_6 = a_7 = 0,$$

20. W_9 curvature tensor if

$$a_0 = 1,\ a_3 = -a_4 = \frac{1}{n-1},\ a_1 = a_2 = a_5 = a_6 = a_7 = 0.$$

1.18 Almost Contact Metric Manifolds

If M^n be an odd dimensional differentiable manifold on which there are defined a real vector valued linear function φ, a 1-form η and a vector field ξ satisfying for arbitrary vectors $X, Y, Z, ...$

$$\varphi^2 X = -X + \eta(X)\xi, \tag{1.18.1}$$

$$\eta(\xi) = 1, \tag{1.18.2}$$

$$\varphi(\xi) = 0, \tag{1.18.3}$$

$$\eta(\varphi X) = 0, \tag{1.18.4}$$

and

$$rank(\varphi) = n - 1, \tag{1.18.5}$$

is called an almost contact manifold (Sasaki, 1965) and the structure (φ, η, ξ) is called an almost contact structure (Hatakeyama et al., 1963; Sasaki and Hatakeyama, 1960, 61).

An almost contact manifold M^n on which a Riemannian metric tensor g satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.18.6}$$

and

$$g(X, \xi) = \eta(X), \tag{1.18.7}$$

is called an almost contact metric manifold and the structure (φ, ξ, η, g) is called an almost contact metric structure (Sasaki,1960).

The fundamental 2-form $'F$ of an almost contact metric manifold M^n is defined by

$$'F(X,Y) = g(\varphi X, Y). \tag{1.18.8}$$

We have

$$'F(X,Y) =' F(\varphi X, \varphi Y), \tag{1.18.9}$$

and

$$'F(X,Y) = -'F(Y,X). \tag{1.18.10}$$

If in an almost contact metric manifold

$$2'F(X,Y) = (\nabla_X \eta)(Y) - (\nabla_Y \eta)(X). \tag{1.18.11}$$

then M^n is called an almost Sasakian manifold.

An almost contact metric manifold is said to be Sasakian manifold if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X \tag{1.18.12}$$

and

$$\nabla_X \xi = -\phi X. \tag{1.18.13}$$

An almost contact metric manifold is called a Kenmotsu manifold if (Kenmotsu, 1972)

$$(\nabla_X \varphi)(Y) = g(\varphi X, Y)\xi - \eta(Y)\varphi(X) \tag{1.18.14}$$

and

$$\nabla_X \xi = X - \eta(X)\xi. \tag{1.18.15}$$

An almost contact metric manifold manifold M^n is trans-Sasakian Manifold (Oubina, 1985) if $\big(M^n \times \mathbb{R}, J, G\big)$ belong to the class ω_4 of the Hermitian manifolds, where G is the product metric on $\big(M^n \times \mathbb{R}\big)$ and J is the almost complex structure on $\big(M^n \times \mathbb{R}\big)$ defined by

$$J\big(U, f\frac{d}{dt}\big) = (\phi Z - f\xi, \eta(U)\frac{d}{dt})$$

for any $U \in \chi(M)$. This may be stated by the relation (Oubina, 1985)

$$\begin{aligned}(\nabla_X\varphi)(Y) &= \alpha\lambda\{g(X,Y)\xi - \eta(Y)X\} \\ &+ \beta\{-\eta(Y)\varphi X + g(Y,\varphi X)\xi\}. \qquad (1.18.16)\end{aligned}$$

From the above relations we have (Blair, 1990)

$$\nabla_X\xi = -\alpha\varphi X + \beta\{X - \eta(X)\xi\}, \qquad (1.18.17)$$

and

$$(\nabla_X\eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y), \qquad (1.18.18)$$

where α and β are non zero constants.

1.19 α-Cosymplectic Manifolds

An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be almost cosymplectic (Goldberg and Yano, 1969) if $d\eta = 0$ and $d\Phi = 0$, where d is the exterior differential operator. The manifold defined by $M = N \times \mathbb{R}$, where N is an almost Kählerian manifold and $\mathbb{R}$ is the real line is the simplest example of almost cosymplectic manifold (Olszak, 1981). An almost contact manifold (M, ϕ, ξ, η) is said to be normal if the Nijenhuis torsion

$$N_\phi(X,Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2(X,Y) + 2d\eta(X,Y)\xi$$

vanishes for any vector fields X and Y. A normal almost cosymplectic manifold is cosymplectic manifold.

An almost contact metric manifold M is said to be almost α-Kenmotsu if $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, α being a non-zero real constant.

If these two classes are joined, we obtain a new notion of an almost α-cosymplectic manifold, which is defined by the following formula

$$d\eta = 0, \qquad d\Phi = 2\alpha\eta \wedge \Phi,$$

for any real number α. A normal almost α-cosymplectic manifold is called an α-cosymplectic manifold. An α-cosymplectic manifold is either cosymplectic under the condition $\alpha = 0$ or α-Kenmotsu under the condition $\alpha \neq 0$, for $\alpha \in \mathbb{R}$.

On such an α-cosymplectic manifold, we have

$$(\nabla_X\phi)Y = \alpha\big[g(\phi X, Y)\xi - \eta(Y)\phi X\big] \tag{1.19.1}$$

and

$$\nabla_X\xi = -\alpha\phi^2 X = \alpha[X - \eta(X)\xi]. \tag{1.19.2}$$

1.20 Generalized Sasakian-space-form

A Sasakian manifold with constant ϕ-sectional curvature c is called a Sasakian-space-form and its curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z &= \frac{c+3}{4}\big[g(Y,Z)X - g(X,Z)Y\big] \\ &+ \frac{c-1}{4}\big[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\big] \\ &+ \frac{c-1}{4}\big[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\big]. \end{aligned} \tag{1.20.1}$$

In 2004, Alegre et al. generalized the Sasakian-space-form by replacing the constant quantities $\frac{c+3}{4}$ and $\frac{c-1}{4}$ with differentiable functions. Such space is called generalized Sasakian-space-form (Alegre et al., 2004).

1.21 Weakly Symmetric Manifolds

A non-flat Riemannian manifold (M^n, g), $(n > 2)$ is called weakly symmetric if the curvature tensor R of type (1,3) satisfies the condition

$$\begin{aligned} \nabla_X R(Y,Z)V = A(X)R(Y,Z)V + B(Y)R(X,Z)V + C(Z)\ R(Y,X)V \\ +D(V)R(Y,Z)X + g(R(Y,Z)V,X)P, \end{aligned} \tag{1.21.1}$$

for all $X, Y, Z, V \in \chi(M)$, where ∇ denotes the Levi-Civita connection on (M^n, g) and A, B, C, D and P are 1-forms and a vector field respectively which are non-zero simultaneously.

A non-flat Riemannian manifold (M^n, g), $(n > 2)$ is called weakly Ricci symmetric if the Ricci tensor S satisfies the condition

$$(\nabla_X)S(Y,Z) = \alpha_1(X)S(Y,Z) + \beta_1(Y)S(X,Z) + \gamma_1(Z)S(Y,X), \tag{1.21.2}$$

where α_1, β_1 and γ_1 are non-zero simultaneously.

1.22 Recurrent Manifolds

Let M^n be an n-dimensional smooth Riemannian manifold and $\chi(M)$ denotes the set of differentiable vector fields on M^n. Let $X, Y \in \chi(M)$; $\nabla_X Y$ denotes the covariant derivative of Y with respect to X and $R(X,Y,Z)$ be the Riemannian curvature tensor of type $(1,3)$. A Riemannian manifold M^n is said be recurrent (Ruse, 1946) if

$$(\nabla_U R)(X,Y,Z) = \alpha(U)R(X,Y,Z), \tag{1.22.1}$$

where α is a non-zero 1-form known as recurrence parameter. If the 1-form α is zero in (1.22.1), then the manifold reduces to symmetric manifold (Singh and Khan,1999).

A Riemannian manifold (M^n, g) is said to be Ricci-recurent if it satisfies the relation (Patterson, 1952)

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) \tag{1.22.2}$$

for all $X, Y, Z \in \chi(M)$, where ∇ denotes the Levi-Civita connection and A is a 1-form on M^n. If the 1-form A vanishes identically on M^n, then a Ricci-recurrent manifold becomes a Ricci-symmetric manifold.

A Riemannian manifold (M^n, g) is a called generalized recurrent Riemannian manifold (De and Guha, 1991) if its curvature tensor R satisfies the condition:

$$(\nabla_X R)(Y,Z)U = A(X)R(Y,Z)U + B(X)[g(Z,U)Y - g(Y,U)Z], \tag{1.22.3}$$

where A and B are two 1-forms, B is non-zero and these are defined by

$$A(X) = g(X, \rho_1), \qquad B(X) = g(X, \rho_2), \tag{1.22.4}$$

ρ_1 and ρ_2 are vector fields associated with 1-forms A and B, respectively.

A Riemannian manifold (M^n, g) is said to be φ-recurrent manifold if there exists a non -zero 1-form A such that

$$\varphi^2((\nabla_X R)(Y,Z)W) = A(X)R(Y,Z)W \tag{1.22.5}$$

for arbitrary vector fields X, Y, Z, W.

A Riemannian manifold (M^n, g) is called generalized φ-recurrent if its curvature tensor R satisfies the condition

$$\begin{aligned} \varphi^2((\nabla_W R)(Y,Z)U) &= A(W)R(Y,Z)U \\ &+ B(W)[g(Z,U)Y - g(Y,U)Z], \end{aligned} \tag{1.22.6}$$

where A and B are two 1-forms, B is non-zero and these are defined by earlier.

1.23 Semi-symmetric Manifolds

A manifold is said to be semi-symmetric and Ricci semi-symmetric (Cartan, 1946) if the Riemannian curvature tensor R and Ricci tensor S satisfies $R.R = 0$ and $R.S = 0$ respectively. That is

$$R(X,Y).R(U,V)W = 0 \tag{1.23.1}$$

and

$$R(X,Y).S(U,V) = 0 \tag{1.23.2}$$

for all $X, Y, U, V, , W \in \chi(M)$.

1.24 Pseudosymmetric manifolds

An n-dimensional Riemannian manifold M, $n > 2$, is called pseudosymmetric manifolds (Deszez, 1992) if $R.R$ and $Q(g, R)$ are linearly dependent, i.e.,

$$R.R = FQ(g,R), \tag{1.24.1}$$

holds on the set $U_R = \{x \in M : Q(g,R) \neq 0 \; at \; x\}$, where F is some function on U_R.

And the manifold is called Ricci pseudosymmetric and Ricci-generalized pseudosymmetric manifold if

$$R.S = f'Q(g,S) \tag{1.24.2}$$

and

$$R.R = fQ(S,R) \tag{1.24.3}$$

holds on the set $U_S = \{x \in M : Q(g,S) \neq 0 \; at \; x\}$ and $U_R = \{x \in M : Q(g,R) \neq$

$0 \; at \; x\big\}$ respectively, where f' and f are some functions on U_S and U_R.

1.25 Group Manifolds

A Riemannian manifold is a group manifold with respect to the quarter-symmetric connection if (Eisenhart, 1933)

$$\begin{aligned} \tilde{R}(U,V)Z &= 0 \\ &\textit{and} \\ (\tilde{\nabla}_X \tilde{T})(U,V) &= 0 \end{aligned} \tag{1.25.1}$$

for all $U, V, Z \in \chi(M)$.

1.26 Review of Literature

Hamilton (1982) studied 3-dimensional manifold with positive Ricci tensor. In the paper, he used the evolution equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.$$

Hamilton (1988) later used this equation for the expression of the Ricci flow , where g_{ij} denotes the metric tensor and R_{ij} denotes the associated Ricci tensor. In 1992, Ivey studied soliton for the Ricci flow to introduce the concept of Ricci soliton.

In 2008, Sinha and Sharma started the study of Ricci soliton in contact manifolds . Later Ricci soliton in contact and almost contact manifolds have been studied by many authors such as: Ricci solitons in contact metric manifolds by Tripathi (2008), Ricci solitons in manifolds with quasi-constant curvature by Bejan (2011), Ricci solitons in Lorentzian α-Sasakian manifolds by Bagewadi (2012), Ricci solitons and gradient Ricci solitons in three-dimensional trans-Sasakian manifolds by Turan et. al. (2012), Ricci solitons in Kenmotsu manifolds by Nagaraja and Premalatha (2012), etc.

η-Ricci soliton was introduced by Cho and Kimura (2009) while studying Ricci solitons of real hypersurfaces in a non-flat complex space form.

In 1967 Blair defined a cosymplectic manifold as a quasi-Sasakian structures satisfying $d\eta = 0$. This is to be noted that the notion of cosymplectic manifold introduced by Libermann (1959) is different from that of Blair (1967). An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be almost cosymplectic (Goldberg, 1969) if $d\eta = 0$ and $d\Phi = 0$, where d is the exterior differential operator. The manifold defined by

$M = N \times R$, where N is an almost Kahlerian manifold and R is a real line is the simplest examples of almost cosymplectic manifolds (Olszak, 1981). An almost contact manifold (M, ϕ, ξ, η) is said to be normal if the Nijenhuis torsion

$$N_\phi(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2(X, Y) + 2d\eta(X, Y)\xi.$$

vanishes for any vector fields X and Y . A normal almost cosymplectic manifolds is cosymplectic manifold.

An almost contact metric manifold M is said to be almost α-Kenmotsu if $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, α being a non-zero real constant.

Kim and Pak (2005) combined almost α-Kenmotsu and almost cosymplectic manifolds into a new class which is called almost α-cosymplectic manifolds, where α is a scalar. If we join these two classes, we obtain a new notion of an almost α-cosymplectic manifold, which is defined by the following formula

$$d\eta = 0, \qquad d\Phi = 2\alpha\eta \wedge \Phi.$$

for any real number α. A normal almost α-cosymplectic manifold is called an α-cosymplectic manifold. An α-cosymplectic manifold is either cosymplectic under the condition $\alpha = 0$ or α-Kenmotsu under the condition $\alpha \neq 0$ for $\alpha \in \mathbb{R}$.

In a Riemannian manifold, a curvature tensor given by $K(X, Y) = R(X, Y, Y, X)$ for an orthonormal pair of vectors (X, Y), is known as the sectional curvature. A Riemannian manifold with constant sectional curvature c is called a real-space-form, and its curvature tensor R satisfies

$$R(X, Y)Z = c\big\{g(Y, Z)X - g(X, Z)Y\big\}.$$

A Sasakian manifold with constant ϕ-sectional curvature c is called a Sasakian-space-form and its curvature tensor R is given by

$$\begin{aligned} R(X, Y)Z &= \frac{c+3}{4}\big[g(Y, Z)X - g(X, Z)Y\big] \\ &+ \frac{c-1}{4}\big[g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\big] \\ &+ \frac{c-1}{4}\big[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\big]. \end{aligned}$$

In 2004, Alegre et al. generalized the Sasakian-space-form by replacing the constant quantities $\dfrac{c+3}{4}$ and $\dfrac{c-1}{4}$ with differentiable functions. Such space is called generalized Sasakian-space-form.

The generalized Sasakian-space-form have been studied by many authors such as Sarkar and De (2010, 2012), De et al. (2012), Singh (2016a, 2016b), De and Majhi (2013, 2015, 2019), Kishor et al. (2017), Alegre and Carriazo (2008, 2018), Akbar and Sarkar (2015), Sular and Ozgur (2011,2014) and many others.

In 2008, Alegre and Carriazo studied structures on generalized Sasakian-space-form and studied generalized Sasakian-space-form admitting trans-Sasakian structure. In 1989, Tanno defined the generalized TanakaWebster connection for contact metric manifolds, which generalized the connection given by Tanaka (1976) and Webster (1978). The generalized TanakaWebster connection have been studied by De de Dios Prez (2015), De (2016) and others.

In 2012 Mantica and Suh defined the Z-tensor which is given by

$$\tilde{\mathbf{Z}}(X,Y) = S(X,Y) + \phi_1 g(X,Y),$$

where ϕ_1 is an arbitrary scalar function and S and g denotes the Ricci tensor and metric tensor respectively. Particularly if $\phi_1 = 0$, the Z-tensor reduces to the Ricci tensor. Later the Z-tensor have been studied by Mallick and De (2016), De and Pal (2014), Mantica et al. (2012a, 2012b), Chaubey (2018) and many others.

The notion of (locally) symmetric manifolds was introduced by Cartan (1926) as a generalization of the notion of a space of constant curvature. An n-dimensional Riemannian manifold M is said to be locally symmetric due to Cartan if its curvature tensor R satisfies $\nabla R = 0$, where ∇ denotes the Levi-Civita connection. Locally symmetric manifolds and recurrent manifolds have been studied by many authors in several ways and to various extent such as weakly symmetric manifolds by Tamssy and Binh (1989), conformally symmetric Ricci-recurrent spaces by Roter (1974), conformally recurrent Ricci-recurrent manifolds by Roter (1982), conformally symmetric manifolds by Chaki and Gupta (1963), pseudo symmetric manifolds introduced by Chaki (1987), recurrent manifolds introduced by Walker (1950), conformally recurrent manifolds by Adati and Miyazawa (1967), Ricci-recurrent manifolds by Patterson (1952), 2-recurrent manifolds by Lichnerowicz (1950), projective 2-recurrent manifolds by Ghosh (1970) and others.

Ricci-recurrent manifolds was introduced by Patterson in 1952. Ricci-recurrent manifolds have been studied by several authors Chaki (1956), Prakash (1962), Roter (1974), Yamaguchi (968) and many others.

According to De et al. (1995), a non-flat Riemannian manifold (M^n, g), $n > 2$ is called generalized Ricci-recurrent if the Ricci tensor R_{ij} is non-zero and satisfies the condition

$$R_{ij,l} = \lambda_l R_{ij} + \mu_l g_{ij},$$

where λ_l and μ_l are non-zero vectors. It is denoted by GR_n. If $\mu_l = 0$, then the manifold GR_n become a Ricci-recurrent manifold R_n. Also De and Guha (1991) introduced a non-flat Riemannian manifold (M^n, g), $n > 2$ called a generalized recurrent manifold. Such a manifold has been denoted by GK_n. If the associated vector μ_l becomes zero, then the manifold GK_n reduces to a recurrent manifold introduced by walker (1950) which is denoted by K_n.

The generalized recurrent and generalized Ricci-recurrent manifolds have been studied by several authors such as Ozgur (2007, 2008a, 2008b), Mallick et. al. (2013), Arslan et. al. (2009) and many others.

Golab (1975) introduced a quarter-symmetric linear connection on a differentiable manifold. A quarter-symmetric metric connection generalized a semi-symmetric connection which is introduced by Friedman and Schouten (Friedman and Schouten, 1924) in 1924. A quarter-symmetric connection is further studied by many authors such as: Barman (Barman, 2015), Prakasha and Vikas (Prakasha, 2015), Singh (2014, 2015a, 2015b), Prasad and Haseeb (Prasad and Haseeb, 2016), Dey et al. (2015, 2017) etc.

Chapter 2

CHARACTERIZATION OF RICCI SOLITONS

In this chapter we studied Ricci soliton and η-Ricci soliton which is a generalization of Ricci soliton in α-cosymplectic manifolds and Sasakian manifolds respectively. We have given the condition for which the Ricci soliton is shrinking, steady or expanding. We also discussed geometrical propertues of η-Ricci soliton.

2.1 Introduction

The notion of Ricci soliton and η-Ricci soliton is given in the equation (1.15.1) and (1.15.3) respectively. Sinha and Sharma (2008) studied K-Contact and (k, μ)-Contact Manifolds which is the first time Ricci soliton is studied in contact manifolds. In this chapter we studied Ricci soliton in α-cosymplectic manifold and η-Ricci soliton in Sasakian manifolds.

Definition 2.1.1 *An almost contact metric manifold is called a Sasakian manifold if and only if (Yano, 1984)*

$$(\nabla_Z \phi)W = g(Z, W)\xi - \eta(W)Z, \qquad \nabla_Z \xi = -\phi Z. \tag{2.1.1}$$

On a Sasakian manifold, we have (Yano, 1984)

$$R(Z, W)\xi = \eta(W)Z - \eta(Z)W, \tag{2.1.2}$$

Singh, J. P. and Lalmalsawma, C. (2018). Ricci solitons in α-cosymplectic manifolds, *Facta Universitatis* **33(3)**, 375-387.

$$R(Z,\xi)X = \eta(X)Z - g(Z,X)\xi, \tag{2.1.3}$$

$$\eta\big(R(Z,W)X\big) = g(W,X)\eta(Z) - g(Z,X)\eta(W), \tag{2.1.4}$$

$$g\big(R(Z,W)\xi,\xi\big) = 0, \tag{2.1.5}$$

$$S(W,\xi) = (n-1)\eta(W), \tag{2.1.6}$$

$$Q\xi = (n-1)\xi. \tag{2.1.7}$$

Now, using the property of Lie derivative we have

$$(\pounds_\xi g)(Z,W) = 0. \tag{2.1.8}$$

From equations (1.15.3) and (2.1.8) we get

$$S(Z,W) = -\mu\eta(Z)\eta(W) - \lambda g(Z,W). \tag{2.1.9}$$

Putting $W = \xi$ in equation (2.1.9) we get

$$S(Z,\xi) = -(\lambda+\mu)\eta(Z), \tag{2.1.10}$$

and

$$Q\xi = -(\lambda+\mu)\xi. \tag{2.1.11}$$

Comparing equation (2.1.7) and equation (2.1.11) we get

$$-(\lambda+\mu) = (n-1). \tag{2.1.12}$$

On an α-cosymplectic manifold M^n, the following relations are held (Ozturk,2010; Ozturk, 2013)

$$R(\xi,X)Y = \alpha^2\big[\eta(Y)X - g(X,Y)\xi\big], \tag{2.1.13}$$

$$R(X,Y)\xi = \alpha^2\big[\eta(X)Y - \eta(Y)X\big], \tag{2.1.14}$$

$$S(\xi,X) = -\alpha^2(n-1)\eta(X), \tag{2.1.15}$$

$$\eta\big(R(X,Y)Z\big) = \alpha^2\big[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\big]. \tag{2.1.16}$$

Using equation (1.19.2) we have

$$\pounds_\xi g(X,Y) = 2\alpha g(X,Y) - 2\alpha\eta(X)\eta(Y). \tag{2.1.17}$$

From equations (1.15.1) and (2.1.17) we get

$$S(X,Y) = \alpha\eta(X)\eta(Y) - (\lambda+\alpha)g(X,Y). \tag{2.1.18}$$

Equation equation (2.1.18) yields

$$QX = \alpha\eta(X)\xi - (\lambda+\alpha)X, \tag{2.1.19}$$

$$S(X,\xi) = -\lambda\eta(X), \tag{2.1.20}$$

$$r = (1-n)\alpha - \lambda n. \tag{2.1.21}$$

Comparing equation (2.1.16) and equation (2.1.20) we get

$$\lambda = \alpha^2(n-1). \tag{2.1.22}$$

Since $\alpha^2 \geq 0$, for $\alpha \in \mathbb{R}$, from equation (2.1.22) we get $\lambda \geq 0$, for all $n \geq 2$. Thus we can state the following:

Lemma 2.1.1 *A Ricci soliton in an n-dimensional α-cosymplectic manifold, $n \geq 2$, is eitheir steady or expanding.*

We have already stated that an α-cosymplectic manifold is either cosymplectic under the condition $\alpha = 0$ or α-Kenmotsu under the condition $\alpha \neq 0$, for $\alpha \in \mathbb{R}$. Thus we can state the following lemmas:

Lemma 2.1.2 *A Ricci soliton in an n-dimensional α-cosymplectic manifold, $n \geq 2$, is steady if and only if it is a cosymplectic manifold.*

Lemma 2.1.3 *A Ricci soliton in an n-dimensional α-cosymplectic manifold, $n \geq 2$, is expanding if and only if it is an α-Kenmotsu manifold.*

2.2 Ricci semi-symmetric α-cosymplectic manifolds, $n \geq 2$

Theorem 2.2.1 *A Ricci semi-symmetric α-cosymplectic manifold, $n \geq 2$, admitting Ricci soliton is cosymplectic manifold.*

Proof: Consider an α-cosymplectic manifold which is Ricci semi-symmetric. Then we have the equation (1.23.2) i.e.

$$R(\xi, X) \cdot S(Y, Z) = 0 \tag{2.2.1}$$

holds in M^n.

From equation (2.2.1) it follows that

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0. \tag{2.2.2}$$

Using equations (2.1.13), (2.1.18) and (2.1.20), we get from equation (2.2.2)

$$\alpha^2 \Big[2\alpha\eta(X)\eta(Y)\eta(Z) - \alpha\eta(Y)g(X, Z) - \alpha\eta(Z)g(X, Y)\Big] = 0,$$

or

$$\alpha^3 \Big[2\eta(X)\eta(Y)\eta(Z) - \eta(Y)g(X, Z) - \eta(Z)g(X, Y)\Big] = 0. \tag{2.2.3}$$

Contracting equation (2.2.3) over X and Y we get

$$\alpha^3(n-1)\eta(Z) = 0. \tag{2.2.4}$$

In general $\eta(Z) \neq 0$. Therefore $\alpha = 0$.
Hence it is Cosymplectic manifold.

Corollary 2.2.1 *A Ricci soliton in a Ricci semi-symmetric α-cosymplectic manifold, $n \geq 2$, is steady.*

Proof: By virtue of Lemma 2.1.2 and the above theorem, it is obvious that the Ricci solitons is steady.

2.3 Pseudo projective semi-symmetric α-cosymplectic manifolds, $n \geq 2$

From the equation (1.16.3) a $(0,4)$ type pseudo-projective curvature tensor $'P$ is

$$\begin{aligned} 'P(X,Y,Z,W) &= a'R(X,Y,Z,W) + b[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)] \\ &- \frac{r}{n}\big(\frac{a}{n-1} + b\big)[g(Y,Z)g(X,W) - g(Y,U)g(Y,W)]. \end{aligned} \tag{2.3.1}$$

where $'R$ is a Riemannian curvature tensor of type $(0,4)$, from (2.3.1) it follows that

$$\sum_{i=1}^{n} {}'P(e_i, Y, Z, e_i) = [a + (n-1)b]\big[S(Y,Z) - \frac{r}{n}g(Y,Z)\big]. \tag{2.3.2}$$

Again from equation (2.3.1) we obtain

$$\eta\big(P(X,Y)Z\big) = \big[a\alpha^2 + \frac{r}{n}\big(\frac{a}{n-1} + b\big) + (\lambda + \alpha)b\big] \times \big[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\big],$$

or

$$\eta\big(P(X,Y)Z\big) = \beta[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)], \tag{2.3.3}$$

where $\beta = \big[a\alpha^2 + \frac{r}{n}\big(\frac{a}{n-1} + b\big) + (\lambda + \alpha)b\big]$.

Theorem 2.3.1 *A peudo projective semi-symmetric α-Kenmotsu manifold, $n \geq 2$, admitting Ricci soliton is an Einstein manifold.*

Proof: Now we assume that the condition

$$R(\xi, X) \cdot P(Y,Z)W = 0 \tag{2.3.4}$$

holds in M.

From equation (2.3.4) it follows that

$$\begin{aligned} R(\xi,X)P(Y,Z)W - P(R(\xi,X)Y,Z)W - P(Y,R(\xi,X)Z)W \\ -P(Y,Z)R(\xi,X)W = 0. \end{aligned} \tag{2.3.5}$$

Using equation (2.1.13) in equation (2.3.5) we find

$$\begin{aligned} \alpha^2\Big[\eta\big(P(Y,Z)W\big)X - \hat{P}(Y,Z,W,X)\xi - \eta(Y)P(X,Z)W \\ +g(X,Y)P(\xi,Z)W - \eta(Z)P(Y,X)W + g(X,Z)P(Y,\xi)W \\ -\eta(W)P(Y,Z)X + g(X,W)P(Y,Z)\xi\Big] = 0, \end{aligned} \tag{2.3.6}$$

where $\hat{P}(Y,Z,W,X) = g\big(X, P(Y,Z)W\big)$.

Taking the inner product of equation (2.3.5) with ξ we get

$$\alpha^2\Big[\eta\big(P(Y,Z)W\big)\eta(X) - \hat{P}(Y,Z,W,X) - \eta(Y)\eta\big(P(X,Z)W\big) \\ +g(X,Y)\eta\big(P(\xi,Z)W\big) - \eta(Z)\eta\big(P(Y,X)W\big) + g(X,Z)\eta\big(P(Y,\xi)W\big) \\ -\eta(W)\eta\big(P(Y,Z)X\big) + g(X,W)\eta\big(P(Y,Z)\xi\big)\Big] = 0. \qquad (2.3.7)$$

By virtue of equation (2.3.3), equation (2.3.7) yields

$$\alpha^2\Big[\hat{P}(Y,Z,W,X) + \beta\big\{g(X,Y)g(Z,W) - g(X,Z)g(Y,W)\big\}\Big] = 0. \qquad (2.3.8)$$

Contracting equation (2.3.8) over X and Y and using equation (2.3.2) we get

$$\alpha^2\Big[[a+(n-1)b]\big\{S(Z,W) - \frac{r}{n}g(Z,W)\big\} + \beta(n-1)g(Z,W)\Big] = 0. \qquad (2.3.9)$$

We suppose that the α-cosymplectic manifold is an α-Kenmotsu manifold i.e., $\alpha \neq 0$. Thus equation (2.3.9) can be written as

$$S(Z,W) = \Big[\frac{r}{n} - \frac{\beta(n-1)}{a+(n-1)b}\Big]g(Z,W),$$

or

$$S(Z,W) = \rho g(Z,W), \qquad (2.3.10)$$

where $\rho = \Big[\frac{r}{n} - \frac{\beta(n-1)}{a+(n-1)b}\Big]$.

Theorem 2.3.2 *A projective semi-symmetric α-cosymplectic manifold, $n \geq 2$, admitting Ricci soliton is a cosymplectic manifold.*

Proof: Contracting equation (2.3.9) over Z and W we get

$$n(n-1)\alpha^2\beta = 0. \qquad (2.3.11)$$

From equation (2.3.11) it follows that

$$\alpha^2\beta = 0,$$

or

$$\alpha^2\big[a\alpha^2 + \frac{r}{n}\big(\frac{a}{n-1} + b\big) + (\lambda+\alpha)b\big] = 0. \qquad (2.3.12)$$

If we put $a = 1$ and $b = -\frac{1}{(n-1)}$ then equation (2.3.1) takes the form

$$\begin{aligned} P(X,Y)Z &= R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y] \\ &= \tilde{P}(X,Y)Z, \end{aligned} \tag{2.3.13}$$

where $\tilde{P}(X,Y)Z$ is the projective curvature tensor and is a particular case of P.

Now putting $a = 1$ and $b = -\frac{1}{(n-1)}$ in equation (2.3.12) and making use of (2.1.22) we get

$$\alpha^3 = 0,$$

or

$$\alpha = 0. \tag{2.3.14}$$

Corollary 2.3.1 *A Ricci soliton in a projective semi-symmetric α-cosymplectic manifold, $n \geq 2$, is steady.*

Proof: By virtue of the Lemma 2.1.2, it is obvious that Ricci-solitons is steady.

2.4 Weyl semi-symmetric α-cosymplectic manifolds, $n > 2$

We consider the Weyl conformal curvature tensor $\tilde{C}$ of type $(1,3)$ which is defined by

$$\begin{aligned} \tilde{C}(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\big[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ -S(X,Z)Y\big] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{2.4.1}$$

where R is a Riemannian curvature tensot of type $(1,3)$.

From which it follows that

$$\sum_{i=1}^{n}{}'\tilde{C}(e_i,Y,Z,e_i) = 0. \tag{2.4.2}$$

Again from equation (2.4.1) we obtain

$$\eta\big(\tilde{C}(X,Y)Z\big) = 0. \tag{2.4.3}$$

Theorem 2.4.1 *A Weyl semi-symmetric α-Kenmotsu manifold, $n > 2$, admitting Ricci soliton is conformally flat.*

Proof: Now we assume that the condition

$$R(\xi, X) \cdot \tilde{C}(Y,Z)W = 0 \tag{2.4.4}$$

holds in M^n.

From equation (2.4.4) it follows that

$$\begin{aligned} R(\xi,X)\tilde{C}(Y,Z)W - \tilde{C}(R(\xi,X)Y,Z)W - \tilde{C}(Y,R(\xi,X)Z)W \\ -\tilde{C}(Y,Z)R(\xi,X)W = 0. \end{aligned} \tag{2.4.5}$$

Using equation (2.1.13) in equation (2.4.5) we find

$$\begin{aligned} \alpha^2\Big[\eta\big(\tilde{C}(Y,Z)W\big)X -' \tilde{C}(Y,Z,W,X)\xi - \eta(Y)\tilde{C}(X,Z)W \\ +g(X,Y)\tilde{C}(\xi,Z)W - \eta(Z)\tilde{C}(Y,X)W + g(X,Z)\tilde{C}(Y,\xi)W \\ -\eta(W)\tilde{C}(Y,Z)X + g(X,W)\tilde{C}(Y,Z)\xi\Big] = 0, \end{aligned} \tag{2.4.6}$$

where $'\tilde{C}(Y,Z,W,X) = g\big(X, \tilde{C}(Y,Z)W\big)$.

Taking the inner product of equation (2.4.6) with ξ we get

$$\begin{aligned} \alpha^2\Big[\eta\big(\tilde{C}(Y,Z)W\big)\eta(X) -' \tilde{C}(Y,Z,W,X) - \eta(Y)\eta\big(\tilde{C}(X,Z)W\big) \\ +g(X,Y)\eta\big(\tilde{C}(\xi,Z)W\big) - \eta(Z)\eta\big(\tilde{C}(Y,X)W\big) + g(X,Z)\eta\big(\tilde{C}(Y,\xi)W\big) \\ -\eta(W)\eta\big(\tilde{C}(Y,Z)X\big) + g(X,W)\eta\big(\tilde{C}(Y,Z)\xi\big)\Big] = 0. \end{aligned} \tag{2.4.7}$$

By virtue of equation equation (2.4.3), equation (2.4.7) yields

$$\alpha^{2\prime}\tilde{C}(Y,Z,W,X) = 0. \tag{2.4.8}$$

We suppose that the α-cosymplectic manifold is an α-Kenmotsu manifold i.e., $\alpha \neq 0$. Then we have

$$'\tilde{C}(Y,Z,W,X) = 0. \tag{2.4.9}$$

2.5 α-cosymplectic manifolds, $n \geq 2$ satisfying $P(\xi, X) \cdot S = 0$

Making use of equations (2.1.13), (2.1.18) and (2.1.20) in equation (2.3.1) we get

$$P(\xi, Y)Z = \big[\alpha^2 a + \frac{r}{n}\big(\frac{a}{n-1} + b\big) + \lambda b\big][\eta(Z)Y - g(Y, Z)\xi] \\ +\alpha b[\eta(Y)\eta(Z)\xi - g(Y, Z)\xi],$$

or

$$P(\xi, Y)Z = \beta[\eta(Z)Y - g(Y, Z)\xi] + \gamma[\eta(Y)\eta(Z)\xi - g(Y, Z)\xi], \tag{2.5.1}$$

where $\beta = \big[\alpha^2 a + \frac{r}{n}\big(\frac{a}{n-1} + b\big) + \lambda b\big]$ and $\gamma = \alpha b$.

Theorem 2.5.1 *If an α-cosymplectic manifold, $n \geq 2$, admitting Ricci soliton and satisfying $P(\xi, X) \cdot S = 0$ is an α-Kenmotsu manifold, then it satisfies $\alpha^2 = -\frac{r}{n}\big(\frac{1}{n-1} + \frac{b}{a}\big)$.*

Proof: Now we consider that the given manifold satisfies

$$P(\xi, X) \cdot S(Y, Z) = 0.$$

From the above equation it follows that

$$S(P\big(\xi, X)Y, Z\big) + S\big(Y, P(\xi, X)Z\big) = 0. \tag{2.5.2}$$

Using equation (2.5.1) in equation (2.5.2) yields

$$\beta\eta(Y)S(X, Z) - \beta g(X, Y)S(\xi, Z) + \gamma\eta(X)\eta(Y)S(\xi, Z) \\ -\gamma g(X, Y)S(\xi, Z) + \beta\eta(Z)S(X, Y) - \beta g(X, Z)S(\xi, Y) \\ +\gamma\eta(X)\eta(Z)S(\xi, Y) - \gamma g(X, Z)S(\xi, Y) = 0. \tag{2.5.3}$$

Making use of equations (2.1.18) and (2.1.20) in equation (2.5.3) we get

$$\big(\alpha\beta - \lambda\gamma\big)[2\eta(X)\eta(Y)\eta(Z) - g(X, Z)\eta(Y) \\ -g(X, Y)\eta(Z)] = 0. \tag{2.5.4}$$

Contracting equation (2.5.4) over X and Y we get

$$\big(\alpha\beta - \lambda\gamma\big)(1 - n)\eta(Z) = 0. \tag{2.5.5}$$

Putting $Z = \xi$ in equation (2.5.5) yields

$$\big(\alpha\beta - \lambda\gamma\big)(1-n) = 0. \tag{2.5.6}$$

From which it follows that

$$\big(\alpha\beta - \lambda\gamma\big) = 0,$$

or

$$\alpha\big[\alpha^2 a + \frac{r}{n}\big(\frac{a}{n-1} + b\big)\big] = 0. \tag{2.5.7}$$

We suppose that the α-cosymplectic manifold is an α-Kenmotsu manifold i.e., $\alpha \neq 0$. Then equation (2.5.7) yields

$$\big[\alpha^2 a + \frac{r}{n}\big(\frac{a}{n-1} + b\big)\big] = 0,$$

or

$$\alpha^2 = -\frac{r}{n}\big(\frac{1}{n-1} + \frac{b}{a}\big). \tag{2.5.8}$$

Corollary 2.5.1 *If a Ricci soliton in an α-cosymplectic manifold, $n \geq 2$, satisfying $P(\xi, X) \cdot S = 0$ is expanding, then $\alpha^2 = -\frac{r}{n}\big(\frac{1}{n-1} + \frac{b}{a}\big)$.*

Proof: By virtue of Lemma 2.1.3, the Ricci solitons is expanding.

Theorem 2.5.2 *An α-cosymplectic manifold, $n \geq 2$, admitting Ricci soliton and satisfying $\tilde{P}(\xi, X) \cdot S = 0$ is a cosymplectic manifold.*

Proof: Putting $a = 1$ and $b = -\frac{1}{(n-1)}$ in equation (2.5.6)

$$\alpha^3 = 0,$$

or

$$\alpha = 0. \tag{2.5.9}$$

Corollary 2.5.2 *A Ricci solitons in an α-cosymplectic manifold, $n \geq 2$, satisfying $\tilde{P}(\xi, X) \cdot S = 0$ is steady.*

Proof: By virtue of the Lemma 2.1.3, it is obvious that the Ricci solitons is steady.

2.6 Gradient Ricci soliton in α-cosymplectic manifolds

Theorem 2.6.1 *If an α-cosymplectic manifold, $n \geq 2$, admitting gradient Ricci soliton is an α-Ketmotsu manifold, then it is an η-Einstein manifold.*

Proof: From equation (1.15.2) we have

$$\nabla\nabla f = S + \lambda g. \tag{2.6.1}$$

This can be written as

$$\nabla_Y Df = QY + \lambda Y, \tag{2.6.2}$$

where D is the gradient operator of g. Using equation (2.6.2) we can obtain

$$R(X, Y)Df = (\nabla_X Q)Y + (\nabla_Y Q)X. \tag{2.6.3}$$

Taking the inner product of equation (2.6.3) with ξ we get

$$g\big(R(X, Y)Df, \xi\big) = g\big((\nabla_X Q)Y, \xi\big) + g\big((\nabla_Y Q)X, \xi\big). \tag{2.6.4}$$

Using equation (1.19.2) and equation (2.1.19) we have

$$g\big((\nabla_\xi Q)Y, \xi\big) = 0, \tag{2.6.5}$$

and

$$g\big((\nabla_Y Q)\xi, \xi\big) = 0. \tag{2.6.6}$$

By virtue of equations (2.6.5) and (2.6.6), equation (2.6.4) yields

$$g\big(R(\xi, Y)Df, \xi\big) = 0. \tag{2.6.7}$$

Again using equation (2.1.13) in equation (2.6.7) we get

$$g\big(R(\xi, Y)Df, \xi\big) = \alpha^2\big[\eta(Y)\eta(Df) - g(Y, Df)\big]. \tag{2.6.8}$$

From equation (2.6.7) and equation (2.6.8) we have

$$\alpha^2\big[\eta(Y)\eta(Df) - g(Y, Df)\big] = 0. \tag{2.6.9}$$

Now we suppose that $\alpha \neq 0$, i.e., the given manifold is α-Kenmotsu manifold. Equation (2.6.9) yields

$$\eta(Y)\eta(Df) = g(Y, Df). \tag{2.6.10}$$

From equation (2.6.10) we obtain

$$Df = (\xi f)\xi. \tag{2.6.11}$$

Using equation (2.6.11) in equation (2.6.2)

$$Y(\xi f)\xi + \alpha(\xi f)\big[Y - \eta(Y)\xi\big] = QY + \lambda Y. \tag{2.6.12}$$

Taking the inner product of equation (2.6.12) with X, we obtain

$$Y(\xi f)\eta(X) + \alpha(\xi f)\big[g(X, Y) - \eta(X)\eta(Y)\big] = S(X, Y) + \lambda g(X, Y). \tag{2.6.13}$$

Putting $X = \xi$ and using equation (2.1.20) in equation (2.6.13) we get

$$Y(\xi f) = S(\xi, Y) + \lambda\eta(Y) = 0. \tag{2.6.14}$$

From equation (2.6.14) it is clear that ξf is constant. Thus using equation (2.6.13) in equation (2.6.14) yields

$$\alpha(\xi f)\big[g(X, Y) - \eta(X)\eta(Y)\big] = S(X, Y) + \lambda g(X, Y),$$

or

$$S(X, Y) = \big[\alpha(\xi f) - \lambda\big]g(X, Y) - \alpha(\xi f)\eta(X)\eta(Y). \tag{2.6.15}$$

Corollary 2.6.1 *If a gradient Ricci soliton in an α-cosymplectic manifold, $n \geq 2$, is expanding, then it is an η-Einstein manifold.*

Proof: By virtue of Lemma 2.1.2 we have and the above theorem, we get the corollary.

2.7 Parallel symmetric tensor field of type $(0, 2)$

Theorem 2.7.1 *In a Sasakian manifold, any symmetric tensor field h of type $(0, 2)$ which is parallel with respect to ∇ is a constant multiple of the metric.*

Proof: Consider a symmetric tensor field h of type $(0,2)$ which is parallel with respect to ∇. Then using Ricci commutation identity we can obtain (Sharma, 1989)

$$h\big(R(U,V)Z,W\big)+h\big(R(U,V)W,Z\big)=0. \tag{2.7.1}$$

Setting $Z=W=\xi$ in equation (2.7.1) and by symmetric property of h we obtain

$$h\big(R(U,V)\xi,\xi\big)=0. \tag{2.7.2}$$

Putting $U=\xi$ in equation (2.7.2) and using equation (2.1.2) we obtain

$$h(V,\xi)-\eta(V)h(\xi,\xi)=0,$$

or

$$h(V,\xi)=\eta(V)h(\xi,\xi). \tag{2.7.3}$$

Again putting $Z=U=\xi$ in equation (2.7.1) we get

$$\eta(V)h(\xi,W)-h(V,W)+g(V,W)h(\xi,\xi)-\eta(W)h(\xi,V)=0. \tag{2.7.4}$$

Using (2.2.3) in equation (2.7.4) we obtain

$$h(V,W)=h(\xi,\xi)g(V,W). \tag{2.7.5}$$

Theorem 2.7.2 *If the symmetric tensor field in a Sasakian manifold is given by $h=\pounds_V g+2S+2\mu\eta\otimes\eta$, then the manifold admits an η-Ricci soliton.*

Proof: We consider a paticular case for the symmetric tensor field h, where h is given by the expression

$$h=\pounds_V g+2S+2\mu\eta\otimes\eta.$$

Then from the above equation we have

$$h(\xi,\xi)=(\pounds_V)g(\xi,\xi)+2S(\xi,\xi)+2\mu\eta(\xi)\eta(\xi). \tag{2.7.6}$$

Using equations (2.1.8) and (2.1.9) in equation (2.7.6)

$$h(\xi,\xi)=-2\lambda. \tag{2.7.7}$$

Using equation (2.7.7) in equation (2.7.5) we obtain

$$h(V,W)=-2\lambda g(V,W), \tag{2.7.8}$$

for all X and Y. Therefore $-2\lambda g = \pounds_V g + 2S + 2\mu\eta \otimes \eta$ defines an η-Ricci soliton.

2.8 Non existence of Ricci semi-symmetric Sasakian manifolds

Theorem 2.8.1 *A Sasakian manifold admitting η-Ricci soliton can not be Ricci semi-symmetric.*

Proof: Consider a Sasakian which is Ricci semi-symmetric. Then we have the equation (1.23.2) i.e.

$$R(X,Y) \cdot S = 0.$$

Now we assume that the condition

$$R(\xi, X) \cdot S(Y,Z) = 0 \tag{2.8.1}$$

holds in M^n.

From equation (2.8.1) we have

$$S(R(\xi,X)Y,Z) + S(Y,R(\xi,X)Z) = 0. \tag{2.8.2}$$

Using equations (2.1.3), (2.1.9) and (2.1.10), we get from equation (2.8.2)

$$\mu\Big[\eta(Y)g(X,Z) + \eta(Z)g(X,Y) - 2\eta(X)\eta(Y)\eta(Z)\Big] = 0. \tag{2.8.3}$$

Putting $Z = \xi$ in equation (2.8.3) we get

$$\mu\Big[g(X,Y) - 2\eta(X)\eta(Y)\Big] = 0,$$

or

$$\mu\Big[g(\phi X, \phi Y)\Big] = 0. \tag{2.8.4}$$

In general $g(\phi X, \phi Y) \neq 0$. Therefore $\mu = 0$ which is not possible.

2.9 Torse forming vector field in Sasakian manifolds

Definition 2.9.1 *A vector field T on Riemannian manifold is said to be a torse-forming vector field if its covariant derivative satisfies satisfies (Yano, 1944)*

$$\nabla_X T = fX + A(X)T,$$

for all vector field X, where f is a smooth function and A is a 1-form.

Theorem 2.9.1 *A Sasakian manifold admitting η-Ricci soliton with a torse-forming vector field is Einstein.*

Proof: Now we suppose that $T = \xi$ in a Sasakian manifold admitting η-Ricci soliton. Thus we have

$$\nabla_X \xi = fX + A(X)\xi. \tag{2.9.1}$$

From equation (2.9.1) we obtain

$$g(\nabla_X \xi, \xi) = f\eta(X) + A(X). \tag{2.9.2}$$

Using equations (1.18.6) and (2.1.1) we have

$$g(\nabla_X \xi, \xi) = 0.$$

Using the above equation in equation (2.9.2) we obtain

$$A(X) = f\eta(X). \tag{2.9.3}$$

From equation (1.15.3) we have

$$g(\nabla_X \xi, Z) + g(X, \nabla_Z \xi) + 2S(X, Z) + 2\lambda g(X, Z) + 2\mu\eta(X)\eta(Z) = 0. \tag{2.9.4}$$

Making use of equations (2.9.1) and (2.9.3) in equation (2.9.4) we get

$$S(X, Z) + (\lambda + f)g(X, Z) + (\mu - f)\eta(X)\eta(Z) = 0. \tag{2.9.5}$$

Using equation (2.1.9) in equation (2.9.5) we get

$$fg(X, Z) = f\eta(X)\eta(Z).$$

Thus we have

$$g(X,Y) = \eta(X)\eta(Z), \tag{2.9.6}$$

provided $f \neq 0$.

Putting $g(X,Z) = \eta(X)\eta(Z)$ in equation (2.1.9) we get

$$S(X,Z) = -(\lambda + \mu)g(X,Z). \tag{2.9.7}$$

Again Using equation (2.1.12) in equation (2.9.7) we get

$$S(X,Z) = (n-1)g(X,Y). \tag{2.9.8}$$

2.10 m-projectively flat Sasakian manifolds

Theorem 2.10.1 *An m-projectively flat Sasakian manifold admitting η-Ricci soliton is $(\psi B)_n$.*

Proof: We suppose that the given manifold is m-projectively flat, we have

$$M(X,Y,Z,W) = 0. \tag{2.10.1}$$

Combining equation (1.16.5) and equation (2.10.1) and using equation (2.1.9) we get

$$\begin{aligned}\hat{R}(X,Y,Z,W) = {} & \frac{\lambda}{n-1}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] \\ & + \frac{\mu}{n-1}[\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\ & + \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z)].\end{aligned} \tag{2.10.2}$$

Equation (2.10.2) can be written as

$$\hat{R}(X,Y,Z,W) = H(Y,W)H(X,Z) - H(Z,W)H(Y,Z), \tag{2.10.3}$$

where $H(X,Y) = \sqrt{\frac{\lambda}{n-1}}g(X,Y) - \frac{\mu}{n-1}\sqrt{\frac{n-1}{\lambda}}\eta(X)\eta(Y)$. By virue of the above equation, the manifold is $(\psi B)_n$ (Chern, 1955).

Theorem 2.10.2 *An m-projectively flat Sasakian manifold admitting η-Ricci soliton is a manifold of quasi-constant curvature.*

Proof: By putting $a = \dfrac{\lambda}{n-1}$ and $b = \dfrac{\mu}{n-1}$ in (2.10.2) it is clear that an m-projectively flat Sasakian manifold, $n > 2$ admitting η-Ricci soliton is of quasi-constant curvature (Bejan, 2011).

2.11 Pseudo projective Ricci semi-symmetric Sasakian manifolds

Definition 2.11.1 *Pseudo projective curvature tensor P of type $(1,3)$ is given by (1.16.3)*

$$\begin{aligned} P(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ &- \frac{r}{n}\Big(\frac{a}{n-1} + b\Big)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{2.11.1}$$

where R is a Riemannian curvature tensot of type $(1,3)$.

Making use of equations (2.1.3), (2.1.9) and (2.1.10) in equation (2.11.1) we get

$$P(\xi,Y)Z = \beta[g(Y,Z)\xi - \eta(Z)Y] + \mu b[\eta(Z)Y - \eta(Y)\eta(Z)\xi], \tag{2.11.2}$$

where $\beta = \big[a - \frac{r}{n}\big(\frac{a}{n-1} + b\big) - \lambda b\big]$.

Theorem 2.11.1 *A Sasakian manifold admitting η-Ricci soliton can not be pseudo projective Ricci semi-symmetric if $b \neq a\Big(\frac{n}{r} - \frac{1}{n-1}\Big)$.*

Proof: Now we consider a pseudo projective Ricci semi-symmetric manifold. Then we have

$$S(P(\xi,X)Y,Z) + S(Y,P(\xi,X)Z) = 0. \tag{2.11.3}$$

Using equation (2.11.2) in equation (2.11.3) yields

$$\begin{aligned} \beta g(X,Y)S(\xi,Z) - \beta\eta(Y)S(X,Z) + \mu b\eta(Y)S(X,Z) \\ -\mu b\eta(X)\eta(Y)S(\xi,Z) + \beta g(X,Z)S(\xi,Y) - \beta\eta(Z)S(X,Y) \\ +\mu b\eta(Z)S(X,Y) - \mu b\eta(X)\eta(Z)S(\xi,Y) = 0. \end{aligned} \tag{2.11.4}$$

Making use of equations (2.1.9) and (2.1.10) in equation (2.11.4) we get

$$\begin{aligned} -\mu\big(\beta + \lambda b\big)[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) \\ -2\eta(X)\eta(Y)\eta(Z)] = 0. \end{aligned} \tag{2.11.5}$$

Contracting equation (2.11.5) over X and Y we get

$$-\mu\big(\beta+\lambda b\big)(n-1)\eta(Z)=0. \tag{2.11.6}$$

Putting $Z=\xi$ in equation (2.11.6) yields

$$-\mu\big(\beta+\lambda b\big)=0,$$

or

$$-\mu\Big(a-\frac{r}{n}(\frac{a}{n-1}+b\big)\Big)=0. \tag{2.11.7}$$

From which it follows that

$$a-\frac{r}{n}(\frac{a}{n-1}+b\big)=0,$$

or

$$b=a\Big(\frac{n}{r}-\frac{1}{n-1}\Big). \tag{2.11.8}$$

Theorem 2.11.2 *A Sasakian manifold admitting η-Ricci soliton can not be projective Ricci semi-symmetric.*

Proof: Again we suppose that P is a projective curvature tensor. Then we have $a=1$ and $b=-\frac{1}{(n-1)}$. Thus from (2.11.7)

$$-\mu=0,$$

or

$$\mu=0, \tag{2.11.9}$$

which is not possible for η-Ricci soliton.

Chapter 3

GENERALIZED SASAKIAN-SPACE-FORM

In this chapter we considered a generalized Sasakian-space-form admitting Sasakian structure, and we called it Sasakian generalized Sasakian-space-form. We studied certain symmetric property of τ-curvature tensor in generalized Sasakian-space-form. Later we considered Generalized Tanaka-Webster connection in the generalized Sasakian-space-form. Certain symmetries of the manifold with respect to the Generalized Tanaka-Webster connection are discussed.

3.1 Introduction

In a generalized Sasakian-space-form $\dfrac{c+3}{4}$ and $\dfrac{c-1}{4}$ from the equation (1.20.1) is replaced by differentiable functions. Thus the equation become

$$\begin{aligned} R(X,Y)Z &= f_1\big[g(Y,Z)X - g(X,Z)Y\big] \\ &+ f_2\big[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\big] \\ &+ f_3\big[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi\big]. \end{aligned}$$

An almost contact metric manifold is called a Sasakian manifold if and only if (Yano and Kon, 1985)

$$(\nabla_X\phi)Y = g(X,Y)\xi - \eta(Y)X, \qquad \nabla_X\xi = -\phi X. \tag{3.1.1}$$

On a Sasakian manifold M^n, the following relations are held (Yano and Kon, 1985)

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{3.1.2}$$

$$R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi, \tag{3.1.3}$$

$$\eta\big(R(X,Y)Z\big) = \eta(X)g(Y,Z) - \eta(Y)g(X,Z), \tag{3.1.4}$$

$$\eta\big(R(X,Y)\xi\big) = 0, \tag{3.1.5}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{3.1.6}$$

$$Q\xi = (n-1)\xi, \tag{3.1.7}$$

$$(\nabla_X \eta) = g(X,\phi Y). \tag{3.1.8}$$

In a generalized Sasakian-space-form the following properties holds (Alegre, et. al,, 20014)

$$\begin{aligned} R(X,Y)Z &= f_1\big[g(Y,Z)X - g(X,Z)Y\big] \\ &+ f_2\big[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\big] \\ &+ f_3\big[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi\big], \end{aligned} \tag{3.1.9}$$

$$\begin{aligned} S(X,Y) &= \big[(n-1)f_1 + 3f_2 - f_3\big]g(X,Y) \\ &- \big[3f_2 + (n-2)f_3\big]\eta(X)\eta(Y), \end{aligned} \tag{3.1.10}$$

$$QX = \big[(n-1)f_1 + 3f_2 - f_3\big]X - \big[3f_2 + (n-2)f_3\big]\eta(X)\xi, \tag{3.1.11}$$

$$S(X,\xi) = (n-1)(f_1 - f_3)\eta(X), \tag{3.1.12}$$

$$Q\xi = (n-1)(f_1 - f_3)\xi, \tag{3.1.13}$$

$$R(X,Y)\xi = (f_1 - f_3)\big\{\eta(Y)X - \eta(X)Y\big\}, \tag{3.1.14}$$

$$R(\xi,Y)Z = (f_1 - f_3)\big\{g(Y,Z)\xi - \eta(Z)Y\big\}, \tag{3.1.15}$$

$$R(\xi,Y)\xi = (f_1 - f_3)\big\{\eta(Y)\xi - Y\big\}. \tag{3.1.16}$$

$$r = n(n-1)f_1 + 3(n-1)f_2 - 2(n-1)f_3, \tag{3.1.17}$$

where $r = \sum\limits_{i=1}^{n} S(e_i, e_i)$ is the scalar curvature.

3.2 On ϕ-τ semisymmetric generalized Sasakian-space-form

Definition 3.2.1 *A generalized Sasakian-space-form is ϕ-τ semisymmetric if the τ-curvature tensor satisfies*

$$\big(\tau(X,Y).\phi Z\big) = 0.$$

Theorem 3.2.1 *In a ϕ-τ semisymmetric generalized Sasakian-space-form, we have $a_0(f_1 - f_3) + a_1(n-1)(f_1 - f_3) + a_4\big\{(n-1)f_1 + 3f_2 - f_3\big\} + a_7 r = 0$.*

Proof: We know that

$$\big(\tau(X,Y).\phi Z\big) = \tau(X,Y)\phi Z - \phi\big(\tau(X,Y)Z\big). \tag{3.2.1}$$

Now

$$
\begin{aligned}
\tau(X,Y)\phi Z &= a_0 R(X,Y)\phi Z + a_1 S(Y,\phi Z)X + a_2 S(X,\phi Z)Y \\
&+ a_3 S(X,Y)\phi Z + a_4 g(Y,\phi Z)QX + a_5 g(X,\phi Z)QY \\
&+ a_6 g(X,Y)Q\phi Z + a_7 r[g(Y,\phi Z)X + g(X,\phi Z)Y]. \qquad (3.2.2)
\end{aligned}
$$

Using equations (3.1.9), (3.1.10) and (3.1.11) in the above equation we get

$$
\begin{aligned}
\tau(X,Y)\phi Z = a_0\Big[f_1\big\{g(Y,\phi Z)X - g(X,\phi Z)Y\big\} \\
+f_2\big\{g(X,\phi^2 Z)\phi Y - g(Y,\phi^2 Z)\phi X + 2g(X,\phi Y)\phi^2 Z\big\} \\
+f_3\big\{g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi\big\}\Big] \\
+a_1\big[(n-1)f_1 + 3f_2 - f_3\big]g(Y,\phi Z)X \\
+a_2\big[(n-1)f_1 + 3f_2 - f_3\big]g(X,\phi Z)Y \\
+a_3\big[\big\{(n-1)f_1 + 3f_2 - f_3\big\}g(X,Y) \\
-\big\{3f_2 + (n-2)f_3\big]\eta(X)\eta(Y)\big\}\big]\phi Z \\
+a_4 g(Y,\phi Z)\big[\big\{(n-1)f_1 + 3f_2 - f_3\big\}X - \big\{3f_2 + (n-2)f_3\big\}\eta(X)\xi\big] \\
+a_5 g(X,\phi Z)\big[\big\{(n-1)f_1 + 3f_2 - f_3\big\}Y - \big\{3f_2 + (n-2)f_3\big\}\eta(Y)\xi\big] \\
+a_6 g(X,Y)\big[\big\{(n-1)f_1 + 3f_2 - f_3\big\}\phi Z\big] \\
+a_7 r[g(Y,\phi Z)X - g(X,\phi Z)Y]. \qquad (3.2.3)
\end{aligned}
$$

Also

$$
\begin{aligned}
\phi\Big(\tau(X,Y)Z\Big) &= \phi\Big(a_0 R(X,Y)Z + a_1 S(Y,Z)X + a_2 S(X,Z)Y \\
&+ a_3 S(X,Y)Z + a_4 g(Y,Z)QX + a_5 g(X,Z)QY \\
&+ a_6 g(X,Y)QZ + a_7 r[g(Y,Z)X + g(X,Z)Y]\Big). \qquad (3.2.4)
\end{aligned}
$$

Using equations (3.1.9), (3.1.10) and (3.1.11) in equation(3.2.4) we get

$$\begin{aligned}
\phi\Big(\tau(X,Y)Z\Big) = a_0\Big[&f_1\{g(Y,Z)\phi X - g(X,Z)\phi Y\}\\
&+f_2\{g(X,\phi Z)\phi^2 Y - g(Y,\phi Z)\phi^2 X + 2g(X,\phi Y)\phi^2 Z\}\\
&+f_3\{\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X\}\Big]\\
&+a_1\Big[\{(n-1)f_1 + 3f_2 - f_3\}g(Y,Z)\\
&-\{3f_2 + (n-2)f_3]\eta(Y)\eta(Z)\}\Big]\phi X\\
&+a_2[\{(n-1)f_1 + 3f_2 - f_3\}g(X,Z)\\
&-\{3f_2 + (n-2)f_3]\eta(X)\eta(Z)\}]\phi Y\\
&+a_3[\{(n-1)f_1 + 3f_2 - f_3\}g(X,Y)\\
&-\{3f_2 + (n-2)f_3]\eta(X)\eta(Y)\}\phi Z\\
&+a_4 g(Y,Z)\{(n-1)f_1 + 3f_2 - f_3\}\phi X\\
&+a_5 g(X,Z)\{(n-1)f_1 + 3f_2 - f_3\}\phi Y\\
&+a_6 g(X,Y)[\{(n-1)f_1 + 3f_2 - f_3\}\phi Z]\\
&+a_7 r[g(Y,Z)\phi X - g(X,Z)\phi Y]. \qquad (3.2.5)
\end{aligned}$$

Using equations (3.2.3) and (3.2.5) in equation (3.2.1) we get

$$\begin{aligned}
\Big(\tau(X,Y).\phi Z\Big) = a_0\Big[&f_1\{g(Y,\phi Z)X - g(X,\phi Z)Y - g(Y,Z)\phi X\\
&+g(X,Z)\phi Y\} + f_2\{g(X,\phi^2 Z)\phi Y - g(Y,\phi^2 Z)\phi X + g(X,\phi Z)\phi^2 Y +\\
&g(Y,\phi Z)\phi^2 X\}\\
&+f_3\{g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi - \eta(X)\eta(Z)\phi Y\\
&+\eta(Y)\eta(Z)\phi X\}\Big]\\
&+a_1[\{(n-1)f_1 + 3f_2 - f_3\}\{g(Y,\phi Z)X - g(Y,Z)\phi X\}\\
&+\{3f_2 + (n-2)f_3\}\eta(Y)\eta(Z)\phi X]\\
&+a_2[\{(n-1)f_1 + 3f_2 - f_3\}\{g(X,\phi Z)Y - g(X,Z)\phi Y\}\\
&+\{3f_2 + (n-2)f_3\}\eta(X)\eta(Z)\phi Y]\\
&+a_4[\{(n-1)f_1 + 3f_2 - f_3\}\{g(Y,\phi Z)X - g(Y,Z)\phi X\}\\
&-\{3f_2 + (n-2)f_3\}g(Y,\phi Z)\eta(X)\xi]\\
&+a_5[\{(n-1)f_1 + 3f_2 - f_3\}\{g(X,\phi Z)Y - g(X,Z)\phi Y\}\\
&-\{3f_2 + (n-2)f_3\}g(X,\phi Z)\eta(Y)\xi]\\
&+a_7 r[g(Y,\phi Z)X - g(X,\phi Z)Y - g(Y,Z)\phi X + g(X,Z)\phi Y]. \qquad (3.2.6)
\end{aligned}$$

Putting $Y = \xi$ in equation (3.2.6) we get

$$\begin{aligned}
\big(\tau(X,\xi).\phi Z\big) &= -a_0(f_1 - f_3)[g(X,\phi Z)\xi + \eta(Z)\phi X] \\
&- a_1(n-1)(f_1 - f_3)\eta(Z)\phi X \\
&+ a_2[(n-1)f_1 + 3f_2 - f_3)]g(X,\phi Z)\xi \\
&- a_4\big\{(n-1)f_1 + 3f_2 - f_3\big\}\eta(Z)\phi X \\
&+ a_5(n-1)(f_1 - f_3)g(X,\phi Z)\xi \\
&- a_7 r[g(X,\phi Z)\xi + \eta(Z)\phi X]. \qquad (3.2.7)
\end{aligned}$$

Again putting $Z = \xi$ in equation (3.2.7) we get

$$\begin{aligned}
\big(\tau(X,\xi).\phi\xi\big) = \Big[&- a_0(f_1 - f_3) - a_1(n-1)(f_1 - f_3) \\
&- a_4\big\{(n-1)f_1 + 3f_2 - f_3\big\} - a_7 r\Big]\phi X. \qquad (3.2.8)
\end{aligned}$$

For ϕ-τ semisymmetry we have

$$\begin{aligned}
&\Big[a_0(f_1 - f_3) + a_1(n-1)(f_1 - f_3) \\
&+ a_4\big\{(n-1)f_1 + 3f_2 - f_3\big\} + a_7 r\Big] = 0. \qquad (3.2.9)
\end{aligned}$$

Corollary 3.2.1 *In a generalized Sasakian-space-form, we have the following conditions and results*

Conditions	*Results*
ϕ *semisymmetric*	$f_1 = f_3$
ϕ-W_0^* *semisymmetric*	$f_1 = f_3$
ϕ-W_1 *semisymmetric*	$f_1 = f_3$
ϕ-W_4 *semisymmetric*	$f_1 = f_3$
ϕ-W_5 *semisymmetric*	$f_1 = f_3$
ϕ-$\dot{C}$ *semisymmetric*	$f_3 = \frac{3f_2}{2-n}$
ϕ-W_2 *semisymmetric*	$f_3 = \frac{3f_2}{2-n}$
ϕ-W_9 *semisymmetric*	$f_3 = \frac{3f_2}{2-n}$
ϕ-M *semisymmetric*	$nf_1 + 3f_2 - 2f_3 = 0$
ϕ-$\hat{C}$ *semisymmetric*	$nf_1 + 3f_2 - 2f_3 = 0$
ϕ-P *semisymmetric*	$f_3 = \frac{3f_2}{2-n}$ *or* $a_0 = (2-n)a_2$
ϕ-C *semisymmetric*	$f_3 = \frac{3f_2}{2-n}$ *or* $a_0 = (1-n)a_2$
ϕ-W_3 *semisymmetric*	$2(n-1)f_1 + 3f_2 - nf_3 = 0$
ϕ-W_7 *semisymmetric*	$(n-1)f_1 + 3f_2 - f_3 = 0$

.

Proof: Using the particular cases of equation (1.17.1) in the equation (3.2.8) we have the following results

$$\big(R(X,\xi).\phi\xi\big) = -(f_1 - f_3)\phi X, \tag{3.2.10}$$

$$\big(C(X,\xi).\phi\xi\big) = \frac{3f_2 + (n-2)f_3}{n}\Big[a_0 + (n-2)a_1\Big]\phi X, \tag{3.2.11}$$

$$\big(\tilde{C}(X,\xi).\phi\xi\big) = 0, \tag{3.2.12}$$

$$\big(\hat{C}(X,\xi).\phi\xi\big) = \frac{nf_1 + 3f_2 - 2f_3}{(n-2)}\phi X, \tag{3.2.13}$$

$$\big(\dot{C}(X,\xi).\phi\xi\big) = \frac{3f_2 + (n-2)f_3}{n}\phi X, \tag{3.2.14}$$

$$\big(P(X,\xi).\phi\xi\big) = \frac{3f_2+(n-2)f_3}{n}\Big[a_0+(n-1)a_1\Big]\phi X, \tag{3.2.15}$$

$$\big(\tilde{P}(X,\xi).\phi\xi\big) = 0, \tag{3.2.16}$$

$$\big(M(X,\xi).\phi\xi\big) = \frac{nf_1+3f_2-2f_3}{2(n-1)}\phi X, \tag{3.2.17}$$

$$\big(W_0(X,\xi).\phi\xi\big) = 0, \tag{3.2.18}$$

$$\big(W_0^*(X,\xi).\phi\xi\big) = -2(f_1-f_3)\phi X, \tag{3.2.19}$$

$$\big((W_1,\xi).\phi\xi\big) = -2(f_1-f_3)\phi X, \tag{3.2.20}$$

$$\big(W_1^*(X,\xi).\phi\xi\big) = 0, \tag{3.2.21}$$

$$\big(W_2(X,\xi).\phi\xi\big) = \frac{3f_2+(n-2)f_3}{(n-1)}\phi X, \tag{3.2.22}$$

$$\big(W_3(X,\xi).\phi\xi\big) = \frac{\big\{-2(n-1)f_1-3f_2+nf_3\big\}}{(n-1)}\phi X, \tag{3.2.23}$$

$$\big(W_4(X,\xi).\phi\xi\big) = -(f_1-f_3)\phi X, \tag{3.2.24}$$

$$\big(W_5(X,\xi).\phi\xi\big) = -(f_1-f_3)\phi X, \tag{3.2.25}$$

$$\big(W_6(X,\xi).\phi\xi\big) = 0, \tag{3.2.26}$$

$$\big(W_7(X,\xi).\phi\xi\big) = -\frac{\big\{(n-1)f_1 + 3f_2 - f_3\big\}}{(n-1)}\phi X, \tag{3.2.27}$$

$$\big(W_8(X,\xi).\phi\xi\big) = 0, \tag{3.2.28}$$

$$\big(W_9(X,\xi).\phi\xi\big) = \frac{3f_2 + (n-2)f_3}{(n-1)}\phi X. \tag{3.2.29}$$

Also we have

Corollary 3.2.2 *In a generalized Sasakian-space-form, we have* $\big(\tilde{C}(X,\xi).\phi\xi\big) = 0$, $\big(\tilde{P}(X,\xi).\phi\xi\big) = 0$, $\big(W_0(X,\xi).\phi\xi\big) = 0$, $\big(W_1^*(X,\xi).\phi\xi\big) = 0$, $\big(W_6(X,\xi).\phi\xi\big) = 0$ *and* $\big(W_8(X,\xi).\phi\xi\big) = 0$.

Proof: From the equations (3.2.12, 3.2.16, 3.2.18, 3.2.21), 3.2.26), 3.2.28) we get the corollary.

De and Sarkar (De and Sarkar, 2010) studied the projective curvature tensor in generalized Sasakian-space-form and proved the theorem “An n-dimensional generalized Sasakian-space-form is projectively semisymmetric if and only if $f_1 = f_3$”.

Corollary 3.2.3 *An n-dimensional ϕ-semisymmetric, ϕ-W_0^* semisymmetric, ϕ-W_1 semisymmetric, ϕ-W_4 semisymmetric and ϕ-W_5 semisymmetric generalized Sasakian-Space-form are projectively semisymmetric.*

Proof: By using the above theorems and corollary 3.2.1, we get the results.

3.3 Generalized Sasakian-space-form satisfying $\tau.\tilde{Z} = 0$

In this section we consider generalized Sasakian-space-form satisfying

$$\big(\tau(X,Y).\tilde{\mathbf{Z}}\big)(U,V) = 0, \tag{3.3.1}$$

where $\tilde{\mathbf{Z}}$ denote the Z-tensor given by equation (1.14.1).

Theorem 3.3.1 *In a generalized Sasakian-space-form satisfying $\tau.\tilde{\mathbf{Z}} = 0$, one of the following is true,*
1. $(a_1 + a_2 + a_3 + a_4 + a_5 + a_6) = 0$,
2. $f_1 = f_3$,
3. $\big\{(n-1)(f_1 - f_3) + \phi_1\big\} = 0$.

Proof: Equation (3.3.1) implies

$$\big(\tau(\xi, X).\tilde{\mathbf{Z}}\big)(Y, \xi) = 0.$$

Now

$$\big(\tau(\xi, X).\tilde{\mathbf{Z}}\big)(Y, \xi) = \tilde{\mathbf{Z}}\big(\tau(\xi, X)Y, \xi\big) + \tilde{\mathbf{Z}}\big(Y, \tau(\xi, X)\xi\big). \tag{3.3.2}$$

Then using the value of $\tilde{\mathbf{Z}}$ in equation (3.3.2) we obtain

$$\begin{aligned}\big(\tau(\xi, X).\tilde{\mathbf{Z}}\big)(Y, \xi) = {}& S\big(\tau(\xi, X)Y, \xi\big) + \phi_1\eta\big(\tau(\xi, X)Y\big) \\ & + S\big(Y, \tau(\xi, X)\xi\big) + \phi_1 g\big(Y, \tau(\xi, X)\xi\big).\end{aligned} \tag{3.3.3}$$

From equation (1.17.1) we have

$$\begin{aligned}\tau(\xi, X)Y &= a_0 R(\xi, X)Y + a_1 S(X, Y)\xi + a_2 S(\xi, Y)X \\ &+ a_3 S(\xi, X)Y + a_4 g(X, Y)Q\xi + a_5 g(\xi, Y)QX \\ &+ a_6 g(\xi, X)QY + a_7 r[g(X, Y)\xi - g(\xi, Y)X].\end{aligned} \tag{3.3.4}$$

Using equations (3.1.10), (3.1.11), (3.1.12), (3.1.13) and (3.1.15) in the above equation we get

$$
\begin{aligned}
\tau(\xi, X)Y &= a_0(f_1 - f_3)\big[g(X,Y)\xi - \eta(Y)X\big] \\
&+ a_1\big[\big\{(n-1)f_1 + 3f_2 - f_3\big\}g(X,Y) \\
&- \big\{3f_2 + (n-2)f_3\big]\eta(X)\eta(Y)\big\}\big]\xi \\
&+ a_2(n-1)(f_1 - f_3)\eta(Y)X \\
&+ a_3(n-1)(f_1 - f_3)\eta(X)Y \\
&+ a_4(n-1)(f_1 - f_3)g(X,Y)\xi \\
&+ a_5\big[\big\{(n-1)f_1 + 3f_2 - f_3\big\}X \\
&- \big[3f_2 + (n-2)f_3\big\}\eta(X)\xi\big]\eta(Y) \\
&+ a_6\big[\big\{(n-1)f_1 + 3f_2 - f_3\big\}Y \\
&- \big\{3f_2 + (n-2)f_3\big\}\eta(Y)\xi\big]\eta(X) \\
&+ a_7 r[g(X,Y)\xi - \eta(Y)X].
\end{aligned} \tag{3.3.5}
$$

Using the above equation we obtain

$$
\begin{aligned}
S\big(\tau(\xi, X)Y, \xi\big) &= a_0(f_1 - f_3)\big[g(X,Y)S(\xi,\xi) - \eta(Y)S(X,\xi)\big] \\
&+ a_1\Big[\big\{(n-1)f_1 + 3f_2 - f_3\big\}g(X,Y) \\
&- \big\{3f_2 + (n-2)f_3\big]\eta(X)\eta(Y)\big\}\Big]S(\xi,\xi) \\
&+ a_2(n-1)(f_1 - f_3)\eta(Y)S(X,\xi) \\
&+ a_3(n-1)(f_1 - f_3)\eta(X)S(Y,\xi) \\
&+ a_4(n-1)(f_1 - f_3)g(X,Y)S(\xi,\xi) \\
&+ a_5\big[\big\{(n-1)f_1 + 3f_2 - f_3\big\}(X,\xi) \\
&- \big[3f_2 + (n-2)f_3\big\}\eta(X)S(\xi,\xi)\big]\eta(Y) \\
&+ a_6\big[\big\{(n-1)f_1 + 3f_2 - f_3\big\}S(Y,\xi) \\
&- \big\{3f_2 + (n-2)f_3\big\}\eta(Y)S(\xi,\xi)\big]\eta(X) \\
&+ a_7 r[g(X,Y)S(\xi,\xi) - \eta(Y)S(X,\xi)].
\end{aligned} \tag{3.3.6}
$$

Using equation (3.1.12) in equation (3.3.6) we obtain

$$\begin{aligned}
S\big(\tau(\xi,X)Y,\xi\big) = a_0(n-1)(f_1-f_3)^2\big[g(X,Y)-\eta(X)\eta(Y)\big]\\
+a_1(n-1)(f_1-f_3)\Big[\big\{(n-1)f_1+3f_2-f_3\big\}g(X,Y)\\
-\big\{3f_2+(n-2)f_3\big]\eta(X)\eta(Y)\big\}\Big]\\
+(a_2+a_3+a_5+a_6)(n-1)^2(f_1-f_3)^2\eta(X)\eta(Y)\\
+a_4(n-1)^2(f_1-f_3)^2 g(X,Y)\\
+a_7(n-1)(f_1-f_3)r[g(X,Y)-\eta(X)\eta(Y)].
\end{aligned} \tag{3.3.7}$$

From equation (3.3.5) we have

$$\begin{aligned}
\eta\big(\tau(\xi,X)Y\big) = a_0(f_1-f_3)\big[g(X,Y)-\eta(X)\eta(Y)\big]\\
+a_1\big[\big\{(n-1)f_1+3f_2-f_3\big\}g(X,Y)-\big\{3f_2+(n-2)f_3\big]\eta(X)\eta(Y)\big\}\big]\\
+(a_2+a_3+a_5+a_6)(n-1)(f_1-f_3)\eta(X)\eta(Y)\\
+a_4(n-1)(f_1-f_3)g(X,Y)+a_7r[g(X,Y)-\eta(X)\eta(Y)].
\end{aligned} \tag{3.3.8}$$

Again from equation (3.3.5) we have

$$\begin{aligned}
\tau(\xi,X)\xi = a_0(f_1-f_3)\big[\eta(X)\xi-X\big]+a_2(n-1)(f_1-f_3))X\\
+(a_1+a_3+a_4+a_6)(n-1)(f_1-f_3)\eta(X)\xi\\
+a_5\big[\big\{(n-1)f_1+3f_2-f_3\big\}X-\big[3f_2+(n-2)f_3\big\}\eta(X)\xi\big]\\
+a_7r[\eta(X)\xi-X].
\end{aligned} \tag{3.3.9}$$

From the above equation we obtain

$$\begin{aligned}
S\big(Y,\tau(\xi,X)\xi\big) = a_0(f_1-f_3)\big[\eta(X)S(Y,\xi)-S(Y,X)\big]\\
+a_2(n-1)(f_1-f_3))S(Y,X)\\
+(a_1+a_3+a_4+a_6)(n-1)(f_1-f_3)\eta(X)S(Y,\xi)\\
+a_5\big[\big\{(n-1)f_1+3f_2-f_3\big\}S(Y,X)\\
-\big[3f_2+(n-2)f_3\big\}\eta(X)S(Y,\xi)\big]\\
+a_7r[\eta(X)S(Y,\xi)-S(Y,X)].
\end{aligned} \tag{3.3.10}$$

Using equations (3.1.10) and (3.1.12) in the above equation we get

$$
\begin{aligned}
S\big(Y, \tau(\xi, X)\xi\big) = {} & a_0(f_1 - f_3)\big\{(n-1)f_1 + 3f_2 - f_3\big\} \times \\
& \big\{\eta(X)\eta(Y) - g(X,Y)\big\} \\
& + a_2(n-1)(f_1 - f_3)\Big[\big\{(n-1)f_1 + 3f_2 - f_3\big\}g(X,Y) \\
& - \big\{3f_2 + (n-2)f_3\big\}\eta(X)\eta(Y)\Big] \\
& + (a_1 + a_3 + a_4 + a_6)(n-1)^2(f_1 - f_3)^2\eta(X)\eta(Y) \\
+ a_5\Big[\big\{(n-1)f_1 + 3f_2 - f_3\big\}&\big[\big\{(n-1)f_1 + 3f_2 - f_3\big\}g(X,Y) \\
& - \big\{3f_2 + (n-2)f_3\big\}\eta(X)\eta(Y)\big] \\
& - (n-1)(f_1 - f_3)\big\{3f_2 + (n-2)f_3\big\}\eta(X)\eta(Y)\Big] \\
& + a_7 r\Big[\big\{(n-1)f_1 + 3f_2 - f_3\big\}\big\{\eta(X)\eta(Y) - g(X,Y)\big\}\Big]. \qquad (3.3.11)
\end{aligned}
$$

Also from equation (3.3.9) we get

$$
\begin{aligned}
g\big(Y, \tau(\xi, X)\xi\big) = {} & a_0(f_1 - f_3)\big[\eta(X)\eta(Y) - g(Y,X)\big] \\
& + a_2(n-1)(f_1 - f_3)g(Y,X) \\
& + (a_1 + a_3 + a_4 + a_6)(n-1)(f_1 - f_3)\eta(X)\eta(Y) \\
+ a_5\big[\big\{(n-1)f_1 + 3f_2 - f_3\big\}&g(Y,X) - \big[3f_2 + (n-2)f_3\big\}\eta(X)\eta(Y)\big] \\
& + a_7 r[\eta(X)\eta(Y) - g(Y,X)]. \qquad (3.3.12)
\end{aligned}
$$

Using equations (3.3.7), (3.3.8), (3.3.11) and (3.3.12) in equation (3.3.3) we get

$$\begin{aligned}\big(\tau(\xi,X).\tilde{\mathbf{Z}}\big)(Y,\xi) = a_0(f_1-f_3)\big\{3f_2+(n-2)f_3\big\}\times\\ \big\{\eta(X)\eta(Y)-g(X,Y)\big\}\\ +(a_1+a_2)(n-1)(f_1-f_3)\Big[\big\{(n-1)f_1+3f_2-f_3\big\}g(X,Y)\\ -\big\{3f_2+(n-2)f_3\big\}\eta(X)\eta(Y)\big\}\Big]\\ +(a_1+a_2+2a_3+a_4+a_5+2a_6)(n-1)^2(f_1-f_3)^2\eta(X)\eta(Y)\\ +a_4(n-1)^2(f_1-f_3)^2g(X,Y)\\ +a_5\Big[\big\{(n-1)f_1+3f_2-f_3\big\}\big[\big\{(n-1)f_1+3f_2-f_3\big\}g(X,Y)\\ -\big\{3f_2+(n-2)f_3\big\}\eta(X)\eta(Y)\big]\\ -(n-1)(f_1-f_3)\big\{3f_2+(n-2)f_3\big\}\eta(X)\eta(Y)\Big]\\ +a_7r\big\{3f_2+(n-2)f_3\big\}\big\{\eta(X)\eta(Y)-g(X,Y)\big\}\\ +\phi_1\Big[a_1\big[\big\{(n-1)f_1+3f_2-f_3\big\}g(X,Y)\\ -\big\{3f_2+(n-2)f_3\big]\eta(X)\eta(Y)\big\}\big]\\ +(a_1+a_2+2a_3+a_4+a_5+2a_6)(n-1)(f_1-f_3)\eta(X)\eta(Y)\\ +(a_2+a_4)(n-1)(f_1-f_3)g(X,Y)\\ +a_5\big[\big\{(n-1)f_1+3f_2-f_3\big\}g(Y,X)\\ -\big[3f_2+(n-2)f_3\big\}\eta(X)\eta(Y)\big]\Big].\end{aligned} \tag{3.3.13}$$

Putting $Y=\xi$ in equation (3.3.13) we get

$$\begin{aligned}\big(\tau(\xi,X).\tilde{\mathbf{Z}}\big)(\xi,\xi) = 2(a_1+a_2+a_3+a_4+a_5+a_6)(n-1)^2(f_1-f_3)^2\eta(X)\\ +\phi_1\Big[2(a_1+a_2+a_3+a_4+a_5+a_6)(n-1)(f_1-f_3)\eta(X)\Big],\end{aligned}$$

or

$$\begin{aligned}\big(\tau(\xi,X).\tilde{\mathbf{Z}}\big)(\xi,\xi) = 2(a_1+a_2+a_3+a_4+a_5+a_6)(n-1)(f_1-f_3)\\ \times\eta(X)\big\{(n-1)(f_1-f_3)+\phi_1\big\}.\end{aligned} \tag{3.3.14}$$

For Sasakian Spaceform Satisfying $\tau.\tilde{\mathbf{Z}}=0$ we have

$$\begin{aligned}2(a_1+a_2+a_3+a_4+a_5+a_6)(n-1)(f_1-f_3)\eta(X)\\ \times\big\{(n-1)(f_1-f_3)+\phi_1\big\}=0,\end{aligned}$$

or

$$(a_1 + a_2 + a_3 + a_4 + a_5 + a_6)(f_1 - f_3)\big\{(n-1)(f_1 - f_3) + \phi_1\big\} = 0. \tag{3.3.15}$$

Thus, eitheir $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 0$, $f_1 - f_3 = 0$ or $(n-1)(f_1 - f_3) + \phi_1 = 0$.

Theorem 3.3.2 *In a generalized Sasakian spaceform satisfying $\tau.S = 0$, either $(a_1 + a_2 + a_3 + a_4 + a_5 + a_6) = 0$ or projectively semisymmetric.*

Proof: It is known that when $\phi_1 = 0$, $\tilde{\mathbf{Z}} = S$. Now putting $\phi_1 = 0$ in equation (3.3.14) we get

$$\begin{aligned}\big(\tau(\xi, X).S\big)(\xi, \xi) = {} & 2(a_1 + a_2 + a_3 + a_4 + a_5 + a_6)\\ & \times (n-1)^2 (f_1 - f_3)^2 \eta(X).\end{aligned} \tag{3.3.16}$$

Therefore $\tau.S = 0$ implies

$$(a_1 + a_2 + a_3 + a_4 + a_5 + a_6)(f_1 - f_3)^2 = 0. \tag{3.3.17}$$

Thus, eitheir $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 0$ or $f_1 - f_3 = 0$. We have mentioned before that "An n-dimensional generalized Sasakian-space-form is projectively semisymmetric if and only if $f_1 = f_3$". And hence the theorem is proved.

Using the particular cases of equation (1.17.1) in equation (3.3.14) we have the following

Corollary 3.3.1 *In a generalized Sasakian-space-form,*
$\big(R(\xi, X).\tilde{\mathbf{Z}}\big)(\xi, \xi)$, $\big(C(\xi, X).\tilde{\mathbf{Z}}\big)(\xi, \xi)$, $\big(\tilde{C}(\xi, X).\tilde{\mathbf{Z}}\big)(\xi, \xi)$,
$\big(\hat{C}(\xi, X).\tilde{\mathbf{Z}}\big)(\xi, \xi)$, $\big(\dot{C}(\xi, X).\tilde{\mathbf{Z}}\big)(\xi, \xi)$, $\big(P(\xi, X).\tilde{\mathbf{Z}}\big)(\xi, \xi)$,
$\big(\tilde{P}(\xi, X).\tilde{\mathbf{Z}}\big)(\xi, \xi)$, and $\big(M(\xi, X).\tilde{\mathbf{Z}}\big)(\xi, \xi)$, all vanishes.

3.4 Generalized Tanaka-Webster connection

The generalized Tanaka-Webster connection $\tilde{\nabla}$ for contact metric manifolds is given by (Tanno, 1989)

$$\tilde{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi(Y) \tag{3.4.1}$$

for all $X, Y \in \chi M$, and ∇ is the Riemannian connection.

Let R and $\tilde{R}$ denotes the Riemannian curvature tensors of Sasakian manifold with respect to ∇ and $\tilde{\nabla}$ respectively. A relation between R and $\tilde{R}$ is given by (De and Ghosh,2016)

$$
\begin{aligned}
\tilde{R}(X,Y)Z &= R(X,Y)Z + \big[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\big]\xi \\
&- g(Y,\phi Z)\phi X + g(X,\phi Z)\phi Y + 2g(Y,\phi X)\phi Z \\
&- \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y. \qquad (3.4.2)
\end{aligned}
$$

Contracting equation (3.4.2) we obtain

$$
\tilde{S}(Y,Z) = S(Y,Z) - g(Y,Z) - (n-3)\eta(X)\eta(Y). \qquad (3.4.3)
$$

Using equations (3.1.9) and (3.1.10) in the above equations we have

$$
\begin{aligned}
\tilde{R}(X,Y)Z &= (f_1 - 1)\big[g(Y,Z)X - g(X,Z)Y\big] \\
&+ f_2\big[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\big] \\
&+ f_3\big[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\
&- g(Y,Z)\eta(X)\xi\big] - g(Y,\phi Z)\phi X + g(X,\phi Z)\phi Y \\
&+ 2g(Y,\phi X)\phi Z - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y, \qquad (3.4.4)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{S}(Y,Z) &= \big[(n-1)f_1 + 3f_2 - f_3\big]g(X,Y) \\
&- \big[3f_2 + (n-2)f_3\big]\eta(X)\eta(Y) \\
&- g(Y,Z) - (n-3)\eta(X)\eta(Y). \qquad (3.4.5)
\end{aligned}
$$

Now we have

$$
\tilde{R}(X,Y)\xi = (f_1 - f_3 - 1)\big\{\eta(Y)X - \eta(X)Y\big\}, \qquad (3.4.6)
$$

$$
\tilde{R}(\xi,X)Y = (f_1 - f_3 - 1)\big\{g(Y,Z)\xi - \eta(Z)Y\big\}, \qquad (3.4.7)
$$

$$
\tilde{R}(\xi,X)\xi = (f_1 - f_3)\big\{\eta(Y)\xi - Y\big\}, \qquad (3.4.8)
$$

$$
\tilde{S}(X,\xi) = (n-1)(f_1 - f_3 - 1)\eta(X), \qquad (3.4.9)
$$

$$
\tilde{S}(\xi,\xi) = (n-1)(f_1 - f_3 - 1). \qquad (3.4.10)
$$

3.5 Semi-symmetric and Ricci semi-symmetric

Theorem 3.5.1 *If a Sasakian generalized Sasakian-space-form is semi-symmetric with respect to generalized Tanaka-Webster connection, we have*

$$\tilde{R}(Y,V)W = (f_1 - f_3 - 1)\{g(V,W)Y - g(Y,W)V\},$$

provided $f_1 - f_3 - 1 \neq 0$.

Proof: Suppose that the Sasakian generalized Sasakian-space-form is semi-symmetric with respect to generalized Tanaka-Webster connection, then from (1.23.1) we get

$$\tilde{R}(X,Y).\tilde{R}(U,V)W = 0. \tag{3.5.1}$$

It is well known that

$$\begin{aligned}\tilde{R}(X,Y).\tilde{R}(U,V)W &= \tilde{R}(X,Y).\tilde{R}(U,V)W - \tilde{R}(\tilde{R}(X,Y)U,V)W \\ &- \tilde{R}(U,\tilde{R}(X,Y)V)W - \tilde{R}(U,V)\tilde{R}(X,Y)W. \end{aligned} \tag{3.5.2}$$

Now setting $X = U = \xi$ in equation (3.5.1) and using equation (3.5.2) we get

$$(f_1 - f_3 - 1)^2\{g(Y,W)V - g(V,W)Y\} + (f_1 - f_3 - 1)\tilde{R}(Y,V)W = 0,$$

which can be written as

$$\tilde{R}(Y,V)W = (f_1 - f_3 - 1)\{g(V,W)Y - g(Y,W)V\}, \tag{3.5.3}$$

provided $f_1 - f_3 - 1 \neq 0$.

Theorem 3.5.2 *In a semi-symmetric Sasakian generalized Sasakian-space-form with respect to generalized Tanaka-Webster connection the Riemannian curvature tensor is given by*

$$\begin{aligned}R(Y,V)W &= (f_1 - f_3 - 1)\{g(V,W)Y - g(Y,W)V\} \\ &- \big[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\big]\xi + g(Y,\phi Z)\phi X \\ &- g(X,\phi Z)\phi Y - 2g(Y,\phi X)\phi Z \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\},\end{aligned}$$

provided $f_1 - f_3 - 1 \neq 0$.

Proof: Now using equation (3.4.2) in equation (3.5.3) we get

$$\begin{aligned} R(Y,V)W &= (f_1 - f_3 - 1)\{g(V,W)Y - g(Y,W)V\} \\ &- \big[g(Y,W)\eta(V) - g(V,W)\eta(Y)\big]\xi + g(V,\phi W)\phi Y \\ &- g(Y,\phi W)\phi V - 2g(V,\phi Y)\phi W \\ &+ \eta(V)\eta(W)Y - \eta(Y)\eta(W)V, \end{aligned} \tag{3.5.4}$$

provided $f_1 - f_3 - 1 \neq 0$.

Theorem 3.5.3 *A semi-symmetric Sasakian generalized Sasakian-space-form with respect to generalized Tanaka-Webster connection is Einstein manifold provided* $f_1 - f_3 - 1 \neq 0$.

Proof: Suppose that the Sasakian generalized Sasakian-space-form is Ricci semi-symmetric with respect to generalized Tanaka-Webster connection, then from (1.23.2) we get

$$\tilde{R}(X,Y).\tilde{S}(U,V) = 0. \tag{3.5.5}$$

It implies

$$\tilde{S}(\tilde{R}(X,Y).U,V) + \tilde{S}(U,\tilde{R}(X,Y)V) = 0. \tag{3.5.6}$$

Setting $X = U = \xi$ in equation (3.5.6) we get

$$(f_1 - f_3 - 1)\{(n-1)(f_1 - f_3 - 1)g(Y,V) - S(Y,V)\} = 0.$$

Which implies

$$S(Y,V) = (n-1)(f_1 - f_3 - 1)g(Y,V), \tag{3.5.7}$$

provided $f_1 - f_3 - 1 \neq 0$.

3.6 Ricci-generalized pseudosymmetric manifold

Theorem 3.6.1 *A Ricci-generalized pseudosymmetric Sasakian generalized Sasakian-space-form with respect to generalized Tanaka-Webster connection is Ricci flat provided* $f \neq 0$ *and* $f_1 - f_3 - 1 \neq 1$.

Proof: Suppose that the Sasakian generalized Sasakian-space-form is Ricci-generalized pseudosymmetric with respect to generalized Tanaka-Webster connection, then from

(1.24.3)

$$\tilde{R}(X,Y).\tilde{R}(U,V)W = fQ(\tilde{S},\tilde{R})(U,V,W;X,Y).$$

This is equivalent to

$$\tilde{R}(X,Y).\tilde{R}(U,V)W = f\{\big((X\wedge_{\tilde{S}} Y).\tilde{S}\big)(U,V)\}, \tag{3.6.1}$$

where $\big((X\wedge_{\tilde{S}} Y)Z\big) = \tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y$ for all X,Y,Z.

Thus we get

$$\begin{aligned} \tilde{R}(X,Y).\tilde{R}(U,V)W &- \tilde{R}(\tilde{R}(U,V)X,Y)W - \tilde{R}(X,\tilde{R}(U,V)Y)W \\ -\tilde{R}(X,Y)\tilde{R}(U,V)W &= f\{(X\wedge_{\tilde{S}} Y)\tilde{R}(U,V)W - \tilde{R}((X\wedge_{\tilde{S}} Y)U,V)W \\ &- \tilde{R}(U,(X\wedge_{\tilde{S}} Y)V)W - \tilde{R}(U,V)(X\wedge_{\tilde{S}} Y)W)\}, \end{aligned}$$

or

$$\begin{aligned} \tilde{R}(X,Y).\tilde{R}(U,V)W &- \tilde{R}(\tilde{R}(U,V)X,Y)W - \tilde{R}(X,\tilde{R}(U,V)Y)W \\ -\tilde{R}(X,Y)\tilde{R}(U,V)W &= f\{\tilde{S}(Y,\tilde{R}(U,V)W)X - \tilde{S}(X,\tilde{R}(U,V)W)Y \\ &- \tilde{S}(Y,U)\tilde{R}(X,V)W + \tilde{S}(X,U)\tilde{R}(Y,V)W \\ &- \tilde{S}(Y,V)\tilde{R}(U,X)W + \tilde{S}(X,V)\tilde{R}(U,Y)W \\ &- \tilde{S}(Y,W)\tilde{R}(U,V)X + \tilde{S}(X,W)\tilde{R}(U,V)Y. \end{aligned} \tag{3.6.2}$$

Setting $X = U = \xi$ in equation (3.6.2) we get

$$\begin{aligned} (f_1-f_3-1)^2 \quad \{ & \quad g(Y,W)V - g(V,W)Y\} + (f_1-f_3-1)\tilde{R}(Y,V)W \\ = & \quad f\Big[(n-1)(f_1-f_3-1)\big\{\tilde{R}(Y,V)W - g(V,W)Y \\ + & \quad g(Y,W)\eta(V)\xi + g(V,Y)\eta(W)\xi\big\} \\ - & \quad \tilde{S}(Y,V)\eta(W)\xi - \tilde{S}(Y,W)\big\{\eta(V)\xi - V\big\}\Big]. \end{aligned} \tag{3.6.3}$$

Again setting $V = \xi$ in equation (3.6.3) we get

$$\begin{aligned} & f(f_1-f_3-1)\big[g(Y,W)\xi - \eta(W)Y\big] \\ = \; & f(f_1-f_3-1)^2\big[g(Y,W)\xi - \eta(W)Y\big]. \end{aligned} \tag{3.6.4}$$

We have either

$$f_1 - f_3 - 1 = 0, \tag{3.6.5}$$

or

$$f_1 - f_3 - 1 = 1, \tag{3.6.6}$$

provided $f \neq 0$.

Setting $W = \xi$ in equation (3.6.3) and using equation (3.6.5) we get

$$S(Y, V) = 0, \tag{3.6.7}$$

for all $Y, V \in \chi M$, provided $f \neq 0$ and $f_1 - f_3 - 1 \neq 1$.

Theorem 3.6.2 *A Ricci-generalized pseudosymmetric Sasakian generalized Sasakian-space-form with respect to generalized Tanaka-Webster connection is Einstein manifold provided $f \neq 0$ and $f_1 - f_3 - 1 \neq 0$.*

Proof: Setting $W = \xi$ in equation (3.6.3) and using equation (3.6.6) we get

$$S(Y, V) = (n-1)g(V, Y), \tag{3.6.8}$$

for all $Y, V \in \chi M$, provided $f \neq 0$ and $f_1 - f_3 - 1 \neq 0$.

3.7 Ricci-pseudosymmetric manifold

Theorem 3.7.1 *A Ricci-pseudosymmetric Sasakian generalized Sasakian-space-form with respect to generalized Tanaka-Webster connection is Einstein manifold provided $f_1 - f_3 - f' - 1 \neq 0$.*

Proof: Suppose that the Sasakian generalized Sasakian-space-form is Ricci-pseudosymmetric with respect to generalized Tanaka-Webster connection, then from equation (1.24.2)

$$\tilde{R}(X, Y).\tilde{S}(U, V) = f'Q(g, \tilde{R})(U, V; X, Y).$$

This is equivalent to

$$\tilde{R}(X, Y).\tilde{S}(U, V) = f'\{\big((X \wedge_g Y).\tilde{S}\big)(U, V)\}, \tag{3.7.1}$$

where $\big((X \wedge_g Y)Z\big) = g(Y, Z)X - g(X, Z)Y$ for all X, Y, Z.

Thus we get

$$\tilde{S}(\tilde{R}(X, Y).U, V) + \tilde{S}(U, \tilde{R}(X, Y)V) = f'\{\tilde{S}\big((X \wedge_g Y)U, V\big) + \tilde{S}\big(U, (X \wedge_g Y)V\big)\},$$

or

$$\tilde{S}(\tilde{R}(X,Y).U,V) + \tilde{S}(U,\tilde{R}(X,Y)V) = f'\{g(Y,U)\tilde{S}(X,V) - g(X,U)\tilde{S}(Y,V) + g(Y,V)\tilde{S}(U,X) - g(X,V)\tilde{S}(U,Y)\}. \quad (3.7.2)$$

Setting $X = U = \xi$ in equation (3.7.2) we get

$$(f_1 - f_3 - f' - 1)\{S(Y,V) - (n-1)(f_1 - f_3 - 1)g(Y,V)\} = 0.$$

Which implies

$$S(Y,V) = (n-1)(f_1 - f_3 - 1)g(Y,V), \quad (3.7.3)$$

for all $Y, V \in \chi M$, provided $f_1 - f_3 - f' - 1 \neq 0$.

Now using the theorem 4.2 from the article, "structures on generalized-Sasakian-space-form" (Alegre and Carriazo, 2008) and the equation (3.7.3) we get the following corollary

Corollary 3.7.1 *An n-dimensional connected Sasakian generalized Sasakian-space-form, ($n \geq 5$), which is Ricci-pseudosymmetric with respect to generalized Tanaka-Webster connection is Ricci flat provided $f' \neq 0$.*

Chapter 4

GENERALIZED PSEUDO-PROJECTIVE RECURRENT MANIFOLDS

In this chapter we considered a semi-Riemannian manifold which is generalized recurrent with respect to the pseudo-projective curvature tensor. We studied a manifold with certain geometrical properties like constant scalar curvature, Ricci-symmetric, conformally flat, Einstein and quasi Einstein manifolds. We also considered the case where the manifold is decomposable. At the end of the chapter examples are given to support the results.

4.1 Introduction

From equation (1.16.3) it follows that the pseudo projective curvature tensor satisfy

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Singh, J. P. and Lalmalsawma, C. (2018). On generalized pseudo-projectively recurrent manifolds, *Journal of the Indian Math. Soc.* **85(3-4)**, 449-469.

$$(i)\sum_{i=1}^{n}\epsilon_i' P(Y,Z,e_i,e_i) = 0 = \sum_{i=1}^{n}\epsilon_i' P(e_i,e_i,U,V),$$
$$(ii)\sum_{i=1}^{n}\epsilon_i' P(e_i,Z,U,e_i) = [a+(n-1)b]\big[S(Z,U) - \frac{r}{n}g(Z,U)\big],$$
$$(iii)\sum_{i=1}^{n}\epsilon_i' P(e_i,Z,e_i,V) = (b-a)\big[S(Z,V) - \frac{r}{n}g(Z,V)\big],$$
$$(iv)\sum_{i=1}^{n}\epsilon_i' P(Y,e_i,U,e_i) = -[a+(n-1)b]\big[S(Y,U) - \frac{r}{n}g(Y,U)\big],$$
$$(v)\sum_{i=1}^{n}\epsilon_i' P(Y,e_i,e_i,V) = (a-b)\big[S(Y,V) - \frac{r}{n}g(Y,V)\big], \tag{4.1.1}$$

where $r = \sum_{i=1}^{n}\epsilon_i S(e_i,e_i)$ is the scalar curvature, $\{e_i\}$ are an orthonormal basis of the tangent space at each point of the semi-Riemannian manifold, where $1 \leq i \leq n$ such that $g(e_i,e_j) = 0$ for $i \neq j$ and $g(e_i,e_i) = \epsilon_i, \epsilon_i = \pm 1$.

4.2 Constant Scalar Curvature

Definition 4.2.1 *A semi-Riemannian manifold (M^n, g) is called generalized pseudo-projectively recurrent manifold if the pseudo-projective curvature tensor of type (0,4), satisfies the condition:*

$$(\nabla_X P)(Y,Z)U = A(X)R(Y,Z)U + B(X)[g(Z,U)Y - g(Y,U)Z] \tag{4.2.1}$$

where A and B are two 1-forms, and B is non-zero. It is denoted by $G\{PP(K_n)\}$.

Theorem 4.2.1 *The scalar curvature r of a generalized pseudo-projectively recurrent manifold is constant if and only if*

$$rA(X) \;=\; nA(LX) - \frac{n(n^2-3n+2)}{2a+(n-2)b}B(X)$$

holds for all vector fields, provided $b - a \neq a + (n-1)b$.

Proof: From equation (1.16.3) we have,

$$\begin{aligned}(\nabla'_X P)(Y,Z,U,V) &= a(\nabla_X R)(Y,Z,U,V)\\ &+ b[(\nabla_X S)(Z,U)g(Y,V) - (\nabla_X S)(Y,U)g(Z,V)].\end{aligned}$$

Making use of equation (4.2.1) in the above equation,

$$\begin{aligned}(\nabla'_X R)(Y,Z,U,V) \quad &= \frac{1}{a}A(X)'P(Y,Z,U,V)\\ &+\frac{1}{a}B(X)\big[g(Z,U)g(Y,V)-g(Y,U)g(Z,V)\big]\\ &-\frac{b}{a}\big[(\nabla_X S)(Z,U)g(Y,V)-(\nabla_X S)(Y,U)g(Z,V)\big]. \quad (4.2.2)\end{aligned}$$

Using second Bianchi's identity in the above equation we get

$$\begin{aligned} & \big[A(X)'P(Y,Z,U,V)+A(Y)'P(Z,X,U,V)+A(Z)'P(X,Y,U,V)\big]\\ + & \Big[B(X)\big\{g(Z,U)g(Y,V)-g(Y,U)g(Z,V)\big\}\\ + & B(Y)\big\{g(X,U)g(Z,V)-g(Z,U)g(X,V)\big\}\\ + & B(Z)\big\{g(Y,U)g(X,V)-g(X,U)g(Y,V)\big\}\Big]\\ - & b\Big[\big\{(\nabla_X S)(Z,U)g(Y,V)-(\nabla_X S)(Y,U)g(Z,V)\big\}\\ + & \big\{(\nabla_Y S)(X,U)g(Z,V)-(\nabla_Y S)(Z,U)g(X,V)\big\}\\ + & \big\{(\nabla_Z S)(Y,U)g(X,V)-(\nabla_Z S)(X,U)g(Y,V)\big\}\Big]=0. \qquad (4.2.3)\end{aligned}$$

Putting $Y=V=e_i$ in equation (4.2.3), where e_i, $1\leq i\leq n$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, we get

$$\begin{aligned} & \{a+(n-1)b\}A(X)\big[S(Z,U)-\frac{r}{n}g(Z,U)\big]+A(\tilde{P}(Z,X)U)\\ - & \{a+(n-1)b\}A(Z)\big[S(X,U)-\frac{r}{n}g(X,U)\big]\\ + & (n-2)[B(X)g(Z,U)-B(Z)g(X,U)]\\ - & (n-2)b[(\nabla_X S)(Z,U)-(\nabla_Z S)(X,U)]=0. \qquad (4.2.4)\end{aligned}$$

Again putting $Z=U=e_i$ in equation (4.2.4), where e_i, $1\leq i\leq n$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, we get

$$\begin{aligned} - & [2a+(n-2)b][A(LX)-\frac{r}{n}A(X)]+(n^2-3n+2)B(X)\\ - & \frac{(n-2)b}{2}dr(X)=0.\end{aligned}$$

From the equation (1.16.3) it is obvious that $a+(n-1)b=0$ if and only if the pseudo projective curvature tensor is a scalar multiple of the projective curvature tensor. In this study, we consider the case of pseudo projective curvature tensor where $a+(n-1)b \neq 0$. Thus

$$\begin{aligned} rA(X) &= nA(LX) - \frac{n(n^2-3n+2)}{2a+(n-2)b}B(X) \\ &+ \frac{n(n-2)b}{2[2a+(n-2)b]}dr(X), \end{aligned} \tag{4.2.5}$$

provided $b-a \neq a+(n-1)b$.

Theorem 4.2.2 *A generalized pseudo-projectively recurrent manifold with constant scalar curvature is a generalized Ricci recurrent manifold or Ricci recurrent manifold, provided* $b-a \neq a+(n-1)b$.

Proof: Now we suppose that the scalar curvature r is constant in a $G\{PP(K_n)\}$, that is $dr=0$. Then from equation (4.2.5) we get

$$rA(X) = nA(LX) - \frac{n(n-1)(n-2)}{2a+(n-2)b}B(X), \tag{4.2.6}$$

provided $b-a \neq a+(n-1)b$.

Putting $Y=V=e_i$ in equation (4.2.2), where e_i, $1 \leq i \leq n$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, we get

$$\begin{aligned} (\nabla_X S)(Z,U) &= \frac{1}{a}[a+(n-1)b]A(X)[S(Z,U) - \frac{r}{n}g(Z,U)] \\ &+ \frac{1}{a}(n-1)B(X)g(Z,U) \\ &- \frac{b}{a}[n(\nabla_X S)(Z,U) - (\nabla_X S)(Z,U)]. \end{aligned}$$

Which implies

$$\begin{aligned} (\nabla_X S)(Z,U) &= A(X)[S(Z,U) - \frac{r}{n}g(Z,U)] \\ &+ \frac{n-1)}{a+(n-1)b}B(X)g(Z,U). \end{aligned} \tag{4.2.7}$$

Since r is constant. so using equation (4.2.6) in equation (4.2.7) yields

$$
\begin{aligned}
(\nabla_X S)(Z,U) &= A(X)S(Z,U) \\
&+ \Big[-A(LX) + \frac{n(n-1)[a+(n-2)b]}{\{2a+(n-2)b\}\{a+(n-1)b\}}B(X)\Big]g(Z,U),
\end{aligned}
$$

provided $b - a \neq a + (n-1)b$.

The above expression can be written as

$$
(\nabla_X S)(Z,U) = A(X)S(Z,U) + D(X)g(Z,U),
$$

where

$$
D(X) = \Big[-A(LX) + \frac{n(n-1)[a+(n-2)b]}{\{2a+(n-2)b\}\{a+(n-1)b\}}B(X)\Big].
$$

Hence the manifold is a generalized Ricci recurrent manifold for $D(X) \neq 0$ and Ricci recurrent manifold for $D(X) = 0$.

4.3 Ricci-Symmetric Manifolds

If $G\{PP(K_n)\}$ is Ricci-symmetric, then $\nabla S = 0$, that is, $\nabla L = 0$. Then the scalar curvature r is constant and $dr = 0$.

Theorem 4.3.1 *A Ricci-symmetric $G\{PP(K_n)\}$ is an Einstein manifold, provided it is not locally symmetric.*

Proof:

Now suppose that $G\{PP(K_n)\}$ is Ricci-symmetric. So we have from equation (4.2.7)

$$
A(X)\big[S(Z,U) - \frac{r}{n}g(Z,U)\big] + \frac{n-1}{a+(n-1)b}B(X)g(Z,U) = 0. \tag{4.3.1}
$$

Again, since r is constant using the value of $B(X)$ from equation (4.2.6) in equation (4.3.1) we get

$$
\begin{aligned}
S(Z,U) &= \frac{1}{(n-2)\{a+(n-1)b\}} \\
&\times \Big[\{a+(n-2)b\}r - \{2a+(n-2)b\}\frac{A(LX)}{A(X)}\Big]g(Z,U).
\end{aligned}
$$

This can bee written as

$$S(Z,U) = \lambda g(Z,U),$$

where $\lambda = \dfrac{1}{(n-2)\{a+(n-1)b\}}\Big[\{a+(n-2)b\}r - \{2a+(n-2)b\}\dfrac{A(LX)}{A(X)}\Big]$ is a scalar.

4.4 Einstein Manifolds

Theorem 4.4.1 *An Einstein $G\{PP(K_n)\}$, $n > 2$ is a GK_n, provided*

$$\frac{Q}{a} \neq \frac{r}{n(n-1)}P,$$

where $A(X) = g(X,P)$ and $B(X) = g(X,Q)$.

Proof: If a $G\{PP(K_n)\}$ is Einstein then the Ricci tensor satisfies

$$S(Z,U) = \frac{r}{n}g(Z,U), \tag{4.4.1}$$

which imply

$$dr(X) = 0$$

and

$$(\nabla_X S)(Y,Z) = 0, \tag{4.4.2}$$

for all X, Y, Z.
Using equations (4.4.1) and (4.4.2) we get from equation (1.16.3)

$$(\nabla'_X P)(Y,Z,U,V) = a(\nabla'_X R)(Y,Z,U,V). \tag{4.4.3}$$

Now using equation (4.4.3) in equation (4.2.1) we have

$$\begin{aligned} & a(\nabla'_X R)(Y,Z,U,V) \\ = \;& A(X)'P(Y,Z,U,V) + B(X)\big[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\big]. \end{aligned} \tag{4.4.4}$$

Again using equation (1.16.3) in equation (4.4.4) ,we get

$$
\begin{aligned}
& a(\nabla'_X R)(Y,Z,U,V) \\
= \; & A(X)\Big[a'R(Y,Z,U,V) + b\big\{S(Z,U)g(Y,V) - S(Y,U)g(Z,V)\big\} \\
- \; & \frac{r}{n}(\frac{a}{n-1} + b)\big\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\big\}\Big] \\
+ \; & B(X)\big\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\big\}. \qquad (4.4.5)
\end{aligned}
$$

Since the manifold is Einstein, so using equation (4.4.1) in equation (4.4.5) we obtain

$$
\begin{aligned}
(\nabla'_X R)(Y,Z,U,V) &= A(X)'R(Y,Z,U,V) \\
&+ \Big\{\frac{B(X)}{a} - \frac{r}{n(n-1)}A(X)\Big\} \\
&\times \big\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\big\},
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
(\nabla'_X R)(Y,Z,U,V) &= A(X)'R(Y,Z,U,V) \\
&+ E(X)\big\{(Z,U)g(Y,V) - g(Y,U)g(Z,V)\big\}, \qquad (4.4.6)
\end{aligned}
$$

where $E(X) = \frac{B(X)}{a} - \frac{r}{n(n-1)}A(X)$. Let the 1-forms A and B be metrically equivalent to the vector fields P and Q, respectively.
From equation (4.4.6) we conclude that an Einstein $G\{PP(K_n)\}$ is a GK_n, provided $\frac{Q}{a} \neq \frac{r}{n(n-1)}P$.

4.5 Conformally flat Manifolds, $n > 3$

Theorem 4.5.1 *A conformally flat $G\{PP(K_n)\}$, $n > 3$, with constant scalar curvature is a $PP(K_n)$.*

Proof: In this section we assume that the manifold $G\{PP(K_n)\}$, $n > 3$ is conformally flat. Then $div\tilde{C} = 0$, where $\tilde{C}$ denotes the Weyl's conformal curvature tensor and '*div*' denotes divergence. Hence we have (Eisenhart, 1949)

$$
(\nabla_X S)(Y,Z) - (\nabla_Z S)(X,Y) = \frac{1}{2(n-1)}\big\{g(Y,Z)dr(X) - g(X,Y)dr(Z)\big\}. \quad (4.5.1)
$$

From equation (4.2.7), we have

$$(\nabla_X S)(Y,Z) = A(X)\big\{S(Y,Z) - \frac{r}{n}g(Y,Z)\big\} + \frac{n-1}{a+(n-1)b}B(X)g(Y,Z). \quad (4.5.2)$$

Using equation (4.5.2) in equation (4.5.1) we obtain

$$\begin{aligned} & A(X)\big\{S(Y,Z) - \frac{r}{n}g(Y,Z)\big\} + \frac{(n-1)}{a+(n-1)b}B(X)g(Y,Z) \\ - \; & A(Z)\big\{S(X,Y) - \frac{r}{n}g(X,Y)\big\} + \frac{(n-1)}{a+(n-1)b}B(Z)g(X,Y) \\ = \; & \frac{1}{2(n-1)}\big\{g(Y,Z)dr(X) - g(X,Y)dr(Z)\big\}. \end{aligned} \quad (4.5.3)$$

Now taking a frame field over X and Z, we have from equation (4.5.2) that

$$\frac{1}{2}dr(Y) = A(LY) - \frac{r}{n}A(Y) + \frac{(n-1)}{a+(n-1)b}B(Y). \quad (4.5.4)$$

Replacing Y by X in equation (4.5.4)we obtain

$$\frac{1}{2}dr(X) = A(LX) - \frac{r}{n}A(X) + \frac{(n-1)}{a+(n-1)b}B(X). \quad (4.5.5)$$

Again taking contraction over Y and Z in equation (4.5.3)

$$\frac{1}{2}dr(X) = -A(LX) + \frac{r}{n}A(X) - \frac{(n-1)^2}{a+(n-1)b}B(X). \quad (4.5.6)$$

Adding equation (4.5.5) and equation (4.5.6) we get

$$dr(X) = -\frac{(n-1)(n-2)}{a+(n-1)b}B(X). \quad (4.5.7)$$

Now we suppose that the scalar curvature r is constant in a $G\{PP(K_n)\}$, that is $dr = 0$. Then from the above equation we get

$$B(X) = 0.$$

Then the $G\{PP(K_n)\}$,$n > 3$ is reduced to $PP(K_n)$.

4.6 Quasi Einstein Manifolds

Theorem 4.6.1 *If in a $G\{PP(K_n)\}$ with constant scalar curvature the associated unit vector field P is a unit concircular vector field whose associated scalar is a non-zero constant, then the manifold reduces to a quasi Einstein manifold.*

Proof: From equation (4.5.2), we have

$$(\nabla_X S)(Y,Z) = A(X)\big\{S(Y,Z) - \frac{r}{n}g(Y,Z)\big\} + \frac{n-1}{a+(n-1)b}B(X)g(Y,Z). \quad (4.6.1)$$

A vector field P on a manifold with a linear connection ∇ is said to be concircular if

$$\nabla_X P = \alpha X + \omega(X)P$$

for every vector field X, where α is a scalar function and ω is a closed 1-form. If the manifold is a semi-Riemannian manifolds and a concircular P satisfies additional assumption that $g(P,P) \equiv 1$, then $g(\nabla_X P, P) = 0$. Consequently we have (De and Pal, 2014)

$$A(LX) = -(n-1)\alpha^2 A(X), \quad (4.6.2)$$

where L is the Ricci operator defined by

$$g(LX,Y) = S(X,Y).$$

The above equation implies

$$S(X,P) = -(n-1)\alpha^2 A(X), \quad (4.6.3)$$

Also we have,

$$\begin{aligned}(\nabla_X S)(Y,P) = & - (n-1)\alpha^3[g(X,Y) - A(X)A(Y)] \\ & - \alpha[S(X,Y) - A(X)S(Y,P)]. \quad (4.6.4)\end{aligned}$$

Putting $Z = P$ in equation (4.6.1) we have

$$\begin{aligned}(\nabla_X S)(Y,P) &= A(X)[S(Y,P) - \frac{r}{n}g(Y,P)] \\ &+ \frac{(n-1)}{a+(n-1)b}B(X)g(Y,P). \quad (4.6.5)\end{aligned}$$

Now using equations (4.6.4) and (4.6.3) in equation (4.6.5)

$$
\begin{aligned}
-(n-1)\alpha^3 g(X,Y) \quad - \quad \alpha S(X,Y) &= -(n-1)\alpha^2 A(X)A(Y) - \frac{r}{n}A(X)A(Y) \\
- \quad \frac{(n-1)}{a+(n-1)b}B(X)A(Y) &+ \frac{b}{a-b}dr(X)A(Y). \qquad (4.6.6)
\end{aligned}
$$

Also we assume that the scalar curvature r is constant in the $G\{PP(K_n)\}$. Hence using equation (4.6.2) in equation (4.2.6)we get

$$
B(X) \quad = \quad -\frac{2a+(n-2)b}{n(n-1)(n-2)}[r+n(n-1)\alpha^2]A(X). \qquad (4.6.7)
$$

Using equation (4.6.7) in equation (4.6.6) we get

$$
\begin{aligned}
S(X,Y) \quad &= \quad -(n-1)\alpha^2 g(X,Y) \\
&+ \quad \frac{r+n(n-1)\alpha^2}{\alpha(n-2)}\Big[\frac{a+(n-2)b}{a+(n-1)b}\Big]A(X)A(Y). \qquad (4.6.8)
\end{aligned}
$$

Since α is a non-zero constant, equation (4.6.8) can be written as

$$
S(X,Y) = pg(X,Y) + qA(X)A(Y).
$$

where $p = -(n-1)\alpha^2$ and $q = \dfrac{r+n(n-1)\alpha^2}{\alpha(n-2)}\Big[\dfrac{a+(n-2)b}{a+(n-1)b}\Big]$ are two non-zero constants as α is non-zero constant. Hence the manifold is a quasi Einstein manifold.

4.7 Decomposable Manifolds

A semi-Riemannian manifold (M^n, g) is said to be decomposable or a product manifold (Schouten, 1954) if it can be expressed as $M_1^p \times M_2^{n-p}$ for some p in the range $2 \leq p \leq (n-2)$, that is, in some coodinate neighbourhood of the semi-Riemannian manifold (M^n, g), the metric can be expressed as

$$
ds^2 = g_{ij}dx^i dx^j = \bar{g}_a b dx^a dx^b + g'_{\alpha\beta}dx^\alpha dx^\beta, \qquad (4.7.1)
$$

where $\bar{g}_a b$ are functions of $x^1, x^2, ..., x^p$ denoted by $\bar{x}$ and $g'_{\alpha\beta}$ are functions of $x^{p+1}, x^{p+2}, ..., x^n$ denoted by x'. Here $a, b, c, ...$ run from 1 to p and $\alpha, \beta, \gamma, ...$ run from $p+1$ to n.

The two parts of equation (4.7.1) are the metrics of M_1^p, $p \geq 2$ and M_2^{n-p}, $p \geq 2$ which are called the components of the decomposable manifold $M^n = M_1^p \times M_2^{n-p}$.

Let (M^n, g) be a semi-Riemannian decomposable manifold such that M_1^p, $p \geq 2$ and M_2^{n-p}, $p \geq 2$ are components of this manifold.

In this section each object denoted by a bar $(^-)$ is assumed to come from M_1 and each object denoted by (') is assumed to come from M_2.

Let $\bar{X}, \bar{Y}, \bar{Z}, \bar{U}, \bar{V} \in \chi(M_1)$ and $X', Y', Z', U', V' \in \chi(M_2)$. Then in a decomposable semi-Riemannian manifold $M^n = M_1^p \times M_2^{n-p}$, $2 \leq p \leq n-2$, the following relations hold (Kruckovic,1957):

$$'R(X', \bar{Y}, \bar{Z}, \bar{U}) = 0 =' R(\bar{X}, Y', \bar{Z}, U') =' R(\bar{X}, Y', Z', U'),$$

$$(\nabla'_{X'} R)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = 0 = (\nabla'_{\bar{X}} R)(\bar{Y}, Z', \bar{U}, V') = (\nabla'_{X'} R)(\bar{Y}, Z', \bar{U}, V'),$$

$$'R(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) =' \bar{R}(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}),$$

$$'R(X', Y', Z', U') =' R'(X', Y', Z', U'),$$

$$S(\bar{X}, \bar{Y}) = \bar{S}(\bar{X}, \bar{Y}); \qquad S(X', Y') = S'(X', Y'),$$

$$(\nabla_{\bar{X}} S)(\bar{Y}, \bar{Z}) = (\bar{\nabla}_{\bar{X}} \bar{S})(\bar{Y}, \bar{Z}); \qquad (\nabla_{X'} S)(Y', Z') = (\nabla'_{X'} S')(Y', Z'),$$

where the meaning of $\bar{X}, \bar{Y}, \bar{Z}$ is different on each side, that is, the left hand side of $S(\bar{X}, \bar{Y}) = \bar{S}(\bar{X}, \bar{Y})$ means the value of the Ricci tenso S on M for $\bar{X}, \bar{Y}, \bar{Z} \in \chi(M_1)$ and the right hand side means the value of the Ricci rensor $\bar{S}$ on M_1 for $\bar{X}, \bar{Y}, \bar{Z} \in \chi(M_1)$. Similarly for X', Y', Z', and $r = \bar{r} + r'$, where, r, $\bar{r}$ and r' are scalar curvatures of M, M_1 and M_2 respectively.

Theorem 4.7.1 *Let (M^n, g) be a semi-Riemannian manifold which is not pseudo-projectively flat, such that $M = M_1^p \times M_2^{n-p}, 2 \leq p \leq n-2$. If (M^n, g) is a $G\{PP(K_n)\}$ and $B(X') = 0$ for all $X' \in \chi M_2$, (resp. $A\bar{X}) = 0$ for all $\bar{X} \in \chi M_1$), them either (i) or (ii) holds.*
(i) $B(X') = 0$ for all $X' \in \chi M_2$, (resp. $A\bar{X}) = 0$ for all $\bar{X} \in \chi M_1$), and hence M_2 (resp. M_1) is Ricci symmetric as well as locally symmetric.
(ii) M_1 (resp. M_2 is pseudo-projectively flat.

Proof: Let us consider a semi-Riemannian manifold (M^n, g), which is a decomposable $G\{PP(K_n)\}$. Then $M^n = M_1^p \times M_2^{n-p}, 2 \leq p \leq n-2$.

Now from equation (1.16.3), we get

$$'P(\bar{X},\bar{Y},\bar{Z},\bar{U}) =' \bar{P}(\bar{X},\bar{Y},\bar{Z},\bar{U}), \tag{4.7.2}$$

$$'P(X',Y',Z',U') =' P'(X',Y',Z',U'),$$

$$'P(Y',\bar{Z},\bar{U},\bar{V}) = 0 =' P(\bar{Y},Z',U',V') =' P(\bar{Y},Z',\bar{U},\bar{V}) =' P(\bar{Y},\bar{Z},U',\bar{V}),$$

$$'P(\bar{Y},Z',U',\bar{V}) = bS(Z',U')g(\bar{Y},\bar{V}) - \frac{r}{n}\Big(\frac{a}{n-1}+b\Big)g(Z',U')g(\bar{Y},\bar{V}), \tag{4.7.3}$$

$$'P(Y',\bar{Z},\bar{U},V') = bS(\bar{Z},\bar{U})g(Y',V') - \frac{r}{n}\Big(\frac{a}{n-1}+b\Big)g(\bar{Z},\bar{U})g(Y',V'), \tag{4.7.4}$$

$$'P(Y',\bar{Z},U',\bar{V}) = -bS(Y',U')g(\bar{Z},\bar{V}) + \frac{r}{n}\Big(\frac{a}{n-1}+b\Big)g(Y',U')g(\bar{Z},\bar{V}),$$

$$'P(\bar{Y},Z',\bar{U},V') = -bS(\bar{Y},\bar{U})g(Z',V') + \frac{r}{n}\Big(\frac{a}{n-1}+b\Big)g(\bar{Y},\bar{U})g(Z',V'),$$

$$(\nabla'_{X'}P)(\bar{Y},\bar{Z},\bar{U},\bar{V}) = 0 = (\nabla'_{\bar{X}}P)(Y',Z',U',V'),$$

Again from equation (4.2.1), we get

$$\begin{aligned}(\nabla'_{\bar{X}}P)(\bar{Y},\bar{Z},\bar{U},\bar{V}) &= A\bar{X})'P(\bar{Y},\bar{Z},\bar{U},\bar{V}) \\ &+ B(\bar{X})[g(\bar{Z},\bar{U})g(\bar{Y},\bar{V}) - g(\bar{Y},\bar{U})g(\bar{Z},\bar{V})],\end{aligned}$$

$$A(X')'P(\bar{Y},\bar{Z},\bar{U},\bar{V}) + B(X')[g(\bar{Z},\bar{U})g(\bar{Y},\bar{V}) - g(\bar{Y},\bar{U})g(\bar{Z},\bar{V})] = 0, \tag{4.7.5}$$

and

$$B_{(\bar{p},p')}(0\oplus v) = 0$$

for every $\bar{p}\in M_1$, $p'\in M_2$ and $v\in T_{p'}M_2$. Also for every $(\bar{p},p')\in M$ from (4.2.1) we obtain

$$(\nabla'_{X'}P)_{(\bar{p},p')}(Y',Z',U',V') = (\nabla'_{X'}{}'P)_{(\bar{p},p')}(Y',Z',U',V'), \tag{4.7.6}$$

and the value of $(\nabla_{X'}P)_{(\bar{p},p')}(Y',Z',U',V')$ does not depend on $\bar{p}\in M_1$ for every $\bar{p}\in M_1$ and $p'\in M_2$.

If possible let $B(X')=0$ for all $X'\in\chi(M_2)$, then from equation (4.7.5) we get

$$A(X')'P(\bar{Y},\bar{Z},\bar{U},\bar{V})=0. \tag{4.7.7}$$

Using equation (4.7.2) in equation (4.7.7) we get

$$A(X')'\bar{P}(\bar{Y},\bar{Z},\bar{U},\bar{V})=0. \tag{4.7.8}$$

If M_1 is not pseudo-projectively flat, that is, $\bar{P}_{\bar{p}_0}\neq 0$ for some $\bar{p}_0\in M_1$, then from equations (4.7.7) and (4.7.8) it follows that

$$A_{(\bar{p},p')}(0\oplus v)=0 \tag{4.7.9}$$

for every $\bar{p}\in M_1$ and $p'\in M_2$ and for every $v\in T_{p'}M_2$. Hence equation (4.2.1) yield

$$(\nabla'_{X'}P)_{(\bar{p},p')}(Y',Z',U',V')=0$$

for every $\bar{p}\in M_1$ and $p'\in M_2$. It follows that if M_1 is not pseudo-projectively flat,then

$$A_{(\bar{p},p')}(X')'P'_{p'}(Y',Z',U',V')=0 \tag{4.7.10}$$

for all $\bar{p}\in M_1$ and $p'\in M_2$.

Now we assume that

$$\begin{aligned}(\nabla'_X P)(Y,Z,U,V) &= \bar{A}(X)'P(Y,Z,U,V)\\ &+ \bar{B}(X)\big\{g(Z,U)g(Y,V)-g(Y,U)g(Z,V)\big\},\end{aligned} \tag{4.7.11}$$

where $\bar{A}$ and $\bar{B}$ are 1-forms. Putting equation (4.7.11) in equation (4.2.1) we get

$$\begin{aligned}&\big\{A(X)-\bar{A}(X)\big\}'P(Y,Z,U,V)\\ +\ &\big\{B(X)-\bar{B}(X)\big\}\big\{g(Z,U)g(Y,V)-g(Y,U)g(Z,V)]\big\}=0.\end{aligned} \tag{4.7.12}$$

Contracting equation (4.7.12) over Y and V, and using equation (4.1.1) we obtain

$$\begin{aligned}\{a+(n-1)b\}\big\{A(X) &- \bar{A}(X)\big\}\big\{S(Z,U)-\frac{r}{n}g(Z,U)\big\}\\ &+ (n-1)\big\{B(X)-\bar{B}(X)\big\}g(Z,U)=0.\end{aligned} \tag{4.7.13}$$

Again contracting equation (4.7.13) over Z and U we get

$$B(X) = \bar{B}(X) \tag{4.7.14}$$

for all $X \in M$. Using equation (4.7.14) in equation (4.7.12) we get

$$A(X) = \bar{A}(X)$$

for all $X \in M$, provided $P(Y, Z, U, V) \neq 0$, that is, if the manifold is not pseudo-projectively flat manifold. Thus the 1-forms A and B in equation (4.2.1) are uniquely determined, provided that the manifold is not pseudo-projectively flat manifold. Hence from equation (4.7.10) we obtain

$$A_{(\bar{p},p')}(X') = 0 \tag{4.7.15}$$

for all $\bar{p} \in M_1$ and $p' \in M_2$.

From (4.7.8) we conclude that either

(i) $A(X') = 0$ for all $X' \in \chi M_2$, or

(ii) M_1 is pseudo-projectively flat.

Also from equation equation (4.2.1) we obtain

$$\begin{aligned}(\nabla'_{X'}P)(\bar{Y}, Z', U', \bar{V}) = A(X')'P(\bar{Y}, Z', U', \bar{V}) + B(X')\big\{g(Z', U')g(\bar{Y}, \bar{V}) \\ -g(\bar{Y}, U')g(Z', \bar{V})\big\}.\end{aligned} \tag{4.7.16}$$

Now we consider the case (i). From equation (4.7.16), it follows that

$$(\nabla'_{X'}P)(\bar{Y}, Z', U', \bar{V}) = 0,$$

which implies by virtue of equation (4.7.3) that,

$$(\nabla_{X'}S)(Z', U') = 0. \tag{4.7.17}$$

Hence the component M_2 is Ricci symmetric. Using equations (4.7.3), (4.7.6), (4.7.9), (4.7.10) and (4.7.15) and $A(X') = 0$, $B(X') = 0$ for all $X' \in \chi M_2$, from equation (4.2.1), we have

$$(\nabla'_{X'}P)(Y', Z', U', V') = 0,$$

and hence

$$\begin{aligned} a(\nabla'_{X'}R)(Y',Z',U',V') &+ b\{(\nabla_{X'}S)(Z',U')g(Y',V') \\ &- (\nabla_{X'}S)(Y',U')g(Z',V')\} = 0, \end{aligned}$$

which yield by virtue of equation (4.7.17) that

$$(\nabla'_{X'}R)(Y',Z',U',V') = 0,$$

that is, the component M_2 is locally symmetric. Similar result can be proved forM_1.

Theorem 4.7.2 *Let (M^n, g) be a semi-Riemannian manifold such that $M = M_1^p \times M_2^{n-p}$, $2 \leq p \leq n-2$. If (M^n, g) is a $G\{PP(K_n)\}$, then M_1 and M_2 are generalized Ricci-recurrent manifolds.*

Proof:

Also from equation (4.2.1) we get

$$\begin{aligned} (\nabla'_{\bar{X}}P)(Y',\bar{Z},\bar{U},V') &= A(\bar{X})'P(Y',\bar{Z},\bar{U},V') \\ +B(\bar{X})\{g(\bar{Z},\bar{U})g(Y',V') &- g(Y',\bar{U})g(\bar{Z},V')\}. \end{aligned} \tag{4.7.18}$$

Using equation (4.7.4) in equation (4.7.18) we get

$$\begin{aligned} b(\nabla_{\bar{X}}S)(\bar{Z},\bar{U})g(Y',V') &= A(\bar{X})\{bS(\bar{Z},\bar{U})g(Y',V') \\ &- \frac{r}{n}(\frac{a}{n-1}+b)g(\bar{Z},\bar{U})g(Y',V')\} \\ &+ B(\bar{X})g(\bar{Z},\bar{U})g(Y',V'). \end{aligned} \tag{4.7.19}$$

Now assume that $g(Y',V') \neq 0$. Then from equation (4.7.19) we get

$$\begin{aligned} (\nabla_{\bar{X}}S)(\bar{Z},\bar{U}) &= A(\bar{X})S(\bar{Z},\bar{U}) \\ &+ \frac{1}{b}\{B(\bar{X}) - \frac{r}{n}(\frac{a}{n-1}+b)A(\bar{X})\}g(\bar{Z},\bar{U}), \end{aligned}$$

which implies

$$(\nabla_{\bar{X}}S)(\bar{Z},\bar{U}) = A(\bar{X})S(\bar{Z},\bar{U}) + C(\bar{X})g(\bar{Z},\bar{U}),$$

where $A(\bar{X})$ and $C(\bar{X}) = \Big\{B(\bar{X}) - \frac{r}{n}(\frac{a}{n-1}+b)A(\bar{X})\Big\}$ are two non zero 1-forms. Thus M_1 is a generalized Ricci-recurrent manifold. Similar result can be proved for M_2.

4.8 Examples

Example 4.8.1

Let us consider a Lorentzian metric g on $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = x^1(dx^1)^2 + x^1(dx^2)^2 + x^1(dx^3)^2 - (dx^4)^2, \tag{4.8.1}$$

where $i, j = 1, 2, 3, 4$. Then the only non-vanishing components of the Christoffel symbols, the Riemannian curvature tensor and the Ricci tensor are:

$$\Gamma^1_{22} = \Gamma^1_{33} = -\frac{1}{2x^1}, \qquad \Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{2x^1},$$

$$R_{1221} = R_{1331} = -\frac{1}{2x^1}, \qquad R_{2332} = \frac{1}{4x^1},$$

and

$$S_{22} = S_{33} = -\frac{1}{4(x^1)^2}, \qquad S_{11} = -\frac{1}{(x^1)^2}, \qquad S_{44} = 0.$$

And the scalar curvature of the resulting manifold (R^4, g) is

$$r = -\frac{3}{2(x^1)^3}.$$

Now, the non vanishing components of pseudo-projective curvature tensor and their covariant derivatives are:

$$\begin{aligned} P_{1212} &= P_{1313} = \frac{3a}{8x^1} + \frac{5b}{8x^1}, \qquad P_{1331} = P_{1221} = -\frac{3a}{8x^1} + \frac{b}{8x^1}, \\ P_{2332} &= -P_{2323} = \frac{3a}{8x^1} + \frac{b}{8x^1}, \end{aligned}$$

$$\begin{aligned} P_{1221,1} &= -\frac{3a}{8(x^1)^2} - \frac{5b}{8(x^1)^2}, \qquad P_{2332,1} = \frac{3a}{8(x^1)^2} - \frac{b}{8(x^1)^2}, \\ P_{2332,1} &= -\frac{3a}{8(x^1)^2} - \frac{b}{8(x^1)^2}, \end{aligned}$$

where ‘,’ denotes the covariant derivative with respect to the metric tensor.

Let us choose the associated 1-forms as follows:

$$A_i(x) = \begin{cases} -\dfrac{1}{x^1}, & for\ i = 1 \\ 0, & otherwise, \end{cases} \tag{4.8.2}$$

$$B_i(x) = 0, \quad for\ i = 1, 2, 3, 4 \tag{4.8.3}$$

at any point $x \in \mathbb{R}^4$. To verify the relation (4.2.1), it is sufficient to check the following equations:

$$P_{1212,1} = A_1 P_{1212} + B_1(g_{22}g_{11} - g_{12}g_{21}), \tag{4.8.4}$$

$$P_{1221,1} = A_1 P_{1221} + B_1(g_{33}g_{22} - g_{23}g_{32}), \tag{4.8.5}$$

$$P_{2332,1} = A_1 P_{2332} + B_1(g_{33}g_{11} - g_{13}g_{31}), \tag{4.8.6}$$

since for the other cases equation (4.2.1) holds trivially. From equations (4.8.2) and (4.8.3) we get

$$\begin{aligned} R.H.S.\ of\ (4.8.4) &= A_1 P_{1212} + B_1(g_{22}g_{11} - g_{12}g_{21}) \\ &= \{-\frac{1}{x^2}\}\{-\frac{3a}{8x^2} - \frac{5b}{8x^2}\} + 0 \\ &= \frac{3a}{8(x^2)^2} + \frac{5b}{8(x^2)^2} \\ &= P_{1212,1} \\ &= L.H.S.\ of\ (4.8.4). \end{aligned}$$

By similar argument it can be shown that equation (4.8.5) and equation (4.8.6) are also true. So $(\mathbb{R}^4, g)$ is a $PP(K_n)$.

Example 4.8.2

Consider a Riemannian space V_n which metric is given by

$$ds^2 = \phi(dx^1)^2 + K_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1 dx^n, \ (n \geq 4), \tag{4.8.7}$$

where $[K_{\alpha\beta}]$ is a symmetric and non singular matrix consisting of constants and ϕ is a function of $x^1, x^2, ..., x^{n-1}$ and independent of x^n, and each Latin index runs over

$1, 2, ..., n$ and each Greek index over $2, 3, ..., n-1$. In the metric considered, the only non-vanishing components of Christoffel symbols, Riemannian curvature tensor and Ricci tensor are [?]

$$\Gamma^{\beta}_{11} = -\frac{1}{2}K^{\alpha\beta}\phi_{.\alpha}, \qquad \Gamma^{n}_{11} = \frac{1}{2}\phi_{.1}, \qquad , \Gamma^{n}_{1\alpha} = \frac{1}{2}\phi_{.\alpha},$$

$$R_{1\alpha\beta 1} = \frac{1}{2}\phi_{.\alpha\beta}, \qquad S_{11} = \frac{1}{2}K^{\alpha\beta}\phi_{.\alpha\beta}, \tag{4.8.8}$$

where '.' denotes the partial differentiation with respect to the coordinates and $K^{\alpha\beta}$ are the elements of the matrix inverse to $[K_{\alpha\beta}]$. Here we consider $K_{\alpha\beta}$ as Kronecker symbol $\delta_{\alpha\beta}$ and

$$\phi = (M_{\alpha\beta} + \delta_{\alpha\beta})x^{\alpha}x^{\beta}e^{(x^1)^2},$$

where $M_{\alpha\beta}$ are constant and satisfy the relations

$$\begin{aligned} M_{\alpha\beta} &= \begin{cases} 0, & for\ \alpha \neq \beta \\ \neq 0, & for\ \alpha = \beta, \end{cases} \\ \sum_{\alpha=1}^{n-1} M_{\alpha\alpha} &= 0. \end{aligned}$$

This is to be noted that the metric with this form of ϕ was considered by De and Gazi [?]. Thus we have

$$\phi_{\alpha\beta} = 2(M_{\alpha\beta} + \delta_{\alpha\beta})e^{(x^1)^2}, \qquad \delta_{\alpha\beta}\delta^{\alpha\beta} = n-2,$$
$$\delta^{\alpha\beta}M_{\alpha\beta} = \sum_{\alpha=1}^{n-1} M_{\alpha\alpha} = 0.$$

Therefore

$$\delta^{\alpha\beta}\phi_{\alpha\beta} = 2(\delta^{\alpha\beta}M_{\alpha\beta} + \delta^{\alpha\beta}\delta_{\alpha\beta})e^{(x^1)^2} = 2(n-2)e^{(x^1)^2}.$$

Since $\phi_{\alpha\beta}$ vanishes for $\alpha \neq \beta$, the only non-zero components of the Riemannian curvature tensor and Ricci tensor by virtue of equation (4.8.8) are

$$R_{1\alpha\alpha 1} = \frac{1}{2}\phi_{.\alpha\alpha} = \big(1 + M_{\alpha\alpha}\big)e^{(x^1)^2},$$

$$S_{11} = \frac{1}{2}\phi_{.\alpha\beta}\delta^{\alpha\beta} = (n-2)e^{(x^1)^2},$$

Also, the scalar curvature $r = 0$. Hence the only non-zero components of the pseudo-projective curvature tensor, and their covariant derivatives are

$$P_{\alpha 11\alpha} = a\big(1 + M_{\alpha\alpha}\big)e^{(x^1)^2} + b(n-2)e^{(x^1)^2},$$

$$P_{\alpha 11\alpha,1} = 2ax^1\big(1 + M_{\alpha\alpha}\big)e^{(x^1)^2} + 2bx^1(n-2)e^{(x^1)^2}.$$

where ',' denotes the covariant derivative with respect to the metric tensor. Let us choose the associated 1-forms as follows:

$$A_i(x) = \begin{cases} 2x^1, & for\ i = 1 \\ 0, & otherwise, \end{cases} \tag{4.8.9}$$

$$B_i(x) = 0, \quad for\ i = 1, 2, 3, 4 \tag{4.8.10}$$

at any point $x \in V_n$. To verify the relation equation (4.2.1), it is sufficient to check the following equation:

$$P_{\alpha 11\alpha,1} = A_1 P_{\alpha 11\alpha} + B_1(g_{11}g_{\alpha\alpha} - g_{\alpha 1}g_{1\alpha}), \tag{4.8.11}$$

$$\begin{aligned} R.H.S.\ of\ equation\ (4.8.11) &= A_1 P_{\alpha 11\alpha} + B_1(g_{11}g_{\alpha\alpha} - g_{\alpha 1}g_{1\alpha}) \\ &= 2x^1\big[a\big(1 + M_{\alpha\alpha}\big)e^{(x^1)^2} + b(n-2)e^{(x^1)^2}\big] \\ &= 2ax^1\big(1 + M_{\alpha\alpha}\big)e^{(x^1)^2} + 2bx^1(n-2)e^{(x^1)^2} \\ &= P_{\alpha 11\alpha,1} \\ &= L.H.S.\ of\ equation\ (4.8.11). \end{aligned}$$

So V_n is a $PP(K_n)$.

Chapter 5

QUARTER SYMMETRIC NON-METRIC CONNECTION IN TRANS-SASAKIAN MANIFOLD

A trans-Sasakian manifold which admits a quarter symmetric non-metric connection is considered in this chapter. The relation between curvature tensor and Ricci tensor with respect to the Riemannian connection and the quarter symmetric non-metric connection are given. Weakly symmetries, locally symmetries, semi-symmetries and recurrency of the manifold are studied in this chapter

5.1 Introduction

In trans-Sasakian manifolds the curvature tensor and Ricci tensor satisfy (De and Mukut, 2003)

$$\begin{aligned} R(U,V)\xi = (\alpha^2-\beta^2)\{\eta(V)U-\eta(U)V\} + 2\alpha\beta\{\eta(V)\phi U-\eta(U)\phi V\} \\ -(U\alpha)\phi V+(V\alpha)\phi U-(U\beta)\phi^2 V+(V\beta)\phi^2 U, \end{aligned} \tag{5.1.1}$$

$$\begin{aligned} R(\xi,V)Z = (\alpha^2-\beta^2)\{g(V,Z)\xi-\eta(Z)V\} + 2\alpha\beta\{g(\phi Z,V)\xi+\eta(Z)\phi V\} \\ +(Z\alpha)\phi V+(Z\beta)\{V-\eta(V)\xi\}+g(\phi Z,V)(grad\alpha)-g(\phi Z,\phi V)(grad\beta), \end{aligned} \tag{5.1.2}$$

Lalmalsawma C. and Singh J. P. (2019). Some curvature properties of trans-Sasakian manifolds admitting a quarter-symmetric non-metric connection, *Bulletin of the Transilvania University of Braov Series III: Mathematics, Informatics, Physics,* **12(61)**, 65-76.

$$R(\xi,V)\xi=(\alpha^2-\beta^2-\xi\beta)\{\eta(V)\xi-V\},$$
$$2\alpha\beta+(\xi\alpha)=0, \tag{5.1.3}$$

$$S(U,\xi)=\{(n-1)(\alpha^2-\beta^2)-(\xi\beta)\}\eta(U)-(n-2)(U\beta)-\big((\phi U)\alpha\big), \tag{5.1.4}$$

$$Q\xi=\{(n-1)(\alpha^2-\beta^2)-(\xi\beta)\}\xi-(n-2)(grad\beta)+\phi(grad\alpha). \tag{5.1.5}$$

5.2 Quarter symmetric non-metric connection in trans-Sasakian manifold

We consider a linear connection $\tilde{\nabla}$ on a trans-Sasakian manifold which is given by

$$\tilde{\nabla}_X U=\nabla_X U+\eta(U)\phi(X). \tag{5.2.1}$$

Thus $\tilde{\nabla}$ is a quarter-symmetric conection on the manifold. Also using equation (5.2.1) we obtain

$$\big(\tilde{\nabla}_X g\big)(U,V)=-g(\phi X,V)\eta(U)-g(\phi X,U)\eta(V),$$

which shows that $\tilde{\nabla}$ is a non-metric connection.

The relation between the Riemannian curvature tensor R and the curvature tensor $\tilde{R}$ with respect to $\tilde{\nabla}$ is given by (Patra and Battacharyya, 2013)

$$\begin{aligned}\tilde{R}(U,V)Z=R(U,V)Z+\alpha\{g(\phi V,Z)\phi U-g(\phi U,Z)\phi V+\eta(U)\eta(Z)V\\ -\eta(V)\eta(Z)U\}+\beta\{g(U,Z)\phi V-g(V,Z)\phi U+2\eta(Z)g(\phi U,V)\xi\}.\end{aligned} \tag{5.2.2}$$

From equation (5.2.2), it follows that

$$\begin{aligned}\tilde{R}(\xi,V)Z=(\alpha^2-\beta^2)\{g(V,Z)\xi-\eta(Z)V\}+2\alpha\beta\{g(\phi Z,V)\xi+\eta(Z)\phi V\}\\ +(Z\alpha)\phi V+g(\phi Z,V)grad\alpha+(Z\beta)\{V-\eta(V)\xi\}\\ -g(\phi Z,\phi V)grad\beta+\alpha\eta(Z)\{V-\eta(V)\xi\}+\beta\eta(Z)\phi V,\end{aligned} \tag{5.2.3}$$

$$\begin{aligned}\tilde{R}(U,V)\xi=(\alpha^2-\beta^2-\alpha)\{\eta(V)U-\eta(U)V\}-(U\alpha)\phi V+(V\alpha)\phi U\\ +(2\alpha\beta\ -\beta)\{\eta(V)\phi U-\eta(U)\phi V\}-(U\beta)\phi^2V+(V\beta)\phi^2U+2\beta g(\phi U,V)\xi,\end{aligned} \tag{5.2.4}$$

$$\tilde{R}(\xi, V)\xi = (\alpha^2 - \beta^2 - \alpha - \xi\beta)\{\eta(U)\xi - U\} + \beta\phi U. \tag{5.2.5}$$

Contracting equation (5.2.2) we obtain (Patra and Battacharyya, 2013)

$$\tilde{S}(U, V) = S(U, V) + \alpha\{g(\phi U, \phi V) - (n-1)\eta(U)\eta(V)\} + \beta g(\phi U, V). \tag{5.2.6}$$

Again from equation (5.2.2), we have

$$\begin{aligned} \tilde{S}(U, \xi) &= \{(n-1)(\alpha^2 - \beta^2 - \alpha) - \xi\beta\}\eta(U) - (n-2)(U\beta) - \big((\phi U)\alpha\big) \\ &= \tilde{S}(\xi, U), \end{aligned} \tag{5.2.7}$$

$$\tilde{S}(\xi, \xi) = (n-1)(\alpha^2 - \beta^2 - \alpha - \xi\beta). \tag{5.2.8}$$

Also from equation (1.18.17) we obtain

$$\tilde{\nabla}_X \xi = -(\alpha - 1)\phi X + \beta\{X - \eta(X)\xi\}. \tag{5.2.9}$$

Again from equation (5.2.6) we obtain

$$\tilde{r} = r, \tag{5.2.10}$$

where $\tilde{r} = \sum_{i=1}^{n} \tilde{S}(e_i, e_i)$ and $r = \sum_{i=1}^{n} S(e_i, e_i)$ are the scalar curvatures.

5.3 Weakly symmetric trans-Sasakian manifolds

Let M^n be a trans-Sasakian manifolds admitting $\tilde{\nabla}$. Suppose that M^n is weakly symmetric, then we have equation (1.21.1)

$$\begin{aligned} (\tilde{\nabla}_X \tilde{R})(U, V)Z = A(X)\tilde{R}(U, V)Z + B(U)\tilde{R}(X, V)Z + C(V)\tilde{R}(U, X)Z \\ + D(Z)\tilde{R}(U, V)X + g(\tilde{R}(U, V)Z, X)P. \end{aligned} \tag{5.3.1}$$

Theorem 5.3.1 *In the given manifolds M^n, the sum of 1-forms A, C and D is given* *by*

$$
\begin{aligned}
&\big[A(X)+C(X)+D(X)\big]\\
&=\frac{2\big[(n-1)\{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\xi\alpha\}-\xi(\xi\beta)\big]\eta(X)}{(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)}\\
&+\frac{\{2\alpha(X\alpha)-2\beta(X\beta)-X\alpha-X(\xi\beta)\}}{(\alpha^2-\beta^2-\alpha-\xi\beta)}\\
&+\frac{2(\alpha-1)\{(X\alpha)-\eta(X)(\xi\alpha)-(n-2)(\phi X)\beta\}}{(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)}\\
&+\frac{2\beta\big[\big((\phi X)\alpha+(n-2)\{(X\beta)-\eta(X)(\xi\beta)\}\big]}{(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)}\\
&-\frac{2(n-2)\big(X(\xi\beta)\big)-2\big((\phi X)(\xi\alpha)\big)}{(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)}\\
&-R\times\frac{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\xi\alpha-\xi(\xi\beta)}{(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)^2}.
\end{aligned}
$$

Proof: Contracting equation (5.3.1) we get

$$
\begin{aligned}
(\tilde{\nabla}_X\tilde{S})(V,Z)&=A(X)\tilde{S}(V,Z))+B(\tilde{R}(X,V)Z)+C(V)\tilde{S}(X,Z)\\
&+D(Z)\tilde{S}(V,X)+E(\tilde{R}(X,Z)V), \qquad (5.3.2)
\end{aligned}
$$

where E is 1-form defined by $E(X)=g(X,P)$. Setting $Z=\xi$ in equation (5.3.2) and making use of equations (5.2.3), (5.2.4), (5.2.6) and (5.2.7), we obtain

$$
\begin{aligned}
&(\tilde{\nabla}_X\tilde{S})(V,\xi)=A(X)\big[(n-1)(\alpha^2-\beta^2-\alpha)-\xi\beta\big]\eta(V)\\
&-\big[(n-2)(V\beta)+\big((\phi V)\alpha\big)\big]+(\alpha^2-\beta^2-\alpha)\{\eta(V)B(X)-\eta(X)B(V)\}\\
&+(2\alpha\beta-\beta)\{\eta(V)B(\phi X)-\eta(X)B(\phi V)\}-(X\alpha)B(\phi V)+(V\alpha)B(\phi X)\\
&-(X\beta)B(\phi^2V)+(V\beta)B(\phi^2X)+2\beta g(\phi X,V)B(\xi)+D(\xi)\tilde{S}(X,V)\\
&+C(V)\big[\{(n-1)(\alpha^2-\beta^2-\alpha)-\xi\beta\}\eta(X)-(n-2)(X\beta)-\big((\phi X)\alpha\big)\big]\\
&-(\alpha^2-\beta^2)\{g(V,X)E(\xi)-\eta(V)E(X)\}-(V\alpha)E(\phi X)\\
&-2\alpha\beta\{g(\phi V,X)E(\xi)+\eta(V)E(\phi X)\}-g(\phi V,X)E(grad\alpha)\\
&-(V\beta)\{E(X)-\eta(X)E(\xi)\}+g(\phi V,\phi X)E(grad\beta)\\
&-\alpha\eta(V)\{E(X)-\eta(X)E(\xi)\}-\beta\eta(V)E(\phi X). \qquad (5.3.3)
\end{aligned}
$$

By properties of linear connection we have

$$
(\tilde{\nabla}_X\tilde{S})(V,\xi)=\tilde{\nabla}_X\tilde{S}(Z,\xi)-\tilde{S}(\tilde{\nabla}_XV,\xi)-\tilde{S}(V,\tilde{\nabla}_X\xi). \qquad (5.3.4)
$$

Combining the equations (5.3.3) and (5.3.4), it follows that

$$\begin{aligned}\tilde{\nabla}_X\tilde{S}(V,\xi)-\tilde{S}(\tilde{\nabla}_XV,\xi)-\tilde{S}(V,\tilde{\nabla}_X\xi)=&-A(X)\big[(n-2)(V\beta)+\big((\phi V)\alpha\big)\big]\\&+A(X)\big[(n-1)(\alpha^2-\beta^2-\alpha)-\xi\beta\big]\eta(V)-(X\alpha)B(\phi V)\\&+(V\alpha)B(\phi X)+(2\alpha\beta\ -\beta)\{\eta(V)B(\phi X)-\eta(X)B(\phi V)\}\\&-(X\beta)B(\phi^2V)+(V\beta)B(\phi^2X)+2\beta g(\phi X,V)B(\xi)\\&+(\alpha^2-\beta^2-\alpha)\{\eta(V)B(X)-\eta(X)B(V)\}+D(\xi)\tilde{S}(X,V)\\&+C(V)\big[\{(n-1)(\alpha^2-\beta^2-\alpha)-\xi\beta\}\eta(X)-(n-2)(X\beta)-\big((\phi X)\alpha\big)\big]\\&-(\alpha^2-\beta^2)\{g(V,X)E(\xi)-\eta(V)E(X)\}-(V\beta)\{E(X)-\eta(X)E(\xi)\}\\&-2\alpha\beta\{g(\phi V,X)E(\xi)+\eta(V)E(\phi X)\}\\&-(V\alpha)E(\phi X)-g(\phi V,X)E(grad\alpha)+g(\phi V,\phi X)E(grad\beta)\\&-\alpha\eta(V)\{E(X)-\eta(X)E(\xi)\}-\beta\eta(V)E(\phi X).\end{aligned}\quad(5.3.5)$$

By setting $X=V=\xi$ in equation (5.3.5) and using equations (1.18.1), (1.18.2), (1.18.3), (5.1.5), (5.2.1) and (5.2.8), we obtain the following equation

$$\begin{aligned}&(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)\big[A(\xi)+C(\xi)+D(\xi)\big]\\&\quad=(n-1)\{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\xi\alpha-\xi(\xi\beta)\}.\end{aligned}\quad(5.3.6)$$

Since $n>2$, the above equation can be written as

$$A(\xi)+C(\xi)+D(\xi)=\frac{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\xi\alpha-\xi(\xi\beta)}{\alpha^2-\beta^2-\alpha-\xi\beta},\quad(5.3.7)$$

provided $\alpha^2-\beta^2-\alpha-\xi\beta\neq 0$.

Again setting $V = \xi$ in equation (5.3.2) and combining with equation (5.3.4) we get

$$\begin{aligned}\tilde{\nabla}_X\tilde{S}(\xi, Z) - \tilde{S}(\tilde{\nabla}_X\xi, Z) - \tilde{S}(\xi, \tilde{\nabla}_X Z) = A(X)(n-1)(\alpha^2 - \beta^2 - \alpha)\eta(Z)\\ -A(X)(\xi\beta)\eta(Z) - A(X)\big[(n-2)(Z\beta) + \big((\phi Z)\alpha\big)\big]\\ -(\alpha^2 - \beta^2)\{g(Z, X)B(\xi) - \eta(Z)B(X)\}\\ -2\alpha\beta\{g(\phi Z, X)B(\xi) + \eta(Z)B(\phi X)\} - (Z\alpha)B(\phi X)\\ -(Z\beta)\{B(X) - \eta(X)B(\xi)\} - g(\phi Z, X)B(grad\alpha)\\ +g(\phi Z, \phi X)B(grad\beta) - \alpha\eta(Z)\{B(X) - \eta(X)B(\xi)\} - \beta\eta(Z)B(\phi X)\\ +C(\xi)\tilde{S}(X, Z) + D(Z)\big[\{(n-1)(\alpha^2 - \beta^2 - \alpha) - \xi\beta\}\eta(X)\\ -(n-2)(X\beta) - \big((\phi X)\alpha\big)\big] + (\alpha^2 - \beta^2 - \alpha)\{\eta(Z)E(X)\\ -\eta(X)E(Z)\} + (2\alpha\beta\ - \beta)\{\eta(Z)E(\phi X) - \eta(X)E(\phi Z)\}\\ -(X\alpha)E(\phi Z) + (Z\alpha)E(\phi X) - (X\beta)E(\phi^2 Z) + (Z\beta)E(\phi^2 X)\\ +2\beta g(\phi X, Z)E(\xi).\end{aligned} \tag{5.3.8}$$

Putting $Z = \xi$ in equation (5.3.8) we get

$$\begin{aligned}A(X)(n-1)(\alpha^2 - \beta^2 - \alpha - \xi\beta) - \beta B(\phi X)\\ +(\alpha^2 - \beta^2 - \alpha - \xi\beta)\{B(X) - \eta(X)B(\xi)\}\\ +\big[\{(n-1)(\alpha^2 - \beta^2 - \alpha) - \xi\beta\}\eta(X) - (n-2)(X\beta) - \big((\phi X)\alpha\big)\big] \times\\ \big[C(\xi) + D(\xi)\big] + (\alpha^2 - \beta^2 - \alpha - \xi\beta)\{E(X) - \eta(X)E(\xi)\}\\ = (n-1)\{2\alpha(X\alpha) - 2\beta(X\beta) - X\alpha - X(\xi\beta)\}\\ +2(n-2)\beta\{(X\beta) - \eta(X)(\xi\beta)\} - 2(n-2)(\alpha - 1)(\phi X)\beta\\ +2\beta\big((\phi X)\alpha\big) + 2(\alpha - 1)\{(X\alpha) - \eta(X)(\xi\alpha)\}.\end{aligned} \tag{5.3.9}$$

Again putting $X = \xi$ in equation (5.3.8) we get

$$\begin{aligned}\big[\{(n-1)(\alpha^2 - \beta^2 - \alpha) - \xi\beta\}\eta(Z) - (n-2)(Z\beta) - \big((\phi Z)\alpha\big)\big] \times\\ \{A(\xi) + C(\xi)\} + D(Z)(n-1)(\alpha^2 - \beta^2 - \alpha - \xi\beta)\\ +\beta E(\phi Z) - (\alpha^2 - \beta^2 - \alpha - \xi\beta)\{E(Z) - \eta(Z)E(\xi)\}\\ = \big[(n-1)\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \xi\alpha\} - \xi(\xi\beta)\big]\eta(Z)\\ -(n-2)\big(Z(\xi\beta)\big) - \big((\phi Z)(\xi\alpha)\big).\end{aligned} \tag{5.3.10}$$

By replacing Z by X, the equation equation (5.3.10) becomes

$$\begin{aligned}
\big[\{(n-1)(\alpha^2-\beta^2-\alpha)-\xi\beta\}\eta(X)-(n-2)(X\beta)-\big((\phi X)\alpha\big)\big]\times\\
\{A(\xi)+C(\xi)\}+D(X)(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)\\
+\beta E(\phi X)-(\alpha^2-\beta^2-\alpha-\xi\beta)\{E(X)-\eta(X)E(\xi)\}\\
=\big[(n-1)\{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\xi\alpha\}-\xi(\xi\beta)\big]\eta(X)\\
-(n-2)\big(X(\xi\beta)\big)-\big((\phi X)(\xi\alpha)\big).
\end{aligned}\tag{5.3.11}$$

Adding the equations equation (5.3.9) and equation (5.3.11) we obtain

$$\begin{aligned}
\big[\{(n-1)(\alpha^2-\beta^2-\alpha)-\xi\beta\}\eta(X)-(n-2)(X\beta)-\big((\phi X)\alpha\big)\big]\times\\
\{A(\xi)+2C(\xi)+D(\xi)\}+(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)\{A(X)+D(X)\}\\
+(\alpha^2-\beta^2-\alpha-\xi\beta)\{B(X)-\eta(X)B(\xi)\}-\beta B(\phi X)\\
=(n-1)\{2\alpha(X\alpha)-2\beta(X\beta)-X\alpha-X(\xi\beta)\}\\
+2(n-2)\beta\{(X\beta)-\eta(X)(\xi\beta)\}-2(n-2)(\alpha-1)(\phi X)\beta+2\beta\big((\phi X)\alpha\big)\\
\big[(n-1)\{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\xi\alpha\}-\xi(\xi\beta)\big]\eta(X)+2(\alpha-1)\{(X\alpha)\\
-\eta(X)(\xi\alpha)\}-(n-2)\big(X(\xi\beta)\big)-\big((\phi X)(\xi\alpha)\big).
\end{aligned}\tag{5.3.12}$$

Now putting $X=\xi$ in equation (5.3.5) we get

$$\begin{aligned}
\big[\{(n-1)(\alpha^2-\beta^2-\alpha)-\xi\beta\}\eta(V)-(n-2)(V\beta)-\big((\phi V)\alpha\big)\big]\times\\
\{A(\xi)+D(\xi)\}+C(V)(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)+\beta B(\phi V)\\
-(\alpha^2-\beta^2-\alpha-\xi\beta)\{B(V)-\eta(V)B(\xi)\}\\
=\big[(n-1)\{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\xi\alpha\}-\xi(\xi\beta)\big]\eta(V)\\
-(n-2)\big(V(\xi\beta)\big)-\big((\phi V)(\xi\alpha)\big).
\end{aligned}\tag{5.3.13}$$

By replacing V by X the equation (5.3.13) becomes

$$\begin{aligned}
\big[\{(n-1)(\alpha^2-\beta^2-\alpha)-\xi\beta\}\eta(X)-(n-2)(X\beta)-\big((\phi X)\alpha\big)\big]\times\\
\{A(\xi)+D(\xi)\}+C(X)(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)+\beta B(\phi X)\\
-(\alpha^2-\beta^2-\alpha-\xi\beta)\{B(X)-\eta(X)B(\xi)\}\\
=\big[(n-1)\{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\xi\alpha\}-\xi(\xi\beta)\big]\eta(X)\\
-(n-2)\big(X(\xi\beta)\big)-\big((\phi X)(\xi\alpha)\big).
\end{aligned}\tag{5.3.14}$$

Adding the equations (5.3.12) and (5.3.14) and making use of equation (5.3.7) we obtain

$$\begin{aligned}
&(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)\big[A(X)+C(X)+D(X)\big]\\
&=2\big[(n-1)\{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\xi\alpha\}-\xi(\xi\beta)\big]\eta(X)\\
&\quad+(n-1)\{2\alpha(X\alpha)-2\beta(X\beta)-X\alpha-X(\xi\beta)\}\\
&\quad+2(\alpha-1)\{(X\alpha)-\eta(X)(\xi\alpha)-(n-2)(\phi X)\beta\}\\
&\quad+2\beta\big[\big((\phi X)\alpha+(n-2)\{(X\beta)-\eta(X)(\xi\beta)\}\big]\\
&\quad-2(n-2)\big(X(\xi\beta)\big)-2\big((\phi X)(\xi\alpha)\big)\\
&\quad-R\times\frac{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\xi\alpha-\xi(\xi\beta)}{\alpha^2-\beta^2-\alpha-\xi\beta},
\end{aligned}\tag{5.3.15}$$

where $R=\big[\{(n-1)(\alpha^2-\beta^2-\alpha)-\xi\beta\}\eta(X)-(n-2)(X\beta)-\big((\phi X)\alpha\big)\big]$, provided $\alpha^2-\beta^2-\alpha-\xi\beta\neq 0$. Since $n>2$, the above equation yields

$$\begin{aligned}
&\big[A(X)+C(X)+D(X)\big]\\
&=\frac{2\big[(n-1)\{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\xi\alpha\}-\xi(\xi\beta)\big]\eta(X)}{(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)}\\
&\quad+\frac{\{2\alpha(X\alpha)-2\beta(X\beta)-X\alpha-X(\xi\beta)\}}{(\alpha^2-\beta^2-\alpha-\xi\beta)}\\
&\quad+\frac{2(\alpha-1)\{(X\alpha)-\eta(X)(\xi\alpha)-(n-2)(\phi X)\beta\}}{(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)}\\
&\quad+\frac{2\beta\big[\big((\phi X)\alpha+(n-2)\{(X\beta)-\eta(X)(\xi\beta)\}\big]}{(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)}\\
&\quad-\frac{2(n-2)\big(X(\xi\beta)\big)-2\big((\phi X)(\xi\alpha)\big)}{(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)}\\
&\quad-R\times\frac{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\xi\alpha-\xi(\xi\beta)}{(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)^2},
\end{aligned}\tag{5.3.16}$$

where $R=\big[\{(n-1)(\alpha^2-\beta^2-\alpha)-\xi\beta\}\eta(X)-(n-2)(X\beta)-\big((\phi X)\alpha\big)\big]$, provided $\alpha^2-\beta^2-\alpha-\xi\beta\neq 0$.

5.4 Weakly Ricci-symmetric trans-Sasakian manifolds

Let M^n be a weakly Ricci symmetric trans-Sasakian manifolds admitting $\tilde{\nabla}$. Then from equation (1.21.2) we have

$$(\tilde{\nabla}_X \tilde{S})(U,V) = B_1(X)\tilde{S}(U,V) + B_2(U)\tilde{S}(X,V) + B_3(V)\tilde{S}(U,X). \tag{5.4.1}$$

Theorem 5.4.1 *In the given manifold M^n the 1-forms M^n B_1, B_2, B_3 are given by*

$$\begin{aligned} B_1(X) = {} & \frac{(n-1)\{2\alpha(X\alpha) - 2\beta(X\beta) - X\alpha - X(\xi\beta)\}}{S} \\ & + \frac{2(n-2)\beta\{(X\beta) - \eta(X)(\xi\beta)\}}{S} - \frac{2(n-2)(\alpha-1)(\phi X)\beta}{S} \\ & + \frac{2\beta\big((\phi X)\alpha\big)}{S} + \frac{2(\alpha-1)\{(X\alpha) - \eta(X)(\xi\alpha)\}}{S} \\ & - \big[\{(n-1)(\alpha^2-\beta^2) - (\xi\beta)\}\eta(X) - (n-2)(X\beta) - \big((\phi X)\alpha\big)\big] \times \Big[\frac{dr(\xi)}{S \times T} \\ & + \frac{(n-2)\big\{(\sigma_2\beta) + (\sigma_3\beta)\big\} + \big((\phi\sigma_2)\alpha\big) + \big((\phi\sigma_3)\alpha\big)}{S \times T} \\ & - \frac{r(n-1)}{S^2 \times T}\big\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \xi\alpha - \xi(\xi\beta)\big\}\Big], \end{aligned}$$

$$\begin{aligned} B_2(X) = {} & \frac{\big[(n-1)\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \xi\alpha\} - \xi(\xi\beta)\big]\eta(X)}{S} \\ & - \frac{(n-2)\big(X(\xi\beta)\big)}{S} - \frac{\big((\phi X)(\xi\alpha)\big)}{S} \\ & - \big[\{(n-1)(\alpha^2-\beta^2) - (\xi\beta)\}\eta(X) - (n-2)(X\beta) - \big((\phi X)\alpha\big)\big] \\ & \times \Big[\frac{dr(\xi)}{2 \times S \times T} + \frac{(n-2)\big\{(\sigma_1\beta) + (\sigma_3\beta)\big\} + \big((\phi\sigma_1)\alpha\big) + \big((\phi\sigma_3)\alpha\big)}{S \times T} \\ & - \frac{r(n-1)}{S^2 \times T}\big\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \xi\alpha - \xi(\xi\beta)\big\}\Big] \end{aligned}$$

and

$$B_3(X) = \frac{\big[(n-1)\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \xi\alpha\} - \xi(\xi\beta)\big]\eta(X)}{S}$$
$$- \frac{(n-2)\big(X(\xi\beta)\big)}{S} - \frac{\big((\phi X)(\xi\alpha)\big)}{S}$$
$$-\big[\{(n-1)(\alpha^2-\beta^2) - (\xi\beta)\}\eta(X) - (n-2)(X\beta) - \big((\phi X)\alpha\big)\big]$$
$$\times\Big[\frac{dr(\xi)}{2\times S\times T} + \frac{(n-2)\big\{(\sigma_1\beta) + (\sigma_2\beta)\big\} + \big((\phi\sigma_1)\alpha\big) + \big((\phi\sigma_2)\alpha\big)}{S\times T}$$
$$-\frac{r(n-1)}{S^2\times T}\big\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \xi\alpha - \xi(\xi\beta)\big\}\Big]$$

respectively.

Proof: Setting $V = \xi$ in the expression (5.4.1) we get

$$(\tilde{\nabla}_X\tilde{S})(U,\xi) = B_1(X)\tilde{S}(U,\xi) + B_2(U)\tilde{S}(X,\xi) + B_3(\xi)\tilde{S}(U,X). \tag{5.4.2}$$

Using equation (5.2.7) in equation (5.4.2) and combining with equation (5.2.4) we get

$$\begin{aligned}
&\tilde{\nabla}_X\tilde{S}(U,\xi) - \tilde{S}(\tilde{\nabla}_X U,\xi) - \tilde{S}(U,\tilde{\nabla}_X\xi)\\
&= B_1(X)\big[\{(n-1)(\alpha^2-\beta^2-\alpha) - \xi\beta\}\eta(U) - (n-2)(U\beta) - \big((\phi U)\alpha\big)\big]\\
&+B_2(U)\big[\{(n-1)(\alpha^2-\beta^2-\alpha) - \xi\beta\}\eta(X) - (n-2)(X\beta) - \big((\phi X)\alpha\big)\big]\\
&+B_3(\xi)\tilde{S}(U,X).
\end{aligned} \tag{5.4.3}$$

Setting $X = U = \xi$ in equation (5.4.3) yields

$$\begin{aligned}
&(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)\big[(\xi)B_1(\xi) + B_2(\xi) + B_3(\xi)\big]\\
&= (n-1)\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \xi\alpha - \xi(\xi\beta)\}.
\end{aligned} \tag{5.4.4}$$

Since $n > 2$, the above equation can be written as

$$B_1(\xi) + B_2(\xi) + B_3(\xi) = \frac{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \xi\alpha - \xi(\xi\beta)}{\alpha^2-\beta^2-\alpha-\xi\beta}, \tag{5.4.5}$$

provided $\alpha^2 - \beta^2 - \alpha - \xi\beta \neq 0$.

From equation (5.4.3) we have

$$\begin{aligned}
&(\tilde{\nabla}_X\tilde{S})(U,\xi) = B_3(\xi)\tilde{S}(U,X)\\
&+B_1(X)\big[\{(n-1)(\alpha^2-\beta^2-\alpha) - \xi\beta\}\eta(U) - (n-2)(U\beta) - \big((\phi U)\alpha\big)\big]\\
&+B_2(U)\big[\{(n-1)(\alpha^2-\beta^2-\alpha) - \xi\beta\}\eta(X) - (n-2)(X\beta) - \big((\phi X)\alpha\big).
\end{aligned} \tag{5.4.6}$$

Contracting equation (5.4.6) over X and U and using equation (5.2.10) we get

$$\begin{aligned}\frac{1}{2}dr(\xi) &= \big[B_1(\xi)+B_2(\xi)\big]\{(n-1)(\alpha^2-\beta^2-\alpha)-\xi\beta\}\\ -(n-2)(\sigma_1\beta)-\big((\phi\sigma_1)\alpha\big)&-(n-2)(\sigma_2\beta)-\big((\phi\sigma_2)\alpha\big)+rB_3(\xi),\end{aligned} \tag{5.4.7}$$

wher σ_1,σ_2 and σ_3 are the vector fields associated with 1-forms B_1, B_2 and B_3 respectively.

Using equation (5.4.5) in equation (5.4.7) we obtain

$$\begin{aligned}\frac{1}{2}dr(\xi) &= \big[B_1(\xi)+B_2(\xi)\big]\{(n-1)(\alpha^2-\beta^2-\alpha)-\xi\beta\}\\ &-(n-2)(\sigma_1\beta)-\big((\phi\sigma_1)\alpha\big)-(n-2)(\sigma_2\beta)-\big((\phi\sigma_2)\alpha\big)\\ &+r\Big[\frac{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\xi\alpha-\xi(\xi\beta)}{\alpha^2-\beta^2-\alpha-\xi\beta}-B_1(\xi)-B_2(\xi)\Big],\end{aligned}$$

or

$$\begin{aligned}\big[B_1(\xi)+B_2(\xi)\big]\{(n-1)(\alpha^2-\beta^2-\alpha)-\xi\beta-r\} &= \frac{1}{2}dr(\xi)\\ +(n-2)\big\{(\sigma_1\beta)+(\sigma_2\beta)\big\}&+\big((\phi\sigma_1)\alpha\big)+\big((\phi\sigma_2)\alpha\big)\\ -r\Big[&\frac{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\xi\alpha-\xi(\xi\beta)}{\alpha^2-\beta^2-\alpha-\xi\beta}\Big],\end{aligned} \tag{5.4.8}$$

provided $\alpha^2-\beta^2-\alpha-\xi\beta\neq 0$.

Similarly we obtain

$$\begin{aligned}\big[B_1(\xi)+B_3(\xi)\big]\{(n-1)(\alpha^2-\beta^2-\alpha)-\xi\beta-r\} &= \frac{1}{2}dr(\xi)\\ +(n-2)\big\{(\sigma_1\beta)+(\sigma_3\beta)\big\}&+\big((\phi\sigma_1)\alpha\big)+\big((\phi\sigma_3)\alpha\big)\\ -r\Big[&\frac{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\xi\alpha-\xi(\xi\beta)}{\alpha^2-\beta^2-\alpha-\xi\beta}\Big],\end{aligned} \tag{5.4.9}$$

and

$$\begin{aligned}\big[B_2(\xi)+B_3(\xi)\big]\{(n-1)(\alpha^2-\beta^2-\alpha)-\xi\beta-r\} &= dr(\xi)\\ +(n-2)\big\{(\sigma_2\beta)+(\sigma_3\beta)\big\}&+\big((\phi\sigma_2)\alpha\big)+\big((\phi\sigma_3)\alpha\big)\\ -r\Big[&\frac{2\alpha(\xi\alpha)-2\beta(\xi\beta)-\xi\alpha-\xi(\xi\beta)}{\alpha^2-\beta^2-\alpha-\xi\beta}\Big],\end{aligned} \tag{5.4.10}$$

provided $\alpha^2-\beta^2-\alpha-\xi\beta\neq 0$.

Setting $U = V = \xi$ in equation (5.4.1) and using equation (5.4.5) we get

$$\begin{aligned} B_1(X)\tilde{S}(\xi,\xi) + \big[B_2(\xi) + B_3(\xi)\big]\tilde{S}(X,\xi) \\ = (n-1)\{2\alpha(X\alpha) - 2\beta(X\beta) - X\alpha - X(\xi\beta)\} \\ +2(n-2)\beta\{(X\beta) - \eta(X)(\xi\beta)\} - 2(n-2)(\alpha-1)(\phi X)\beta \\ +2\beta\big((\phi X)\alpha\big) + 2(\alpha-1)\{(X\alpha) - \eta(X)(\xi\alpha)\}. \end{aligned} \tag{5.4.11}$$

Using equations (5.1.4) and (5.4.10) in the equation (5.4.11) we get

$$\begin{aligned} B_1(X) = \frac{(n-1)\{2\alpha(X\alpha) - 2\beta(X\beta) - X\alpha - X(\xi\beta)\}}{S} \\ +\frac{2(n-2)\beta\{(X\beta) - \eta(X)(\xi\beta)\}}{S} - \frac{2(n-2)(\alpha-1)(\phi X)\beta}{S} \\ +\frac{2\beta\big((\phi X)\alpha\big)}{S} + \frac{2(\alpha-1)\{(X\alpha) - \eta(X)(\xi\alpha)\}}{S} \\ -\big[\{(n-1)(\alpha^2-\beta^2) - (\xi\beta)\}\eta(X) - (n-2)(X\beta) - \big((\phi X)\alpha\big)\big] \times \Big[\frac{dr(\xi)}{S\times T} \\ +\frac{(n-2)\big\{(\sigma_2\beta) + (\sigma_3\beta)\big\} + \big((\phi\sigma_2)\alpha\big) + \big((\phi\sigma_3)\alpha\big)}{S\times T} \\ -\frac{r(n-1)}{S^2\times T}\big\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \xi\alpha - \xi(\xi\beta)\big\}\Big], \end{aligned} \tag{5.4.12}$$

where $S = (n-1)(\alpha^2 - \beta^2 - \alpha - \xi\beta)$, $T = (n-1)(\alpha^2 - \beta^2 - \alpha) - \xi\beta - r$ provided $S, T \neq 0$.

By following the same step we calculate

$$\begin{aligned} B_2(X) = \frac{\big[(n-1)\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \xi\alpha\} - \xi(\xi\beta)\big]\eta(X)}{S} \\ -\frac{(n-2)\big(X(\xi\beta)\big)}{S} - \frac{\big((\phi X)(\xi\alpha)\big)}{S} \\ -\big[\{(n-1)(\alpha^2-\beta^2) - (\xi\beta)\}\eta(X) - (n-2)(X\beta) - \big((\phi X)\alpha\big)\big] \\ \times\Big[\frac{dr(\xi)}{2\times S\times T} + \frac{(n-2)\big\{(\sigma_1\beta) + (\sigma_3\beta)\big\} + \big((\phi\sigma_1)\alpha\big) + \big((\phi\sigma_3)\alpha\big)}{S\times T} \\ -\frac{r(n-1)}{S^2\times T}\big\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \xi\alpha - \xi(\xi\beta)\big\}\Big], \end{aligned} \tag{5.4.13}$$

and

$$B_3(X) = \frac{\big[(n-1)\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \xi\alpha\} - \xi(\xi\beta)\big]\eta(X)}{S}$$
$$- \frac{(n-2)\big(X(\xi\beta)\big)}{S} - \frac{\big((\phi X)(\xi\alpha)\big)}{S}$$
$$-\big[\{(n-1)(\alpha^2-\beta^2) - (\xi\beta)\}\eta(X) - (n-2)(X\beta) - \big((\phi X)\alpha\big)\big]$$
$$\times\Big[\frac{dr(\xi)}{2\times S\times T} + \frac{(n-2)\big\{(\sigma_1\beta)+(\sigma_2\beta)\big\} + \big((\phi\sigma_1)\alpha\big) + \big((\phi\sigma_2)\alpha\big)}{S\times T}$$
$$- \frac{r(n-1)}{S^2\times T}\big\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \xi\alpha - \xi(\xi\beta)\big\}\Big], \qquad (5.4.14)$$

where $S = (n-1)(\alpha^2 - \beta^2 - \alpha - \xi\beta)$, $T = (n-1)(\alpha^2-\beta^2-\alpha) - \xi\beta - r$ provided $S, T \neq 0$.

Adding equation (5.4.12), equation (5.4.13) and equation (5.4.14) we get

$$B_1(X) + B_2(X) + B_3(X) = \frac{(n-1)\{2\alpha(X\alpha) - 2\beta(X\beta) - X\alpha - X(\xi\beta)\}}{S}$$
$$+ \frac{2(n-2)\beta\{(X\beta) - \eta(X)(\xi\beta)\}}{S} - \frac{2(n-2)(\alpha-1)(\phi X)\beta}{S}$$
$$+ \frac{2\beta\big((\phi X)\alpha\big)}{S} + \frac{2(\alpha-1)\{(X\alpha) - \eta(X)(\xi\alpha)\}}{S}$$
$$\frac{2\big[(n-1)\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \xi\alpha\} - \xi(\xi\beta)\big]\eta(X)}{S}$$
$$- \frac{2(n-2)\big(X(\xi\beta)\big)}{S} - \frac{2\big((\phi X)(\xi\alpha)\big)}{S}$$
$$-\big[\{(n-1)(\alpha^2-\beta^2) - (\xi\beta)\}\eta(X) - (n-2)(X\beta) - \big((\phi X)\alpha\big)\big]$$
$$\times\Big[\frac{2dr(\xi)}{S\times T} + \frac{2(n-2)\big\{(\sigma_1\beta)+(\sigma_2\beta)+(\sigma_3\beta)\big\}}{S\times T}$$
$$+ \frac{2\big\{\big((\phi\sigma_1)\alpha\big) + \big((\phi\sigma_2)\alpha\big) + \big((\phi\sigma_3)\alpha\big)\big\}}{S\times T}$$
$$- \frac{3(n-1)r}{S^2\times T}\big\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \xi\alpha - \xi(\xi\beta)\big\}\Big]. \qquad (5.4.15)$$

Corollary 5.4.1 *In the given Symmetric manifold M^n, the sum of the 1-forms B_1, B_2, B_3 is given by (5.1.5).*

5.5 Locally symmetric trans-Sasakian manifolds

Definition 5.5.1 *(Cartan, 1926) A Riemannian manifold is said to be (locally) symmetric if the Riemannian curvature tensor R satisfies $\nabla R = 0$.*

Theorem 5.5.1 *If β is a non-zero constant in a locally symmetric trans Sasakian manifold with respect to $\tilde{\nabla}$, then it is locally symmetric with respect to ∇ provided $\alpha \neq 0, \frac{1}{2}$.*

Proof: Consider a trans-Sasakian manifold which is symmetric with respect to $\tilde{\nabla}$. Then we have

$$(\tilde{\nabla}_X \tilde{R})(U,V)Z = 0 \tag{5.5.1}$$

for all $X, U, V, Z \in \chi(M)$.

By properties of $\tilde{\nabla}$ we have

$$\begin{aligned}(\tilde{\nabla}_X \tilde{R})(U,V)Z &= \tilde{\nabla}_X \tilde{R}(U,V)Z - \tilde{R}(\tilde{\nabla}_X U, V)Z \\ &\quad -\tilde{R}(U, \tilde{\nabla}_X V)Z - \tilde{R}(U,V)\tilde{\nabla}_X Z.\end{aligned} \tag{5.5.2}$$

Using equation (5.2.2) in equation (5.5.2) we obtain

$$\begin{aligned}(\tilde{\nabla}_X \tilde{R})(U,V)Z &= (\tilde{\nabla}_X R)(U,V)Z + (X\alpha)\big[g(\phi V, Z)\phi U - g(\phi U, Z)\phi V \\ &\quad +\eta(U)\eta(Z)V - \eta(V)\eta(Z)U\big] + (X\beta)\big[g(U,Z)\phi V - g(V,Z)\phi U \\ &\quad +2\eta(Z)g(\phi U, V)\xi\big] + \alpha\Big[\big\{(\tilde{\nabla}_X g)(\phi V, Z) + g\big((\tilde{\nabla}_X \phi)V, Z\big)\big\}\phi U \\ &\quad +g(\phi V, Z)(\tilde{\nabla}_X \phi)U - \big\{(\tilde{\nabla}_X g)(\phi U, Z) - g\big((\tilde{\nabla}_X \phi)U, Z\big)\big\}\phi V \\ &\quad -g(\phi U, Z)(\tilde{\nabla}_X \phi)V + (\tilde{\nabla}_X \eta)(Z)\big\{\eta(U)V - \eta(V)U\big\} \\ &\quad +\eta(Z)\big\{(\tilde{\nabla}_X \eta)(U)V - (\tilde{\nabla}_X \eta)(V)U\big\}\Big] + \beta\Big[(\tilde{\nabla}_X g)(U,Z)\phi V - (\tilde{\nabla}_X g)(V,Z)\phi U \\ &\quad +g(U,Z)(\tilde{\nabla}_X \phi)V - g(V,Z)(\tilde{\nabla}_X \phi)U + 2(\tilde{\nabla}_X \eta)(Z)g(\phi U, V)\xi \\ &\quad +2\eta(Z)\{(\tilde{\nabla}_X g)(\phi U, V) + g\big((\tilde{\nabla}_X \phi)U, V\big)\}\xi\Big].\end{aligned} \tag{5.5.3}$$

Again using equation (5.2.1) in equation (5.5.3) we obtain

$$
\begin{aligned}
(\tilde{\nabla}_X\tilde{R})(U,V)Z &= (\nabla_X R)(U,V)Z + (X\alpha)\big[g(\phi V,Z)\phi U - g(\phi U,Z)\phi V \\
&+\eta(U)\eta(Z)V - \eta(V)\eta(Z)U\big] + (X\beta)\big[g(U,Z)\phi V - g(V,Z)\phi U \\
&+2\eta(Z)g(\phi U,V)\xi\big] + \alpha\Big[\big\{(\tilde{\nabla}_X g)(\phi V,Z) + g\big((\tilde{\nabla}_X\phi)V,Z\big)\big\}\phi U \\
&+g(\phi V,Z)(\tilde{\nabla}_X\phi)U - \big\{(\tilde{\nabla}_X g)(\phi U,Z) - g\big((\tilde{\nabla}_X\phi)U,Z\big)\big\}\phi V \\
&-g(\phi U,Z)(\tilde{\nabla}_X\phi)V + (\tilde{\nabla}_X\eta)(Z)\big\{\eta(U)V - \eta(V)U\big\} \\
+\eta(Z)\big\{(\tilde{\nabla}_X\eta)(U)V &- (\tilde{\nabla}_X\eta)(V)U\big\}\Big] + \beta\Big[(\tilde{\nabla}_X g)(U,Z)\phi V - (\tilde{\nabla}_X g)(V,Z)\phi U \\
&+g(U,Z)(\tilde{\nabla}_X\phi)V - g(V,Z)(\tilde{\nabla}_X\phi)U + 2(\tilde{\nabla}_X\eta)(Z)g(\phi U,V)\xi \\
&+2\eta(Z)\{(\tilde{\nabla}_X g)(\phi U,V) + g\big((\tilde{\nabla}_X\phi)U,V\big)\}\xi\Big] + \eta\big(R(U,V)Z\big)\phi X \\
&-\eta(U)R(\phi X,V)Z - \eta(V)R(U,\phi X)Z - \eta(Z)R(U,V)\phi X. \quad (5.5.4)
\end{aligned}
$$

Combining equation (5.5.1) and equation (5.5.4)

$$
\begin{aligned}
&(\nabla_X R)(U,V)Z + (X\alpha)\big[g(\phi V,Z)\phi U - g(\phi U,Z)\phi V \\
&+\eta(U)\eta(Z)V - \eta(V)\eta(Z)U\big] + (X\beta)\big[g(U,Z)\phi V - g(V,Z)\phi U \\
&+2\eta(Z)g(\phi U,V)\xi\big] + \alpha\Big[\big\{(\tilde{\nabla}_X g)(\phi V,Z) + g\big((\tilde{\nabla}_X\phi)V,Z\big)\big\}\phi U \\
&+g(\phi V,Z)(\tilde{\nabla}_X\phi)U - \big\{(\tilde{\nabla}_X g)(\phi U,Z) - g\big((\tilde{\nabla}_X\phi)U,Z\big)\big\}\phi V \\
&-g(\phi U,Z)(\tilde{\nabla}_X\phi)V + (\tilde{\nabla}_X\eta)(Z)\big\{\eta(U)V - \eta(V)U\big\} \\
&+\eta(Z)\big\{(\tilde{\nabla}_X\eta)(U)V - (\tilde{\nabla}_X\eta)(V)U\big\}\Big] + \beta\Big[(\tilde{\nabla}_X g)(U,Z)\phi V - (\tilde{\nabla}_X g)(V,Z)\phi U \\
&+g(U,Z)(\tilde{\nabla}_X\phi)V - g(V,Z)(\tilde{\nabla}_X\phi)U + 2(\tilde{\nabla}_X\eta)(Z)g(\phi U,V)\xi \\
&+2\eta(Z)\{(\tilde{\nabla}_X g)(\phi U,V) + g\big((\tilde{\nabla}_X\phi)U,V\big)\}\xi\Big] + \eta\big(R(U,V)Z\big)\phi X \\
&-\eta(U)R(\phi X,V)Z - \eta(V)R(U,\phi X)Z - \eta(Z)R(U,V)\phi X = 0. \quad (5.5.5)
\end{aligned}
$$

Setting $X = U = Z = \xi$ in equation (5.5.5) and using equations (5.1.3), (5.2.1), (5.2.5) and (5.2.9), we obtain the following equation

$$\big(2\alpha(\xi\alpha) - 2\beta(\xi\beta) - \xi\alpha - \xi(\xi\beta)\big)\big(\eta(V)\xi - V\big) + (\xi\beta)\phi V = 0. \quad (5.5.6)$$

If β is a non-zero constant, then equation (5.5.6) become

$$(2\alpha - 1)(\xi\alpha)\big(\eta(V)\xi - V\big) = 0,$$

or

$$2\alpha\beta(1 - 2\alpha)\big(\eta(V)\xi - V\big) = 0.$$

From the above equation we have

$$\eta(V)\xi - V = 0 \tag{5.5.7}$$

for all $V \in \chi(M)$, provided $\alpha \neq 0, \frac{1}{2}$.

Now from equation (5.5.7) we obtain

$$\phi V = 0 \tag{5.5.8}$$

for all $V \in \chi(M)$, provided $\alpha \neq 0, \frac{1}{2}$.

Using equations (5.5.7) and (5.5.8) in equation (5.5.4) we get

$$(\tilde{\nabla}_X \tilde{R})(U, V)Z = (\nabla_X R)(U, V)Z. \tag{5.5.9}$$

Thus

$$(\tilde{\nabla}_X \tilde{R})(U, V)Z = 0 \Rightarrow (\nabla_X R)(U, V)Z = 0.$$

Theorem 5.5.2 *If a trans Sasakian manifold is locally symmetric with respect to $\tilde{\nabla}$, then $\tilde{\nabla} = \nabla$ if and only if β is a non-zero constant, provided $\alpha \neq 0, \frac{1}{2}$.*

Proof: Again using equation (5.5.8) in equation (5.2.1) we get

$$\tilde{\nabla}_X U = \nabla_X U, \tag{5.5.10}$$

provided $\alpha \neq 0, \frac{1}{2}$.

5.6 Ricci semi-symmetric trans-Sasakian manifolds

Theorem 5.6.1 *In a Ricci semi-symmetric trans Sasakian manifold with respect to $\tilde{\nabla}$, $\beta = 0$ if and only if α is constant, provided $\alpha \neq 0, 1$.*

Proof: Let M^n be a Ricci semi-symmetric trans-Sasakian manifold with respect to $\tilde{\nabla}$, we have

$$\tilde{R}(U, V).\tilde{S}(Z, W) = 0 \tag{5.6.1}$$

for all $U, V, Z, W \in \chi(M)$.

From equation (5.6.2) we have

$$\tilde{S}(\tilde{R}(U, V)Z, W) + \tilde{S}(Z, \tilde{R}(U, V)W) = 0. \tag{5.6.2}$$

Using equation (5.2.2) in equation (5.6.2 we get

$$\begin{aligned}\tilde{S}(R(U,V)Z,W)+\tilde{S}(Z,R(U,V)W)+\alpha\{g(\phi V,Z)S(\phi U,W)\\ -g(\phi U,Z)S(\phi V,W)+\eta(U)\eta(Z)S(V,W)-\eta(V)\eta(Z)S(U,W)\}\\ +\beta\{g(U,Z)S(\phi V,W)-g(V,Z)S(\phi U,W)+2\eta(Z)g(\phi U,V)S(\xi,W)\}\\ +\alpha\{g(\phi V,W)S(Z,\phi U)-g(\phi U,W)S(Z,\phi V)+\eta(U)\eta(W)S(Z,V)\\ -\eta(V)\eta(W)S(Z,U)\}+\beta\{g(U,W)S(Z,\phi V)-g(V,W)S(Z,\phi U)\\ +2\eta(W)g(\phi U,V)S(Z,\xi)\}=0.\end{aligned} \tag{5.6.3}$$

Setting $U=Z=W=\xi$ in equation (5.6.3) yields

$$\begin{aligned}(\alpha^2-\beta^2-\alpha-\xi\beta)\big[(n-2)\big\{(V\beta)-(\xi\beta)\eta(V)\big\}+\big((\phi V)\alpha\big)\big]\\ -\beta\big[(n-2)\big((\phi V)\beta\big)-(V\alpha)+\eta(V)(\xi\beta)\}=0.\end{aligned} \tag{5.6.4}$$

For $\beta=0$ we have

$$(\alpha^2-\alpha)\big((\phi V)\alpha\big)=0,$$

or

$$\big((\phi V)\alpha\big)=0 \tag{5.6.5}$$

for all $V\in M$, provided $\alpha\neq 0,1$. That is if $\beta=0$ then α is constant.

Conversely we suppose that α is a non zero constant. From equation (5.1.3) we have

$$\beta=0. \tag{5.6.6}$$

We know that a tran Sasakian manifold of type $(\alpha,0)$ is α-Sasakian manifold. Thus we have

Corollary 5.6.1 *A Ricci semi-symmetric trans Sasakian manifold with respect to $\tilde{\nabla}$ is α-Sasakian manifold if and only if α is constant.*

5.7 Generalized recurrent trans-Sasakian manifolds

Theorem 5.7.1 *In a $G\{(K_n)TS\}$ the relation between the 1-forms A and B is given by*

$$\begin{aligned} A(X)(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)+B(X)(n-1) \\ =(n-1)\{2\alpha(X\alpha)-2\beta(X\beta)-X\alpha-X(\xi\beta)\} \\ +2(n-2)\beta\{(X\beta)-\eta(X)(\xi\beta)\}-2(n-2)(\alpha-1)(\phi X)\beta \\ +2\beta\big((\phi X)\alpha\big)+2(\alpha-1)\{(X\alpha)-\eta(X)(\xi\alpha)\}. \end{aligned}$$

Proof: We consider a generalized recurrent trans Sasakian manifold with respect to $\tilde{\nabla}$ and we denote the manifold by $G\{(K_n)TS\}$

$$(\tilde{\nabla}_X\tilde{R})(U,V)Z=A(X)\tilde{R}(U,V)Z+B(X)\big[g(V,Z)U-g(U,Z)V\big]. \tag{5.7.1}$$

Contracting equation (5.7.1) we get

$$(\tilde{\nabla}_X\tilde{S})(V,Z)=A(X)\tilde{S}(V,Z)+B(X)(n-1)g(V,Z). \tag{5.7.2}$$

It is well known that

$$(\tilde{\nabla}_X\tilde{S})(V,Z)=\tilde{\nabla}_X\tilde{S}(V,Z)-\tilde{S}(\tilde{\nabla}_XV,Z)-\tilde{S}(V,\tilde{\nabla}_XZ). \tag{5.7.3}$$

Setting $V=Z=\xi$ in the above equation and using equations (5.2.7), (5.2.8), (5.2.9) we get

$$\begin{aligned} (\tilde{\nabla}_X\tilde{S})(\xi,\xi)=(n-1)\{2\alpha(X\alpha)-2\beta(X\beta)-X\alpha-X(\xi\beta)\} \\ +2(n-2)\beta\{(X\beta)-\eta(X)(\xi\beta)\}-2(n-2)(\alpha-1)(\phi X)\beta \\ +2\beta\big((\phi X)\alpha\big)+2(\alpha-1)\{(X\alpha)-\eta(X)(\xi\alpha)\}. \end{aligned} \tag{5.7.4}$$

Again setting $V=Z=\xi$ in equation (5.7.2) and using equations (5.7.4) and (5.2.8) we get

$$\begin{aligned} A(X)(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)+B(X)(n-1) \\ =(n-1)\{2\alpha(X\alpha)-2\beta(X\beta)-X\alpha-X(\xi\beta)\} \\ +2(n-2)\beta\{(X\beta)-\eta(X)(\xi\beta)\}-2(n-2)(\alpha-1)(\phi X)\beta \\ +2\beta\big((\phi X)\alpha\big)+2(\alpha-1)\{(X\alpha)-\eta(X)(\xi\alpha)\}. \end{aligned} \tag{5.7.5}$$

Theorem 5.7.2 *In a $G\{(K_n)TS\}$ the expression for A and B is given by equations*

$$\begin{aligned} A(X)\big[(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)-\frac{r}{n}\big]+\frac{dr(X)}{n} \\ =(n-1)\{2\alpha(X\alpha)-2\beta(X\beta)-X\alpha-X(\xi\beta)\} \\ +2(n-2)\beta\{(X\beta)-\eta(X)(\xi\beta)\}-2(n-2)(\alpha-1)(\phi X)\beta \\ +2\beta\big((\phi X)\alpha\big)+2(\alpha-1)\{(X\alpha)-\eta(X)(\xi\alpha)\} \end{aligned}$$

and

$$\begin{aligned} B(X)\big[(n-1)-\frac{n(n-1)^2(\alpha^2-\beta^2-\alpha-\xi\beta)}{r}\big] \\ =(n-1)\{2\alpha(X\alpha)-2\beta(X\beta)-X\alpha-X(\xi\beta)\} \\ +2(n-2)\beta\{(X\beta)-\eta(X)(\xi\beta)\}-2(n-2)(\alpha-1)(\phi X)\beta \\ +2\beta\big((\phi X)\alpha\big)+2(\alpha-1)\{(X\alpha)-\eta(X)(\xi\alpha)\}\beta \\ -\frac{dr(X)}{r}(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta) \end{aligned}$$

respectively.

Proof: Now contracting equation (5.7.2) over V and Z we get

$$dr(X)=rA(X)+n(n-1)B(X). \tag{5.7.6}$$

Using equation (5.7.5) from equation (5.7.6) to eliminate B we have

$$\begin{aligned} A(X)\big[(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta)-\frac{r}{n}\big]+\frac{dr(X)}{n} \\ =(n-1)\{2\alpha(X\alpha)-2\beta(X\beta)-X\alpha-X(\xi\beta)\} \\ +2(n-2)\beta\{(X\beta)-\eta(X)(\xi\beta)\}-2(n-2)(\alpha-1)(\phi X)\beta \\ +2\beta\big((\phi X)\alpha\big)+2(\alpha-1)\{(X\alpha)-\eta(X)(\xi\alpha)\}. \end{aligned} \tag{5.7.7}$$

Using equation (5.7.6) in equation (5.7.7) we get

$$\begin{aligned} B(X)\big[(n-1)-\frac{n(n-1)^2(\alpha^2-\beta^2-\alpha-\xi\beta)}{r}\big] \\ =(n-1)\{2\alpha(X\alpha)-2\beta(X\beta)-X\alpha-X(\xi\beta)\} \\ +2(n-2)\beta\{(X\beta)-\eta(X)(\xi\beta)\}-2(n-2)(\alpha-1)(\phi X)\beta \\ +2\beta\big((\phi X)\alpha\big)+2(\alpha-1)\{(X\alpha)-\eta(X)(\xi\alpha)\}\beta \\ -\frac{dr(X)}{r}(n-1)(\alpha^2-\beta^2-\alpha-\xi\beta). \end{aligned} \tag{5.7.8}$$

Theorem 5.7.3 *If in a $G\{(K_n)TS\}$ the scalar curvature is constant, then the 1-forms*

A satisfies

$$A(\tilde{Q}X) = \frac{r}{n}A(X).$$

Proof: Again contracting equation (5.7.2) over X and Z we get

$$\frac{1}{2}dr(V) = A(\tilde{Q}V) + (n-1)B(V), \tag{5.7.9}$$

where $\tilde{Q}$ is the Ricci operator with respect $\tilde{\nabla}$.

Replacing V by X in equation (5.7.9)

$$dr(X) = 2A(\tilde{Q}X) + 2(n-1)B(X). \tag{5.7.10}$$

Again from equations (5.7.10) and (5.7.6) we obtain

$$A(\tilde{Q}X) = \frac{r}{n}A(X) + \frac{n-2}{2n}dr(X). \tag{5.7.11}$$

If the scalar curvature is constant i.e $dr(X) = 0$, equation (5.7.11) become

$$A(\tilde{Q}X) = \frac{r}{n}A(X). \tag{5.7.12}$$

5.8 Group manifolds

Theorem 5.8.1 *A trans Sasakian manifold is group manifold with respect to the quarter symmetric connection if and only if $\tilde{R}(U,V)Z = 0$, provided $\beta \neq 0$.*

Proof: Now we suppose that the curvature satisfies

$$\tilde{R}(U,V)Z = 0 \tag{5.8.1}$$

for all $U, V, Z \in \chi(M)$. From the above equation it is clear that

$$\tilde{S}(V,Z) = 0 \tag{5.8.2}$$

for all $V, Z \in \chi(M)$.

Setting $U = Z = \xi$ in equation (5.8.1) we get

$$(\alpha^2 - \beta^2 - \alpha - \xi\beta)\{\eta(V)\xi - V\} + \beta\phi V = 0. \tag{5.8.3}$$

Again setting $V = Z = \xi$ in equation (5.8.2) we get

$$(n-1)(\alpha^2 - \beta^2 - \alpha - \xi\beta) = 0. \tag{5.8.4}$$

Since $n \geq 3$, the above equation becomes

$$(\alpha^2 - \beta^2 - \alpha - \xi\beta) = 0. \tag{5.8.5}$$

Using equation (5.8.5) in equation (5.8.3) we get

$$\beta\phi V = 0 \tag{5.8.6}$$

for all $V \in \chi(M)$.

From equation (5.8.6) we have

$$\phi V = 0 \tag{5.8.7}$$

for all $V \in \chi(M)$, provided $\beta \neq 0$.

Using equation (5.2.1) we obtain the torsion tensor with respect to the quarter symmetric connection as

$$\tilde{T}(U, V) = \eta(V)\phi U - \eta(U)\phi V, \tag{5.8.8}$$

for all $V \in \chi(M)$.

Using equation (5.8.7) in equation (5.8.8) we obtain

$$(\tilde{\nabla}_X \tilde{T})(U, V) = 0 \tag{5.8.9}$$

for all $U, V \in \chi(M)$, provided $\beta \neq 0$.

By virtue of equations (5.2.1) and (5.8.7) we have

Corollary 5.8.1 *If a trans Sasakian manifold is a group manifold with respect to $\tilde{\nabla}$, then $\tilde{\nabla} = \nabla$, provided $\beta \neq 0$.*

Moreover we obtain

$$\tilde{R}(U, V)Z = R(U, V)Z \tag{5.8.10}$$

and

$$\tilde{T}(U, V) = T(U, V) \tag{5.8.11}$$

for all $U, V, Z \in \chi(M)$. Thus we have

Corollary 5.8.2 *If a trans Sasakian manifold is a group manifold with respect to $\tilde{\nabla}$, then it is a group manifold with respect to ∇, provided $\beta \neq 0$.*

Chapter 6

SUMMARY AND CONCLUSION

In the present thesis, we studied analytic properties of some almost contact manifolds. Using appropriate curvature conditions, we developed theorem for the manifolds under the given conditions.

Chapter 1 gives the basic definitions and formula of differential geometry. It is the general introduction of the studies. It Includes definitions of different curvature tensors, contact manifolds, symmetric connection, symmetries of manifold, recurrence manifolds, Ricci Solitons and Generalized Sasakian-space-form which are used in the studies. Literature reviews is also included in this chapter.

Chapter 2 is dedicated to the studies of Ricci solitons. We studied Ricci soliton in α-cosymplectic manifolds and η-Ricci soliton in Sasakian manifolds. We gives the conditions for which the Ricci soliton is steady or expanding in α-cosymplectic manifolds. We also studied Ricci semi-symmetric, pseudo projective semi-symmetric, weyl semi-symmetric, pseudo projective Ricci semi-symmetric, gradient Ricci soliton in α-cosymplectic manifolds and obtain some important geometrical properties. We have proved that Ricci solitons in Ricci semi-symmetric Sasakian manifolds can not exist. Also we studied η-Ricci soliton in torse forming vector field, m-projectively flat and Pseudo projective Ricci semi-symmetric Sasakian manifold.

In Chapter 3, we studied some properties of generalized Sasakian-space-form. We studied τ-curvature tensor in generalized Sasakian-space-form and obtained particular cases of the τ-curvature tensor. Using relation between the Riemannian curvature tensor and the curvature tensor with respect to the Generalized Tanaka-Webster connection in generalized Sasakian-space-form, we obtain necessary condition for Semi-symmetric and Ricci semi-symmetric, Ricci-generalized pseudosymmetric and Ricci pseudosymmetric manifolds.

We studied generalized recurrent manifolds in chaper 4. We studied generalized pseudo-projective recurrent manifolds and obtained the necessary and sufficient condition for the scalar curvature to be a constant. We proved that a Ricci symmetric generalized pseudo-projective recurrent manifolds is an Einstein manifold provided it is not locally symmetric. We also proved that an Einstein generalized pseudo-projective recurrent manifolds is a pseudo-projective recurrent manifolds provided it satisfy some condition. Again we proved that a conformally flat with constant scalar curvature is a pseudo-projective recurrent manifolds. Also we obtained some geometric properties for decomposable generalized pseudo-projective recurrent manifolds. Finally we given two examples to support our results.

In chapter 5, we studied quarter-symmetric non-metric connection in trans-Sasakian manifolds. We obtained the relations between the 1-forms in the weakly Ricci-symmetric, weakly Ricci-symmetric and generalized recurrent trans-Sasakian manifolds. We obtained the necessary condition for locally symmetric trans-Sasakian manifolds. Also we obtained the necessary and sufficient condition for Ricci semi-symmetric and group trans-Sasakian manifolds.

Finally, we concluded that the whole work of this thesis give some geometrical properties of manifolds, which are symmetric properties of contact manifolds, certain connection in contact manifolds, properties of certain curvature tensors in contact manifold and recurrent properties in both contact and semi-Riemannian manifold. In particular, the studies of Ricci solitons and Generalized Sasakian-space-forms have wide range of applications in the theory of relativity.

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Appendices

(A) LIST OF RESEARCH PUBLICATIONS

(1) Lalmalsawma, C. and Singh, J.P.(2018) On almost pseudo m-projectively symmetric manifolds, *Novi Sad Journal of Mathematics* 48(2), 83-97 .

(2) Singh, J.P. and Lalmalsawma, C. (2018) Ricci solitons in α-cosymplectic manifold, *Facta Universitatis* 33(3), 375-387.

(3) Singh, J.P. and Lalmalsawma, C. (2018) On generalized pseudo-projectively recurrent manifolds, *Journal of the Indian Mathematical Society* 85(3-4), 449-469.

(4) Lalmalsawma, C. and Singh, J.P.(2019) Some curvature properties of Trans-Sasakian manifolds admitting a quarter-symmetric non-metric connection, *Bulletin of the Transilvania University of Braov Series III Mathematics, Informatics, Physics* 12(61), 65-76.

(5) Lalmalsawma, C. and Singh, J.P. Symmetries of Sasakian generalized Sasakian-space-form admitting generalized Tanaka-Webster connection, *Tamkang Journal of Mathematics*. Accepted.

(B) CONFERENCES/ SEMINARS/ WORKSHOPS

(1) Attended "Second Mizoram Mathematics Congress" organized by Mizoram Mathematics Society (MMS) in collaboration with Department of Mathematics, Pachhunga University College and Department of Mathematics and Computer Science, Mizoram University, Aizawl - 796004, Mizoram, on 13^{th} and 14^{th} August, 2015.

(2) Attended "National Conference on Application of Mathematics" organized by Department of Mathematics and Computer Science, Mizoram University, Aizawl, Mizoram, on 25^{th} and 26^{th} February, 2016.

(3) Attended "Mizoram Science Congress 2016" organized by MAS and co., Mizoram University, Aizawl, Mizoram, on 13^{th} and 14^{th} October, 2016.

(4) Presented a paper "Ricci solitons in α-cosymplectic manifold" in Mizoram Science Congress 2018 organized by MISTIC and co., on 4^{th} and 5^{th} October, 2018.

(5) Presented a paper "Some curvature properties of Trans-Sasakian manifolds admitting a quarter-symmetric non-metric connection" in Multidisciplinary International Seminar On "A perspective of Global Research Process: Presented Scenario and Future Challenges" organized by Manipur University, Manipur, on 19^{th} and 20^{th} January, 2019.

(6) Attended Instructional School for Teachers "Mathematical Modelling in Continuum Mechanics and Ecology" organized by National Centre for Mathematics, Mizoram University, Aizawl, Mizoram, on $3^{rd} - 15^{th}$ June, 2019.

(7) Attended "National Workshop On Ethics in Research and Preventing Plagiarism (ERPP 2019)" organized by Department of Physics, School of Physical Sciences, Mizoram University, Aizawl, Mizoram, on 3^{rd} October, 2019.

(8) Attended Webinar on "Mathematical modeling in infectious diseases: Its relevance in time of covid" and "Binary recurrent sequences and its arithmetic" organized by Department of Mathematics and computer science, School of Physical Sciences, Mizoram University, Aizawl, Mizoram, on 11^{th} June, 2020.

(9) Presented a paper "Certain Properties of T-curvature Tensor in Generalized Sasakian-space-form" in the 2^{nd} annual convention of "North East (India) Academy of Science and Technology (NEAST) and International Seminar on Recent Advances in Science and Technology (ISRAST)" organized by NEAST, on 16^{th} to 18^{th} December, 2020.

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B.Sc Mathematics	MZU	2013	II	48.3
M.Sc Mathematics	MZU	2015	I	74.1
SLET	SLET commission, Assam	2015	-	-
NET	CSIR-UGC	2016	NET(JRF)	-

(C. LALMALSAWMA)

PARTICULARS OF THE CANDIDATE

NAME OF CANDIDATE	: C. LALMALSAWMA
DEGREE	: DOCTOR OF PHILOSOPHY
DEPARTMENT	: MATHEMATICS AND COMPUTER SCIENCE
TITLE OF THESIS	: AN ANALYTICAL STUDY OF CERTAIN ALMOST CONTACT MANIFOLDS
DATE OF ADMISSION	: 07.08.2015

APPROVAL OF RESEARCH PROPOSAL:

1. BOS : 17.10.2016
2. SCHOOL BOARD : 26.10.2016

MZU REGISTRATION NO.	: 1631 of 2009-10
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ABSTRACT

AN ANALYTICAL STUDY OF CERTAIN ALMOST CONTACT MANIFOLDS

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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DECEMBER, 2020

AN ANALYTICAL STUDY OF CERTAIN ALMOST CONTACT MANIFOLDS

By

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Submitted
In partial fulfillment of the requirement of the Degree of
Doctor of Philosophy in Mathematics of Mizoram University,
Aizawl

ABSTRACT

Contact geometry is the study of contact structure on smooth manifolds.The root of contact geometry can be traced back to the 19^{th} century as a tool to study a system of differential equation. Mostly focused on the study of almost contact manifold, the thesis is divided in to six chapters which are

1.Introduction.

2.Characterization of Ricci solitons.

3.Generalized Sasakian-space-form.

4. Generalized pseudo-projective recurrent manifolds.

5. Quarter symmetric non-metric connection in trans-Sasakian manifolds.

6. Summary and conclusion.

Chapter 1 is the General Introduction of the problems which includes basic definitions, some mathematical tools like contraction and covariant derivative, different curvature tensors, certain curvature conditionof manifolds and review of literatures.

In Chapter 2, we studied Ricci soliton which is a generalizatio of we consider α-cosymplectic and Sasakian manifolds which admit Ricci soliton and η-Ricci soliton respectively. We have shown that the Ricci soliton in α-cosymplectic cannot be shrinking. We showed how the curvature properties of the manifolds are related to the status of the Ricci solitoni.e, steady or shrinking. We have also proved that Sasakian manifolds admitting Ricci soliton cannot be Ricci semi-symmetric.

In Chapter 3, Generalized Sasakian-space-form with τ-curvature tensor and generalized Sasakian-space-form admitting Generalized Tanaka-Webster connection is studied. In the latter case we considered generalized Sasakian-space-form admitting a Sasakian structure and we called it a Sasakian generalized Sasakian-space-form. Some curvature conditions are considered and we obtained results for different curvature tensor by putting particular values of the τ-curvature tensor. We have proved that Ricci-pseudosymmetric and Ricci-generalized pseudosymmetricSasakian generalized Sasakian-space-formwith respect to generalized Tanaka-Webster connection are Einstein manifolds with conditions.

In Chapter 4, generalized recurrent manifolds with respect to the pseudo-projective curvature tensor has been studied. Manifolds with constant scalar curvature, Ricci-symmetric manifolds, Einstein manifolds, conformally flat manifolds, quasi Einstein manifolds and Decomposable manifolds with generalized pseudo-projective recurrence have been considered. Finally examples are given to support results.

In Chapter 5, we studied quarter-symmetric non-metric connection in trans-Sasakian manifold. The relation between 1-forms in weakly symmetric, weakly Ricci-symmetric and generalized recurrenttrans-Sasakian manifold admitting quarter-symmetric non-metric connection have been obtained. We have proved that with a given condition, a locally symmetric trans-Sasakian manifold the Riemannian connection and the quarter-symmetric non-metric connection are equal. We have also proved that in the given manifold, the Riemannian connection and the quarter-symmetric non-metric connection are equal for non-zero β.

In Chapter 6, we summarized the problem and results of the research work in the Conclusion.